Rumson-Fair Haven Regional High School Curriculum

Course: *Multivariable Calculus*
Staff Writers: Krishna Kanuga
Supervisor: Jon Pennetti
Approved: August 2026

Section I: Course Description

Multivariable Calculus is an advanced course that equates to a college-level *Calculus* 3 class. The course covers vectors and the geometry of space, vector-valued functions, functions of several variables, partial derivatives, and multiple integrals. Students will make extensive use of a graphing calculator.

Section II: NJSLS: New Jersey Student Learning Standards/Learning Objectives

1. ***2023 New Jersey Student Learning Standards – Mathematics:***
   - "A New Jersey education in Mathematics builds quantitatively and analytically literate citizens prepared to meet the demands of college and career, and to engage productively in an information-driven society; …A high-quality mathematics education fosters a population that…leverages data in decision-making and as a lens for discussing, analyzing, and responding to practical questions, persists to make sense of and model problems arising in everyday life, society, and the workplace, thinks critically and strategically to assess quantitative relationships and to solutions to complex problems, employs precise reasoning and constructs viable arguments to deduce conclusions, recognize false statements and assess peers' reasoning, interprets, evaluates and critiques the mathematics embedded in social, scientific and commercial systems, as well as the claims made in the private and public sectors, communicates precisely when conveying, representing, and justifying both qualitative and quantitative perspectives."
2. ***2023 New Jersey Student Learning Standards - English Language Arts:***
   - "A New Jersey education in English Language Arts builds readers, writers, and communicators prepared to meet the demands of college and career and to engage as productive American citizens with global responsibilities. …Students will develop the necessary skills in reading, writing, speaking, and listening that are the foundations for creative and purposeful expression in language read rich, challenging texts that build their knowledge of the world, grow their confidence and identities as readers, and develop critical thinking skills and vocabulary necessary for long-term success[; e]ngage in regular, meaningful, writing authentic tasks, exploring valued topics, writing for impact and expression, and sharing their work with others (including authentic audiences) leverage complex texts and digital media to develop comprehension, active listening, and discussion skills ground daily writing and discussion in evidence, fostering an ability to read critically, build arguments, cite evidence, and communicate ideas to contribute meaningfully as productive citizens evaluate the reliability, credibility, and perspective of authors and speakers across all forms of media express ideas and knowledge through a variety of modalities and media, and serve as effective communicators who purposefully read, write, and speak across multiple disciplines [and l]earn to persist in reading complex texts, establishing lifelong habits to read voluntarily for pleasure, for further education, for information on public policy, and for advancement in the workplace."
3. ***Standard 8.1 (Computer Science) and 8.2 (Design Thinking)* of the *2020 NJSLS*:**
   - "The 'Intent and Spirit of the Computer Science and Design Thinking Standards' is to focus on deep understanding of concepts that enable students to think critically and systematically about leveraging technology to solve local and global issues. Authentic learning experiences that enable students to apply content knowledge, integrate concepts across disciplines, develop computational thinking skills, acquire and incorporate varied perspectives, and communicate with diverse audiences about the use and effects of computing prepares New Jersey students for college and careers."
4. ***Standard 9.4 (Life Literacies and Key Skills)* of the *2020 NJSLS*:**
   - "This standard outlines key literacies and technical skills such as critical thinking, global and cultural awareness, and technology literacy* that are critical for students to develop to live and work in an interconnected global economy."

     Climate Change: "The state of New Jersey has mandated instruction in, "Climate Change across all content areas, leveraging the passion students have shown for this critical issue and providing them opportunities to develop a deep understanding of the science behind the changes and to explore the solutions our world desperately needs."
5. ****Amistad Law: N.J.S.A. 18A 52:16A-88:***
   - "The inclusion of lessons and resources/texts dealing with the African slave trade, slavery in America, the vestiges of slavery in this country and the contributions of African-Americans to our society will be implemented in English and Social Studies courses in accordance with state law: "Every board of education

shall incorporate the information regarding the contributions of African-Americans to our country in an appropriate place in the curriculum of elementary and secondary school students."

6. ***Holocaust Law: N.J.S.A. 18A 35-28:***
    - "The inclusion of lessons and resources/texts that enable pupils to identify and analyze applicable theories concerning human nature and behavior; to understand that genocide is a consequence of prejudice and discrimination; and to understand that issues of moral dilemma and conscience have a profound impact on life will be implemented in English and Social Studies courses in accordance with state law: "Every board of education shall include instruction on the Holocaust and genocides in an appropriate place in the curriculum of all elementary and secondary school pupils. The instruction shall further emphasize the personal responsibility that each citizen bears to fight racism and hatred whenever and wherever it happens."
7. ***LGBT and Disabilities Law: N.J.S.A. 18A:35-4.35:***
    - "A transformative approach to the inclusion of lessons and resources/texts on the contributions and issues concerning the LGBTQ+ population and people with disabilities will be implemented across all core subjects in accordance with state law: "A board of education shall include instruction on the political, economic, and social contributions of persons with disabilities and lesbian, gay, bisexual, and transgender people, in an appropriate place in the curriculum of middle school and high school students as part of the district's implementation of the New Jersey Student Learning Standards (N.J.S.A.18A:35-4.36). A board of education shall have policies and procedures in place pertaining to the selection of instructional materials to implement the requirements of N.J.S.A. 18A:35-4.35."
8. ****Asian American and Pacific Islanders Legislation: N.J.S.A 4021/A6100:***
    - "The inclusion of lessons and resources/texts on the history and contributions of Asian Americans and Pacific Islanders, will enable New Jersey's schools to provide a curriculum that reflects the diversity of our state. In accordance with state law: "A board of education shall include instruction on the history and contributions of Asian Americans and Pacific Islanders in an appropriate place in the curriculum of students in grades kindergarten through as part of the school district's implementation of the New Jersey Student Learning Standards in Social Studies."
9. Acquisition/development/refinement of the higher-order critical thinking skills aligned with the *Revised Bloom's Taxonomy of Cognitive Objectives*

Section III: Curriculum Modifications

Multivariable Calculus curriculum is subject to case-by-case modifications to support/advance the needs of all students, including special education students, English language learners, gifted students, and those at risk of school failure. These modifications are based on Individualized Learning Programs (IEPs), recommendations made by the district's Multi-Language Learners (MLL) coordinator, feedback from members of the Intervention & Referral Services Team (*I&RS*) for at-risk students, and 504 Plans.

Coursework and assessments will be modified on an individual basis for students when necessary. Modifications may include but are not limited to those outlined on the Modifications/Accommodations for Mathematics Courses chart.

Section IV: Preparation for Standardized Testing

Instruction in *Multivariable Calculus* is aligned with the requirements of state and national standardized assessments, including the *NJGPA, NJSLA*, the *ACT*, the *PSAT,* and the *SAT*.

Section V: Curriculum Pacing Guide

Curriculum Pacing Guide	
Course Title: *Multivariable Calculus*	**Grade Level:** 12
Unit I: Parametric Equations and Polar Coordinates	Weeks 1-5

Unit II: Vectors and the Geometry of Space	Weeks 6-10
Unit III: Vector-Valued Functions & Motion in Space	Weeks 11-16
Unit IV: Partial Derivatives	Weeks 17-27
Unit V: Multiple Integrals	Weeks 28-33
Unit VI: Integrals and Vector Fields	Weeks 34-40

Section VI: Primary Texts and Year-Long Instructional Resources

The following texts and instructional resources are employed in *Multivariable Calculus*:

- *Common Sense Education* (www.commonsense.org)
- *OpenSTAX Calculus 3*, Strang, Edwin, et al.
- *Calculus Early Transcendentals,* 14th Edition: By Thomas, Hass, Heil, Weir

Section VII: Grading Formula and Assessment Modes

Marking period grades in *Multivariable Calculus* are determined via a percentage weighting model. The specific grading categories and weightings of each will be determined prior to the start of each academic year and will be published in the posted/distributed course syllabi.

Assessments in *Multivariable Calculus* vary greatly in format, scope/content/skills assessed, and alternative assessments; differentiation in assessments and choice will be incorporated as appropriate. Preliminary assessments of each format will be used as benchmarks, and summative assessments will be created/revised collaboratively each year and planned by members of the *Multivariable Calculus* instructional team to inform future learning and to measure student growth.

Section VII: Unit Templates

The following unit templates have been established for the *Multivariable Calculus* Curriculum by the *Multivariable Calculus* instructional team:

Unit I: Parametric Equations and Polar Coordinates
Unit Summary
In this unit, students will study new ways to define curves in the plane. Instead of thinking of a curve as the graph of a function or equation, they will consider a more general way of thinking of a curve as the path of a moving particle whose position is changing over time. Then each of the x- and y-coordinates of the particle's position becomes a function of a third variable t. The way in which points in the plane themselves are described can be changed by using polar coordinates rather than the rectangular or Cartesian system. Both of these new tools are useful for describing motion, like that of planets and satellites, or projectiles moving in the plane or space. In addition, students will review the geometric definitions and standard equations of parabolas, ellipses, and hyperbolas. These curves are called conic sections, or conics, and model the paths traveled by projectiles, planets, or any other object moving under the sole influence of a gravitational or electromagnetic force.
Standards/Core Ideas/Performance Expectations

The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:*

- *2023 New Jersey Student Learning Standards: Mathematics*
  - MP.1-8, N.CN.B.5
- *2023 New Jersey Student Learning Standards English Language Arts*
  - L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6
- *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking*
  - 8.1.12.DA.5-6
- *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills*
  - 9.4.12.CI.1, 9.4.12.CT.1-2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings
• What is a parametrization of a curve in the xy-plane? Does a function $y = f(x)$ always have a parametrization? Are parametrizations of a curve unique? • What is a cycloid? What are typical parametric equations for cycloids? What physical properties account for the importance of cycloids? • What is the formula for the slope dy/dx of a parametrized curve x = f(t), y = g(t)? When does the formula apply, and when can d^2y/dx^2 be found? • How do students find the area bounded by a parametrized curve and one of the coordinate axes? • How do students find the length of a smooth parametrized curve x = f(t), y = g(t)? • How do students find the area of the surface generated by revolving a curve x = f(t), y = g(t), about the x-axis? • What are polar coordinates? What equations relate polar coordinates to Cartesian coordinates? Why might one coordinate system be changed to the other? • How is the area of a region bounded by two polar curves in the polar coordinate plane found? • How is the length of a polar curve found, and under what conditions?	• For any curve, the set of points (x,y) = (f(t), g(t)), where x=f(t) and y=g(t), is a parametric curve. A curve can have several parametrizations. Any continuous function can be parametrized, but parametrizations are never unique, as infinite sets of equations can trace the same path at different speeds or directions. • A cycloid is the graceful, arching curve traced by a specific point on the rim of a circular wheel as the wheel rolls along a flat, straight surface without slipping. Typical parametric equations describe the horizontal and vertical positions using the wheel's radius and the angle the wheel has rotated through. Cycloids are famous in physics because they represent the curve of fastest descent under gravity (the brachistochrone problem) and the curve on which a pendulum will take the exact same amount of time to swing regardless of its starting height (the tautochrone problem). • The formula for the slope is calculated by taking the derivative of the y-component with respect to the parameter and dividing it by the derivative of the x-component with respect to the parameter. This formula applies perfectly as long as the denominator (the x-derivative) is not zero. The second derivative can be found if the first derivative curve is itself smooth enough to be differentiated again, but it is critical to remember to divide the result by the x-derivative one more time to account for the chain rule. • The area bounded by a parametrized curve is found by taking the standard integral formula for area under a curve and substituting the parametric expression for y and the differential expression for dx (which is the derivative of x multiplied by dt), then integrating over the starting and ending values of the parameter t. • The arc length is found by essentially integrating the "speed" of the tracing point along the curve. Students take the square root of the sum of the squared derivatives of both the x and y components, and then integrate this expression over the parameter's interval. This requires the curve to be smooth (the derivatives exist, are continuous, and are not simultaneously zero) and that the curve doesn't backtrack or retrace itself. • To find the surface area of a revolution, multiply the arc length expression by the circumference of the circle that a point on the curve traces as it revolves. If revolving around the x-axis, they integrate 2pi times the y-coordinate (which serves as the radius of revolution) multiplied by that same square root of the sum of squared derivatives, over the parameter's interval. • Polar area is found by sweeping out tiny, pie-shaped wedges from the origin. The area bounded between two polar curves is found by taking one-half the integral of the difference between the square of the outer curve's radius and the square of the inner curve's radius, integrating with respect to the angle between the starting and ending angles.

	• The length of a polar curve is found by integrating a slightly modified arc length formula: the square root of the sum of the radius squared and the derivative of the radius squared (with respect to the angle). This applies as long as the derivative is continuous and the curve is traced exactly once as the angle sweeps across the given interval.

Evidence of Learning

Formative & Alternative Assessments:	Benchmark & Summative Assessments:	Resources Needed:
• Classwork • Homework • Desmos activity • Conics activity • Geogebra activity • Individual student check-ins with teacher	• Quiz on Parametrization (Benchmark) • Quiz on Calculus with Parametric Curves (Benchmark) • Quiz on Calculating the Surface Area (Benchmark) • Quiz on Conics (Benchmark) • Summative I	• Desmos • Geogebra

Unit II: Vectors and the Geometry of Space

Unit Summary

This unit is foundational to the study of *Multivariable Calculus*. To apply Calculus in many real-world situations and in higher mathematics, students will need analytic geometry to describe three-dimensional space. To accomplish this objective, three-dimensional coordinate systems and vectors will be introduced. Building on what is already known about coordinates in the xy-plane, coordinates in space will be established by adding a third axis that measures the distance above and below the xy-plane. Then students will define vectors and use them to study the analytic geometry of space. Vectors provide simple ways to define equations for lines, planes, curves, and surfaces in space. These geometric concepts are used throughout the remainder of the text to study motion in space and the Calculus of functions of several variables and vector fields, with their many important applications in science, engineering, operations research, economics, and higher mathematics.

Standards/Core Ideas/Performance Expectations

The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:*

- *2023 New Jersey Student Learning Standards: Mathematics*
  - MP.1-8, N.VM.A.3, B.4b, B.5a & C.11, F.IF.C.7
- *2023 New Jersey Student Learning Standards English Language Arts*
  - L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6
- *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking*
  - 8.1.12.DA.5-6
- *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills*
  - 9.4.12.CI.1, 9.4.12.CT.1-2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings
• When do directed line segments in the plane represent the same vector? • How are vectors added and subtracted geometrically? Algebraically? • How are a vector's magnitude and direction found? • If a vector is multiplied by a positive scalar, how is the result related to the original vector? • What is the dot product (scalar product) of two vectors? Which algebraic laws are satisfied by dot products? When is the dot	• Two vectors are equal if they have the same magnitude and direction. • Geometrically, vectors are added by the parallelogram law of addition. Algebraically, vectors are added or subtracted by adding or subtracting their components. • The magnitude of a vector can be found by finding the root of the sum of the squares of the components of the vector. Dividing a vector by its magnitude gives its direction. • Multiplying a vector by a scalar changes the length of a vector if the scalar is positive. If the scalar is negative, then the scalar multiplication also reverses the direction of the vector. • The dot product of any two vectors is commutative. If two vectors are orthogonal, then their dot product vanishes. • The dot product gives the geometric projection of one vector in the direction of a second vector. • The cross product can be used to calculate the torque exerted by a force.

product of two vectors equal to zero? ● What geometric interpretation does the dot product have? ● What is the cross product (vector product) of two vectors? Which algebraic laws are satisfied by cross products, and which are not? When is the cross product of two vectors equal to zero? ● What geometric or physical interpretations do cross products have? ● What is the determinant formula for calculating the cross product of two vectors relative to the Cartesian i, j, k-coordinate system? ● How does one find equations for lines, line segments, and planes in space? Can one express a line in space by a single equation? A plane? ● How is distance from a point to a line in space found? From a point to a plane? ● What are box products? What significance do they have? How are they evaluated? ● How is the intersection of two lines in space found? A line and a plane? Two planes? ● How are equations for spheres in space found? ● What is a cylinder? ● What are quadric surfaces?	The cross product is not commutative. If two vectors are parallel, then their cross product vanishes. ● Geometrically, the magnitude (length) of the cross product vector is exactly equal to the area of the parallelogram formed by the two original vectors. Physically, it is used to represent concepts like torque (rotational force) and angular momentum, where direction matters relative to a plane. ● Relative to the Cartesian i, j, k system, the cross product is calculated by setting up a three-by-three matrix. The top row contains the unit vectors i, j, and k. The middle row contains the components of the first vector, and the bottom row contains the components of the second vector. Evaluating the determinant of this matrix yields the new vector. ● Equations for lines, line segments, and planes are as follows: Lines: Must have a known point and a direction vector. A line cannot be defined by a single equation in 3D; it requires a set of parametric equations. Segments: A line segment uses the same parametric equations as a line but restricts the parameter to a specific interval. Planes: Need a point on the plane and a normal vector (a vector perpendicular to the plane). Unlike a line, a plane can be expressed by a single linear equation ● The distance from a point to a line is found by calculating the area of a parallelogram (using the cross product of a vector from the line to the point and the line's direction vector) and dividing by the line's direction vector magnitude. The distance from a point to a plane is found by projecting a vector (from the plane to the point) onto the plane's normal vector using the dot product. ● A box product, or scalar triple product, is the dot product of one vector with the cross product of two others. Geometrically, its absolute value represents the volume of a slanted three-dimensional box (a parallelepiped) defined by the three vectors. It is evaluated by computing the determinant of a three-by-three matrix made entirely of the components of the three vectors. ● Intersections can be found as follows: Two lines: Set their parametric equations equal to each other and see if there is a common set of parameters that satisfies all three coordinate equations. Line and a plane: Substitute the line's parametric expressions for x, y, and z directly into the single equation of the plane, and solve for the line's parameter. Two planes: Because non-parallel planes intersect in a line, their two equations are solved as a system. Typically, one variable is set to be a parameter (like let z = t) and solve for the other two variables in terms of that parameter to get the line's equations. ● A sphere's equation is derived directly from the 3D distance formula. It states that the squared distance from any point on the sphere to the center is equal to the squared radius ● In calculus, a cylinder is a three-dimensional surface formed by taking a two-dimensional curve and sweeping it infinitely along a straight line. ● Quadric surfaces are three-dimensional surfaces defined by second-degree equations.

Evidence of Learning		
Formative & Alternative Assessments: ● Classwork ● Homework ● Weekly topical quizzes ● Individual student check-ins with teacher	**Benchmark & Summative Assessments:** ● Quiz on Vectors ● Quiz on Dot product ● Quiz on Cross Product ● Quiz on Lines ● Quiz on Planes ● Unit Test	**Resources Needed:** ● Desmos ● Geogebra

Unit III: Vector-Valued Functions & Motion in Space

Unit Summary

In this unit, students will combine what they have learned about vectors and the geometry of space with the earlier study of functions. Here, the calculus of vector-valued functions will be introduced. The domains of these functions are sets of real numbers, as before, but their ranges consist of vectors, not scalars. When a vector-valued function changes, the change can occur in both magnitude and direction, so the derivative is itself a vector. The integral of a vector-valued function is also a vector. The calculus of these functions is used to describe the paths and motions of objects moving in a plane or in space, so their velocities and accelerations are given by vectors. New scalars that quantify the turning and twisting in the path of an object moving in space will also be introduced.

Standards/Core Ideas/Performance Expectations

The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:*

- *2023 New Jersey Student Learning Standards: Mathematics*
  - MP.1-8, N.VM.A.3 & B.4b
- *2023 New Jersey Student Learning Standards: English Language Arts*
  - L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6
- *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking*
  - 8.1.12.DA.5-6
- *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills*
  - 9.4.12.CI.1, 9.4.12.CT.1-2, 9.4.12.TL.3

Unit Essential Questions

- What are the rules for differentiating and integrating vector functions?
- How is it important to define and calculate the velocity, speed, direction of motion, and acceleration of a body moving along a sufficiently differentiable space curve?
- What is special about the derivatives of vector functions of constant length?
- How does one define and calculate the length of a segment of a smooth space curve? What mathematical assumptions are involved in the definition?
- What is a differentiable curve's unit tangent vector?
- What is a plane curve's principal normal vector? When is it defined? Which way does it point?
- What is a curve's binormal vector?
- How is this vector related to the curve's torsion?
- What are Kepler's laws?

Unit Enduring Understandings

- Differentiating and integrating vector functions are performed component by component, where standard rules apply, along with product rules for dot and cross products
- Velocity and acceleration are found by using derivatives when given the position of an object. The direction of motion is a unit tangent vector.
- The derivative of a vector function of constant length is orthogonal to the vector function at every point.
- The length of a segment of a smooth space curve is an integral that assumes the curve is smooth, continuous, and non-zero, and that the curve is traversed exactly once without backtracking.
- The unit vector in the direction of the velocity vector of a curve is the Unit Tangent Vector. The derivative of the unit tangent vector with respect to the arc length is the curvature of the curve. A straight line has zero curvature. The curvature of a circle is constant.
- The principal normal vector is the ratio of the derivative of the unit tangent vector to the curvature. It points perpendicular to the tangent vector and lies in the plane of the curve.
- The binormal vector is the cross product of the tangent and normal vectors.
- The torsion is defined in terms of the dot product of the normal vector and the derivative of the binormal vector with respect to the arc length.
- Kepler's Laws are the following:
  - A planet's path is an ellipse with the sun at one focus.
  - The radius vector from the sun to a planet sweeps out equal areas in equal times.
  - The orbital period and the orbit's semimajor axis are related by a derived equation.

Evidence of Learning

Formative & Alternative Assessments:	Benchmark & Summative Assessments:	Resources Needed:
• Classwork • Homework • Weekly topical quizzes	• Quiz on Limits and Differentiation of Curves in Space	• [Desmos] • [Geogebra]

● Kepler's laws activity ● Desmos vector derivative activity ● Individual student check-ins with teacher	● Quiz on Integrals of Vector Functions in Space ● Quiz on Projectile Motion ● Quiz on Curvature ● Unit Test	

Unit IV: Partial Derivatives
Unit Summary
In this unit, the basic ideas of single-variable differential calculus to functions of several variables will be extended. Their derivatives are more varied and interesting because of the different ways the variables can interact. The applications of these derivatives are also more varied than for single-variable calculus, and in the next unit, students will see that the same is true for integrals involving several variables. Students will be able to define, graph, and identify the basic properties of the limits of a real-valued function of two or three independent variables. Students will use their knowledge of derivatives to work through the Chain rule for up to three independent variables.
Standards/Core Ideas/Performance Expectations
The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:* ● *2023 New Jersey Student Learning Standards: Mathematics* ○ MP.1-8, F.IF.B.6 ● *2023 New Jersey Student Learning Standards English Language Arts* ○ L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6 ● *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking* ○ 8.1.12.DA.5-6 ● *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills* ○ 9.4.12.CI.1, 9.4.12.CT.1-2, 9.4.12.TL.3

Unit Essential Questions	**Unit Enduring Understandings**
● What is a real-valued function of two independent variables? Three independent variables? ● What does it mean for sets in the plane or in space to be open? Closed? ● How can the values of a function f(x, y) of two independent variables graphically be displayed? How does one do the same for a function f(x, y, z) of three independent variables? ● What does it mean for a function f(x, y) to have limit L as (x, y) approaches (x_0 , y_0)? What are the basic properties of limits of functions of two independent variables? ● When is a function of two (three) independent variables continuous at a point in its domain? ● What can be said about algebraic combinations and composites of continuous functions? ● What is the two-path test for nonexistence of limits? ● How are the partial derivatives of a function f(x, y) defined? How are they interpreted and calculated? ● How does the relation between first	● A function of two variables is a rule that assigns a unique real number z = f(x,y) to each ordered pair (x,y) in a domain. A function of three variables assigns a real number w = f(x,y,z) to each ordered triplet (x,y,z). ● An open set is one where every point has a small neighborhood (a disk or ball) that is completely contained within the set (it does not contain its boundary). A closed set contains all of its boundary points. ● To display a function graphically, such as f(x,y), values are displayed as a 3D surface z = f(x,y) or via 2D contour maps (level curves f(x,y) = c). For f(x,y,z), we cannot graph in 4D, so we display them using 3D level surfaces f(x,y,z) = c. ● The limit is L if f(x,y) gets arbitrarily close to L as (x,y) gets sufficiently close to (x0,y0) from any direction. The properties of limits are that they obey the same sum, difference, constant multiple, product, and quotient rules (if the denominator does not equal 0) as single-variable limits. ● A function is continuous at a point P_o if the limit of the function as it approaches P_o exists and is exactly equal to the function's value at that point. ● Algebraic combinations are sums, differences, products, multiples, quotients (where the denominator does not equal 0), and compositions of continuous functions are also continuous on their domains. ● The two-path test for nonexistence of limits is that if a function f(x, y) has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then the limit as (x,y) approaches (x_0,y_0) of f(x, y) does not exist. ● The partial derivative of f(x, y) with respect to x at the point (x_0, y_0) is $\frac{\delta f}{\delta x}\|(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$, provided the limit exists.

partial derivatives and continuity of functions of two independent variables differ from the relation between first derivatives and continuity for real-valued functions of a single independent variable? ● What is the Mixed Derivative Theorem for mixed second-order partial derivatives? ● What does it mean for a function $f(x, y)$ to be differentiable? ● What does the Increment Theorem say about differentiability? ● What is the general Chain Rule? ● What is the gradient vector of a differentiable function $f(x, y)$? How is it related to the function's directional derivatives? ● How is the tangent line at a point on a level curve of a differentiable function $f(x, y)$ found? ● How can directional derivatives be used to estimate change? ● How does one linearize a function $f(x, y)$ of two independent variables at a point (x_0, y_0)? Why might a student want to do this? How does one linearize a function of three independent variables? ● What can be said about the accuracy of linear approximations of functions of two (three) independent variables? ● How does one define local maxima, local minima, and saddle points for a differentiable function $f(x, y)$? ● What derivative tests are available for determining the local extreme values of a function $f(x, y)$? ● What are Lagrange multipliers?. ● How does Taylor's formula for a function $f(x, y)$ generate polynomial approximations and error estimates?	● A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without the function being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. ● The Mixed Derivative Theorem states that if $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$. ● A function, f, is differentiable if its partial derivatives exist at every point in its domain. ● The Increment Theorem states that if the partial derivatives of a function are continuous throughout an open region, then the function is differentiable at every point of the open region. If a function is differentiable at a point, then the function is continuous at that point. ● The chain rule is used to differentiate composite functions in 2 and 3 variables ● The gradient vector points in the direction of the maximum rate of increase. The directional derivative is the dot product of the gradient vector in the direction of the vector. ● The tangent line at a point on a level curve of a differentiable function $f(x, y)$ is orthogonal to level curves/surfaces. Use partial derivatives to find the tangent line and the tangent plane at a point on a level curve. ● Directional derivatives can be used to estimate the change in the value of a differentiable function f when a small distance ds moves from a point P_0 in a particular direction $\mathbf{u}$; use the formula $df = (\nabla f\,\|_P . \mathbf{u})\, ds$ ● Partial derivatives are used to linearize a differentiable function. Students would do this to obtain an expression for the error incurred in replacing a function by its linearization. If one moves from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change $df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$ in the linearization of f is called the total differential of f. ● The accuracy of linear approximations increases as the distance to the point of tangency decreases. ● Let $f(x, y)$ be defined on a region R containing the point (a, b). Then $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b). $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b). $f(x, y)$ has a saddle point at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. ● The second derivative test is used to find local extreme values. ● Lagrange Multipliers calculate the maxima and minima of a function that is constrained to lie within some particular subset of the plane. ● Taylor's formula expands a function into a polynomial of partial derivatives. Error is estimated using a remainder term evaluated at an intermediate point.

Evidence of Learning

Formative & Alternative Assessments:	**Benchmark & Summative Assessments:**	**Resources Needed:**
● Classwork ● Homework ● Weekly topical quizzes ● Desmos activity-two variables ● Individual student check-ins with teacher	● Quiz on Properties of Functions of Two or Three Variables ● Quiz on Limits and Continuity ● Quiz on Partial Derivatives ● Quiz on Chain Rule ● Quiz on Directional Derivatives	● Desmos ● Geogebra

	- Quiz on Tangent Planes and Differentials - Quiz on Local Extrema and Saddle Points - Quiz on Absolute Maxima - Quiz on Lagrange Multipliers - Quiz on Partial Derivatives with Constrained Variables - Summative II	

Unit V: Multiple Integration
Unit Summary
In this unit, the double integral of a function of two variables $f(x, y)$ over a region in the plane as the limit of approximating Riemann sums is defined. Just as a single integral represents signed area, so does a double integral represent signed volume. Double integrals can be evaluated using the Fundamental Theorem of Calculus studied earlier, but now the evaluations are done twice by integrating with respect to each of the variables x and y in turn. Double integrals can be used to find areas of more general regions in the plane than those encountered earlier. Moreover, just as the Substitution Rule could simplify finding single integrals, we can sometimes use polar coordinates to simplify computing a double integral. More general substitutions for evaluating double integrals as well will be studied. Triple integrals for a function of three variables $f(x, y, z)$ over a region in space will be defined. Triple integrals can be used to find volumes of still more general regions in space, and their evaluation is like that of double integrals with yet a third evaluation. Cylindrical or spherical coordinates can sometimes be used to simplify the calculation of a triple integral, and we investigate those techniques. Double and triple integrals have several additional applications, such as calculating the average value of a multivariable function and finding moments and centers of mass for more general regions than those encountered before.
Standards/Core Ideas/Performance Expectations
The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:*   *2023 New Jersey Student Learning Standards: Mathematics*   - MP.1-8, N.CN.B.4, A.CED.A.2   *2023 New Jersey Student Learning Standards English Language Arts*   - L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6   *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking*   8.1.12.DA.5-6   *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills*   9.4.12.CI.1, 9.4.12.CT.1-2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings
- Define the double integral of a function of two variables over a bounded region in the coordinate plane. - How are double integrals evaluated as iterated integrals? - Does the order of integration matter? - How are the limits of integration determined? - What is the triple integral of a function f(x, y, z) over a bounded region in space? - How are triple integrals in rectangular coordinates evaluated? How are the limits of integration determined? - How are triple integrals defined in cylindrical and spherical coordinates? Why might one prefer working in one	- The double integral is the limit of a Riemann sum, representing the net signed volume between the surface and the region. - Double integrals are evaluated by integrating with respect to one variable while holding the other constant, then integrating the result. - Order of integration: By Fubini's Theorem, the order of integration does not matter for continuous functions over bounded, measurable regions. - The limits for double integrals are found by drawing the region, and the inner limits are the bottom and top boundary curves, and the outer limits are the constant leftmost and rightmost points a and b. Use Fubini's theorem to evaluate double integrals as iterated integrals. - The triple integral of a function is the limit of a Riemann sum over a 3D region divided into small sub-boxes. - Triple integrals in rectangular coordinates are evaluated as three nested iterated integrals, working from the innermost integral outward. The Innermost limits are the lower and upper bounding surfaces. The outer two limits are determined by projecting the 3D region onto the corresponding 2D plane.

of these coordinate systems to working in rectangular coordinates? ● How are triple integrals in cylindrical and spherical coordinates evaluated? How are the limits of integration found? ● How are substitutions in double integrals pictured as transformations of two-dimensional regions? ● How are substitutions in triple integrals pictured as transformations of three-dimensional regions?	● Triple Integrals in cylindrical and spherical coordinates substitute the proper formulas. They are preferred when regions have axial symmetry (cylinders, paraboloids) or central symmetry (spheres, cones), turning complex boundary equations into simple constant limits. ● Evaluated as iterated integrals with the specific dV multipliers. Limits are found by visualizing the solid: for cylindrical, trace the height z, radius r, and angle theta; for spherical, trace the distance from the origin rho, the angle dropped from the positive z-axis phi, and the rotation around the z-axis theta. ● A 2D substitution is pictured as a mapping that transforms a simple region in a plane into a complex region. The scale factor is the absolute value of the Jacobian determinant. ● A 3D substitution transforms a simple volume into a complex volume, scaling the volume differential dV by the absolute value of the 3x3 Jacobian determinant.

Evidence of Learning		
Formative & Alternative Assessments: ● Classwork ● Homework ● Weekly topical quizzes ● Desmos activity ● Individual student check-ins with teacher	**Benchmark & Summative Assessments:** ● Quiz on Double Integrals in Rectangular Regions ● Quiz on Double Integrals in General Regions ● Quiz on Double Integrals using Polar Coordinates ● Quiz on Triple Integrals ● Quiz on Applications of Integrals ● Quiz on Triple Integrals in Cylindrical and Spherical Coordinates ● Quiz on Substitutions in Multiple Integrals ● Unit Test	**Resources Needed:** ● Desmos ● Geogebra

Unit VI: Integrals and Vector Fields
Unit Summary
In this unit, the theory of integration over coordinate lines and planes to general curves and surfaces in space will be extended. The resulting theory of line and surface integrals gives powerful mathematical tools for science and engineering. Line integrals are used to find the work done by a force in moving an object along a path, and to find the mass of a curved wire with variable density. Surface integrals are used to find the rate of flow of a fluid across a surface. The fundamental theorems of vector integral Calculus will be presented, and their mathematical consequences and physical applications will be discussed. In the final analysis, the key theorems are shown as generalized interpretations of the Fundamental Theorem of Calculus.
Standards/Core Ideas/Performance Expectations
The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:* ● *2023 New Jersey Student Learning Standards: Mathematics* ○ MP.1-8, A.CED.A.2 ● *2023 New Jersey Student Learning Standards English Language Arts* ○ L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6 ● *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking* ○ 8.1.12.DA.5-6 ● *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills* ○ 9.4.12.CI.1, 9.4.12.CT.1-2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings

• How is a line integral evaluated? • How can line integrals be used to find the centers of wires? • What is a vector field? • What is the line integral of a vector field? • What is a gradient field? • What is the Fundamental Theorem of Line Integrals, and how does it relate to the Fundamental Theorem of Calculus? • How can one tell when a field is conservative? • What is special about path-independent fields? • What is a potential function? • What is a differential form? How does one test for exactness? • What is Green's Theorem? • How can the area of a parametrized surface in space be calculated? • What is the curl of a vector field? • What is Stokes' Theorem? • What is the Divergence Theorem? • How do Green's Theorem, Stokes' Theorem, and the Divergence Theorem relate to the Fundamental Theorem of Calculus for ordinary single integrals?	• To evaluate a line integral, first find a smooth parametrization of the curve and then evaluate the integral. • To calculate the total mass of a wire lying along a curve in space, or to find work done by a variable force acting along such a curve, use a line integral. • A vector field is a function that assigns a vector to each point in its domain. • The line integral of a vector field determines the flow or circulation of the vector field along the curve and also the work done in moving an object from the point A=r(a) to the point B=r(b). • The gradient field is the gradient vector of a differentiable scalar-valued function at a point that gives the direction of greatest increase of the function. • Let C be a smooth curve joining points A and B in a plane or space that is parametrized by r(t). The Fundamental Theorem of Line Integrals says that if *f* is a differentiable function with a continuous gradient vector $F = \nabla f$ field on a domain D containing C, then $\int_C Fdr = f(B) - f(A)$. • A field is considered conservative when the principle of conservation of energy holds. • Path-independent fields are special fields where the integral's value is the same for all paths from A to B. • A field F is conservative if and only if it is the gradient field of a scalar function *f*. The function *f* is then called the potential function for F. • Any expression *M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz* is a differential form. • A differential form is considered exact if $Mdx + Ndy + Pdz = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy + \frac{\delta f}{\delta z}dz = df$. The component test can be used to determine the exactness of a differential form. • Green's Theorem relates a line integral around a simple closed curve C to a double integral over the plane region D bounded by C. • To find the area of a parametrized surface in space, evaluate a double integral. • The length of the curl vector measures the rate of rotation. • Stokes' Theorem states that the line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface. • The Divergence Theorem says that under suitable conditions, the outward flux of a vector field across a closed surface equals the triple integral of the divergence of the field over the three-dimensional region enclosed by the surface. • The Fundamental Theorem of Calculus, Green's Theorem, Stokes' Theorem, and the Divergence Theorem all say the integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.
Evidence of Learning	

Formative & Alternative Assessments:	Benchmark & Summative Assessments:	Resources Needed:
● Classwork ● Homework ● Weekly topical quizzes ● Desmos activity ● GeoGebra line integrals ● Individual student check-ins with teacher	● Quiz on Line Integrals ● Quiz on Work, Circulation and Flux ● Quiz on Conservative Fields ● Quiz on Green's Theorem ● Quiz on Surface Integrals and Area ● Quiz on Stoke's Theorem ● Quiz on Divergence Theorem ● Summative III	● Desmos ● Geogebra

Section X: Unit Reflection

The *Multivariable Calculus* Instructional Team must confer upon the completion of each instructional unit in the *Multivariable Calculus* curriculum and rate the degrees to which the instructional units meet performance criteria established by the New Jersey Department of Education using the Unit Reflection Form. Completed unit reflection forms should be used by the instructional team for future modifications within the *Multivariable Calculus* instructional curriculum.

Unit Reflection Form: ***Multivariable Calculus***			
Lesson Activities:	**Strongly**	**Moderately**	**Weakly**
Foster student use of technology as a tool to develop critical thinking, creativity, and innovation skills;			
Are challenging and require higher-order thinking and problem-solving skills;			
Allow for student choice;			
Provide scaffolding for acquiring targeted knowledge/skills;			
Integrate modern, global perspectives, especially those regarding diversity, genocide, global issues, and historical ones regarding racial relations;			
Integrate 21st century skills;			
Provide opportunities for interdisciplinary connection and transfer of knowledge and skills;			
Are varied to address different student learning styles and preferences;			
Are differentiated based on student needs;			
Are student-centered with the teacher acting as a facilitator and co-learner during the teaching and learning process;			
Provide means for students to demonstrate knowledge and skills and progress in meeting learning goals and objectives;			

Provide opportunities for student reflection and self-assessment;			
Provide data to inform and adjust instruction to better meet the varying needs of learners.			

Appendix

Writing Instruction and the RFH Community

Writing instruction should happen across the RFH Community. Writing across the curriculum is a philosophy that advances the belief that writing is a method of learning. Since all departments are committed to helping students learn, writing must be used as a methodology to advance student learning.

Each academic discipline has its own unique conventions, formats and structures. It is the responsibility of each department to agree upon domain-specific writing praxes, model them for students, and require them to utilize them on a consistent basis. Students must understand that acceptable writing in one domain may not be acceptable writing in another area. The development of domain-specific writing skills supports the overall development of the student writer because all writing is grounded in the writing situation: audience, context, purpose, subject, and writer. Representatives from the academic disciplines must share their domain-specific writing praxes with each other, identify intersections, and determine how to address perceived gaps that limit student learning.

Students must experience writing situations that help them learn how to think creatively and critically and communicate effectively in the academic disciplines. Writing instruction, regardless of the academic discipline, must always reinforce student understanding of the writing situation. When students experience writing situations, they must study examples of domain-specific writing in order to understand how writers communicate in discipline-related contexts. This does not mean information embedded in textbooks. Domain-specific writing is writing that is used to inform and influence readers as it draws them into an established circle of discourse. Students must use these non-fiction texts to develop the close reading skills that will shape their own writing. Focused engagement with domain-specific writing should not be limited to basic reading comprehension and topical understanding. It must also include the analysis of the writing situation that is represented in the text: audience, context, purpose, subject, and writer. The close reading of well-written texts—regardless of the domain—will show students the importance of writing mechanics, diction, and syntax. The development of close reading skills will also help the students grow in terms of their ability to construct and advance independent and original claims that are well-supported by evidence. Domain-specific writing is grounded in positioning of claims and the effective use of evidence.

The final written product is important; nevertheless, the learning that results in this production must not be devalued. The writing process is not limited to the basic steps of planning, drafting, revising, and editing/proofreading. It is a complex sequence of critical and creative thinking and writing that leads to the production of a text that provides evidence of learning and understanding. Students must ultimately develop the ability to self-assess the effectiveness of their writing as a representation of the writing situation. Without the use of models that evidence learning and understanding, students will not develop the ability to self-assess their own work—the true outcome of the writing process.

What types of writing situations should RFH students engage in?

RFH students should engage in writing situations across the curriculum that require them to:

- write to improve mechanical proficiency, diction usage, and syntactical sophistication
- write to narrate, describe, and reflect
- write to summarize and report
- write to classify and define
- write to explain how process leads to an outcome
- write to compare, contrast and evaluate
- write to speculate on cause and effect
- write to propose solutions and solve problems
- write to analyze

These writing situations should be positioned in a coordinated, developmental sequence that extends <u>across</u> the academic disciplines.

Upon Completion of Grade 12, RFH students must be ready to transition to the following writing situations:

- write to analyze
- write to persuade (argument)

The core foci of first-year college writing courses are analysis and argument. These courses orient the students to the demands and expectations of writing for the academic culture of college. At colleges/universities with carefully coordinated writing programs, students must demonstrate proficiency in analysis and argument before they transition to upper level courses that require them to engage in the following writing situation:

- write to investigate (research)