ESL
SCIENCE
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RAHWAY PUBLIC SCHOOLS

CURRICULUM & INSTRUCTION

Content Area: Mathematics

Course: Advanced Placement Precalculus

Grade Level: 10-12

This curriculum is part of the Educational Program of Studies of the Rahway Public Schools.

ACKNOWLEDGMENTS

Jeffery Kurczeski,

Program Supervisor of 7-12 Math & Science and 9-12 Business & Technology Education

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Mrs. Lee Andrea Garvin, Director of Curriculum and Instruction

Subject/Course Title:
Advanced Placement Precalculus
Grades 10-12

Date of Board Adoption:
August 25, 2026

RAHWAY PUBLIC SCHOOLS CURRICULUM

Advanced Placement Precalculus: Grades 10-12

PACING GUIDE

Unit	Title	Pacing
1	Polynomial and Rational Functions	8 weeks
2	Exponential and Logarithmic Functions	9 weeks
3	Trigonometric and Polar Functions	10 weeks
4	Functions Involving Parameters, Vectors, and Matrices	7 weeks

ACCOMMODATIONS

504 Accommodations:

- Provide scaffolded vocabulary and vocabulary lists.
- Provide extra visual and verbal cues and prompts.
- Provide adapted/alternate/excerpted versions of the text and/or modified supplementary materials.
- Provide links to audio files and utilize video clips.
- Provide graphic organizers and/or checklists.
- Provide modified rubrics.
- Provide a copy of teaching notes, especially any key terms, in advance.
- Allow additional time to complete assignments and/or assessments.
- Provide shorter writing assignments.
- Provide sentence starters.
- Utilize small group instruction.
- Utilize Think-Pair-Share structure.
- Check for understanding frequently.
- Have student restate information.
- Support auditory presentations with visuals.
- Weekly home-school communication tools (notebook, daily log, phone calls or email messages).
- Provide study sheets and teacher outlines prior to assessments.
- Quiet corner or room to calm down and relax when anxious.
- Reduction of distractions.
- Permit answers to be dictated.
- Hands-on activities.
- Use of manipulatives.
- Assign preferential seating.
- No penalty for spelling errors or sloppy handwriting.
- Follow a routine/schedule.
- Provide student with rest breaks.
- Use verbal and visual cues regarding directions and staying on task.
- Assist in maintaining agenda book.

IEP Accommodations:

- Provide scaffolded vocabulary and vocabulary lists.
- Differentiate reading levels of texts (e.g., Newsela).
- Provide adapted/alternate/excerpted versions of the text and/or modified supplementary materials.
- Provide extra visual and verbal cues and prompts.
- Provide links to audio files and utilize video clips.
- Provide graphic organizers and/or checklists.
- Provide modified rubrics.
- Provide a copy of teaching notes, especially any key terms, in advance.
- Provide students with additional information to supplement notes.
- Modify questioning techniques and provide a reduced number of questions or items on tests.
- Allow additional time to complete assignments and/or assessments.
- Provide shorter writing assignments.
- Provide sentence starters.
- Utilize small group instruction.
- Utilize Think-Pair-Share structure.
- Check for understanding frequently.
- Have student restate information.
- Support auditory presentations with visuals.
- Provide study sheets and teacher outlines prior to assessments.
- Use of manipulatives.
- Have students work with partners or in groups for reading, presentations, assignments, and analyses.
- Assign appropriate roles in collaborative work.
- Assign preferential seating.
- Follow a routine/schedule.

Gifted and Talented Accommodations:

- Differentiate reading levels of texts (e.g., Newsela).
- Offer students additional texts with higher lexile levels.
- Provide more challenging and/or more supplemental readings and/or activities to deepen understanding.
- Allow for independent reading, research, and projects.
- Accelerate or compact the curriculum.
- Offer higher-level thinking questions for deeper analysis.
- Offer more rigorous materials/tasks/prompts.
- Increase number and complexity of sources.
- Assign group research and presentations to teach the class.
- Assign/allow for leadership roles during collaborative work and in other learning activities.

ELL Accommodations:

- Provide extended time.
- Assign preferential seating.
- Assign peer buddy who the student can work with.
- Check for understanding frequently.
- Provide language feedback often (such as grammar errors, tenses, subject-verb agreements, etc…).
- Have student repeat directions.
- Make vocabulary words available during classwork and exams.
- Use study guides/checklists to organize information.
- Repeat directions.
- Increase one-on-one conferencing.
- Allow student to listen to an audio version of the text.
- Give directions in small, distinct steps.
- Allow copying from paper/book.
- Give student a copy of the class notes.

	● Provide written and oral instructions. ● Differentiate reading levels of texts (e.g., Newsela). ● Shorten assignments. ● Read directions aloud to student. ● Give oral clues or prompts. ● Record or type assignments. ● Adapt worksheets/packets. ● Create alternate assignments. ● Have student enter written assignments in criterion, where they can use the planning maps to help get them started and receive feedback after it is submitted. ● Allow student to resubmit assignments. ● Use small group instruction. ● Simplify language. ● Provide scaffolded vocabulary and vocabulary lists. ● Demonstrate concepts possibly through the use of visuals. ● Use manipulatives. ● Emphasize critical information by highlighting it for the student. ● Use graphic organizers. ● Pre-teach or pre-view vocabulary. ● Provide student with a list of prompts or sentence starters that they can use when completing a written assignment. ● Provide audio versions of the textbooks. ● Highlight textbooks/study guides. ● Use supplementary materials. ● Give assistance in note taking ● Use adapted/modified textbooks. ● Allow use of computer/word processor. ● Allow student to answer orally, give extended time (time-and-a-half). ● Allow tests to be given in a separate location (with the ESL teacher). ● Allow additional time to complete assignments and/or assessments. ● Read question to student to clarify. ● Provide a definition or synonym for words on a test that do not impact the validity of the exam. ● Modify the format of assessments. ● Shorten test length or require only selected test items. ● Create alternative assessments. ● On an exam other than a spelling test, don’t take points off for spelling errors.

UNIT 1 OVERVIEW

Content Area: Mathematics

Unit Title: Polynomial and Rational Functions

Target Course/Grade Level: Advanced Placement Precalculus/Grades 10-12

Unit Summary: In Unit 1, students develop understanding of two key function concepts while exploring polynomial and rational functions. The first concept is covariation, or how output values change in tandem with changing input values. The second concept is rates of change, including average rate of change, rate of change at a point, and changing rates of change. The central idea of a function as a rule for relating two simultaneously changing sets of values provides students with a vital tool that has many applications, in nature, human society, and business and industry. For example, the idea of crop yield increasing but at a decreasing rate or the efficacy of a medicine decreasing but at an increasing rate are important understandings that inform critical decisions.

Approximate Length of Unit: 8 weeks

LEARNING TARGETS

NJ Student Learning Standards:

F.IF.A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F.IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F.IF.B.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

F.IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function $h(n)$ gives the number of person-hours it takes to assemble engines in a factory, then the positive integers would be an appropriate domain for the function.

F.IF.B.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

F.IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

F.IF.C.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

F.BF.B.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, $f(x + k)$ and for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

College Board Standards:

1.1.A Describe how the input and output values of a function vary together by comparing function values.
1.1.B Construct a graph representing two quantities that vary with respect to each other in a contextual scenario.
1.2.A Compare the rates of change at two points using average rates of change near the points.
1.2.B Describe how two quantities vary together at different points and over different intervals of a function.
1.3.A Determine the average rates of change for sequences and functions, including linear, quadratic, and other function types.
1.3.B Determine the change in the average rates of change for linear, quadratic, and other function types.
1.4.A Identify key characteristics of polynomial functions related to rates of change.
1.5.A Identify key characteristics of a polynomial function related to its zeros when suitable factorizations are available or with technology.
1.5.B Determine if a polynomial function is even or odd.
1.6.A Describe end behaviors of polynomial functions.
1.7.A Describe end behaviors of rational functions.
1.8.A Determine the zeros of rational functions.
1.9.A Determine vertical asymptotes of graphs of rational functions.
1.10.A Determine holes in graphs of rational functions.
1.11.A Rewrite polynomial and rational expressions in equivalent forms.
1.11.B Determine the quotient of two polynomial functions using long division.
1.11.C Rewrite the repeated product of binomials using the binomial theorem.
1.12.A Construct a function that is an additive and/or multiplicative transformation of another function.
1.13.A Identify an appropriate function type to construct a function model for a given scenario.
1.13.B Describe assumptions and restrictions related to building a function model.
1.14.A Construct a linear, quadratic, cubic, quartic, polynomial of degree n, or related piecewise-defined function model.
1.14.B Construct a rational function model based on a context.
1.14.C Apply a function model to answer questions about a data set or contextual scenario.

Career Readiness, Life Literacies, and Key Skills:

9.4.12.CI.1 Demonstrate the ability to reflect, analyze, and use creative skills and ideas.
9.4.12.CT.1 Identify problem-solving strategies used in the development of an innovative product or practice.
9.4.12.CT.2 Explain the potential benefits of collaborating to enhance critical thinking and problem solving.
9.4.12.IML.2 Evaluate digital sources for timeliness, accuracy, perspective, credibility of the source, and relevance of information, in media, data, or other resources.
9.4.12.TL.1 Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specific task.

Interdisciplinary Connections and Standards:

ELA

L.SS.11–12.1 Demonstrate command of the system and structure of the English language when writing or speaking.

L.VL.11–12.3 Determine or clarify the meaning of unknown and multiple-meaning words and phrases based on grades 11–12 reading and content, including technical meanings, choosing flexibly from a range of strategies.

W.AW.11–12.1 Write arguments to support claims in an analysis of substantive topics or texts, using valid reasoning and relevant and sufficient evidence.

W.IW.11–12.2 Write informative/explanatory texts (including the narration of historical events, scientific procedures/ experiments, or technical processes) to examine and convey complex ideas, concepts, and information clearly and accurately through the effective selection, organization, and analysis of content.

W.WR.11–12.5 Conduct short as well as more sustained research projects to answer a question (including a self-generated question) or solve a problem; narrow or broaden the inquiry when appropriate; synthesize multiple sources on the subject, demonstrating understanding of the subject under investigation.

SL.PE.11–12.1 Initiate and participate effectively in a range of collaborative discussions (one-on-one, in groups, and teacher-led) with peers on grades 11–12 topics, texts, and issues, building on others' ideas and expressing their own clearly and persuasively.

SL.II.11–12.2 Integrate multiple sources of information presented in diverse formats and media (e.g., visually, quantitatively, orally) in order to make informed decisions and solve problems, evaluating the credibility and accuracy of each source and noting any discrepancies among the data.

SL.PI.11–12.4 Present information, findings and supporting evidence clearly, concisely, and logically. The content, organization, development, and style are appropriate to task, purpose, and audience.

SL.AS.11–12.6 Adapt speech to a variety of contexts and tasks, demonstrating a command of formal English when indicated or appropriate.

Science

HS-ETS1-2 Design a solution to a complex real-world problem by breaking it down into smaller, more manageable problems that can be solved through engineering.

Unit Understandings:

Students will understand that…

- Covariation describes how output values change in tandem with changing input values.
- A function is a rule for relating two simultaneously changing sets of values, which is a vital tool with applications in nature, human society, and business and industry.
- There are different types of rates of change, including average rate of change, rate of change at a point, and changing rates of change.

Unit Essential Questions:

- How do we model the intensity of light from its source?
- How can I use data and graphs to figure out the best time to purchase event tickets?
- How can we adjust known projectile motion models to account for changes in conditions?

Knowledge and Skills:

Students will know...

1.1A.1 A function is a mathematical relation that maps a set of input values to a set of output values such that each input value is mapped to exactly one output value. The set of input values is called the domain of the function, and the set of output values is called the range of the function. The variable representing input values is called the independent variable, and the variable representing output values is called the dependent variable. Two functions, f and g are equal if they have the same domain, and for every input value a in that same domain, both functions yield the same output value ($f(a) = g(a)$).

1.1.A.2 The input and output values of a function vary in tandem according to the function rule, which can be expressed graphically, numerically, analytically, or verbally. The image of an input value is the single output value yielded by the function rule. The preimage of an output value is the set of input values for which the function rule yields that output value.

1.1.A.3 A function is increasing over an interval of its domain if, as the input values increase, the output values always increase. That is, for all a and b in the interval, if $a < b$, then $f(a) < f(b)$.

1.1.A.4 A function is decreasing over an interval of its domain if, as the input values increase, the output values always decrease. That is, for all a and b in the interval, if $a < b$, then $f(a) > f(b)$.

1.1.B.1 The graph of a function displays a set of input-output pairs and shows how the values of the function's input and output values vary.

1.1.B.2 A verbal description of the way aspects of phenomena change together can be the basis for constructing a graph.

1.1.B.3 The graph of a function is concave up on intervals in which the rate of change is increasing.

1.1.B.4 The graph of a function is concave down on intervals in which the rate of change is decreasing.

1.1.B.5 The graph intersects the x-axis when the output value is zero. The corresponding input values are said to be zeros of the function. * **EXCLUSION STATEMENT** - Discriminating between open intervals and closed intervals as it relates to intervals on which a function is increasing or intervals on which a function is decreasing is outside the scope of the AP Precalculus course and exam.

1.2.A.1 The average rate of change of a function over an interval of the function's domain is the constant rate of change that yields the same change in the output values as the function yielded on that interval of the function's domain. It is the ratio of the change in the output values to the change in input values over that interval.

1.2.A.2 The rate of change of a function at a point quantifies the rate at which output values would change were the input values to change at that point. The rate of change at a point can be approximated by the average rates of change of the function over small intervals containing the point, if such values exist.

1.2.A.3 The rates of change at two points can be compared using average rate of change approximations over sufficiently small intervals containing each point, if such values exist.

1.2.B.1 Rates of change quantify how two quantities vary together.

1.2.B.2 A positive rate of change indicates that as one quantity increases or decreases, the other quantity does the same.

1.2.B.3 A negative rate of change indicates that as one quantity increases, the other decreases.

1.3.A.1 For a linear function, the average rate of change over any length input-value interval is constant.

1.3.A.2 For a quadratic function, the average rates of change over consecutive equal-length input-value intervals can be given by a linear function.

1.3.A.3 The average rate of change of a function f over the closed interval $[a, b]$ is the slope of the secant line from the point $(a, f(a))$ to $(b, f(b))$.

1.3.B.1 For a linear function, since the average rates of change over consecutive equal-length input value intervals can be given by a constant function, these average rates of change for a linear function are changing at a rate of zero.

1.3.B.2 For a quadratic function, since the average rates of change over consecutive equal-length input value intervals can be given by a linear function, these average rates of change for a quadratic function are changing at a constant rate.

1.3.B.3 When the average rate of change over equal length input-value intervals is increasing for all small-length intervals, the graph of the function is concave up. When the average rate of change over equal-length input-value intervals is decreasing for all small-length intervals, the graph of the function is concave down.

1.4.A.1 A nonconstant polynomial function of x is any function representation that is equivalent to the analytical form $p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$, where n is a positive integer, a_i is a

real number for each i from 0 to n, and a_n is nonzero. The polynomial has degree n, the leading term is $a_n x^n$, and the leading coefficient is a_n. A nonzero constant is also a polynomial function of degree zero.

1.4.A.2 Where a polynomial function switches between increasing and decreasing, or at the included endpoint of a polynomial with a restricted domain, the polynomial function will have a local, or relative, maximum or minimum output value. If any local maximum is greater than all other function output values, then that local maximum is a global, or absolute, maximum. Similarly, if any local minimum is less than all other function output values, then that local minimum is a global, or absolute, minimum.

1.4.A.3 Between every two distinct real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or local minimum.

1.4.A.4 A polynomial function of even degree has either a global maximum or a global minimum. For a quadratic function, the global maximum or global minimum occurs at the vertex.

1.4.A.5 Points of inflection of a polynomial function occur at input values where the rate of change of the function changes from increasing to decreasing or from decreasing to increasing. This occurs where the graph of a polynomial function changes from concave up to concave down or from concave down to concave up.

1.5.A.1 If a is a complex number and $p(a) = 0$, then a is called a zero of the polynomial function p, or a root of $p(x) = 0$. If a is a real number, then $(x - a)$ is a linear factor of p if and only if a is a zero of p.

1.5.A.2 If a linear factor $(x - a)$ is repeated n times, the corresponding zero of the polynomial function has a multiplicity n. A polynomial function of degree n has exactly n complex zeros when counting multiplicities.

1.5.A.3 If a is a real zero of a polynomial function p, then the graph of $y = p(x)$ has an x- intercept at the point $(a, 0)$. Consequently, real zeros of a polynomial can be endpoints for intervals satisfying polynomial inequalities.

1.5.A.4 If $a + bi$ is a non-real zero of a polynomial function p, then its conjugate $a - bi$ is also a zero of p.

1.5.A.5 If the real zero, a, of a polynomial function has even multiplicity, then the signs of the output values are the same for input values near $x = a$. For these polynomial functions, the graph will be tangent to the x-axis at $x = a$.

1.5.A.6 The degree of a polynomial function can be found by examining the successive differences of the output values over equal-interval input values. The degree of the polynomial function is equal to the least value n for which the successive nth differences are constant.

1.5.B.1 An *even* function is graphically symmetric over the line $x = 0$ and analytically has the property $f(-x) = f(x)$. If n is even, then a polynomial of the form $p(x) = a_n x^n$, where $n \geq 1$ and $a_n \neq 0$, is an even function.

1.5.B.2 An *odd* function is graphically symmetric about the point (0, 0) and analytically has the property $f(-x) = -f(x)$. If n is odd, then a polynomial of the form $p(x) = a_n x^n$, where $n \geq 1$ and $a_n \neq 0$, is an odd function.

1.6.A.1 As input values of a nonconstant polynomial function increase without bound, the output values will either increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to \infty} p(x) = \infty$ or $\lim_{x \to \infty} p(x) = -\infty$

1.6.A.2 As input values of a nonconstant polynomial function decrease without bound, the output values will either increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to -\infty} p(x) = \infty$ or $\lim_{x \to -\infty} p(x) = \infty$

1.6.A.3 The degree and sign of the leading term of a polynomial determines the end behavior of the polynomial function, because as the input values increase or decrease without bound, the values of the leading term dominate the values of all lower-degree terms.

1.7.A.1 A rational function is analytically represented as a quotient of two polynomial functions and gives a measure of the relative size of the polynomial function in the numerator compared to the polynomial function in the denominator for each value in the rational function's domain.

1.7.A.2 The end behavior of a rational function will be affected most by the polynomial with the greater degree, as its values will dominate the values of the rational function for input values of large magnitude. For input values of large magnitude, a polynomial is dominated by its leading term. Therefore, the end behavior of a rational function can be understood by examining the corresponding quotient of the leading terms.

1.7.A.3 If the polynomial in the numerator dominates the polynomial in the denominator for input values of large magnitude, then the quotient of the leading terms is a nonconstant polynomial, and the original rational function has the end behavior of that polynomial. If that polynomial is linear, then the graph of the rational function has a slant asymptote parallel to the graph of the line.

1.7.A.4 If neither polynomial in a rational function dominates the other for input values of large magnitude, then the quotient of the leading terms is a constant, and that constant indicates the location of a horizontal asymptote of the graph of the original rational function.

1.7.A.5 If the polynomial in the denominator dominates the polynomial in the numerator for input values of large magnitude, then the quotient of the leading terms is a rational function with a constant in the numerator and nonconstant polynomial in the denominator, and the graph of the original rational function has a horizontal asymptote at $y = 0$.

1.7.A.6 When the graph of a rational function r has a horizontal asymptote at $y = b$, where b is a constant, the output values of the rational function get arbitrarily close to b and stay arbitrarily close to b as input values increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to \infty} r(x) = b$ or $\lim_{x \to -\infty} r(x) = b$.

1.8.A.1 The real zeros of a rational function correspond to the real zeros of the numerator for such values in its domain.

1.8.A.2 The real zeros of both polynomial functions of a rational function r are endpoints or asymptotes for intervals satisfying the rational function inequalities $r(x) \geq 0$ or $r(x) \leq 0$.

1.9.A.1 If the value a is a real zero of the polynomial function in the denominator of a rational function and is not also a real zero of the polynomial function in the numerator, then the graph of the rational function has a vertical asymptote at $x = a$. Furthermore, a vertical asymptote also occurs at $x = a$ if the multiplicity of a as a real zero in the denominator is greater than its multiplicity as a real zero in the numerator.

1.9.A.2 Near a vertical asymptote, $x = a$, of a rational function, the values of the polynomial function in the denominator are arbitrarily close to zero, so the values of the rational function r increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to a^+} r(x) = \infty$ or $\lim_{x \to a^+} r(x) = -\infty$ for input values near a and greater than a, and $\lim_{x \to a^-} r(x) = \infty$ or $\lim_{x \to a^-} r(x) = -\infty$ for input values near a and less than a.

1.10.A.1 If the multiplicity of a real zero in the numerator is greater than or equal to its multiplicity in the denominator, then the graph of the rational function has a hole at the corresponding input value.

1.10.A.2 If the graph of a rational function r has a hole at $x = c$, then the location of the hole can be determined by examining the output values corresponding to input values sufficiently close to c. If input values sufficiently close to c correspond to output values arbitrarily close to L, then the hole is located at the point with coordinates (c, L). The corresponding mathematical notation is $\lim_{x \to c} r(x) = L$. It should be noted that $\lim_{x \to c^-} r(x) = \lim_{x \to c^+} r(x) = \lim_{x \to c} r(x) = L$.

1.11.A.1 Because the factored form of a polynomial or rational function readily provides information about real zeros, it can reveal information about x-intercepts, asymptotes, holes, domain, and range.

1.11.A.2 The standard form of a polynomial or rational function can reveal information about end behaviors of the function.

1.11.A.3 The information extracted from different analytic representations of the same polynomial or rational function can be used to answer questions in context.

1.11.B.1 Polynomial long division is an algebraic process similar to numerical long division involving a quotient and remainder. If the polynomial f is divided by the polynomial g, then f can be rewritten as $f(x) = g(x)q(x) + r(x)$, where q is the quotient, r is the remainder, and the degree of r is less than the degree of g.

1.11.B.2 The result of polynomial long division is helpful in finding equations of slant asymptotes for graphs of rational functions.

1.11.C.1 The binomial theorem utilizes the entries in a single row of Pascal's Triangle to more easily expand expressions of the form $(a + b)^n$, including polynomial functions of the form $p(x) = (x + c)^n$, where c is a constant.

1.12.A.1 The function $g(x) = f(x) + k$ is an additive transformation of the function f that results in a vertical translation of the graph of f by k units.

1.12.A.2 The function $g(x) = f(x + h)$ is an additive transformation of the function f that results in a horizontal translation of the graph of f by $-h$ units.

1.12.A.3 The function $g(x) = af(x)$, where $a \neq 0$, is a multiplicative transformation of the function f that results in a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the x-axis.

1.12.A.4 The function $g(x) = f(bx)$, where $b \neq 0$, is a multiplicative transformation of the function f that results in a horizontal dilation of the graph of f by a factor of $\left|\frac{1}{b}\right|$. If $b < 0$, the transformation involves a reflection over the y-axis.

1.12.A.5 Additive and multiplicative transformations can be combined, resulting in combinations of horizontal and vertical translations and dilations. The set of points prior to the transformations is called the preimage of the transformations, and the set of points after the transformations is called the image of the transformations.

1.12.A.6 The domain and range of a function that is a transformation of a parent function may be different from those of the parent function.

1.13.A.1 Linear functions model data sets or aspects of contextual scenarios that demonstrate roughly constant rates of change.

1.13.A.2 Quadratic functions model data sets or aspects of contextual scenarios that demonstrate roughly linear rates of change, or data sets that are roughly symmetric with a unique maximum or minimum value.

1.13.A.3 Geometric contexts involving area or two dimensions can often be modeled by quadratic functions. Geometric contexts involving volume or three dimensions can often be modeled by cubic functions.

1.13.A.4 Polynomial functions model data sets or contextual scenarios with multiple real zeros or multiple maxima or minima.

1.13.A.5 A polynomial function of degree n models data sets or contextual scenarios that demonstrate roughly constant nonzero nth differences.

1.13.A.6 A polynomial function of degree n or less can be used to model a graph of $n + 1$ points with distinct input values.

1.13.A.7 A piecewise-defined function consists of a set of functions defined over nonoverlapping domain intervals and is useful for modeling a data set or contextual scenario that demonstrates different characteristics over different intervals.

1.13.B.1 A model may have underlying assumptions about what is consistent in the model.

1.13.B.2 A model may have underlying assumptions about how quantities change together.

1.13.B.3 A model may require domain restrictions based on mathematical clues, contextual clues, or extreme values in the data set.

1.13.B.4 A model may require range restrictions, such as rounding values, based on mathematical clues, contextual clues, or extreme values in the data set.
1.14.A.1 A model can be constructed based on restrictions identified in a mathematical or contextual scenario.
1.14.A.2 A model of a data set or a contextual scenario can be constructed using transformations of the parent function.
1.14.A.3 A model of a data set can be constructed using technology and regressions, including linear, quadratic, cubic, and quartic regressions.
1.14.A.4 A piecewise-defined function model can be constructed through a combination of modeling techniques.
1.14.B.1 Data sets and aspects of contextual scenarios involving quantities that are inversely proportional can often be modeled by rational functions. For example, the magnitudes of both gravitational force and electromagnetic force between objects are inversely proportional to the objects' squared distance.
1.14.C.1 A model can be used to draw conclusions about the modeled data set or contextual scenario, including answering key questions and predicting values, rates of change, average rates of change, and changing rates of change. Appropriate units of measure should be extracted or inferred from the given context.

Students will be able to...
1.A Solve equations and inequalities represented analytically,with and without technology.
1.B Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.
1.C Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data,with and without technology.
2.A Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.
2.B Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
3.A Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.B Apply numerical results in a given mathematical or applied context.
3.C Support conclusions or choices with a logical rationale or appropriate data.

EVIDENCE OF LEARNING

Assessment:
What evidence will be collected and deemed acceptable to show that students truly "understand"?

- End of Unit Common Assessment
  - AP Classroom FRQs (Free Response Questions)
  - AP Classroom MCQs (Multiple Choice Questions)
- Flippedmath Unit 1A Assessments
  - AP Precalculus Mastery Checks - Unit 1A
  - AP Precalculus Tests - Unit 1A
- Flippedmath Unit 1B Assessments
  - AP Precalculus Mastery Checks - Unit 1B
  - AP Precalculus Tests - Unit 1B
- Khan Academy Exercises
- Daily observation of student work
- Student/group presentations

Learning Activities:

What differentiated learning experiences and instruction will enable all students to achieve the desired results?

- Mixed Six Worksheet: Working in small groups, students will complete tasks that approach Unit 1 concepts from multiple perspectives:
    - Recalling factual knowledge
    - Executing mathematical procedures
    - Classifying mathematical objects
    - Providing proofs and justifications
    - Extending mathematical concepts
    - Analyzing and critiquing fallacies
- AP Classroom Hunt and Discussion: Working in pairs, students will investigate available AP Classroom resources.
- AP Precalculus Unit 1 MCQ Strategies practice/discussion: First as a class, then in small groups, students will explore MCQ questions and focus on strategies to answer questions.
- AP Precalculus FRQ#1 Strategies practice/discussion: First as a class, then in small groups, students will explore FRQ#1 questions and focus on strategies to answer questions.
- AP Precalculus FRQ#2 Strategies practice/discussion: First as a class, then in small groups, students will explore FRQ#2 questions and focus on strategies to answer questions.
- Point or No Point FRQ#1: In small groups students will examine sample FRQ#1 answers and explain why each answer would or would not have received a point from the AP Scorers.
- Point or No Point FRQ#2: In small groups students will examine sample FRQ#2 answers and explain why each answer would or would not have received a point from the AP Scorers.
- Students are given a list of variables such as time, temperature, speed, cost, intensity, length,height,volume, and air flow. The teacher sketches the graph of a function curve with unlabeled axes. In pairs, students develop a situation and story involving two of the variables that can be modeled by the given curve.
- Students are given phrases such as “the function f is positive, decreasing, and concave down”, “the function is increasing with a decreasing rate of change”, “the function has an average rate of change of -4 on the interval $[2, 5]$”, “the rates of change of a function are constant”, and “the polynomial function is even and has a local minimum at $x = 2$”. Students construct a graph that would be consistent with each phrase and compare their results.
- Each student is given cards containing different rational functions in analytical representations. Have students use a calculator to graph the function and then record the intercepts on the card as well as limit expressions to describe the function’s end behavior and behavior at each vertical or horizontal asymptote (e.g. $\lim_{x \to 3^+} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 2$). In pairs, students take turns reading their limit statements to each other. Without seeing the actual rational function and without using a calculator, students will try to sketch the function’s graph and then check and discuss. Have students rotate to form new pairs and repeat.
- Students are presented with a nonconstant polynomial or rational function in analytical representations, and they then translate the expression into a variety of representations: constructing a graph, writing the expression as a product of linear factors when possible, and verbally describing characteristics such as real zeros, x-intercepts, asymptotes, and holes. Then have students check their graphs using technology.
- Students are given statements and need to classify each statement as whether the statement is always true, sometimes true, or never true. They should justify or explain their choices. Sample statements include: “The graphs of all rational functions have a horizontal asymptote”, “If $f(x) < 3$ and $f(3) > 0$ the polynomial function f must have a zero between $x = 1$ and $x = 3$” , and “The graphs of rational functions have holes and vertical asymptotes”.
- Students are given graphs of polynomial and rational functions. Students are then asked to graph a transformation of one of the provided graphs, such as a vertical dilation by a factor of 3 and a

horizontal translation of 2 units. Students will then switch with a peer and try to write the new expression for the function transformation. Students then have time to discuss the new function expressions and adjust as needed.

- While studying polynomial functions, students will complete a card-sorting activity in which the cards contain different representations of polynomial functions: analytical, graphical, tabular, and verbal. Students will match the four representations of each given function and use finite differences with the tabular representation to confirm that the degree of the polynomial matches its graph.
- Students will be presented with the analytical representation of various rational functions. For each function, students will state the reasons why the graph has either a horizontal or slant asymptote, determine the equation of the slant or horizontal asymptote, and describe the end behavior for each function using appropriate mathematical language.
- Students will be determining extrema using the graphing capabilities of a graphing calculator. Students will find relative and absolute extrema over closed intervals and across all real numbers. Students will practice this skill through a Desmos exploration activity and using a practice worksheet in class.
- In a homework assignment, students will rewrite rational functions in a form that allows them to determine the function's removable and non-removable discontinuities without technology. In class the next day, they will work with partners to answer the question: When is it helpful to rewrite a rational function in an equivalent form, and when is it not? What forms of rational functions are most helpful to determine certain essential characteristics of the function (such as end behavior, discontinuities, etc.)?
- In a group activity, students are given a table of values of a population as a function of time in which the time intervals are of varying lengths; the students compute the average rate of change of the population on each interval and apply their results to refute or support the claim that the population is increasing at an increasing rate. The teacher walks around the room ensuring that each student is using appropriate mathematical language in making their arguments.
- In a classroom task, pairs of students are each given three functions and three scenarios, each pair of which represents the temperature of an object as a function of time. The functions and scenarios are not given in pairs to the students, so the students' task is to determine which function corresponds to which scenario. Students will present a function-scenario match and, using appropriate mathematical language, provide a rationale for the match by identifying features of the function and scenario.
- A worksheet gives students practice problems that cannot be solved analytically in which they must use their calculators to find the roots of functions, find where two functions intersect, find the extrema of functions, and find solutions to equations. All calculator tools used must give answers with the number of correct decimal places required on the AP Exam.
- In a classroom activity, students will use graphs to estimate the x-intercepts of polynomial functions. Then, they will algebraically confirm the zeros of the function by solving the polynomial equation without using a calculator.
- As a classroom activity, students are given a table with rows numbered 1–12. Each row of the table describes a characteristic of a function such as a zero, a vertical asymptote, a horizontal asymptote, an end behavior, or the x-value of a hole. Each student is assigned three of these characteristics using a dodecahedral die. Each student must write the formula for a function having those three characteristics. If such a function does not exist, the student must explain why and is given three new characteristics. This process is repeated until each student has a viable set of characteristics for which they will build a function. In the last step of this activity, in pairs, students will provide a rationale as to how their function displays the three characteristics.

RESOURCES

Teacher Resources:

- AP Classroom
- Flipped Math
- Khan Academy
- Desmos Calculator

Equipment Needed:

- Laptop/Chromebook
- Fast and reliable internet for teacher/students
- Newline Board/Projector
- Document Camera
- Connection cables
- Printer
- Graphing Calculators

UNIT 2 OVERVIEW

Content Area: Mathematics

Unit Title: Exponential and Logarithmic Functions

Target Course/Grade Level: Advanced Placement Precalculus/Grades 10-12

Unit Summary: In Unit 2, students build an understanding of exponential and logarithmic functions. Exponential and logarithmic function models are widespread in the natural and social sciences. When an aspect of a phenomenon changes proportionally to the existing amount, exponential and logarithmic models are employed to harness the information. Exponential functions are key to modeling population growth, radioactive decay, interest rates, and the amount of medication in a patient. Logarithmic functions are useful in modeling sound intensity and frequency, the magnitude of earthquakes, the pH scale in chemistry, and the working memory in humans. The study of these two function types touches careers in business, medicine, chemistry, physics, education, and human geography, among others.

Approximate Length of Unit: 9 weeks

LEARNING TARGETS

NJ Student Learning Standards:

A.REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

F.BF.A.1 Write a function that describes a relationship between two quantities.

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

b. Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

c. Compose functions. For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.

F.BF.B.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, $f(x + k)$ and for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

F.BF.B.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2x^3$ or $f(x) = \frac{x+1}{x-1}$ for $x \neq 1$.

b. Verify by composition that one function is the inverse of another.

c. Read values of an inverse function from a graph or a table, given that the function has an inverse.

d. Produce an invertible function from a non-invertible function by restricting the domain.

F.BF-B.5 Use the inverse relationship between exponents and logarithms to solve problems involving logarithms and exponents.

F.IF.A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F.IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F.IF.B.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

F.IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function $h(n)$ gives the number of person-hours it takes to assemble engines in a factory, then the positive integers would be an appropriate domain for the function.

F.IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

F.IF.C.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.02)^{\frac{t}{10}}$, and classify them as representing exponential growth or decay.

F.LE.A.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.

a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

F.LE.A.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

F.LE.A.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

F.LE.A.4 Understand the inverse relationship between exponents and logarithms. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

F.LE.B.5 Interpret the parameters in a linear or exponential function in terms of a context.

College Board Standards:

2.1.A Express arithmetic sequences found in mathematical and contextual scenarios as functions of the whole numbers.
2.1.B Express geometric sequences found in mathematical and contextual scenarios as functions of the whole numbers.
2.2.A Construct functions of the real numbers that are comparable to arithmetic and geometric sequences.
2.2.B Describe similarities and differences between linear and exponential functions.
2.3.A Identify key characteristics of exponential functions.
2.4.A Rewrite exponential expressions in equivalent forms.
2.5.A Construct a model for situations involving proportional output values over equal-length input-value intervals.
2.5.B Apply exponential models to answer questions about a data set or contextual scenario.
2.6.A Construct linear, quadratic, and exponential models based on a data set.
2.6.B Validate a model constructed from a data set.
2.7.A Evaluate the composition of two or more functions for given values.
2.7.B Construct a representation of the composition of two or more functions.
2.7.C Rewrite a given function as a composition of two or more functions.
2.8.A Determine the input-output pairs of the inverse of a function.
2.8.B Determine the inverse of a function on an invertible domain.
2.9.A Evaluate logarithmic expressions.
2.10.A Construct representations of the inverse of an exponential function with an initial value of 1.
2.11.A Identify key characteristics of logarithmic functions.
2.12.A Rewrite logarithmic expressions in equivalent forms.
2.13.A Solve exponential and logarithmic equations and inequalities.
2.13.B Construct the inverse function for exponential and logarithmic functions.
2.14.A Construct a logarithmic function model.
2.15.A Determine if an exponential model is appropriate by examining a semi-log plot of a data set.
2.15.B Construct the linearization of exponential data.

Career Readiness, Life Literacies, and Key Skills:

9.4.12.CI.1 Demonstrate the ability to reflect, analyze, and use creative skills and ideas.
9.4.12.CT.1 Identify problem-solving strategies used in the development of an innovative product or practice.
9.4.12.CT.2 Explain the potential benefits of collaborating to enhance critical thinking and problem solving.
9.4.12.IML.2 Evaluate digital sources for timeliness, accuracy, perspective, credibility of the source, and relevance of information, in media, data, or other resources.
9.4.12.TL.1 Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specific task.

Interdisciplinary Connections and Standards:

ELA

L.SS.11–12.1 Demonstrate command of the system and structure of the English language when writing or speaking.
L.VL.11–12.3 Determine or clarify the meaning of unknown and multiple-meaning words and phrases based on grades 11–12 reading and content, including technical meanings, choosing flexibly from a range of strategies.
W.AW.11–12.1 Write arguments to support claims in an analysis of substantive topics or texts, using valid reasoning and relevant and sufficient evidence.
W.IW.11–12.2 Write informative/explanatory texts (including the narration of historical events, scientific procedures/ experiments, or technical processes) to examine and convey complex ideas, concepts, and information clearly and accurately through the effective selection, organization, and analysis of content.

W.WR.11–12.5 Conduct short as well as more sustained research projects to answer a question (including a self-generated question) or solve a problem; narrow or broaden the inquiry when appropriate; synthesize multiple sources on the subject, demonstrating understanding of the subject under investigation.

SL.PE.11–12.1 Initiate and participate effectively in a range of collaborative discussions (one-on-one, in groups, and teacher-led) with peers on grades 11–12 topics, texts, and issues, building on others' ideas and expressing their own clearly and persuasively.

SL.II.11–12.2 Integrate multiple sources of information presented in diverse formats and media (e.g., visually, quantitatively, orally) in order to make informed decisions and solve problems, evaluating the credibility and accuracy of each source and noting any discrepancies among the data.

SL.PI.11–12.4 Present information, findings and supporting evidence clearly, concisely, and logically. The content, organization, development, and style are appropriate to task, purpose, and audience.

SL.AS.11–12.6 Adapt speech to a variety of contexts and tasks, demonstrating a command of formal English when indicated or appropriate.

Science

HS-ETS1-2 Design a solution to a complex real-world problem by breaking it down into smaller, more manageable problems that can be solved through engineering.

Unit Understandings:

Students will understand that…

- When an aspect of a phenomenon changes proportionally to the existing amount, exponential and logarithmic models are employed to harness the information.
- Exponential functions are key to modeling population growth, radioactive decay, interest rates, and the amount of medication in a patient.
- Logarithmic functions are useful in modeling sound intensity and frequency, the magnitude of earthquakes, the pH scale in chemistry, and the working memory in humans.

Unit Essential Questions:

- How can I make a single model that merges the interest I earn from my bank with the taxes that are due so I can know how much I will have in the end?
- How can we adjust the scale of distance for a model of planets in the solar system so the relationships among the planets are easier to see?
- If different functions can be used to model data, how do we pick which one is best?

Knowledge and Skills:

Students will know...

2.1.A.1 A sequence is a function from the whole numbers to the real numbers. Consequently, the graph of a sequence consists of discrete points instead of a curve.

2.1.A.2 Successive terms in an arithmetic sequence have a common difference, or constant rate of change.

2.1.A.3 The general term of an arithmetic sequence with a common difference d is denoted by a_n and is given by $a_n = a_0 + dn$, where a_0 is the initial value, or by $a_n = a_k + d(n - k)$, where a_k is the kth term of the sequence.

2.1.B.1 Successive terms in an increasing or decreasing geometric sequence have a common ratio, or constant proportional change.

2.1.B.2 The general term of an increasing or decreasing geometric sequence with a common ratio r is denoted by g_n and is given by $g_n = g_0 r^n$, where g_0 is the initial value, or by $g_n = g_k r^{(n-k)}$, where g_k is the kth term of the sequence.

2.1.B.3 An increasing arithmetic sequence increases equally with each step, whereas an increasing geometric sequence of positive values increases by a larger amount with each successive step.

2.2.A.1 Linear functions of the form $f(x) = b + mx$ are similar to arithmetic sequences of the form $a_n = a_0 + dn$, as both can be expressed as an initial value (b or a_0) plus repeated addition of a constant rate of change, the slope (m or d).

2.2.A.2 Similar to arithmetic sequences of the form $a_n = a_k + d(n - k)$, which are based on a known difference, d, and a kth term, linear functions can be expressed in the form $f(x) = y_i + m(x - x_i)$ based on a known slope, m, and a point, (x_i, y_i).

2.2.A.3 Exponential functions of the form $f(x) = ab^x$ are similar to geometric sequences of the form $g_n = g_0 r^n$, as both can be expressed as an initial value (a or g_0) times repeated multiplication by a constant ratio (b or r).

2.2.A.4 Similar to geometric sequences of the form $g_n = g_k r^{(n-k)}$, which are based on a known ratio, r, and a kth term, exponential functions can be expressed in the form $f(x) = y_i b^{(x-x_i)}$ based on a known ratio, b, and a point, (x_i, y_i).

2.2.A.5 Sequences and their corresponding functions may have different domains.

2.2.B.1 Over equal-length input-value intervals, if the output values of a function change at constant rate, then the function is linear; if the output values of a function change proportionally, then the function is exponential. For an exponential function, the known ratio cannot be 0 or 1.

2.2.B.2 Linear functions of the form $f(x) = b + mx$ and exponential functions of the form $f(x) = ab^x$ can both be expressed analytically in terms of an initial value and a constant involved with change. However, linear functions are based on addition, while exponential functions are based on multiplication.

2.2.B.3 Arithmetic sequences, linear functions, geometric sequences, and exponential functions all have the property that they can be determined by two distinct sequence or function values.

2.3.A.1 The general form of an exponential function is $f(x) = ab^x$, with the initial value a, where $a \neq 0$, and the base b, where $b > 0$, and $b \neq 1$. When $a > 0$ and $b > 1$, the exponential function is said to demonstrate exponential growth. When $a > 0$ and $0 < b < 1$, the exponential function is said to demonstrate exponential decay.

2.3.A.2 When the natural numbers are input values in an exponential function, the input value specifies the number of factors of the base to be applied to the function's initial value. The domain of an exponential function is all real numbers.

2.3.A.3 Because the output values of exponential functions in general form are proportional over equal-length input-value intervals, exponential functions are always increasing or always decreasing, and their graphs are always concave up or always concave down. Consequently, exponential functions do not have extrema except on a closed interval, and their graphs do not have points of inflection.

2.3.A.4 If the output values of the function f are not proportional over equal-length input-value intervals, but the output values of an additive transformation of f are proportional over equal-length input-value intervals, then f can be modeled by an additive transformation of an exponential function.

2.3.A.5 For an exponential function in general form, as the input values increase or decrease without bound, the output values will increase or decrease without bound or will get arbitrarily close to zero. That is, for an exponential function in general form, $\lim_{x \to \pm\infty} ab^x = \infty$, $\lim_{x \to \pm\infty} ab^x = -\infty$, $\lim_{x \to \pm\infty} ab^x = 0$.

2.4.A.1 The product property for exponents states that $b^m b^n = b^{(m+n)}$. Graphically, this property implies that every horizontal translation of an exponential function, $f(x) = b^{(x+k)}$, is equivalent to a vertical dilation, $f(x) = b^{(x+k)} = b^x b^k = ab^x$, where $a = b^k$.

2.4.A.2 The power property for exponents states $\left(b^m\right)^n = b^{(mn)}$. Graphically, this property implies that every horizontal dilation of an exponential function, $f(x) = b^{(cx)}$ is equivalent to a change of the base of an exponential function, $f(x) = \left(b^c\right)^x$, where b^c is a constant and $c \neq 0$.

2.4.A.3 The negative exponent property states that $b^{-n} = \frac{1}{b^n}$.

2.4.A.4 The value of an exponential expression involving an exponential unit fraction, such as $b^{\frac{1}{k}}$ where k is a natural number, is the kth root of b, when it exists.

2.5.A.1 Exponential functions model growth patterns where successive output values over equallength input-value intervals are proportional. When the input values are whole numbers, exponential functions model situations of repeated multiplication of a constant to an initial value.

2.5.A.2 A constant may need to be added to the dependent variable values of a data set to reveal a proportional growth pattern.

2.5.A.3 An exponential function model can be constructed from an appropriate ratio and initial value or from two input-output pairs. The initial value and the base can be found by solving a system of equations resulting from the two input-output pairs.

2.5.A.4 Exponential function models can be constructed by applying transformations to $f(x) = ab^x$ based on characteristics of a contextual scenario or data set.

2.5.A.5 Exponential function models can be constructed for a data set with technology using exponential regressions.

2.5.A.6 The natural base e, which is approximately 2.718, is often used as the base in exponential functions that model contextual scenarios.

2.5.B.1 For an exponential model in general form $f(x) = ab^x$, the base of the exponent, b, can be understood as a growth factor in successive unit changes in the input values and is related to a percent change in context.

2.5.B.2 Equivalent forms of an exponential function can reveal different properties of the function. For example, if d represents number of days, then the base of $f(d) = 2^d$ indicates that the quantity increases by a factor of 2 every day, but the equivalent form $f(d) = \left(2^7\right)^{\frac{d}{7}}$ indicates that the quantity increases by a factor of 2^7 every week.

2.5.B.3 Exponential models can be used to predict values for the dependent variable, depending on the contextual constraints on the domain, and may be used to determine values of the independent variable.

2.6.A.1 Two variables in a data set that demonstrate a slightly changing rate of change can be modeled by linear, quadratic, and exponential function models.

2.6.A.2 Models can be compared based on contextual clues and applicability to determine which model is most appropriate.

2.6.B.1 For a given value of an independent variable, a residual of a regression is the actual value of the dependent variable value minus the value predicted by the function regression model. A residual plot is a graphical representation of the residuals of a regression, with the residuals on the vertical axis and the independent variable on the horizontal axis. A model is justified as appropriate for a data set if the residual plot appears without pattern.

2.6.B.2 For given values of an independent variable, the errors in a model can relate to the signed differences in the values of the dependent variable predicted by the function regression model and the actual values or the unsigned absolute values of the residuals. Errors may reveal that the model underestimates or overestimates actual values on a given input value interval. The data set and context can provide clues as to whether underestimates or overestimates are preferred on a given input value interval.

2.7.A.1 If f and g are functions, the composite function $f \circ g$ maps a set of input values to a set of output values such that the output values of g are used as input values of f. For this reason, the domain of the composite

function is restricted to those input values of g for which the corresponding output value is in the domain of f. $(f \circ g)(x)$ can also be represented as $f(g(x))$.

2.7.A.2 Values for the composite function $f \circ g$ can be calculated or estimated from the graphical, numerical, analytical, or verbal representations of f and g by using output values from g as input values for f.

2.7.A.3 The composition of functions is not commutative; that is, $f \circ g$ and $g \circ f$ are typically different functions; therefore, $f(g(x))$ and $g(f(x))$ are typically different values.

2.7.A.4 If the function $f(x) = x$ is composed with any function g, the resulting composite function is the same as g ; that is, $g(f(x)) = f(g(x)) = g(x)$. The function $f(x) = x$ is called the *identity function*. When composing two functions, the identity function acts in the same way as 0, the additive identity, when adding two numbers, and 1, the multiplicative identity, when multiplying two numbers.

2.7.B.1 Function composition is useful for relating two quantities that are not directly related by an existing formula.

2.7.B.2 When analytic representations of the functions f and g are available, an analytic representation of $f(g(x))$ can be constructed by substituting $g(x)$ for every instance of x in f.

2.7.B.3 A numerical or graphical representation of $f \circ g$ can often be constructed by calculating or estimating values for $(x, f(g(x)))$.

2.7.C.1 Functions given analytically can often be decomposed into less complicated functions. When properly decomposed, the variable in one function should replace each instance of the function with which it was composed.

2.7.C.2 An additive transformation of a function, f , that results in vertical and horizontal translations can be understood as the composition of $g(x) = x + k$ with f.

2.7.C.3 A multiplicative transformation of a function, f , that results in vertical and horizontal dilations can be understood as the composition of $g(x) = kx$ with f.

2.8.A.1 On a specified domain, a function, f , has an inverse function, or is invertible, if each output value of f is mapped from a unique input value. The domain of a function may be restricted in many ways to make the function invertible.

2.8.A.2 An inverse function can be thought of as a reverse mapping of the function. An inverse function, f^{-1}, maps the output values of a function, f, on its invertible domain to their corresponding input values; that is, if $f(a) = b$, then $f^{-1}(b) = a$. Alternately, on its invertible domain, if a function consists of input-output pairs (a, b), then the inverse function consists of input-output pairs (b, a).

2.8.B.1 The composition of a function, f , and its inverse function, f^{-1}, is the identity function; that is, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

2.8.B.2 On a function's invertible domain, the function's range and domain are the inverse function's domain and range, respectively. The inverse of the table of values of $y = f(x)$ can be found by reversing the input-output pairs; that is, (a, b) corresponds to (b, a).

2.8.B.3 The inverse of the graph of the function $y = f(x)$ can be found by reversing the roles of the x- and y -axes; that is, by reflecting the graph of the function over the graph of the identity function $h(x) = x$.

2.8.B.4 The inverse of the function can be found by determining the inverse operations to reverse the mapping. One method for finding the inverse of the function f is reversing the roles of x and y in the equation $y = f(x)$, then solving for $y = f^{-1}(x)$.

2.8.B.5 In addition to limiting the domain of a function to obtain an inverse function, contextual restrictions may also limit the applicability of an inverse function.

2.9.A.1 The logarithmic expression $\log_b c$ is equal to, or represents, the value that the base b must be exponentially raised to in order to obtain the value c. That is, $\log_b c = a$ if and only if $b^a = c$, where a and c are constants, $b > 0$, and $b \neq 1$. (when the base of a logarithmic expression is not specified, it is understood as the common logarithm with a base of 10)

2.9.A.2 The values of some logarithmic expressions are readily accessible through basic arithmetic while other values can be estimated through the use of technology.

2.9.A.3 On a logarithmic scale, each unit represents a multiplicative change of the base of the logarithm. For example, on a standard scale, the units might be 0, 1, 2, ..., while on a logarithmic scale, using logarithm base 10, the units might be $10^0, 10^1, 10^2, \ldots$

2.10.A.1 The general form of a logarithmic function is $f(x) = a\log_b x$, with base b, where $b > 0$, $b \neq 1$, and $a \neq 0$.

2.10.A.2 The way in which input and output values vary together have an inverse relationship in exponential and logarithmic functions. Output values of general-form exponential functions change proportionately as input values increase in equal-length intervals. However, input values of general-form logarithmic functions change proportionately as output values increase in equal-length intervals. Alternately, exponential growth is characterized by output values changing multiplicatively as input values change additively, whereas logarithmic growth is characterized by output values changing additively as input values change multiplicatively.

2.10.A.3 $f(x) = \log_b x$ and $g(x) = b^x$, where $b > 0$ and $b \neq 1$, are inverse functions. That is, $g(f(x)) = f(g(x)) = x$.

2.10.A.4 The graph of the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$, is a reflection of the graph of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, over the graph of the identity function $h(x) = x$.

2.10.A.5 If (s, t) is an ordered pair of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, then (t, s) is an ordered pair of the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$

2.11.A.1 The domain of a logarithmic function in general form is any real number greater than zero, and its range is all real numbers.

2.11.A.2 Because logarithmic functions are inverses of exponential functions, logarithmic functions are also always increasing or always decreasing, and their graphs are either always concave up or always concave down. Consequently, logarithmic functions do not have extrema except on a closed interval, and their graphs do not have points of inflection.

2.11.A.3 The additive transformation function $g(x) = f(x + k)$, where $k \neq 0$, of a logarithmic function f in general form does not have input values that are proportional over equal-length output-value intervals. However, if the input values of the additive transformation function, $g(x) = f(x + k)$, of any function f are proportional over equal-length output value intervals, then f is logarithmic.

2.11.A.4 With their limited domain, logarithmic functions in general form are vertically asymptotic to $x = 0$, with an end behavior that is unbounded. That is, for a logarithmic function in general form, $\lim_{x \to 0^+} a\log_b x = \pm\infty$ and $\lim_{x \to \infty} a\log_b x = \pm\infty$.

2.12.A.1 The product property for logarithms states that $\log_b(xy) = \log_b x + \log_b y$. Graphically, this property implies that every horizontal dilation of a logarithmic function, $f(x) = \log_b(kx)$, is equivalent to a vertical translation, $f(x) = \log_b(kx) = \log_b k + \log_b x = a + \log_b x$ where $a = \log_b k$.

2.12.A.2 The power property for logarithms states that $\log_b x^n = n\log_b x$. Graphically, this property implies that raising the input of a logarithmic function to a power, $f(x) = \log_b x^k$, results in a vertical dilation, $f(x) = \log_b x^k = k\log_b x$.

2.12.A.3 The change of base property for logarithms states that $\log_b x = \frac{\log_a x}{\log_a b}$, where $a > 0$ and $a \neq 1$. This implies that all logarithmic functions are vertical dilations of each other.

2.12.A.4 The function $f(x) = \ln x$ is a logarithmic function with the natural base e; that is, $\ln x = \log_e x$.

2.13.A.1 Properties of exponents, properties of logarithms, and the inverse relationship between exponential and logarithmic functions can be used to solve equations and inequalities involving exponents and logarithms.

2.13.A.2 When solving exponential and logarithmic equations found through analytical or graphical methods, the results should be examined for extraneous solutions precluded by the mathematical or contextual limitations.

2.13.A.3 Logarithms can be used to rewrite expressions involving exponential functions in different ways that may reveal helpful information. Specifically, $b^x = c^{(\log_c b)(x)}$

2.13.B.1 The function $f(x) = ab^{(x+h)} + k$ is a combination of additive transformations of an exponential function in general form. The inverse of $y = f(x)$ can be found by determining the inverse operations to reverse the mapping.

2.13.B.2 The function $f(x) = a\log_b(x + h) + k$ is a combination of additive transformations of a logarithmic function in general form. The inverse of $y = f(x)$ can be found by determining the inverse operations to reverse the mapping.

2.14.A.1 Logarithmic functions are inverses of exponential functions and can be used to model situations involving proportional growth, or repeated multiplication, where the input values change proportionally over equal length output-value intervals. Alternately, if the output value is a whole number, it indicates how many times the initial value has been multiplied by the proportion.

2.14.A.2 A logarithmic function model can be constructed from an appropriate proportion and a real zero or from two input-output pairs.

2.14.A.3 Logarithmic function models can be constructed by applying transformations to $f(x) = a\log_b x$ based on characteristics of a context or data set.

2.14.A.4 Logarithmic function models can be constructed for a data set with technology using logarithmic regressions.

2.14.A.5 The natural logarithm function is often useful in modeling real-world phenomena.

2.14.A.6 Logarithmic function models can be used to predict values for the dependent variable, and they may be used to determine values of the independent variable.

2.15.A.1 In a semi-log plot, one of the axes is logarithmically scaled with logarithm base n, where $n > 1$. When the y-axis of a semi-log plot is logarithmically scaled, data or functions that demonstrate exponential characteristics will appear linear.

2.15.A.2 If for large input values of a bivariate data set the logarithms of the output values trend linear, then transformations of an exponential function can be used to model the data set. Therefore, an advantage of semi-log plots is that the dependent variable values do not need to be transformed by a constant to reveal that transformations of an exponential function would produce an appropriate model.

2.15.B.1 Techniques used to model linear functions can be applied to a semi-log graph.

2.15.B.2 For an exponential model of the form $y = ab^x$, the corresponding linear model for the semilog plot is $y = (\log_n b)x + \log_n a$, where $n > 0$ and $n \neq 1$. Specifically, the linear rate of change is $\log_n b$, and the initial linear value is $\log_n a$.

Students will be able to...

1.A Solve equations and inequalities represented analytically, with and without technology.

1.B Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

1.C Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data,with and without technology.

2.A Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

2.B Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
3.A Describe the characteristics of a function with varying levels of precision, depending on the functionrepresentation and available mathematical tools.
3.B Apply numerical results in a given mathematical or applied context.
3.C Support conclusions or choices with a logical rationale or appropriate data.

EVIDENCE OF LEARNING

Assessment:
What evidence will be collected and deemed acceptable to show that students truly "understand"?

- End of Unit Common Assessment
  - AP Classroom FRQs (Free Response Questions)
  - AP Classroom MCQs (Multiple Choice Questions)
- Flippedmath Unit 2A Assessments
  - AP Precalculus Mastery Checks - Unit 2A
  - AP Precalculus Tests- Unit 2A
- Flippedmath Unit 2B Assessments
  - AP Precalculus Mastery Checks - Unit 2B
  - AP Precalculus Tests - Unit 2B
- Khan Academy Exercises
- Daily observation of student work
- Student/group presentations

Learning Activities:
What differentiated learning experiences and instruction will enable all students to achieve the desired results?

- Mixed Six Worksheet: Working in small groups, students will complete tasks that approach Unit 2 concepts from multiple perspectives:
  - Recalling factual knowledge
  - Executing mathematical procedures
  - Classifying mathematical objects
  - Providing proofs and justifications
  - Extending mathematical concepts
  - Analyzing and critiquing fallacies
- AP Precalculus Unit 2 MCQ Strategies practice/discussion: First as a class, then in small groups, students will explore MCQ questions and focus on strategies to answer questions.
- AP Precalculus FRQ#1 Strategies practice/discussion: First as a class, then in small groups, students will explore FRQ#1 questions and focus on strategies to answer questions.
- AP Precalculus FRQ#2 Strategies practice/discussion: First as a class, then in small groups, students will explore FRQ#2 questions and focus on strategies to answer questions.
- AP Precalculus FRQ#4 Strategies practice/discussion: First as a class, then in small groups, students will explore FRQ#4 questions and focus on strategies to answer questions.
- Point or No Point FRQ#1: In small groups students will examine sample FRQ#1 answers and explain why each answer would or would not have received a point from the AP Scorers.
- Point or No Point FRQ#2: In small groups students will examine sample FRQ#2 answers and explain why each answer would or would not have received a point from the AP Scorers.

- Point or No Point FRQ#4: In small groups students will examine sample FRQ#4 answers and explain why each answer would or would not have received a point from the AP Scorers.
- Five students randomly choose different integers between −6 and 6 that will be used as input values for a function. In groups, students determine appropriate output values for the five input values that illustrate a linear function relationship, then those that illustrate an exponential function relationship. The groups defend their decisions to the rest of the class by explaining how they know their values represent linear and exponential functions.
- Have students pair up and attempt to walk (in a straight line from a specific starting point) using the following walking patterns:
    - i) at a constant rate of change of 2 feet per second, based on the model $d = 2t$
    - ii) at an increasing rate of change based on the model $d = t^2$.

  Ask each pair to describe the differences in their two experiences. Follow up with applied questions such as
    - "How far would you walk in 10 seconds? in 33 seconds?" or
    - "Using each model, how long would it take you to travel around Earth (~25,000 miles)?, travel to the Moon(~239,000miles)?"
- Student A and Student B are both provided with a context that can be modeled using exponential functions such as doubling or halving a certain substance. Without looking at each other's work, each student generates a graph and an expression to model the exponential function. Then, the students switch papers to review each other's solutions.
- Students are given a sheet of paper containing several logarithmic functions represented analytically and their corresponding graphs, in mixed order, in a separate column. In small groups, students match the corresponding representations and take turns explaining how they know each is matched correctly.
- Students will use the calculator to calculate both exponential and quadratic regressions to model sets of data, such as the growth of a bacteria colony over a one-month period. Students will also use the calculator to produce the residual plots for each model and then conclude which model is more appropriate for the data.
- As a classroom activity, students will be given contextual scenarios representing exponential, and logarithmic functions. Based on contextual clues and given input-output values, students will select and construct an appropriate model without the use of a calculator. Two input-output pairs will be used to determine the parameters of the model. Students will write a justification of the function type selected for each contextual scenario.
- During a group activity, students are given a data set and are asked to decide whether it should be modeled by a linear, quadratic, exponential, or logarithmic function. They then use regression to build that model and graph the residual plots to determine if their choice of model was appropriate.
- In a classroom activity, groups of students are each given a set of 24 index cards. The 24 shuffled cards consist of six sets of the graphical, numerical, verbal, and analytic representations of six exponential functions. In the groups, students sort the cards into the six functions in four representations
- In a group project, students collect their own data from some real-world situation and attempt to choose an appropriate model (such as linear or exponential) for the data. They then use the calculator to compute a regression for the data, and finally they use the calculator to graph the residuals in order to attempt to determine if their chosen model was appropriate.
- Students are given an instructional activity in which they are presented with a logarithmic and exponential function that are inverses (the inverse relationship is not revealed). Using the calculator, they complete a guided inquiry in which they identify the intercepts, search for extrema, and determine points of intersection of the two functions to discover the inverse relationship between the exponential and logarithmic functions.

- In a classroom activity, students assume the title of social media manager for the day and leverage a graphing calculator to model data about the spread of a TikTok video using both a quadratic and exponential regression. Then, they analyze residual graphs to determine the most accurate model for the data set.
- On a worksheet, students are presented with 6–8 tables of values that result from different increasing linear, exponential, and logarithmic functions. Students are not told what type of function corresponds to each table. Students work in groups to decide what type of function corresponds to each table, providing a rationale based on how output values are changing with respect to input values. After verifying the function type with the class, students build the appropriate functions.
- In a homework assignment, students are provided with characteristics of a "mystery function" such as the number and type of zeros, the end behavior, and the locations of extrema. They build a function model from those criteria, without using regressions, and craft a justification for their model. The following day in class, they compare their models and justifications with their classmates' models and justifications, and the class decides on the strongest model of the "mystery function."
- Students will study exponential growth of two populations. Students will find intersection points for the pairs of exponential functions, using the calculator's equation solver, and interpret the results in context.

RESOURCES

Teacher Resources:

- AP Classroom
- Flipped Math
- Khan Academy
- Desmos Calculator

Equipment Needed:

- Laptop/Chromebook
- Fast and reliable internet for teacher/students
- Newline Board/Projector
- Document Camera
- Connection cables
- Printer
- Graphing Calculators

UNIT 3 OVERVIEW

Content Area: Mathematics

Unit Title: Trigonometric and Polar Functions

Target Course/Grade Level: Advanced Placement Precalculus/Grades 10-12

Unit Summary: In Unit 3, students explore trigonometric functions and their relation to the angles and arcs of a circle. Since their output values repeat with every full revolution around the circle, trigonometric functions are ideal for modeling periodic, or repeated pattern phenomena, such as: the highs and lows of a wave, the blood pressure produced by a heart, and the angle from the North Pole to the Sun year to year. Furthermore, periodicity is found in human inventions and social phenomena. For example, moving parts of an analog clock are modeled by a trigonometric function with respect to each other or with respect to time; traffic flow at an intersection over the course of a week demonstrates daily periodicity; and demand for a particular product over the course of a year falls into an annually repeating pattern. Polar functions, which are also explored in this unit, have deep ties to trigonometric functions as they are both based on the circle. Polar functions are defined on the polar coordinate system that uses the circular concepts of radii and angles to describe location instead of the rectangular concepts of left-right and up-down, which students worked with previously. Trigonometry serves as the bridge between the two systems.

Approximate Length of Unit: 10 weeks

LEARNING TARGETS

NJ Student Learning Standards:

F.TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F.TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.A.3 Use special triangles to determine geometrically the values of sine, cosine, tangent for $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$, and use the unit circle to express the values of sine, cosines, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

F.TF.A.4 Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F.TF.B.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

F.TF.B.6 Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

F.TF.B.7 Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

F.TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$, given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

F.TF.C.9 Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

College Board Standards:

3.1.A Construct graphs of periodic relationships based on verbal representations.
3.1.B Describe key characteristics of a periodic function based on a verbal representation.
3.2.A Determine the sine, cosine, and tangent of an angle using the unit circle.
3.3.A Determine coordinates of points on a circle centered at the origin.
3.4.A Construct representations of the sine and cosine functions using the unit circle.
3.5.A Identify key characteristics of the sine and cosine functions.
3.6.A Identify the amplitude, vertical shift, period, and phase shift of a sinusoidal function.
3.7.A Construct sinusoidal function models of periodic phenomena.
3.8.A Construct representations of the tangent function using the unit circle.
3.8.B Describe key characteristics of the tangent function.
3.8.C Describe additive and multiplicative transformations involving the tangent function.
3.9.A Construct analytical and graphical representations of the inverse of the sine, cosine, and tangent functions over a restricted domain.
3.10.A Solve equations and inequalities involving trigonometric functions.
3.11.A Identify key characteristics of functions that involve quotients of the sine and cosine functions.
3.12.A Rewrite trigonometric expressions in equivalent forms with the Pythagorean identity.
3.12.B Rewrite trigonometric expressions in equivalent forms with sine and cosine sum identities.
3.12.C Solve equations using equivalent analytic representations of trigonometric functions.
3.13.A Determine the location of a point in the plane using both rectangular and polar coordinates.
3.14.A Construct graphs of polar functions.
3.15.A Describe characteristics of the graph of a polar function.

Career Readiness, Life Literacies, and Key Skills:

9.4.12.CI.1 Demonstrate the ability to reflect, analyze, and use creative skills and ideas.
9.4.12.CT.1 Identify problem-solving strategies used in the development of an innovative product or practice.
9.4.12.CT.2 Explain the potential benefits of collaborating to enhance critical thinking and problem solving.
9.4.12.IML.2 Evaluate digital sources for timeliness, accuracy, perspective, credibility of the source, and relevance of information, in media, data, or other resources.
9.4.12.TL.1 Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specific task.

Interdisciplinary Connections and Standards:

ELA

L.SS.11–12.1 Demonstrate command of the system and structure of the English language when writing or speaking.
L.VL.11–12.3 Determine or clarify the meaning of unknown and multiple-meaning words and phrases based on grades 11–12 reading and content, including technical meanings, choosing flexibly from a range of strategies.
W.AW.11–12.1 Write arguments to support claims in an analysis of substantive topics or texts, using valid reasoning and relevant and sufficient evidence.
W.IW.11–12.2 Write informative/explanatory texts (including the narration of historical events, scientific procedures/ experiments, or technical processes) to examine and convey complex ideas, concepts, and information clearly and accurately through the effective selection, organization, and analysis of content.
W.WR.11–12.5 Conduct short as well as more sustained research projects to answer a question (including a self-generated question) or solve a problem; narrow or broaden the inquiry when appropriate; synthesize multiple sources on the subject, demonstrating understanding of the subject under investigation.
SL.PE.11–12.1 Initiate and participate effectively in a range of collaborative discussions (one-on-one, in groups, and teacher-led) with peers on grades 11–12 topics, texts, and issues, building on others' ideas and expressing their own clearly and persuasively.

SL.II.11–12.2 Integrate multiple sources of information presented in diverse formats and media (e.g., visually, quantitatively, orally) in order to make informed decisions and solve problems, evaluating the credibility and accuracy of each source and noting any discrepancies among the data.
SL.PI.11–12.4 Present information, findings and supporting evidence clearly, concisely, and logically. The content, organization, development, and style are appropriate to task, purpose, and audience.
SL.AS.11–12.6 Adapt speech to a variety of contexts and tasks, demonstrating a command of formal English when indicated or appropriate.

Science

HS-ETS1-2 Design a solution to a complex real-world problem by breaking it down into smaller, more manageable problems that can be solved through engineering.

Unit Understandings:

Students will understand that…

- Trigonometric functions are ideal for modeling periodic, or repeated pattern phenomena, such as: the highs and lows of a wave, the blood pressure produced by a heart, and the angle from the North Pole to the Sun year to year.
- Periodicity is found in human inventions and social phenomena. For example, moving parts of an analog clock are modeled by a trigonometric function with respect to each other or with respect to time; traffic flow at an intersection over the course of a week demonstrates daily periodicity; and demand for a particular product over the course of a year falls into an annually repeating pattern.
- Polar functions are defined on the polar coordinate system that uses the circular concepts of radii and angles to describe location instead of the rectangular concepts of left-right and up-down.

Unit Essential Questions:

- Since energy usage goes up and down through the year, how can I use trends in data to predict my monthly electricity billswhenIgetmyfirst apartment?
- How do we model aspects of circular and spinning objects without using complex equations from the xy-rectangular-based coordinate system?
- How does right triangle trigonometry from geometry relate to trigonometric functions?

Knowledge and Skills:

Students will know...

3.1.A.1 A periodic relationship can be identified between two aspects of a context if, as the input values increase, the output values demonstrate a repeating pattern over successive equal-length intervals.
3.1.A.2 The graph of a periodic relationship can be constructed from the graph of a single cycle of the relationship
3.1.B.1 The period of the function is the smallest positive value k such that $f(x + k) = f(x)$ for all x in the domain. Consequently, the behavior of a periodic function is completely determined by any interval of width k.
3.1.B.2 The period can be estimated by investigating successive equal-length output values and finding where the pattern begins to repeat.
3.1.B.3 Periodic functions take on characteristics of other functions, such as intervals of increase and decrease, different concavities, and various rates of change. However, with periodic functions, all characteristics found in one period of the function will be in every period of the function.
3.2.A.1 In the coordinate plane, an angle is in standard position when the vertex coincides with the origin and one ray coincides with the positive x-axis. The other ray is called the terminal ray. Positive and negative angle measures indicate rotations from the positive x-axis in the counterclockwise and clockwise direction, respectively. Angles in standard position that share a terminal ray differ by an integer number of revolutions.
3.2.A.2 When considering a circle centered at the origin, the radian measure of an angle in standard position is the ratio of the length of the arc of the circle that the angle subtends to the radius of that same circle. For a

unit circle, which has radius 1, one radian is the measure of the angle subtended at the center of the circle by an arc that has length 1.

3.2.A.3 Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The sine of the angle is the ratio of the vertical displacement of P from the x-axis to the distance between the origin and point P. Therefore, for a unit circle, the sine of the angle is the y-coordinate of point P.

3.2.A.4 Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The cosine of the angle is the ratio of the horizontal displacement of P from the y-axis to the distance between the origin and point P. Therefore, for a unit circle, the cosine of the angle is the x-coordinate of point P.

3.2.A.5 Given an angle in standard position, the tangent of the angle is the slope, if it exists, of the terminal ray. Because the slope of the terminal ray is the ratio of the vertical displacement to the horizontal displacement over any interval, the tangent of the angle is the ratio of the y-coordinate to the x-coordinate of the point at which the terminal ray intersects the unit circle; alternately, it is the ratio of the angle's sine to its cosine.

3.3.A.1 Given an angle of measure θ in standard position and a circle with radius centered at the origin, there is a point, P, where the terminal ray intersects the circle. The coordinates of point P are $(r\cos\theta, r\sin\theta)$.

3.3.A.2 The geometry of isosceles right and equilateral triangles, while attending to the signs of the values based on the quadrant of the angle, can be used to find exact values for the cosine and sine of angles that are multiples of $\frac{\pi}{4}$, and $\frac{\pi}{6}$ radians and whose terminal rays do not lie on an axis.

3.4.A.1 Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The sine function, $f(\theta) = \sin\theta$, gives the y-coordinate, or vertical displacement from the x-axis, of point P. The domain of the sine function is all real numbers.

3.4.A.2 As the input values, or angle measures, of the sine function increase, the output values oscillate between -1 and 1, taking every value in between and tracking the vertical displacement of points on the unit circle from the x-axis.

3.4.A.3 Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The cosine function, $f(\theta) = \cos\theta$, gives the x-coordinate, or horizontal displacement from the y-axis, of point P. The domain of the cosine function is all real numbers.

3.4.A.4 As the input values, or angle measures, of the cosine function increase, the output values oscillate between -1 and 1, taking every value in between and tracking the horizontal displacement of points on the unit circle from the y-axis.

3.5.A.1 A sinusoidal function is any function that involves additive and multiplicative transformations of $f(\theta) = \sin\theta$. The sine and cosine functions are both sinusoidal functions, with $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$.

3.5.A.2 The period and frequency of a sinusoidal function are reciprocals. The period of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ is 2π, and the frequency is $\frac{1}{2\pi}$.

3.5.A.3 The amplitude of a sinusoidal function is half the difference between its maximum and minimum values. The amplitude of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ is 1.

3.5.A.4 The midline of the graph of a sinusoidal function is determined by the average, or arithmetic mean, of the maximum and minimum values of the function. The midline of the graphs of $y = \sin\theta$ and $y = \cos\theta$ is $y = 0$.

3.5.A.5 As input values increase, the graphs of sinusoidal functions oscillate between concave down and concave up.

3.5.A.6 The graph of $y = \sin\theta$ has rotational symmetry about the origin and is therefore an odd function. The graph of $y = \cos\theta$ has reflective symmetry over the y-axis and is therefore an even function.

3.6.A.1 Functions that can be written in the form $f(\theta) = a\sin(b(\theta + c)) + d$ or $g(\theta) = a\cos(b(\theta + c)) + d$, where a, b, c, and d are real numbers, $a \neq 0$, and $b \neq 0$, are sinusoidal

functions and are transformations of the sine and cosine functions. Additive and multiplicative transformations are the same for both sine and cosine because the cosine function is a phase shift of the sine function by $-\frac{\pi}{2}$ units.

3.6.A.2 The graph of the additive transformation $g(\theta) = \sin\theta + d$ of the sine function $f(\theta) = \sin\theta$ is a vertical translation of the graph of f , including its midline, by d units. The same transformation of the cosine function yields the same result.

3.6.A.3 The graph of the additive transformation $g(\theta) = \sin(\theta + c)$ of the sine function $f(\theta) = \sin\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units. The same transformation of the cosine function yields the same result

3.6.A.4 The graph of the multiplicative transformation $g(\theta) = a\sin\theta$ of the sine function $f(\theta) = \sin\theta$ is a vertical dilation of the graph of f and differs in amplitude by a factor of $|a|$. The same transformation of the cosine function yields the same result.

3.6.A.5 The graph of the multiplicative transformation $g(\theta) = \sin(b\theta)$ of the sine function $f(\theta) = \sin\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\left|\frac{1}{b}\right|$. The same transformation of the cosine function yields the same result.

3.6.A.6 The graph of $y = f(\theta) = a\sin(b(\theta + c)) + d$ has an amplitude of $|a|$ units, a period of $\left|\frac{1}{b}\right|2\pi$ units, a midline vertical shift of d units from $y = 0$, and a phase shift of $-c$ units. The same transformations of the cosine function yield the same results.

3.7.A.1 The smallest interval of input values over which the maximum or minimum output values start to repeat, that is, the input-value interval between consecutive maxima or consecutive minima, can be used to determine or estimate the period and frequency for a sinusoidal function model.

3.7.A.2 The maximum and minimum output values can be used to determine or estimate the amplitude and vertical shift for a sinusoidal function model.

3.7.A.3 An actual pair of input-output values can be compared to pairs of input-output values produced by a sinusoidal function model to determine or estimate a phase shift for the model.

3.7.A.4 Sinusoidal function models can be constructed for a data set with technology by estimating key values or using sinusoidal regressions.

3.7.A.5 Sinusoidal functions that model a data set are frequently only useful over their contextual domain, can be used to predict values of the dependent variable from values of the independent variable, and may be used to determine values of the independent variable from values of the dependent variable.

3.8.A.1 Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The tangent function, $f(\theta) = \tan\theta$, gives the slope of the terminal ray.

3.8.A.2 Because the slope of the terminal ray is the ratio of the change in the y-values to the change in the x-values between any two points on the ray, the tangent function is also the ratio of the sine function to the cosine function. Therefore, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, where $\cos\theta \neq 0$.

3.8.B.1 Because the slope values of the terminal ray repeat every one-half revolution of the circle, the tangent function has a period of π.

3.8.B.2 The tangent function demonstrates periodic asymptotic behavior at input values $\theta = \frac{\pi}{2} + k\pi$, for integer values of k, because $\cos\theta = 0$ at those values.

3.8.B.3 The tangent function increases and its graph changes from concave down to concave up between consecutive asymptotes.

3.8.C.1 The graph of the additive transformation $g(\theta) = \tan\theta + d$ of the tangent function $f(\theta) = \tan\theta$ is a vertical translation of the graph of f and the line containing its points of inflection by d units.

3.8.C.2 The graph of the additive transformation $g(\theta) = \tan(\theta + c)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units.

3.8.C.3 The graph of the multiplicative transformation $g(\theta) = a\tan\theta$ of the tangent function $f(\theta) = \tan\theta$ is a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the x-axis.

3.8.C.4 The graph of the multiplicative transformation $g(\theta) = \tan(b\theta)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\left|\frac{1}{b}\right|$. If $b < 0$, the transformation involves a reflection over the y-axis.

3.8.C.5 The graph of $y = f(\theta) = a\tan(b(\theta + c)) + d$ is a vertical dilation of the graph of $y = \tan\theta$ by a factor of $|a|$, has a period of $\left|\frac{1}{b}\right|\pi$ units, is a vertical shift of the line containing the points of inflection of the graph of $y = \tan\theta$ by d units, and is a phase shift of $-c$ units.

3.9.A.1 For inverse trigonometric functions, the input and output values are switched from their corresponding trigonometric functions, so the output value of an inverse trigonometric function is often interpreted as an angle measure and the input is a value in the range of the corresponding trigonometric function.

3.9.A.2 The inverse trigonometric functions are called arcsine, arccosine, and arctangent (also represented as $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$). Because the corresponding trigonometric functions are periodic, they are only invertible if they have restricted domains.

3.9.A.3 In order to define their respective inverse functions, the domain of the sine function is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the cosine function to $[0, \pi]$, and the tangent function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3.10.A.1 Inverse trigonometric functions are useful in solving equations and inequalities involving trigonometric functions, but solutions may need to be modified due to domain restrictions.

3.10.A.2 Because trigonometric functions are periodic, there are often infinitely many solutions to trigonometric equations.

3.10.A.3 In trigonometric equations and inequalities arising from a contextual scenario, there is often a domain restriction that can be implied from the context, which limits the number of solutions.

3.11.A.1 The secant function, $f(\theta) = \sec\theta$, is the reciprocal of the cosine function, where $\cos\theta \neq 0$.

3.11.A.2 The cosecant function, $f(\theta) = \csc\theta$, is the reciprocal of the sine function, where $\sin\theta \neq 0$.

3.11.A.3 The graphs of the secant and cosecant functions have vertical asymptotes where cosine and sine are zero, respectively, and have a range of $(-\infty, -1] \cup [1, \infty)$.

3.11.A.4 The cotangent function, $f(\theta) = \cot\theta$, is the reciprocal of the tangent function, where $\tan\theta \neq 0$. Equivalently, $\cot\theta = \frac{\cos\theta}{\sin\theta}$, where $\sin\theta \neq 0$.

3.11.A.5 The graph of the cotangent function has vertical asymptotes for domain values where $\tan\theta = 0$ and is decreasing between consecutive asymptotes.

3.12.A.1 The Pythagorean Theorem can be applied to right triangles with points on the unit circle at coordinates $(\cos\theta, \sin\theta)$, resulting in the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$.

3.12.A.2 The Pythagorean identity can be algebraically manipulated into other forms involving trigonometric functions, such as $\tan^2\theta = \sec^2\theta - 1$, and can be used to establish other trigonometric relationships, such as $\arcsin x = \arccos\left(\sqrt{1 - x^2}\right)$, with appropriate domain restrictions.

3.12.B.1 The sum identity for sine is $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

3.12.B.2 The sum identity for cosine is $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

3.12.B.3 The sum identities for sine and cosine can also be used as difference and double-angle identities.

3.12.B.4 Properties of trigonometric functions, known trigonometric identities, and other algebraic properties can be used to verify additional trigonometric identities.

3.12.C.1 A specific equivalent form involving trigonometric expressions can make information more accessible.

3.12.C.2 Equivalent trigonometric forms may be useful in solving trigonometric equations and inequalities.

3.13.A.1 The polar coordinate system is based on a grid of circles centered at the origin and on lines through the origin. The positive x-axis is called the polar axis. Polar coordinates of a point are defined as an ordered pair, (r, θ). The value of θ is given by the measure of an angle in standard position whose terminal ray lies in a line that passes through the point, and r is the radial displacement of the point from the origin. Positive values of r indicate radial displacement from the origin along the terminal ray of the angle θ. Negative values of r indicate radial displacement from the origin along the line containing the terminal ray of the angle θ, but in the opposite direction of the terminal ray of the angle θ. In the polar coordinate system, the same point can be represented many ways with combinations of positive and negative values of r and θ.

3.13.A.2 The coordinates of a point in the polar coordinate system, (r, θ), can be converted to coordinates in the rectangular coordinate system, (x, y), using $x = r\cos\theta$ and $y = r\sin\theta$.

3.13.A.3 The coordinates of a point in the rectangular coordinate system, (x, y), can be converted to coordinates in the polar coordinate system, (r, θ), using $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$ for $x > 0$ or $\theta = \arctan\left(\frac{y}{x}\right) + \pi$ for $x < 0$.

3.13.A.4 A complex number can be understood as a point in the complex plane and can be determined by its corresponding rectangular or polar coordinates. When the complex number has the rectangular coordinates (a, b), it can be expressed as $a + bi$. When the complex number has polar coordinates (r, θ), it can be expressed as $(r\cos\theta) + i(r\sin\theta)$.

3.14.A.1 The graph of the function $r = f(\theta)$ in polar coordinates consists of input-output pairs of values where the input values are angle measures and the output values are radii.

3.14.A.2 The domain of the polar function $r = f(\theta)$, given graphically, can be restricted to a desired portion of the function by selecting endpoints corresponding to the desired angle and radius.

3.14.A.3 When graphing polar functions in the form of $r = f(\theta)$, changes in input values relate to changes in angle measures from the polar axis, and changes in output values relate to changes in signed radial displacements from the origin.
3.15.A.1 If a polar function, $r = f(\theta)$, is positive and increasing or negative and decreasing, then the distance between the point with polar coordinates $(f(\theta), \theta)$ and the origin is increasing.
3.15.A.2 If a polar function, $r = f(\theta)$, is positive and decreasing or negative and increasing, then the distance between the point with polar coordinates $(f(\theta), \theta)$ and the origin is decreasing.
3.15.A.3 For a polar function, $r = f(\theta)$, if the function changes from increasing to decreasing or decreasing to increasing on an interval, then the function has a relative extremum on the interval corresponding to a point relatively closest to or farthest from the origin.
3.15.A.4 The average rate of change of r with respect to θ over an interval of θ is the ratio of the change in the signed radius values to the change in θ over an interval of θ. Graphically, the average rate of change indicates the rate at which the signed radius is changing per radian.
3.15.A.5 The average rate of change of r with respect to θ over an interval of θ can be used to estimate values of the function within the interval.

Students will be able to...
1.A Solve equations and inequalities represented analytically,with and without technology.
1.B Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.
1.C Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data,with and without technology.
2.A Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.
2.B Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
3.A Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.B Apply numerical results in a given mathematical or applied context.
3.C Support conclusions or choices with a logical rationale or appropriate data.

EVIDENCE OF LEARNING

Assessment:
What evidence will be collected and deemed acceptable to show that students truly "understand"?

- End of Unit Common Assessment
  - AP Classroom FRQs (Free Response Questions)
  - AP Classroom MCQs (Multiple Choice Questions)
- Flippedmath Unit 3A Assessments
  - AP Precalculus Mastery Checks - Unit 3A
  - AP Precalculus Tests - Unit 3A
- Flippedmath Unit 3B Assessments
  - AP Precalculus Mastery Checks - Unit 3B
  - AP Precalculus Tests - Unit 3B
- Khan Academy Exercises
- Daily observation of student work
- Student/group presentations

Learning Activities:

What differentiated learning experiences and instruction will enable all students to achieve the desired results?

- Mixed Six Worksheet: Working in small groups, students will complete tasks that approach Unit 3 concepts from multiple perspectives:
    - Recalling factual knowledge
    - Executing mathematical procedures
    - Classifying mathematical objects
    - Providing proofs and justifications
    - Extending mathematical concepts
    - Analyzing and critiquing fallacies
- AP Precalculus Unit 3 MCQ Strategies practice/discussion: First as a class, then in small groups, students will explore MCQ questions and focus on strategies to answer questions.
- AP Precalculus FRQ#3 Strategies practice/discussion: First as a class, then in small groups, students will explore FRQ#3 questions and focus on strategies to answer questions.
- AP Precalculus FRQ#4 Strategies practice/discussion: First as a class, then in small groups, students will explore FRQ#4 questions and focus on strategies to answer questions.
- Point or No Point FRQ#3: In small groups students will examine sample FRQ#3 answers and explain why each answer would or would not have received a point from the AP Scorers.
- Point or No Point FRQ#4: In small groups students will examine sample FRQ#4 answers and explain why each answer would or would not have received a point from the AP Scorers.
- Students are given analytical trigonometric functions paired with graphs on a single sheet of paper or on index cards. Some of the graphs are correct, and others are incorrect. In pairs, students identify the function for which the analytical and graphical representations are consistent and those for which the representations are inconsistent. For graphs that are in error, students construct the appropriate graph.
- Students are given a polar function. In pairs, students use their graphing calculator to determine intervals on which the function is positive and increasing, positive and decreasing, negative and increasing, and negative and decreasing. Students compare their answers and verify their results. Students then write, describing how the radius is changing as the angle is changing on each interval, and present this wording to the class.
- Procedural and Symbolic Fluency: While studying equivalent expressions of trigonometric functions, students will work with partners on an activity that involves rewriting analytical representations of trigonometric expressions. The activity will require the use of trigonometric identities. Students will verify their step-by-step work in Desmos by inputting each step and confirming that the new form of the expression matched the prior representation.
- Identify numerical solutions to equations in one variable: Students will solve trigonometric equations by finding where both functions intersect the x-axis on a calculator or by using the calculator's equation-solver capability. This activity will be completed via a classwork activity and a homework worksheet.
- As a classroom activity, students will be given contextual scenarios representing sinusoidal functions. Based on contextual clues and given input-output values, students will select and construct an appropriate model without the use of a calculator. The amplitude, midline, period, and phase shift given in context will inform model construction. Students will write a justification of the function type selected for each contextual scenario.
- An exam question gives students information on some cyclic phenomena, including maximal and minimal values and time between these values. Students must write formulas for functions that model these phenomena and justify how their functions meet the given criteria.
- As a classroom activity, students are given six tables of values. Each table contains 15 ordered pairs of values. For each table, students calculate each of the following regressions: linear, quadratic, cubic, quartic, exponential, logarithmic, and sinusoidal. Students choose an appropriate

model for each set of data and validate the decision using the residual plots. In some cases, more than one model may be appropriate.

- In a homework assignment, teachers provide students with the characteristics of a function such as the number and type of zeros, the end behavior, and the locations of extrema. Students build a function model from those criteria, without using regressions, including a justification for their model. The next day in class, they compare their models and justifications with their classmates' models and justifications, and the class picks the most accurate function model.
- The class is presented with sinusoidal functions that model the daily high temperatures and daily low temperatures for five different cities. A homework assignment asks students to decide in which city the cost to keep a house at a comfortable temperature would be greatest and least, and provide a rationale for their choice using appropriate sinusoidal function-related mathematical language.
- In a classroom activity, students will use the calculator to calculate a sinusoidal regression on monthly electricity usage in a town over a 12-month period of time. Students will use the calculator to produce the residual plot and determine the months for which the model underestimates the actual values and for which months the model overestimates the actual values. Using the absolute values of the residuals, students will determine the greatest difference between the predicted and actual value for the 12 data points.

RESOURCES

Teacher Resources:

- AP Classroom
- Flipped Math
- Khan Academy
- Desmos Calculator

Equipment Needed:

- Laptop/Chromebook
- Fast and reliable internet for teacher/students
- Newline Board/Projector
- Document Camera
- Connection cables
- Printer
- Graphing Calculators

UNIT 4 OVERVIEW

Content Area: Mathematics

Unit Title: Functions Involving Parameters, Vectors, and Matrices

Target Course/Grade Level: Advanced Placement Precalculus/Grades 10-12

Unit Summary: In Unit 4, students explore function types that expand their understanding of the function concept. Parametric functions have multiple dependent variables' values paired with a single input variable or parameter. Modeling scenarios with parametric functions allows students to explore change in terms of components. This component-based understanding is important not only in calculus but in all fields of the natural and social sciences where we seek to understand one aspect of a phenomenon independent of other confounding aspects. Another major function type in this unit involves matrices mapping a set of input vectors to output vectors. The capacity to map large quantities of vectors instantaneously is the basis for vector-based computer graphics. While students may see their favorite video game character trip and fall or seemingly move closer or farther away, matrices implement a rotation on a set of vectors or a dilation on a set of vectors. The power of matrices to map vectors is not limited to graphics but to any system that can be expressed in terms of components of vectors such as electrical systems, network connections, and regional population distribution changes over time. Vectors and matrices are also powerful tools of data science as they can be used to model aspects of complex scientific and social science phenomena.

Approximate Length of Unit: 7 weeks

LEARNING TARGETS

NJ Student Learning Standards:

N.VM.A.1 (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes

N.VM.A.2 (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N.VM.A.3 (+) Solve problems involving velocity and other quantities that can be represented by vectors.

N.VM.B.4 (+) Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction as $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w})$, where $(-\mathbf{w})$ is the additive inverse of $\mathbf{w}$, with the same magnitude as $\mathbf{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N.VM.B.5 (+) Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(\mathbf{v}_x, \mathbf{v}_y) = (c\mathbf{v}_x, c\mathbf{v}_y)$

b. Compute the magnitude of a scalar multiple $c\mathbf{v}$ using $||c\mathbf{v}|| = |c|\mathbf{v}$. Compute the direction of $c\mathbf{v}$ knowing that when $|c|\mathbf{v} \neq 0$, the direction of $c\mathbf{v}$ is either along $\mathbf{v}$ (for $c > 0$) or against $\mathbf{v}$ (for $c < 0$)

N.VM.C.6 (+) Perform operations on matrices and use matrices in application. Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N.VM.C.7 (+) Multiply matrices by scalars to produce new matrices, e.g. as when all of the payoffs in a game are doubled.

N.VM.C.8 (+) Add, subtract, and multiply matrices of appropriate dimensions.

N.VM.C.9 (+)Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

N.VM.C.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N.VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N.VM.C.12 (+)Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

College Board Standards:

4.1.A Construct a graph or table of values for a parametric function represented analytically.

4.2.A Identify key characteristics of a parametric planar motion function that are related to position.

4.3.A Identify key characteristics of a parametric planar motion function that are related to direction and rate of change.

4.4.A Express motion around a circle or along a line segment parametrically.

4.5.A Construct a graph of an equation involving two variables.

4.5.B Determine how the two quantities related in an implicitly defined function vary together

4.6.A Represent conic sections with horizontal or vertical symmetry analytically.

4.7.A Represent a curve in the plane parametrically.

4.7.B Represent conic sections parametrically.

4.8.A Identify characteristics of a vector.

4.8.B Determine sums and products involving vectors.

4.8.C Determine a unit vector for a given vector

4.8.D Determine angle measures between vectors and magnitudes of vectors involved in vector addition.

4.9.A Represent planar motion in terms of vector-valued functions.

4.10.A Determine the product of two matrices.

4.11.A Determine the inverse of a 2×2 matrix.

4.11.B Apply the value of the determinant to invertibility and vectors.

4.12.A Determine the output vectors of a linear transformation using a 2×2 matrix.

4.13.A Determine the association between a linear transformation and a matrix.

4.13.B Determine the composition of two linear transformations.

4.13.C Determine the inverse of a linear transformation.

4.14.A Construct a model of a scenario involving transitions between two states using matrices.

4.14.B Apply matrix models to predict future and past states for n transition steps.

Career Readiness, Life Literacies, and Key Skills:

9.4.12.CI.1 Demonstrate the ability to reflect, analyze, and use creative skills and ideas.

9.4.12.CT.1 Identify problem-solving strategies used in the development of an innovative product or practice.

9.4.12.CT.2 Explain the potential benefits of collaborating to enhance critical thinking and problem solving.

9.4.12.IML.2 Evaluate digital sources for timeliness, accuracy, perspective, credibility of the source, and relevance of information, in media, data, or other resources.

9.4.12.TL.1 Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specific task.

Interdisciplinary Connections and Standards:

ELA

L.SS.11–12.1 Demonstrate command of the system and structure of the English language when writing or speaking.

L.VL.11–12.3 Determine or clarify the meaning of unknown and multiple-meaning words and phrases based on grades 11–12 reading and content, including technical meanings, choosing flexibly from a range of strategies.

W.AW.11–12.1 Write arguments to support claims in an analysis of substantive topics or texts, using valid reasoning and relevant and sufficient evidence.

W.IW.11–12.2 Write informative/explanatory texts (including the narration of historical events, scientific procedures/ experiments, or technical processes) to examine and convey complex ideas, concepts, and information clearly and accurately through the effective selection, organization, and analysis of content.

W.WR.11–12.5 Conduct short as well as more sustained research projects to answer a question (including a self-generated question) or solve a problem; narrow or broaden the inquiry when appropriate; synthesize multiple sources on the subject, demonstrating understanding of the subject under investigation.

SL.PE.11–12.1 Initiate and participate effectively in a range of collaborative discussions (one-on-one, in groups, and teacher-led) with peers on grades 11–12 topics, texts, and issues, building on others' ideas and expressing their own clearly and persuasively.

SL.II.11–12.2 Integrate multiple sources of information presented in diverse formats and media (e.g., visually, quantitatively, orally) in order to make informed decisions and solve problems, evaluating the credibility and accuracy of each source and noting any discrepancies among the data.

SL.PI.11–12.4 Present information, findings and supporting evidence clearly, concisely, and logically. The content, organization, development, and style are appropriate to task, purpose, and audience.

SL.AS.11–12.6 Adapt speech to a variety of contexts and tasks, demonstrating a command of formal English when indicated or appropriate.

Science

HS-ETS1-2 Design a solution to a complex real-world problem by breaking it down into smaller, more manageable problems that can be solved through engineering.

Unit Understandings:

Students will understand that…

- Modeling scenarios with parametric functions allows students to explore change in terms of components.
- The capacity to map large quantities of vectors instantaneously is the basis for vector-based computer graphics. While students may see their favorite video game character trip and fall or seemingly move closer or farther away, matrices implement a rotation on a set of vectors or a dilation on a set of vectors
- The power of matrices to map vectors is not limited to graphics but to any system that can be expressed in terms of components of vectors such as electrical systems, network connections, and regional population distribution changes over time.

Unit Essential Questions:

- How can we determine when the populations of species in an ecosystem will be relatively steady?
- How can we analyze the vertical and horizontal aspects of motion independently?
- How does high resolution computer-generated imaging achieve smooth and realistic motion on screen with so many pixels?

Knowledge and Skills:

Students will know...

4.1.A.1 A parametric function in $\mathbb{R}^2$, the set of all ordered pairs of two real numbers, consists of a set of two parametric equations in which two dependent variables, x and y, are dependent on a single independent variable, t, called the parameter.

4.1.A.2 Because variables x and y are dependent on the independent variable, t, the coordinates $\left(x_i, y_i\right)$ at time t_i can be written as functions of t and can be expressed as the single parametric function $f(t) = (x(t), y(t))$, where in this case x and y are names of two functions.

4.1.A.3 A numerical table of values can be generated for the parametric function $f(t) = (x(t), y(t))$, by evaluating x_i and y_i at several values of t_i within the domain.

4.1.A.4 A graph of a parametric function can be sketched by connecting several points from the numerical table of values in order of increasing value of t.

4.1.A.5 The domain of the parametric function f is often restricted, which results in start and end points on the graph of f.

4.2.A.1 A parametric function given by $f(t) = (x(t), y(t))$, can be used to model particle motion in the plane. The graph of this function indicates the position of a particle at time t.

4.2.A.2 The horizontal and vertical extrema of a particle's motion can be determined by identifying the maximum and minimum values of the functions $x(t)$ and $y(t)$, respectively.

4.2.A.3 The real zeros of the function $x(t)$ correspond to y-intercepts, and the real zeros of $y(t)$ correspond to x -intercepts.

4.3.A.1 As the parameter increases, the direction of planar motion of a particle can be analyzed in terms of x and y independently. If $x(t)$ is increasing or decreasing, the direction of motion is to the right or left, respectively. If $y(t)$ is increasing or decreasing, the direction of motion is up or down, respectively.

4.3.A.2 At any given point in the plane, the direction of planar motion may be different for different values of t.

4.3.A.3 The same curve in the plane can be parametrized in different ways and can be traversed in different directions with different parametric functions.

4.3.A.4 Over a given interval $[t_1, t_2]$ within the domain, the average rate of change can be computed for $x(t)$ and $y(t)$ independently. The ratio of the average rate of change of y to the average rate of change of x gives the slope of the graph between the points on the curve corresponding to t_1 and t_2, so long as the average rate of change of $x(t) \neq 0$.

4.4.A.1 A complete counterclockwise revolution around the unit circle that starts and ends at (1,0) and is centered at the origin can be modeled by $(x(t), y(t)) = (\cos t, \sin t)$ with domain $0 \leq t \leq 2\pi$.

4.4.A.2 Transformations of the parametric function $(x(t), y(t)) = (\cos t, \sin t)$ can model any circular path traversed in the plane.

4.4.A.3 A linear path along the line segment from the point (x_1, y_1) to the point (x_2, y_2), can be parametrized many ways, including using an initial position (x_1, y_1) and rates of change for x with respect to t and y with respect to t.

4.5.A.1 An equation involving two variables can implicitly describe one or more functions.

4.5.A.2 An equation involving two variables can be graphed by finding solutions to the equation.

4.5.A.3 Solving for one of the variables in an equation involving two variables can define a function whose graph is part or all of the graph of the equation.

4.5.B.1 For ordered pairs on the graph of an implicitly defined function that are close together, if the ratio of the change in the two variables is positive, then the two variables simultaneously increase or both decrease; conversely, if the ratio is negative, then as one variable increases, the other decreases.

4.5.B.2 The rate of change of x with respect to y or of y with respect to x can be zero, indicating vertical or horizontal intervals, respectively

4.6.A.1 A parabola with vertex (h, k) can, if $a \neq 0$, be represented analytically as $x - h = a(y - k)^2$ if it opens left or right, or as $y - k = a(x - h)^2$ if it opens up or down.

4.6.A.2 An ellipse centered at (h, k), with horizontal radius a and vertical radius b can be represented analytically as $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. A circle is a special case of an ellipse where $a = b$.

4.6.A.3 A hyperbola centered at (h, k), with vertical and horizontal lines of symmetry can be represented algebraically as $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for a hyperbola opening left and right, or as $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ for a hyperbola opening up and down. The asymptotes are $y - k = \pm \frac{b}{a}(x - h)$.

4.7.A.1 A parametrization $(x(t), y(t))$ for an implicitly defined function will, when $x(t)$ and $y(t)$ are substituted for x and y, respectively, satisfy the corresponding equation for every value of t in the domain.

4.7.A.2 If f is a function of x, then $y = f(x)$ can be parametrized as $(x(t), y(t)) = (t, f(t))$. If f is invertible, its inverse can be parametrized as $(x(t), y(t)) = (f(t), t)$ for an appropriate interval of t.

4.7.B.1 A parabola can be parametrized in the same way that any equation that can be solved for x or y can be parametrized. Equations that can be solved for x can be parametrized as $(x(t), y(t)) = (f(t), t)$ by solving for x and replacing y with t. Equations that can be solved for y can be parametrized as $(x(t), y(t)) = (t, f(t))$ by solving for y and replacing x with t.

4.7.B.2 An ellipse can be parametrized using the trigonometric functions $x(t) = h + a\cos t$ and $y(t) = k + b\sin t$ for $0 \leq t \leq 2\pi$.

4.7.B.3 A hyperbola can be parametrized using trigonometric functions. For a hyperbola that opens left and right, the functions are $x(t) = h + a\sec t$ and $y(t) = k + b\tan t$ for $0 \leq t \leq 2\pi$. For a hyperbola that opens up and down, the functions are $x(t) = h + a\tan t$ and $y(t) = k + b\sec t$ for $0 \leq t \leq 2\pi$.

4.8.A.1 A vector is a directed line segment. When a vector is placed in the plane, the point at the beginning of the line segment is called the tail, and the point at the end of the line segment is called the head. The length of the line segment is the magnitude of the vector.

4.8.A.2 A vector $\overline{P_1P_2}$ with two components can be plotted in the xy-plane from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$. The vector is identified by a and b, where $a = x_2 - x_1$ and $b = y_2 - y_1$. The vector can be expressed as $\langle a, b\rangle$. A zero vector $\langle 0, 0\rangle$ is the trivial case when $P_1 = P_2$.

4.8.A.3 The direction of the vector is parallel to the line segment from the origin to the point with coordinates (a, b). The magnitude of the vector is the square root of the sum of the squares of the components.

4.8.A.4 For a vector represented geometrically in the plane, the components of the vector can be found using trigonometry.

4.8.B.1 The multiplication of a constant and a vector results in a new vector whose components are found by multiplying the constant by each of the components of the original vector. The new vector is parallel to the original vector.

4.8.B.2 The sum of two vectors in $\mathbb{R}^2$ is a new vector whose components are found by adding the corresponding components of the original vectors. The new vector can be represented graphically as a vector whose tail corresponds to the tail of the first vector and whose head corresponds to the head of the second vector when the second vector's tail is located at the first vector's head.

4.8.B.3 The dot product of two vectors is the sum of the products of their corresponding components. That is, $\langle a_1, b_1\rangle \bullet \langle a_2, b_2\rangle = a_1a_2 + b_1b_2$.

4.8.C.1 A unit vector is a vector of magnitude 1. A unit vector in the same direction as a given nonzero vector can be found by scalar multiplying the vector by the reciprocal of its magnitude.

4.8.C.2 The vector $\langle a, b\rangle$ can be expressed as $a\vec{i} + b\vec{j}$ in $\mathbb{R}^2$, where i and j are unit vectors in the x and y directions, respectively. That is, $\vec{i} = \langle 1, 0\rangle$ and $\vec{j} = \langle 0, 1\rangle$.

4.8.D.1 The dot product is geometrically equivalent to the product of the magnitudes of the two vectors and the cosine of the angle between them. Therefore, if the dot product of two nonzero vectors is zero, then the vectors are perpendicular.

4.8.D.2 The Law of Sines and Law of Cosines can be used to determine side lengths and angle measures of triangles formed by vector addition.

4.9.A.1 The position of a particle moving in a plane that is given by the parametric function $f(t) = (x(t), y(t))$ may be expressed as a vector-valued function, $p(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$ or $p(t) = \langle x(t), y(t) \rangle$. The magnitude of the position vector at time t gives the distance of the particle from the origin.

4.9.A.2 The vector-valued function $v(t) = \langle x(t), y(t) \rangle$ can be used to express the velocity of a particle moving in a plane at different times, t. At time t, the sign of $x(t)$ indicates if the particle is moving right or left, and the sign of $y(t)$ indicates if the particle is moving up or down. The magnitude of the velocity vector at time t gives the speed of the particle.

4.10.A.1 An $n \times m$ matrix is an array consisting of n rows and m columns.

4.10.A.2 Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second matrix. The product of the matrices is a new matrix in which the component in the *i*th row and *j*th column is the dot product of the *i*th row of the first matrix and the *j*th column of the second matrix.

4.11.A.1 The identity matrix, I, is a square matrix consisting of 1s on the diagonal from the top left to bottom right and 0s everywhere else.

4.11.A.2 Multiplying a square matrix by its corresponding identity matrix results in the original square matrix.

4.11.A.3 The product of a square matrix and its inverse, when it exists, is the identity matrix of the same size.

4.11.A.4 The inverse of a 2×2 matrix, when it exists, can be calculated with or without technology.

4.11.B.1 The determinant of the matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. The determinant can be calculated with or without technology and is denoted det(A) .

4.11.B.2 If a 2×2 matrix consists of two column or row vectors from $\mathbb{R}^2$, then the nonzero absolute value of the determinant of the matrix is the area of the parallelogram spanned by the vectors represented in the columns or rows of the matrix. If the determinant equals 0, then the vectors are parallel.

4.11.B.3 The square matrix A has an inverse if and only if det(A) ≠ 0.

4.12.A.1 A linear transformation is a function that maps an input vector to an output vector such that each component of the output vector is the sum of constant multiples of the input vector components.

4.12.A.2 A linear transformation will map the zero vector to the zero vector.

4.12.A.3 A single vector in $\mathbb{R}^2$ can be expressed as a 2×1 matrix. A set of n vectors in $\mathbb{R}^2$ can be expressed as a $2 \times n$ matrix.

4.12.A.4 For a linear transformation, L, from $\mathbb{R}^2$ to $\mathbb{R}^2$, there is a unique 2×2 matrix, A, such that $L(\vec{v}) = \mathrm{A}\vec{v}$ for vectors in $\mathbb{R}^2$. Conversely, for a given 2×2 matrix, A, the function $L(\vec{v}) = \mathrm{A}\vec{v}$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

4.12.A.5 Multiplication of a 2×2 transformation matrix, A, and a $2 \times n$ matrix of n input vectors gives a $2 \times n$ matrix of the n output vectors for the linear transformation $L(\vec{v}) = \mathrm{A}\vec{v}$

4.13.A.1 The linear transformation mapping $\langle x, y \rangle$ to $\langle a_{11}x + a_{12}y, a_{21}x + a_{22}y \rangle$ is associated with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

4.13.A.2 The mapping of the unit vectors in a linear transformation provides valuable information for determining the associated matrix.

4.13.A.3 The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is associated with a linear transformation of vectors that rotates every vector an angle θ counterclockwise about the origin.

4.13.A.4 The absolute value of the determinant of a 2×2 transformation matrix gives the magnitude of the dilation of regions in $\mathbb{R}^2$ under the transformation.
4.13.B.1 The composition of two linear transformations is a linear transformation.
4.13.B.2 The matrix associated with the composition of two linear transformations is the product of the matrices associated with each linear transformation.
4.13.C.1 Two linear transformations are inverses if their composition maps any vector to itself.
4.13.C.2 If a linear transformation, L is given by $L(\vec{v}) = \mathrm{A}\vec{v}$, then its inverse transformation is given by $L^{-1}(\vec{v}) = \mathrm{A}^{-1}\vec{v}$, where A^{-1} is the inverse of the matrix A
4.14.A.1 A contextual scenario can indicate the rate of transitions between states as percent changes. A matrix can be constructed based on these rates to model how states change over discrete intervals.
4.14.B.1 The product of a matrix that models transitions between states and a corresponding state vectorcanpredictfuturestates.
4.14.B.2 Repeated multiplication of a matrix that models the transitions between states and corresponding resultant state vectors can predict the steady state, a distribution between states that does not change from one step to the next.
4.14.B.3 The product of the inverse of a matrix that models transitions between states and a corresponding state vector can predict past states.

Students will be able to...
1.A Solve equations and inequalities represented analytically,with and without technology.
1.B Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.
1.C Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data,with and without technology.
2.A Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.
2.B Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
3.A Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.B Apply numerical results in a given mathematical or applied context.
3.C Support conclusions or choices with a logical rationale or appropriate data.

EVIDENCE OF LEARNING

Assessment:
What evidence will be collected and deemed acceptable to show that students truly "understand"?

- End of Unit Common Assessment
  - AP Classroom FRQs (Free Response Questions)
  - AP Classroom MCQs (Multiple Choice Questions)
- Flippedmath Unit 4A Assessments
  - AP Precalculus Mastery Checks - Unit 4A
  - AP Precalculus Tests - Unit 4A
- Flippedmath Unit 4B Assessments
  - AP Precalculus Mastery Checks - Unit 4B
  - AP Precalculus Tests - Unit 4B

- Khan Academy Exercises
- Daily observation of student work
- Student/group presentations

Learning Activities:

What differentiated learning experiences and instruction will enable all students to achieve the desired results?

- Mixed Six Worksheet: Working in small groups, students will complete tasks that approach Unit 4 concepts from multiple perspectives:
    - Recalling factual knowledge
    - Executing mathematical procedures
    - Classifying mathematical objects
    - Providing proofs and justifications
    - Extending mathematical concepts
    - Analyzing and critiquing fallacies
- In pairs, students discuss why the graphs of parametric functions can fail the vertical line test (and yet they are functions), but the graphs of many other previously studied function types must pass the vertical line test.
- Students are put into groups of 3-4 and are given a paper with a parametric function used to model planar motion. Each group gets a different function. Each person in the group writes a question on the paper that can be answered based on the given parametric function. On a separate blank sheet of paper, the group members verify that each question is answerable. Then, each group trades their question paper with another group. Each group works together to answer the questions on the received paper.
- Students are provided with four graphs: a parabola that opens up or opens down with the vertex at the origin, a circle centered at the origin, an ellipse not centered at the origin, and a hyperbola with center at the origin. In groups, students develop a reason for why each of the four graphs does not belong to the set of four. (For example: The parabola is the only one that can be explicitly defined with y as a function of x; the circle is the only one with infinitely many lines of symmetry; the ellipse does not belong because it is not symmetric to the y-axis; the hyperbola does not belong because it has two disconnected pieces.) This type of activity can be repeated with four different graphs of a parabola, a circle, an ellipse, and a hyperbola, where students come up with the graphs.
- Using information about the definition of the dot product, students construct an argument to explain why two nonzero vectors are perpendicular if, and only if, their dot product is equal to zero.
- In pairs, students create a graphic organizer to highlight the similarities and differences between linear functions and linear transformations.
- In a classroom assignment, students use a graphing calculator to compute products, inverses, and determinants of matrices.
- While studying conic sections, students will complete a card-sorting activity in which the cards contain different representations of given conic sections. For each conic section, there will be one card with a verbal description of the figure, one card with a graph of the conic, one card containing a set of ordered pairs for the conic, one card with an implicitly defined analytic equation, and one card with the conic expressed parametrically. After students sort the cards, they will provide a rationale for why each set of five cards represents the same conic section
- Students are given a worksheet containing two-by-two matrices. Students will find the determinant of each matrix by hand. If the determinant of the matrix is not zero, students will use the calculator to find the inverse of the matrix. Students will then multiply the original matrix by its inverse on the calculator. Then multiply the inverse by the original matrix to verify they are inverses.

RESOURCES

Teacher Resources

- AP Classroom
- Flipped Math
- Khan Academy
- Desmos Calculator

Equipment Needed:

- Laptop/Chromebook
- Fast and reliable internet for teacher/students
- Newline Board/Projector
- Document Camera
- Connection cables
- Printer
- Graphing Calculators