PreCalculus Honors Cover Page

Content Area: **Mathematics**
Course(s): **Pre Calculus Honors**
Time Period:
Length: **180 days**
Status: **Published**

Course Overview: Objectives, benchmarking and screening plan

PreCalculus Honors is a rigorous, fast-paced, full-year course designed for students who have completed Algebra 2 honors. It emphasizes the higher-level mathematical thinking and analytical reasoning necessary to pursue Advanced Placement Calculus and/or Statistics the following year. By exploring mathematical relationships through multiple perspectives, verbal, algebraic, graphical, and numeric, the curriculum fosters creative problem-solving and deep critical thinking. Through a combination of independent and collaborative tasks, alongside intensive assessments and projects requiring out-of-class commitment, students build a strong foundation in functions and trigonometry essential for future success in STEM fields.

Course Name, Length, Date of Revision and Curriculum Writer

PreCalculus Honors

Full Year Course

6/9/26

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Table of Contents

Unit 1: Polynomial and Rational Functions

Content Area: **Mathematics**
Course(s): **Pre Calculus Honors**
Time Period: **1st Marking Period**
Length: **8 weeks**
Status: **Published**

Summary of the Unit

Unit 1 serves as a bridge between intermediate algebra and the foundational concepts of Calculus. Students will first review and master diverse algebraic methods for solving linear, quadratic, rational, and radical equations. The curriculum then shifts to an advanced, multi-faceted exploration of functions. Students will explore the fundamental behavior of polynomial and rational functions by focusing on the core concepts of covariation and rates of change. The curriculum transitions from basic function notation and domain analysis to advanced techniques, such as identifying complex asymptotic behavior. Students will analyze functions both algebraically and graphically, focusing on domain, range, extrema, average rates of change, and increasing or decreasing intervals. Additionally, students learn to manipulate expressions and utilize transformations to graph functions both manually and with technological assistance. Beyond abstract calculation, the unit emphasizes real-world modeling to show how these mathematical rules apply to practical scenarios in business, science, and industry. By analyzing relative extrema and optimization, students develop the critical thinking skills necessary to interpret changing data trends and make informed, data-driven decisions.

Enduring Understandings

- Identify functions, and find their domain and range.
- Use function notation and evaluate functions abstractly and graphically.
- Use functions to model and solve real-life problems.
- Find the zeros of a function
- Analyze functions by determining intervals on which they are increasing, decreasing, or constant and determining their relative minimum and relative maximum values
- Analyze polynomial functions by finding all zeros using the Fundamental Theorem of Algebra, factoring, and graphing them, both manually and using technology.
- Identify even and odd functions.
- Analyze polynomial and rational functions by finding all zeros, and asymptotes (horizontal, vertical, and slanted) and graphing them, both manually and using technology.

- Rewrite polynomial and rational expressions in equivalent forms
- Recognize graphs of common functions, and use vertical, and horizontal shifts, reflections, and non-rigid transformations to graph them
- Use functions to model and solve real life problems
- Use polynomial and rational functions to model and solve real-life problems including ones with minimum and maximum values.

Essential Questions

- How do you determine the most efficient factoring strategy (e.g., GCF, grouping, difference of squares, or cubes) to factor a polynomial completely?
- Why must certain values be excluded from the domain of a rational expression, and how does factoring help simplify operations like multiplication and addition?
- When solving quadratic, radical, or rational equations, how do you evaluate which algebraic method is the most efficient and accurate?
- How can you determine if a relation is a function?
- How are functions and their graphs related?
- How can technology be used to investigate the properties of families of functions and their graphs?
- What does the degree of a polynomial tell you about its related polynomial function?
- For a polynomial function, how are factors, zeros, and x-intercepts related?
- How do the characteristics of graphs relate to their corresponding equations?
- How can you identify where a function is increasing, decreasing, or constant?
- How can algebra help us get information about a graph from an equation?
- How can I identify the characteristics of a rational function?
- What does the degree of a polynomial tell you about its related polynomial function?
- How can you rewrite polynomial and rational exponents and write them in equivalent form?
- How do you write equations and draw graphs for the simple transformations of functions?
- How do rational functions model real-world problems and their solutions?
- How do polynomial functions model real-world problems and their solutions?
- What does the average rate of change tell us about a function over a specific interval [a, b], and how does it connect to the slope of a line?
- How do algebraic and graphical methods (like the horizontal line test) help determine if a function is invertible, and what is the relationship between their input-output pairs?
- How does the degree of the numerator compared to the denominator determine a rational function's end behavior and the presence of horizontal or slant asymptotes?
- How do the multiplicities of zeros in the numerator versus the denominator reveal whether a rational function will have a vertical asymptote, a hole, or an x-intercept?

Unit Summative Assessment and Alternate Assessment Options

Suggested Formative/Summative Classroom Assessments

- Complete classwork Learning vertically at whiteboard
- Short/Extended Constructed Response Items
- Multiple-Choice Items (where multiple answer choices may be correct)
- Quizzes
- Journal Entries/Reflections/Quick-Writes
- Accountable talk
- Projects
- Observation
- Graphic Organizers/ Concept Mapping
- Presentations
- Teacher-Student and Student-Student Conferencing
- WebAssign Problem Sets WebAssign Instructor Help
- Error analysis tasks
- Task based station assessments
- Homework
- Students will take formal assessments, such as tests and quizzes, to assess knowledge of concepts learned throughout the unit.
- Unit 1 summative assessment taken at the end of the unit
- Students will also demonstrate mastery through various assessment criteria included in the unit such as do nows, exit slips, graded classwork activities and assignments, and/or projects.

Resources

- Classpad.net (Casio): https://classpad.net/us/
- Desmos: https://www.desmos.com/
- Geogebra: https://www.geogebra.org/?lang=en
- Math Open Reference: https://www.mathopenref.com/
- TI Education (Texas Instruments): https://education.ti.com/en

- WolframAlpha: https://www.wolframalpha.com/
- Wolfram MathWorld: https://mathworld.wolfram.com/
- Khan Academy: https://www.khanacademy.org/math/precalculus
- Digital Mathematics Word Wall: http://www.mathwords.com/index_adv_alg_precal.htm
- Extra Notes for Pre-Calculus Content: https://sites.google.com/a/evergreenps.org/ms-griffin-s-math-classes/updates
- Review Documents for Pre-Calculus: https://sites.google.com/site/dgrahamcalculus/trigpre-calculus/trig-pre-calculus-worksheets
- Pre-Calculus IXL Topics and Resources: https://www.ixl.com/math/precalculus
- Classroom Challenges to Support Teachers in Formative Assessments: http://map.mathshell.org/materials/lessons.php?gradeid=24
- Applications of Function Models: https://www.ck12.org/algebra/Applications-of-Function-Models/lesson/Applications-of-Function-Models-BSC-ALG/?referrer=featured_content
- Statistics Education Web (STEW). http://www.amstat.org/education/STEW/
- The Data and Story Library (DASL). http://lib.stat.cmu.edu/DASL/
- WebAssign Instructor Resources
- WebAssign Student Resources
- Cengage Learning: PreCalculus with Limits - A Graphing Approach, Sixth Edition

Unit Plan

Topic/Selection Timeframe	General Objectives	Instructional Activities	Formative and Summative Assessments
Factoring Polynomials (2 days)	SWBAT determine the best method to factor polynomials	Students will add, subtract, and multiply polynomials. Students will apply multiple factoring strategies such as the greatest common factor, factoring by grouping, product sum, difference of perfect squares, sum and difference of cubes to factor polynomials completely.	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to analyze a polynomial and identify the most efficient factoring technique to factor completely. Closure
Rational Expressions (3 days)	SWBAT simplify, multiply, divide, add and subtract rational expressions. SWBAT simplify complex fractions.	Students will specify numbers that must be excluded from the domain of a rational expression. Students will simplify, multiply, and divide rational expressions by factoring each term and canceling out like factors. Students will add and subtract rational expressions by using the least common denominator.	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to simplify, multiply, and

			divide rational expressions by factoring and simplifying. Through questioning students will be able to add and subtract rational expressions by successfully identify the least common denominator. Closure Prerequisite Skills Test # 1
Equations (4 days)	SWBAT use the best method to a solve an equation.	Students will solve linear equations in one variable and solve a formula for a variable. Students will solve quadratic equations by factoring, square roots, completing the square, and the quadratic formula. Students will determine of these methods which is the most efficient. Students will solve radical equations by isolating the radical and then raising each side to the power suggested by the index. Students will solve rational equations with variables in the denominator by using the least common denominator.	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to analyze an equation and identify the best method to solve it. Closure Prerequisite skills test # 2

1.1 - Change in Tandem (3 days)	SWBAT determine whether a relation is a function. SWBAT evaluate a function. SWBAT determine the domain and range of a function from a graph. SWBAT identify intercepts of a function both graphically and algebraically.	Students will use the definition of a function to determine whether a graph or an equation represents a function. Students will evaluate functions by substituting the input value into each variable. Students will recall the domain is the x value and the range is the y value. Students will analyze functions graphically and write the domain and range. Students will analyze graphs to identify the x and y intercepts. Students will find x intercept(s) algebraically by substituting y = 0 and the y - intercept algebraically by substituting x = 0 .	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to analyze and identify functions from graphical, numerical, analytical, and verbal representations. Closure
1.2 - Polynomial Functions (5 days)	SWBAT identify intervals on which a function increases, decreases, or is constant. SWBAT use the graph of a function to determine relative maxima and/or minima. SWBAT identify even and odd functions.	Students will determine intervals of increase, decrease, or constant by visual inspection. Students will determine relative maxima/minima by visual inspection. Students will determine whether a function is even, odd, or neither by using symmetry or algebraic tests.	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to analyze a graph and identify relative extrema , intervals of increase/decrease, and symmetry.

			Closure Quiz Chapter 1.1 - 1.2
1.3 - Average Rate of Change (3 days)	SWBAT calculate the slope of a line. SWBAT to write the equation of a line in point-slope and slope - intercept form. SWBAT find a function's average rate of change.	Given two points, students will use the slope formula to calculate the slope of a line. Given a point and slope or two points on a line, students will write linear equations in both point slope and slope intercept forms. Students will find a functions average rate of change on the closed interval [a, b] by applying the formula (f(b) - f(a)) / (b - a).	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to calculate the average rate of change on the interval [a, b] Closure 1.3 quiz
1.4 - Recognize graphs of common functions, and use vertical, and horizontal shifts, reflections, and non rigid transformations to graph them (6 days)	SWBAT analyze the effect of the coefficients on the graph of a function. SWBAT use and apply vertical and horizontal shifts, reflections, and horizontal and vertical dilations. SWBAT graph functions involving a sequence of	Students will analyze the effect of each variable af(b(x-h))+k has on the graph of a function. Students will translate parent functions vertically and horizontally. Students will reflect parent functions over the x and y axis. Students will dilate functions both horizontally and vertically by multiplying either x or y values respectively.	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to identify information from

	transformations.	Students will identify transformations by analyzing a function graph. Students will graph functions involving a sequence of transformations.	graphical, numerical, analytical, and verbal representations or construct a model with and without technology Closure Topic 1.4 Quiz
1.5 - Rational Functions and End Behaviors (5 days)	SWBAT Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. SWBAT Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.	Each student is given cards containing different rational functions in analytical representations. Have students use a calculator to graph the function and then record the intercepts on the card as well as limit expressions to describe the function's end behavior and behavior at each vertical or horizontal asymptote (e.g., lim as x approaches − ∞ f (x) = 3 , lim as x approaches 2 f (x) = ∞). In pairs, students take turns reading their limit statements to each other. Without seeing the actual rational function and without using a calculator, students will try to sketch the function's graph and then check and discuss. Have students rotate to form new pairs and repeat.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to identify information from graphical, numerical, analytical, and verbal representations or construct a model with and without technology Closure

1.6 - Equivalent Representations of Rational Expressions, Vertical Asymptotes, Holes, and Zeros. (9 days)	SWBAT express rational functions in analytically equivalent forms that are useful in a given mathematical or applied context. SWBAT describe the characteristics of a rational function with varying levels of precision, depending on the function representation.	Students are presented with a non constant polynomial or rational function in analytical representations, and they then translate into a variety of representations: constructing a graph, writing the product of linear factors (x- a) when possible, and describing characteristics such as real zeros, x - intercepts, asymptotes, and holes. Students will check their graphs using technology.	Circulate and monitor student progress as they are working on classwork. Have students complete problems vertically at the board. Peer/self assessments Through questioning students will be able to identify information from graphical, numerical, analytical, and verbal representations or construct a model with and without technology. Closure Quiz 1.5 and 1.6

Standards for Course Content Area and Cross Content Standards Addressed

ELA.L.SS.6.1.A	Ensure that pronouns are in the proper case (subjective, objective, possessive).
MA.9-12.1.1.A.1	A function is a mathematical relation that maps a set of input values to a set of output values such that each input value is mapped to exactly one output value. The set of input values is called the domain of the function, and the set of output values is called the range of the function. The variable representing input values is called the independent variable, and the variable representing output values is called the dependent variable.
MA.9-12.1.1.A.2	The input and output values of a function vary in tandem according to the function rule, which can be expressed graphically, numerically, analytically, or verbally.
MA.9-12.1.1.A.3	A function is increasing over an interval of its domain if, as the input values increase, the output values always increase. That is, for all a and b in the interval, if $a < b$, then $f(a) < f(b)$.

MA.9-12.1.1.A.4	A function is decreasing over an interval of its domain if, as the input values increase, the output values always decrease. That is, for all a and b in the interval, if $a < b$, then $f(a) > f(b)$.
MA.9-12.1.1.B.1	The graph of a function displays a set of input-output pairs and shows how the values of the function's input and output values vary.
MA.9-12.1.1.B.2	A verbal description of the way aspects of phenomena change together can be the basis for constructing a graph.
MA.9-12.1.1.B.5	The graph intersects the x-axis when the output value is zero. The corresponding input values are said to be zeros of the function.
MA.9-12.1.2.A.2	The rate of change of a function at a point quantifies the rate at which output values would change were the input values to change at that point. The rate of change at a point can be approximated by the average rates of change of the function over small intervals containing the point, if such values exist.
MA.9-12.1.2.B.1	Rates of change quantify how two quantities vary together.
MA.9-12.1.2.B.2	A positive rate of change indicates that as one quantity increases or decreases, the other quantity does the same.
MA.9-12.1.2.B.3	A negative rate of change indicates that as one quantity increases, the other decreases.
MA.9-12.1.4.A.1	A nonconstant polynomial function of x is any function representation that is equivalent to the analytical form $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$, where n is a positive integer, a_i is a real number for each i from 1 to n, and a_n is nonzero. The polynomial has degree n, the leading term is $a_n x^n$, and the leading coefficient is a_n. A constant is also a polynomial function of degree zero.
MA.9-12.1.4.A.2	Where a polynomial function switches between increasing and decreasing, or at the included endpoint of a polynomial with a restricted domain, the polynomial function will have a local, or relative, maximum or minimum output value. Of all local maxima, the greatest is called the global, or absolute, maximum. Likewise, the least of all local minima is called the global, or absolute, minimum.
MA.9-12.1.4.A.3	Between every two distinct real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or local minimum.
MA.9-12.1.4.A.4	Polynomial functions of an even degree will have either a global maximum or a global minimum.
MA.9-12.1.4.A.5	Points of inflection of a polynomial function occur at input values where the rate of change of the function changes from increasing to decreasing or from decreasing to increasing. This occurs where the graph of a polynomial function changes from concave up to concave down or from concave down to concave up.
MA.9-12.1.7.A.1	A rational function is analytically represented as a quotient of two polynomial functions and gives a measure of the relative size of the polynomial function in the numerator compared to the polynomial function in the denominator for each value in the rational function's domain.
MA.9-12.1.7.A.2	The end behavior of a rational function will be affected most by the polynomial with the greater degree, as its values will dominate the values of the rational function for input values of large magnitude. For input values of large magnitude, a polynomial is dominated by its leading term. Therefore, the end behavior of a rational function can be understood by examining the corresponding quotient of the leading terms.
MA.9-12.1.7.A.3	If the polynomial in the numerator dominates the polynomial in the denominator for input values of large magnitude, then the quotient of the leading terms is a nonconstant polynomial, and the original rational function has the end behavior of that polynomial. If

	that polynomial is linear, then the graph of the rational function has a slant asymptote parallel to the graph of the line.
MA.9-12.1.7.A.4	If neither polynomial in a rational function dominates the other for input values of large magnitude, then the quotient of the leading terms is a constant, and that constant indicates the location of a horizontal asymptote of the graph of the original rational function.
MA.9-12.1.7.A.5	If the polynomial in the denominator dominates the polynomial in the numerator for input values of large magnitude, then the quotient of the leading terms is a rational function with a constant in the numerator and nonconstant polynomial in the denominator, and the graph of the original rational function has a horizontal asymptote at y = 0.
MA.9-12.1.7.A.6	When the graph of a rational function r has a horizontal asymptote at $y = b$, where b is a constant, the output values of the rational function get arbitrarily close to b and stay arbitrarily close to b as input values increase or decrease without bound. The corresponding mathematical notation is $lim\ [x \rightarrow \infty]\ r(x) = b$ or $lim\ [x \rightarrow -\infty]\ r(x) = b$.
MA.9-12.1.8.A.1	The real zeros of a rational function correspond to the real zeros of the numerator for such values in its domain.
MA.9-12.1.9.A.1	If the value a is a real zero of the polynomial function in the denominator of a rational function and is not also a real zero of the polynomial function in the numerator, then the graph of the rational function has a vertical asymptote at $x = a$. Furthermore, a vertical asymptote also occurs at $x = a$ if the multiplicity of a as a real zero in the denominator is greater than its multiplicity as a real zero in the numerator.
MA.9-12.1.9.A.2	Near a vertical asymptote, $x = a$, of a rational function, the values of the polynomial function in the denominator are arbitrarily close to zero, so the values of the rational function r increase or decrease without bound. The corresponding mathematical notation is $lim\ [x \rightarrow a^+]\ r(x) = \infty$ or $lim\ [x \rightarrow a^+]\ r(x) = -\infty$ for input values near a and greater than a, and $lim\ [x \rightarrow a^-]\ r(x) = \infty$ or $lim\ [x \rightarrow a^-]\ r(x) = -\infty$ for input values near a and less than a.
MA.9-12.1.10.A.1	If the multiplicity of a real zero in the numerator is greater than or equal to its multiplicity in the denominator, then the graph of the rational function has a hole at the corresponding input value.
MA.9-12.1.10.A.2	If the graph of a rational function r has a hole at $x = c$, then the location of the hole can be determined by examining the output values corresponding to input values sufficiently close to c. If input values sufficiently close to c correspond to output values arbitrarily close to L, then the hole is located at the point with coordinates (c, L). The corresponding mathematical notation is $lim\ [x \rightarrow c]\ r(x) = L$. It should be noted that $lim\ [x \rightarrow c^-]\ r(x) = lim\ [x \rightarrow c^+]\ r(x) = lim\ [x \rightarrow c]\ r(x) = L$.
MA.9-12.1.11.A.1	Because the factored form of a polynomial or rational function readily provides information about real zeros, it can reveal information about x-intercepts, asymptotes, holes, domain, and range.
MA.9-12.1.11.A.2	The standard form of a polynomial or rational function can reveal information about end behaviors of the function.
MA.9-12.2.7.A	Evaluate the composition of two or more functions for given values.
MA.9-12.2.7.A.3	The composition of functions is not commutative; that is, $f \circ g$ and $g \circ f$ are typically different functions; therefore, $f(g(x))$ and $g(f(x))$ are typically different values.
MA.9-12.2.7.B.1	Function composition is useful for relating two quantities that are not directly related by an existing formula.
MA.9-12.2.7.C.2	An additive transformation of a function, f, that results in vertical and horizontal

	translations can be understood as the composition of $g(x) = x + k$ with f.
MA.9-12.2.7.C.3	A multiplicative transformation of a function, f, that results in vertical and horizontal dilations can be understood as the composition of $g(x) = kx$ with f.
MA.9-12.2.8.B	Determine the inverse of a function on an invertible domain.
MA.9-12.2.8.B.1	The composition of a function, f, and its inverse function, f^{-1}, is the identity function; that is, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Suggested Modifications for Students with Disabilities, 504 eligible, Multilingual Learners, At Risk Students and Gifted Students

Special Education Students:

- Visual Aids and Manipulatives: Provide visual aids such as graphic organizers, anchor charts, and manipulatives like algebra tiles or graphing boards to help students visualize concepts and relationships.
- Simplified Instructions: Break down complex concepts into smaller, more manageable steps with clear and concise instructions.
- Extra Practice and Scaffolding: Offer additional practice problems with varying difficulty levels, along with scaffolding (hints, prompts, worked examples) to support students as they progress.
- Alternative Assessments: Provide alternative assessment options, such as oral presentations, projects, or portfolios, to demonstrate understanding.
- Individualized Instruction: Tailor instruction to meet individual needs, considering learning styles, strengths, and weaknesses.

Multilingual Learners:

- Visual Support: Incorporate visuals, such as diagrams, graphs, and real-world examples, to enhance understanding of vocabulary and concepts.
- Simplified Language: Use clear and concise language, avoiding jargon and idioms. Provide a glossary of key terms with definitions and translations.
- Bilingual Resources: Offer bilingual dictionaries, textbooks, or online resources to support students' understanding.
- Collaborative Learning: Encourage peer-to-peer interaction and group work to provide opportunities for language practice and collaborative problem-solving.
- Differentiated Instruction: This method tailors instruction to meet individual language proficiency levels, providing additional support for students who need it.

Gifted Students:

- Extension Activities: Provide opportunities for enrichment and extension beyond the standard curriculum, such as exploring advanced topics, independent research projects, or problem-solving challenges.
- Inquiry-Based Learning: Encourage students to ask questions, explore concepts independently, and develop their solutions.
- Higher-Order Thinking Skills: Challenge students with complex problems that require critical thinking, analysis, and synthesis.
- Acceleration: Consider accelerating gifted students to higher-level math courses if they demonstrate readiness.
- Mentorship: Pair gifted students with mentors who can guide and support their academic pursuits.

Computer Sci Design Thinking

CS.9-12.8.1.12.AP.1	Design algorithms to solve computational problems using a combination of original and existing algorithms.
CS.9-12.8.1.12.AP.2	Create generalized computational solutions using collections instead of repeatedly using simple variables.
CS.9-12.8.1.12.AP.5	Decompose problems into smaller components through systematic analysis, using constructs such as procedures, modules, and/or objects.
CS.9-12.8.1.12.DA.1	Create interactive data visualizations using software tools to help others better understand real world phenomena, including climate change.
CS.9-12.8.1.12.DA.6	Create and refine computational models to better represent the relationships among different elements of data collected from a phenomenon or process.
CS.9-12.8.2.12.ED.5	Evaluate the effectiveness of a product or system based on factors that are related to its requirements, specifications, and constraints (e.g., safety, reliability, economic considerations, quality control, environmental concerns, manufacturability, maintenance and repair, ergonomics).
CS.9-12.8.2.12.ED.6	Analyze the effects of changing resources when designing a specific product or system (e.g., materials, energy, tools, capital, labor).
CS.9-12.8.2.12.ITH.1	Analyze a product to determine the impact that economic, political, social, and/or cultural factors have had on its design, including its design constraints.

Career Readiness, Life Literacies and Key Skills Practice

WRK.9.2.12.CAP.2	Develop college and career readiness skills by participating in opportunities such as structured learning experiences, apprenticeships, and dual enrollment programs.
TECH.9.4.12.CI.1	Demonstrate the ability to reflect, analyze, and use creative skills and ideas (e.g., 1.1.12prof.CR3a).
TECH.9.4.12.CI.3	Investigate new challenges and opportunities for personal growth, advancement, and transition (e.g., 2.1.12.PGD.1).
TECH.9.4.12.CT	Critical Thinking and Problem-solving
TECH.9.4.12.CT.1	Identify problem-solving strategies used in the development of an innovative product or practice (e.g., 1.1.12acc.C1b, 2.2.12.PF.3).
TECH.9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
TECH.9.4.12.TL.1	Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specified task (e.g., W.11-12.6.).
TECH.9.4.12.IML.1	Compare search browsers and recognize features that allow for filtering of information.

Unit 2: Exponential and Logarithmic Functions

Content Area: **Mathematics**
Course(s):
Time Period: **2nd Marking Period**
Length:
Status: **Published**

Summary of the Unit

In Unit 2, students will expand their toolkit of functions beyond polynomials to explore the behavior, graphs, properties, and applications of exponential, and logarithmic functions. Students will analyze and graph rational functions, identifying key features such as intercepts, asymptotes, discontinuities, domain, range, and end behavior. To bridge these concepts, students will explore arithmetic and geometric sequences as discrete models of change, establishing a foundation for analyzing linear and exponential patterns. Students will then explore exponential functions as models of growth and decay. They graph and transform exponential functions, examine the effects of parameters on function behavior, and interpret key characteristics. Building on their understanding of inverse relationships, students will investigate logarithmic functions as inverses of exponential functions. Throughout the unit, students will solve exponential and logarithmic equations using algebraic techniques, inverse operations, and technology when appropriate. They will develop mathematical models to represent and analyze real-world data, selecting and justifying appropriate function types based on context. Emphasis is placed on interpreting solutions. Exponential and logarithmic function models are widespread in the natural and social sciences. Exponential functions are key to modeling population growth, radioactive decay, interest rates, and the amount of medication in a patient. Logarithmic functions are useful in modeling sound intensity and frequency, the magnitude of earthquakes, the pH scale in chemistry, and the working memory in humans. The study of these two function types touches careers in business, medicine, chemistry, physics, education, and geography.

Enduring Understandings

- Recognize, write, and find nth terms of arithmetic and geometric sequences.
- Use arithmetic and geometric sequences to model real world problems.
- Describe similarities and differences between linear and exponential functions.
- Construct functions that are comparable to arithmetic and geometric sequences.
- Solve real world applications using arithmetic and geometric sequences.
- Identify key characteristics of exponential functions.
- Use properties of exponents to simplify and evaluate expressions.
- Apply properties of logarithms to evaluate expressions and graph functions.
- Understand the characteristics and effects of constants in the exponential model y = ab^x.
- Apply exponential models about data sets in contextual scenarios and construct a model.
- Construct exponential function models using regression equations and technology.
- Evaluate the composition of two or more functions for given values.
- Rewrite a given function as a composition of two ore more functions
- Determine the inverse of a function on an invertible domain.
- Identify key characteristics of logarithmic functions

- Rewrite logarithmic expressions in equivalent forms.
- Solve exponential and logarithmic equations and inequalities.
- Construct the inverse of exponential functions.

Essential Questions

- How do we use common differences and common ratios to distinguish between arithmetic and geometric patterns, and how do they help us model discrete real-world scenarios?
- How can I make a single model that merges the interest I earn from my bank with the taxes that are due so I can know how much I will have in the end?
- How can we adjust the scale of distance for a model of planets in the solar system so the relationships among the planets are easier to see?
- If different functions can be used to model data, how do we pick which one is best?
- Why is the number e important?
- What are the key characteristics of an exponential function?
- How do the key characteristics of an exponential function graph—such as domain, range, continuity, and horizontal asymptotes—change depending on whether the base represents growth or decay?
- How can algebraic properties of exponents be applied to transform parent exponential functions, and how do these transformations alter their graphic behavior?
- How do I use exponential functions to model exponential behavior?
- What is the mathematical relationship between an exponential expression and a logarithm, and what does a single unit change on a logarithmic scale represent?
- How does reflecting an exponential graph over the identity line y = x visually and algebraically prove the inverse relationship between exponential and logarithmic functions?
- Why is the domain of a general-form logarithmic function restricted to numbers greater than zero, and how does this constraint dictate its vertical asymptotic behavior?
- Why is the domain of a general-form logarithmic function restricted to numbers greater than zero, and how does this constraint dictate its vertical asymptotic behavior?
- What is the relationship between an exponential expression and a logarithm?

Unit Summative Assessment and Alternate Assessment Options

Suggested Formative/Summative Classroom Assessments

- Complete classwork Learning vertically at whiteboard
- Short/Extended Constructed Response Items

- Multiple-Choice Items (where multiple answer choices may be correct)
- Quizzes
- Journal Entries/Reflections/Quick-Writes
- Accountable talk
- Projects
- Observation
- Graphic Organizers/ Concept Mapping
- Presentations
- Teacher-Student and Student-Student Conferencing
- WebAssign Problem Sets WebAssign Instructor Help
- Error analysis tasks
- Task based station assessments
- Homework
- Students will take formal assessments, such as tests and quizzes, to assess knowledge of concepts learned throughout the unit.
- Unit 2 summative assessment taken at the end of the unit
- Students will also demonstrate mastery through various assessment criteria included in the unit such as do nows, exit slips, graded classwork activities and assignments, and/or projects.

Resources

- Classpad.net (Casio): https://classpad.net/us/
- Desmos: https://www.desmos.com/
- Geogebra: https://www.geogebra.org/?lang=en
- Math Open Reference: https://www.mathopenref.com/
- TI Education (Texas Instruments): https://education.ti.com/en
- WolframAlpha: https://www.wolframalpha.com/
- Wolfram MathWorld: https://mathworld.wolfram.com/
- Khan Academy: https://www.khanacademy.org/math/precalculus
- Digital Mathematics Word Wall: http://www.mathwords.com/index_adv_alg_precal.htm

- Extra Notes for Pre-Calculus Content: https://sites.google.com/a/evergreenps.org/ms-griffin-s-math-classes/updates
- Review Documents for Pre-Calculus: https://sites.google.com/site/dgrahamcalculus/trigpre-calculus/trig-pre-calculus-worksheets
- Pre-Calculus IXL Topics and Resources: https://www.ixl.com/math/precalculus
- Classroom Challenges to Support Teachers in Formative Assessments: http://map.mathshell.org/materials/lessons.php?gradeid=24
- Applications of Function Models: https://www.ck12.org/algebra/Applications-of-Function-Models/lesson/Applications-of-Function-Models-BSC-ALG/?referrer=featured_content
- Statistics Education Web (STEW). http://www.amstat.org/education/STEW/
- The Data and Story Library (DASL). http://lib.stat.cmu.edu/DASL/
- WebAssign Instructor Resources
- WebAssign Student Resources
- Cengage Learning: PreCalculus with Limits - A Graphing Approach, Sixth Edition

Topic/Selection Timeframe	General Objectives	Instructional Activities	Benchmarks/Assessments
2.1 - Changes in Arithmetic and Geometric Sequences (7 days)	SWBAT recognize, write, and find nth terms of arithmetic and geometric sequences. SWBAT use arithmetic and geometric sequences to model real-world problems.	Students will identify the common difference and first term of arithmetic sequences written in a variety of ways and use that to write a rule. Students will model arithmetic sequences from contextual and mathematical scenarios. Students will identify the common ratio and the first term of geometric sequences written in a variety of ways and use that information to write a rule. Students will model geometric sequences from contextual and mathematical scenarios.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to write rules for arithmetic and geometric sequences and construct a model with and without technology Closure Quiz 2.1
2.2 - Changes in Linear and Exponential Functions (2 days)	SWBAT describes similarities and differences between linear and exponential functions. SWBAT constructs functions that are comparable to arithmetic and	Students will write linear functions and understand how it is similar to an explicit rule of an arithmetic sequence. Students will write rules for exponential functions and understand how it is similar to an explicit rule of geometric	Circulate and monitor student progress as they are working on classwork. Have students complete problems on the board. Through questioning, students will be able to identify similarities and differences between arithmetic sequences and linear functions and geometric sequences and exponential

	geometric sequences. SWBAT to solve real world applications using arithmetic and geometric sequences.	sequences. Students will understand the sequences and their corresponding functions may have different domains. Students will model geometric and arithmetic sequences from mathematical and contextual scenarios.	functions. Students will also be able to construct a model with and without technology Closure
2.3 – Exponential Functions (2 days)	SWBAT to identify key characteristics of exponential functions.	Students will investigate the basic behaviors of exponential functions and their graphs. Students will analyze exponential graphs for key characteristics, such as domain, range, end behavior, continuity, symmetry, asymptotes, and intervals of increase and decrease.	Circulate and monitor student progress as they are working on classwork. Have students complete problems on the board. Through questioning students will be able to identify key characteristics of exponential functions. Closure
2.4 - Exponential Function Manipulation (2 days)	SWBAT use properties of exponents to simplify and evaluate expressions. SWBAT apply properties of logarithms to evaluate expressions and graph functions. SWBAT understand the characteristics and effects of constants in the	Students will simplify functions using rules of exponents. Students will apply transformations of functions to exponential functions. Students will graph these functions and identify key characteristics.	Circulate and monitor student progress as they are working on classwork. Have students complete problems on the board. Through questioning students will be able to apply transformations to graph exponential functions and analyze key characteristics. Closure

	exponential model y = ab^x.		
2.5 - Exponential Function Context and Data Modeling (3 days)	SWBAT construct a model for scenarios involving proportional output values related to real life scenarios. SWBAT apply exponential models about data sets in contextual scenarios. SWBAT construct exponential function models using regression equations and technology.	Students will identify if a situation or equation represents exponential growth or decay. Students will write equations for exponential models from an appropriate ratio and initial value or from two input output pairs by solving a system of equations. Students will construct exponential function models for a data set with technology using exponential regressions. Students will apply exponential models to answer questions about a data set or contextual scenario.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to construct a model for scenarios involving exponential functions, apply exponential models about data sets ,and construct exponential function models with and without technology. Closure Sec 2.2 - 2.5
2.6 - Composite Functions (3 days)	SWBAT evaluate the composition of two or more functions for given values. SWBAT construct a representation of the composition of two or more functions. SWBAT rewrite a given function as a composition of two or more functions.	Students will perform function operations. Students will find a composition of one function with another. Students will rewrite a given function as a composite of two or more functions. Students will recognize that the composition of functions is not commutative.	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to construct new functions using transformations and compositions that may be useful in modeling contexts, criteria , or data with or without technology Closure

2.7 - Inverse Functions (4 days)	SWBAT determine the input - output pairs of the inverse of a function. SWBAT determine the inverse of a function on an invertible domain.	Students will verify that two functions are inverses both algebraically and graphically. Students will use the horizontal line test to determine if the functions are one to one. Students will find inverse functions both informally and algebraically. Students will use function notation to describe real world scenarios. Students will analyze and choose the appropriate method to find an inverse of a relation or function.	Circulate and monitor progress as students are working on classwork examples. Have students complete problems vertically at the whiteboard. Peer/self assessments Through questioning students will be able to find the inverse of a function both algebraically and graphically and determine if a function is invertible. Closure 2.6 - 2.7 quiz
2.8 - Logarithmic Expressions (2 days)	SWBAT to evaluate logarithmic expressions.	Students will apply the properties of logarithms to evaluate expressions with and without technology. Students will use properties of exponents and logarithms to simplify and evaluate expressions. Students will understand that each unit of the logarithmic scale is a multiplicative change of base 10,	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to express functions, equations, or expressions in analytically equivalent forms. Closure

2.9 - Inverses of Exponential Functions (2 days)	SWBAT construct representations of the inverse of an exponential function with an initial value of 1.	Students will understand the way in which input and output values vary together have an inverse relationship in exponential and logarithmic functions. Students will show the compositions of exponential functions and log functions are inverses. Students will show the graph of an exponential function reflected over the line y = x is the graph of a logarithmic function.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to construct new functions using transformations, compositions, inverses, or regressions, with and without technology. Closure
2.10 - Logarithmic Functions (1 day)	SWBAT identify key characteristics of logarithmic functions.	Students will understand the domain of logarithmic functions is any real number greater than zero. Students will understand that just like exponential functions, logarithmic functions are always increasing or decreasing and are either always concave up or concave down. Students will understand how to apply transformations to logarithmic graphs.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to describe the characteristics of a function with varying levels of precision. Closure

2.11 - Logarithmic Function Manipulation (4 days)	SWBAT rewrite logarithmic expressions in equivalent forms.	Students will apply the properties of logarithms to evaluate expressions and graph functions. Students will use properties of exponents and logarithms to simplify and evaluate expressions Students will recognize graphic translations of logarithms related to the product, power, and change of base property.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to express functions, equations, or expressions in analytically equivalent forms. Closure Quiz 2.8 - 2.11
2.12 - Exponential and Logarithmic Equations (7 days)	SWBAT solve exponential and logarithmic equations and inequalities. SWBAT construct the inverse function for exponential and logarithmic functions.	Students will use properties of exponents and logarithms to solve equations. When solving exponential and logarithmic equations students will analyze for extraneous solutions. Students will use inverse operations to write exponential and logarithmic equations that reverse the mapping.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to solve equations and inequalities represented analytically with and without technology. Closure Quiz 2.12

Standards for Course Content Area and Cross Content Standards Addressed

MA.9-12.2.1.A.1	A sequence is a function from the whole numbers to the real numbers. Consequently, the graph of a sequence consists of discrete points instead of a curve.
MA.9-12.2.1.A.2	Successive terms in an arithmetic sequence have a common difference, or constant rate of change.
MA.9-12.2.1.A.3	The general term of an arithmetic sequence with a common difference d is denoted by a_n and is given by $a_n = a_0 + dn$, where a_0 is the initial value, or by $a_n = a_k + d(n - k)$, where a_k is the kth term of the sequence.
MA.9-12.2.1.B.1	Successive terms in a geometric sequence have a common ratio, or constant proportional change.
MA.9-12.2.1.B.2	The general term of a geometric sequence with a common ratio r is denoted by g_n and is given by $g_n = g_0r^n$, where g_0 is the initial value, or by $g_n = g_kr^{(n - k)}$ where g_k is the kth term of the sequence.
MA.9-12.2.1.B.3	Increasing arithmetic sequences increase equally with each step, whereas increasing geometric sequences increase by a larger amount with each successive step.
MA.9-12.2.2.A.1	Linear functions of the form $f(x) = b + mx$ are similar to arithmetic sequences of the form $a_n = a_0 + dn$, as both can be expressed as an initial value (b or a_0) plus repeated addition of a constant rate of change, the slope (m or d).
MA.9-12.2.2.A.3	Exponential functions of the form $f(x) = ab^x$ are similar to geometric sequences of the form $g_n = g_0r^n$, as both can be expressed as an initial value (a or g_0) times repeated multiplication by a constant proportion (b or r).
MA.9-12.2.2.A.4	Similar to geometric sequences of the form $g_n = g_kr^{(n - k)}$, which are based on a known ratio, r, and a kth term, exponential functions can be expressed in the form $f(x) = y_ir^{(x - x_i)}$ based on a known ratio, r, and a point, (x_i, y_i).
MA.9-12.2.2.A.5	Sequences and their corresponding functions may have different domains.
MA.9-12.2.2.B.1	Over equal-length input-value intervals, if the output values of a function change at constant rate, then the function is linear; if the output values of a function change proportionally, then the function is exponential.
MA.9-12.2.2.B.2	Linear functions of the form $f(x) = b + mx$ and exponential functions of the form $f(x) = ab^x$ can both be expressed analytically in terms of an initial value and a constant involved with change. However, linear functions are based on addition, while exponential functions are based on multiplication.
MA.9-12.2.2.B.3	Arithmetic sequences, linear functions, geometric sequences, and exponential functions all have the property that they can be determined by two distinct sequence or function values.
MA.9-12.2.3.A.1	The general form of an exponential function is $f(x) = ab^x$, with the initial value a, where $a \neq 0$, and the base b, where $b > 0$, and $b \neq 1$. When $a > 0$ and $b > 1$, the exponential function is said to demonstrate exponential growth. When $a > 0$ and $0 < b < 1$, the exponential function is said to demonstrate exponential decay.
MA.9-12.2.3.A.2	When the natural numbers are input values in an exponential function, the input value specifies the number of factors of the base to be applied to the function's initial value. The domain of an exponential function is all real numbers.
MA.9-12.2.3.A.3	Because the output values of exponential functions in general form are proportional over equal-length input-value intervals, exponential functions are always increasing or always decreasing, and their graphs are always concave up or always concave down. Consequently, exponential functions do not have extrema except on a closed interval, and their graphs do not have points of inflection.

MA.9-12.2.3.A.4	If the values of the additive transformation function $g(x) = f(x) + k$ of any function f are proportional over equal-length input-value intervals, then f is exponential.
MA.9-12.2.3.A.5	For an exponential function in general form, as the input values increase or decrease without bound, the output values will increase or decrease without bound or will get arbitrarily close to zero. That is, for an exponential function in general form, $lim\ [x \rightarrow \pm\infty]\ ab^x = \infty$, $lim\ [x \rightarrow \pm\infty]\ ab^x = -\infty$, $lim\ [x \rightarrow \pm\infty]\ ab^x = 0$.
MA.9-12.2.4.A.1	The product property for exponents states that $b^m b^n = b^{(m+n)}$. Graphically, this property implies that every horizontal translation of an exponential function, $f(x) = b^{(x+k)}$, is equivalent to a vertical dilation, $f(x) = b^{(x+k)} = b^x b^k = ab^x$, where $a = b^k$.
MA.9-12.2.4.A.2	The power property for exponents states that $(b^m)^n = b^{(mn)}$. Graphically, this property implies that every horizontal dilation of an exponential function, $f(x) = b^{(cx)}$, is equivalent to a change of the base of an exponential function, $f(x) = (b^c)^x$, where b^c is a constant and $c \neq 0$.
MA.9-12.2.4.A.3	The negative exponent property states that $b^{-n} = 1/b^n$.
MA.9-12.2.4.A.4	The value of an exponential expression involving an exponential unit fraction, such as $b^{(1/k)}$ where k is a natural number, is the kth root of b, when it exists.
MA.9-12.2.5.A.1	Exponential functions model growth patterns where successive output values over equal-length input-value intervals are proportional. When the input values are whole numbers, exponential functions model situations of repeated multiplication of a constant to an initial value.
MA.9-12.2.5.A.2	A constant may need to be added to the dependent variable values of a data set to reveal a proportional growth pattern.
MA.9-12.2.5.A.3	An exponential function model can be constructed from an appropriate ratio and initial value or from two input-output pairs. The initial value and the base can be found by solving a system of equations resulting from the two input-output pairs.
MA.9-12.2.5.A.4	Exponential function models can be constructed by applying transformations to $f(x) = ab^x$ based on characteristics of a contextual scenario or data set.
MA.9-12.2.5.A.5	Exponential function models can be constructed for a data set with technology using exponential regressions.
MA.9-12.2.5.A.6	The natural base e, which is approximately 2.718, is often used as the base in exponential functions that model contextual scenarios.
MA.9-12.2.5.B.1	For an exponential model in general form $f(x) = ab^x$, the base of the exponent, b, can be understood as a growth factor in successive unit changes in the input values and is related to a percent change in context.
MA.9-12.2.5.B.2	Equivalent forms of an exponential function can reveal different properties of the function. For example, if d represents number of days, then the base of $f(d) = 2^d$ indicates that the quantity increases by a factor of 2 every day, but the equivalent form $f(d) = 2^{7(d7)}$ indicates that the quantity increases by a factor of 2^7 every week.
MA.9-12.2.5.B.3	Exponential models can be used to predict values for the dependent variable, depending on the contextual constraints on the domain.
MA.9-12.2.6.A.1	Two variables in a data set that demonstrate a slightly changing rate of change can be modeled by linear, quadratic, and exponential function models.
MA.9-12.2.6.A.2	Models can be compared based on contextual clues and applicability to determine which model is most appropriate.
MA.9-12.2.6.B.1	A model is justified as appropriate for a data set if the graph of the residuals of a regression, the residual plot, appear without pattern.

MA.9-12.2.6.B.2	The difference between the predicted and actual values is the error in the model. Depending on the data set and context, it may be more appropriate to have an underestimate or overestimate for any given interval.
MA.9-12.2.7.A.1	If f and g are functions, the composite function $f \circ g$ maps a set of input values to a set of output values such that the output values of g are used as input values of f. For this reason, the domain of the composite function is restricted to those input values of g for which the corresponding output value is in the domain of f. $(f \circ g)(x)$ can also be represented as $f(g(x))$.
MA.9-12.2.7.A.2	Values for the composite function $f \circ g$ can be calculated or estimated from the graphical, numerical, analytical, or verbal representations of f and g by using output values from g as input values for f.
MA.9-12.2.7.A.4	If the function $f(x) = x$ is composed with any function g, the resulting composite function is the same as g; that is, $g(f(x)) = f(g(x)) = g(x)$. The function $f(x) = x$ is called the identity function. When composing two functions, the identity function acts in the same way as 0, the additive identity, when adding two numbers and 1, the multiplicative identity, when multiplying two numbers.
MA.9-12.2.7.B.1	Function composition is useful for relating two quantities that are not directly related by an existing formula.
MA.9-12.2.7.B.2	When analytic representations of the functions f and g are available, an analytic representation of $f(g(x))$ can be constructed by substituting $g(x)$ for every instance of x in f.
MA.9-12.2.7.B.3	A numerical or graphical representation of $f \circ g$ can often be constructed by calculating or estimating values for $(x, f(g(x)))$.
MA.9-12.2.7.C.1	Functions given analytically can often be decomposed into less complicated functions. When properly decomposed, the variable in one function should replace each instance of the function with which it was composed.
MA.9-12.2.7.C.2	An additive transformation of a function, f, that results in vertical and horizontal translations can be understood as the composition of $g(x) = x + k$ with f.
MA.9-12.2.7.C.3	A multiplicative transformation of a function, f, that results in vertical and horizontal dilations can be understood as the composition of $g(x) = kx$ with f.
MA.9-12.2.8.B.1	The composition of a function, f, and its inverse function, f^{-1}, is the identity function; that is, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.
MA.9-12.2.8.B.3	The inverse of the graph of the function $y = f(x)$ can be found by reversing the roles of the x- and y-axes; that is, by reflecting the graph of the function over the graph of the identity function $h(x) = x$.
MA.9-12.2.8.B.5	In addition to limiting the domain of a function to obtain an inverse function, contextual restrictions may also limit the applicability of an inverse function.
MA.9-12.2.9.A.1	The logarithmic expression $log\_b\ c$ is equal to, or represents, the value that the base b must be exponentially raised to in order to obtain the value c. That is, $log\_b\ c = a$ if and only if $b^a = c$, where a and c are constants, $b > 0$, and $b \neq 1$. (when the base of a logarithmic expression is not specified, it is understood as the common logarithm with a base of 10)
MA.9-12.2.9.A.2	The values of some logarithmic expressions are readily accessible through basic arithmetic while other values can be estimated through the use of technology.
MA.9-12.2.9.A.3	On a logarithmic scale, each unit represents a multiplicative change of the base of the logarithm. For example, on a standard scale, the units might be 0, 1, 2, …, while on a logarithmic scale, using logarithm base 10, the units might be 10^0, 10^1, 10^2, ….

MA.9-12.2.10.A.1	The general form of a logarithmic function is $f(x) = a\ log\_b\ x$, with base b, where $b > 0$, $b \neq 1$, and $a \neq 0$.
MA.9-12.2.10.A.2	The way in which input and output values vary together have an inverse relationship in exponential and logarithmic functions. Output values of general-form exponential functions change proportionately as input values increase in equal-length intervals. However, input values of general-form logarithmic functions change proportionately as output values increase in equal-length intervals. Alternately, exponential growth is characterized by output values changing multiplicatively as input values change additively, whereas logarithmic growth is characterized by output values changing additively as input values change multiplicatively.
MA.9-12.2.10.A.3	$f(x) = log\_b\ x$ and $g(x) = b^x$, where $b > 0$ and $b \neq 1$, are inverse functions. That is, $g(f(x)) = f(g(x)) = x$.
MA.9-12.2.10.A.4	The graph of the logarithmic function $f(x) = log\_b\ x$, where $b > 0$ and $b \neq 1$, is a reflection of the graph of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, over the graph of the identity function $h(x) = x$.
MA.9-12.2.10.A.5	If (s, t) is an ordered pair of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, then (t, s) is an ordered pair of the logarithmic function $f(x) = log\_b\ x$, where $b > 0$ and $b \neq 1$.
MA.9-12.2.11.A.1	The domain of a logarithmic function in general form is any real number greater than zero, and its range is all real numbers.
MA.9-12.2.11.A.4	With their limited domain, logarithmic functions in general form are vertically asymptotic to $x = 0$, with an end behavior that is unbounded. That is, for a logarithmic function in general form, $lim\ [x \to 0^+]\ a\ log\_b\ x = \pm\infty$ and $lim\ [x \to \infty]\ a\ log\_b\ x = \pm\infty$.
MA.9-12.2.12.A.1	The product property for logarithms states that $log\_b\ (xy) = log\_b\ x + log\_b\ y$. Graphically, this property implies that every horizontal dilation of a logarithmic function, $f(x) = log\_b\ (kx)$, is equivalent to a vertical translation, $f(x) = log\_b\ (kx) = log\_b\ k + log\_b\ x = a + log\_b\ x$, where $a = log\_b\ k$.
MA.9-12.2.12.A.3	The change of base property for logarithms states that $log\_b\ x = (log_a\ x)/(log_a\ b)$, where $a > 0$ and $a \neq 1$. This implies that all logarithmic functions are vertical dilations of each other.
MA.9-12.2.12.A.4	The function $f(x) = \ln x$ is a logarithmic function with the natural base e; that is, $\ln x = \log_e x$.

Suggested Modifications for Students with Disabilities, 504 eligible, Multilingual Learners, At Risk Students and Gifted Students

Special Education Students:

- Visual Aids and Manipulatives: Provide visual aids such as graphic organizers, anchor charts, and manipulatives like algebra tiles or graphing boards to help students visualize concepts and relationships.
- Simplified Instructions: Break down complex concepts into smaller, more manageable steps with clear and concise instructions.
- Extra Practice and Scaffolding: Offer additional practice problems with varying difficulty levels, along with scaffolding (hints, prompts, worked examples) to support students as they progress.
- Alternative Assessments: Provide alternative assessment options, such as oral presentations, projects, or portfolios, to demonstrate understanding.
- Individualized Instruction: Tailor instruction to meet individual needs, considering learning styles, strengths, and weaknesses.

Multilingual Learners:

- Visual Support: Incorporate visuals, such as diagrams, graphs, and real-world examples, to enhance understanding of vocabulary and concepts.
- Simplified Language: Use clear and concise language, avoiding jargon and idioms. Provide a glossary of key terms with definitions and translations.
- Bilingual Resources: Offer bilingual dictionaries, textbooks, or online resources to support students' understanding.
- Collaborative Learning: Encourage peer-to-peer interaction and group work to provide opportunities for language practice and collaborative problem-solving.
- Differentiated Instruction: This method tailors instruction to meet individual language proficiency levels, providing additional support for students who need it.

Gifted Students:

- Extension Activities: Provide opportunities for enrichment and extension beyond the standard curriculum, such as exploring advanced topics, independent research projects, or problem-solving challenges.
- Inquiry-Based Learning: Encourage students to ask questions, explore concepts independently, and develop their solutions.
- Higher-Order Thinking Skills: Challenge students with complex problems that require critical thinking, analysis, and synthesis.
- Acceleration: Consider accelerating gifted students to higher-level math courses if they demonstrate readiness.
- Mentorship: Pair gifted students with mentors who can guide and support their academic pursuits.

Computer Sci Design Thinking

CS.9-12.8.1.12.DA.1	Create interactive data visualizations using software tools to help others better understand real world phenomena, including climate change.
CS.9-12.8.1.12.DA.5	Create data visualizations from large data sets to summarize, communicate, and support different interpretations of real-world phenomena.
CS.9-12.8.1.12.DA.6	Create and refine computational models to better represent the relationships among different elements of data collected from a phenomenon or process.
CS.9-12.8.2.12.ED.5	Evaluate the effectiveness of a product or system based on factors that are related to its requirements, specifications, and constraints (e.g., safety, reliability, economic considerations, quality control, environmental concerns, manufacturability, maintenance and repair, ergonomics).
CS.9-12.8.2.12.ITH.1	Analyze a product to determine the impact that economic, political, social, and/or cultural factors have had on its design, including its design constraints.

Career Readiness, Life Literacies and Key Skills Practice

Business & Finance Connection

Name of Task: Corporate Supply Chains and Logistics Scaling

NJSLS Alignments: 9.1.12.C.3, 9.4.12.CI.1, W.11-12.1

Context: A digital streaming startup tracks user acquisition using an exponential model. Meanwhile, an automated fulfillment center optimizes warehouse inventory using a logistic growth curve to avoid over-saturation. The active storage utilization percentage over time (in weeks) is modeled by the function:

$U(t) = 100/(1 + 9e^{-0.35t})$

a. Find the initial capacity utilization of the warehouse at $t = 0$.

b. Identify the horizontal asymptotes of $U(t)$ by evaluating the end behavior as t approaches to infinity. Interpret what this constraint means to a logistics manager monitoring physical warehouse space.

c. The facility triggers an automated infrastructure expansion order the exact week capacity utilization reaches 85%. Set up an inequality and solve algebraically using natural logarithms (ln) to determine the exact week the expansion is triggered.

d. An analyst named Sam notes: "Because this function contains a negative exponent property $e^{(-0.35t)}$, warehouse inventory must be shrinking over time." Write an argument to support or reject Sam's conclusion by evaluating the structural composition of the entire denominator.

Career Awareness & Financial Literacy Connection

Name of Task: The Cost of Waiting (Retirement Modeling)

NJSLS Alignments: 9.1.12.CDM.1, 9.1.12.FP.4

Context: To demonstrate the power of continuous compound interest ($A = Pe^{(rt)}$, a corporate human resources department provides two retirement pathway simulations to onboarding employees:

- Pathway A (Early Investor): Invests $5,000 annually starting at age 22, earning an average of 6.5% compounded continuously, but completely stops adding money at age 32—leaving the principal to sit and compound untouched for the next 33 years until retirement at age 65.

- Pathway B (Late Investor): Waits until age 32 to start investing, then deposits $5,000 annually at the exact same 6.5% continuously compounded rate every single year until retirement at age 65.

a. Write an exponential function to model the growth of Pathway A's asset value after age 32, where t represents the number of years left compounding untouched until retirement.

b. Using a graphing calculator or spreadsheet tool, determine the total value of both portfolios at age 65. Who contributed more out-of-pocket principal? Who ended up with a larger final net worth?

c. Use inverse operations to determine the exact interest rate a Late Investor would need to match the Early Investor's final balance if they only have 20 years to save.

d. Reflective Practice: Write a short summary explaining how the properties of exponential function bases justify the classic financial literacy advice: "Time in the market beats timing the market."

TECH.9.4.12.CI	Creativity and Innovation
TECH.9.4.12.CI.3	Investigate new challenges and opportunities for personal growth, advancement, and transition (e.g., 2.1.12.PGD.1).

TECH.9.4.12.CT.1	Identify problem-solving strategies used in the development of an innovative product or practice (e.g., 1.1.12acc.C1b, 2.2.12.PF.3).
TECH.9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
TECH.9.4.12.TL.1	Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specified task (e.g., W.11-12.6.).
TECH.9.4.12.IML.1	Compare search browsers and recognize features that allow for filtering of information.
TECH.9.4.12.IML.3	Analyze data using tools and models to make valid and reliable claims, or to determine optimal design solutions (e.g., S-ID.B.6a., 8.1.12.DA.5, 7.1.IH.IPRET.8).

Unit 3: Trigonometric and Polar Functions

Content Area: **Mathematics**
Course(s):
Time Period: **3rd Marking Period**
Length: **8 weeks**
Status: **Published**

Summary of the Unit

In this unit, students extend their understanding of geometry and functions through the study of trigonometry, learning to analyze and describe different types of angles, angular movement, and their relation to real-life phenomena. The unit begins with right triangle trigonometry, where students apply their knowledge of trigonometric ratios to solve for unknown side lengths and angles. Students then transition to a broader understanding of angles through the development of radian measure, exploring the relationship between arc length and angle measure, sketching angles in standard position, and identifying the properties of positive and negative angles. Through these investigations, students develop a deeper understanding of angles as rotational quantities and establish connections between geometric and algebraic representations. Building on this foundation, students construct and analyze the unit circle to find exact values of the six trigonometric functions, connecting it to the coordinate plane to explain periodic behavior. The unit concludes with the study of trigonometric graphs, where students analyze and graph the sine, cosine, and tangent functions, identifying key features such as amplitude, period, phase shift, vertical shifts, domain, and range. They will investigate transformations of these functions, use mathematical models to make predictions about real-world periodic phenomena, and apply both trigonometric relationships and algebraic techniques to solve equations. Furthermore, students will extend their trigonometric foundation to the polar coordinate system, learning to plot polar functions and convert between rectangular and polar representations. Throughout the unit, students will consistently utilize multiple representations, mathematical reasoning, and technology to model and communicate complex relationships involving angles, functions, and coordinate systems.

Enduring Understandings

Measure angles in standard position on the coordinate plane and their properties.

A radian is an angle measure with an arc length of one radius.

Label the angles on the unit circle in radians using proportional reasoning.

Use special right triangles to determine the coordinates at key points on the unit circle. Evaluate sine, cosine, and tangent for key angles on the unit circle.

Construct graphs of the sine and cosine functions by observing that as the input values, or angle measures, of the sine function increase, the output values of sine and cosine oscillate between −1 and 1.

Relate additive transformations with horizontal/vertical transformation and multiplicative transformation with dilation and use these transformations to graph transformed sine and cosine functions.

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology

Essential Questions

- How does right triangle trigonometry from geometry relate to trigonometric functions?
- How can we use mathematics to identify and model patterns that repeat over time?
- How do the geometric properties of special right triangles serve as the building blocks for the entire unit circle?
- What happens to our trigonometric coordinates when we scale a circle's radius away from 1?
- How does the continuous motion of traveling around a circle translate into a wave-like graphical representation?
- What causes the domain of sine and cosine functions to be infinite while their range remains restricted?
- What structural properties define a function as "sinusoidal"?
- How do the concepts of even and odd functions manifest visually in trigonometric graphs?
- How do algebraic changes to a trigonometric equation predictably stretch, compress, or shift its corresponding wave?
- Why is understanding the "midline" and "amplitude" essential when graphing a transformed function?
- How can sinusoidal functions be used to forecast and model real-world fluctuations like tidal waves or energy usage?
- Why does the tangent function behave so differently (producing asymptotes and an altered period) compared to sine and cosine?
- What is the fundamental difference between what a trigonometric function inputs/outputs versus what an inverse trigonometric function inputs/outputs?
- Why do trigonometric equations often have an infinite number of solutions, and how do we determine which solutions matter in a given context?
- In what ways do real-world scenarios involving heights and distances rely on our perspective of looking up (elevation) versus looking down (depression)?
- How does the Pythagorean Theorem evolve into a tool for rewriting and simplifying complex trigonometric expressions?
- How does changing our frame of reference from a rectangular grid (x, y) to a circular grid (r, θ) alter how we define location?
- Why can a single point in space have infinitely many different names in the polar coordinate system?
- How do changes in an angle's measure translate into a dynamic distance from the origin when plotting a polar function?
- How can we mathematically determine when a polar graph is spiraling closer to or moving farther away from its center point?

Unit Summative Assessment and Alternate Assessment Options

- Suggested Formative/Summative Classroom Assessments
- Describe Learning Vertically
- Identify Key Building Blocks

- Make Connections (between and among crucial building blocks)
- Short/Extended Constructed Response Items
- Multiple-Choice Items (where multiple answer choices may be correct)
- Drag and Drop Items
- Use of Equation Editor
- Quizzes
- Journal Entries/Reflections/Quick-Writes
- Accountable talk
- Projects
- Portfolio
- Observation
- Graphic Organizers/ Concept Mapping
- Presentations
- Teacher-Student and Student-Student Conferencing
- WebAssign Problem Sets WebAssign Instructor Help
- Homework
- Students will take formal assessments, such as tests and quizzes, to assess knowledge of concepts learned throughout the unit.
- Students will also demonstrate mastery through various assessment criteria included in the unit such as do nows, exit slips, graded classwork activities and assignments, and/or projects.

Resources

- Classpad.net (Casio): https://classpad.net/us/
- Desmos: https://www.desmos.com/
- Geogebra: https://www.geogebra.org/?lang=en
- Math Open Reference: https://www.mathopenref.com/
- TI Education (Texas Instruments): https://education.ti.com/en
- WolframAlpha: https://www.wolframalpha.com/
- Wolfram MathWorld: https://mathworld.wolfram.com/
- Khan Academy: https://www.khanacademy.org/math/precalculus
- Digital Mathematics Word Wall: http://www.mathwords.com/index_adv_alg_precal.htm
- Extra Notes for Pre-Calculus Content: https://sites.google.com/a/evergreenps.org/ms-griffin-s-math-classes/updates
- Review Documents for Pre-Calculus: https://sites.google.com/site/dgrahamcalculus/trigpre-

calculus/trig-pre-calculus-worksheets

- Pre-Calculus IXL Topics and Resources: https://www.ixl.com/math/precalculus
- Classroom Challenges to Support Teachers in Formative Assessments: http://map.mathshell.org/materials/lessons.php?gradeid=24
- Applications of Function Models: https://www.ck12.org/algebra/Applications-of-Function-Models/lesson/Applications-of-Function-Models-BSC-ALG/?referrer=featured_content
- Statistics Education Web (STEW). http://www.amstat.org/education/STEW/
- The Data and Story Library (DASL). http://lib.stat.cmu.edu/DASL/
- WebAssign Instructor Resources
- WebAssign Student Resources
- Cengage Learning: PreCalculus with Limits - A Graphing Approach, Sixth Edition

Unit Plan

Topic/Selection Timeframe	General Objectives	Instructional Activities	Formative and Summative Assessments
3.1 Periodic Phenomena (2 days)	SWBAT Construct graphs of periodic relationships based on verbal representations. SWBAT Describe key characteristics of a periodic function based on a verbal representation.	Students will determine how to measure angles in standard position on the coordinate plane and their properties. Students will determine that a radian is an angle measure with an arc length of one radius. Students will label the angles on the unit circle in radians using proportional reasoning.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to determine coordinates of unit circle. Closure Quiz sec 3.1
3.2 Sine Cosine Tangent (3 days)	SWBAT Determine the sine, cosine, and tangent of an angle using the unit circle.	Students will use special right triangles to determine the coordinates at key points on the unit circle. Students will evaluate sine, cosine, and tangent for key angles on the unit circle. Students will find coordinates of points on circles where $r \neq 1$.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to evaluate sine, cosine, and tangent for key angles on the unit circle. Closure

3.3 Sine and Cosine Function Values (3 days)	SWBAT Determine coordinates of points on a circle centered at the origin.	Students will determine that in a unit circle, the sine and cosine ratios correspond to the y-value and x-value, respectively, of the point where the terminal ray intersects the circle. Students then will be able to use symmetry to identify relationships between the sine and cosine values of angles in all four quadrants.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to determine values of any angle in the unit circle. Closure Quiz sec3.2 - 3.3
3.4 Sine and Cosine Function Graphs (3 days)	Construct representations of the sine and cosine functions using the unit circle.	Students would be able to construct graphs of the sine and cosine functions by observing that as the input values, or angle measures, of the sine function increase, the output values of sine and and cosine oscillate between −1 and 1, taking every value in between and tracking the vertical distance of points on the unit circle from the x-axis.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to construct graphs of the sine and cosine functions using values from the unit circle. Closure
3.5 Sinusoidal Functions (3 days)	SWBAT Identify key characteristics of the sine and cosine functions.	Students would be able to determine that a sinusoidal function is any function that involves additive and multiplicative transformations of sin x. The sine and cosine functions are both sinusoidal functions. The period and frequency of a sinusoidal function are reciprocals. The amplitude of a sinusoidal function is half the difference between its maximum and minimum values. The midline of the graph of a sinusoidal function is determined by	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to Identify key characteristics for the parent functions y=sin x and y=cos x, including domain, amplitude, midline, period, and symmetry. Closure

		the average, or arithmetic mean, of the maximum and minimum values of the function. The midline of the graphs of y = sinθ and y = cosθ is y = 0. As input values increase, the graphs of sinusoidal functions oscillate between concave down and concave up. The graph of y = sinx has rotational symmetry about the origin and is therefore an odd function. The graph of y = cosx has reflective symmetry over the y-axis and is therefore an even function.	
3.6 Sinusoidal Function Transformations (3 days)	SWBAT Identify the amplitude, vertical shift, period, and phase shift of a sinusoidal function.	Students will determine how the amplitude, period, domain, range, and midline of sinusoidal functions are affected by transformations. Students would be able relate additive transformations with horizontal/vertical transformation and and multiplicative transformation with dilation and use these transformations to graph transformed sine and cosine functions.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to graph transformed sine and cosine functions given an equation . Closure Quiz Sec 3.4 - 3.6
3.7 Sinusoidal Function Context and Data Modeling (2 days)	SWBAT Construct sinusoidal function models of periodic phenomena.	Students will determine how to use trigonometric ratios and angle measured relative to the north-south line to solve problems using navigations. Students will Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. Students would be able to describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to Interpret a sinusoidal function's period, amplitude, midline, and range in context be able to construct a trigonometric model based on data points and key features. Closure

3.8 The Tangent Function (2 days)	SWBAT Construct representations of the tangent function using the unit circle. SWBAT Describe key characteristics of the tangent function. SWBAT Describe additive and multiplicative transformations involving the tangent function.	Students would be able to construct graphs of the tangent functions by observing that as the input values, or angle measures, of the tangent function increase. The asymptote equations are found when the function in the denominator is zero.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to evaluate tangent for key angles on the unit circle. Closure Test sec 3.4 - 3.8
3.9 Inverse Trigonometric Functions (3 days)	SWBAT Construct analytical and graphical representations of the inverse of the sine, cosine, and tangent functions over a restricted domain.	SWBAT understand that inverse trigonometric functions input ratios and output angles. The input and output values are switched from their corresponding trigonometric functions. SWBAT understand through graphical representation why and how the domains of sine, cosine, and tangent must be restricted to create an inverse function. SWBAT evaluate inverse trig expressions.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to evaluate inverse of sine, cosine, and tangent for key points on the unit circle. Closure
3.10 Trigonometric Equations (3 days)	SWBAT Solve equations involving trigonometric functions.	SWBAT extend the process of inverse operations to trigonometric equations. SWBAT understand that using the unit circle will give infinite solutions to a trigonometric equation which may need to be restricted based on context and that an inverse trig function gives only one solution that may need to be expanded using symmetry.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to find solutions given trigonometric equations. Closure Quiz Sec 3.9-3.10
3.11 The Secant, Cosecant, and Cotangent Functions (3 days)	SWBAT Identify key characteristics of functions that involve quotients of the sine and cosine functions.	SWBAT define the secant, cosecant, and cotangent functions as the reciprocal of the cosine, sine, and tangent functions, respectively. SWBAT understand how the zeros, vertical asymptotes, and range are related for a trigonometric function and its reciprocal function.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to Identify key characteristics for the parent functions y=csc x and y=sec x and cotangent including domain, period, asymptotes and symmetry. Closure

3.12 Equivalent Representations of Trigonometric Functions (3 days)	SWBAT Rewrite trigonometric expressions in equivalent forms with the Pythagorean identity. SWBAT Rewrite trigonometric expressions in equivalent forms with sine and cosine sum identities. SWBAT Solve equations using equivalent analytic representations of trigonometric functions.	SWBAT explore relationships between all six trigonometric functions, including the Pythagorean identities. SWBAT use identities to establish and verify other trigonometric relationships and solve trigonometric equations	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to verify trigonometric relationships. Closure Quiz: sec 3.11- 3.12
3.13 Trigonometry and Polar Coordinates (3 days)	SWBAT Determine the location of a point in the plane using both rectangular and polar coordinates.	SWBAT understand that polar coordinates give an alternate method for locating points using a distance from the origin and an angle from the positive x-axis. SWBAT use coterminal angles and reflected radii to name polar points in multiple ways. SWBAT convert between polar and rectangular coordinates.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to plot coordinates, and convert between polar and rectangular coordinates. Closure
3.14 Polar Function Graphs (3 days)	SWBAT Construct graphs of polar functions.	SWBAT understand that polar functions input angle measures and output radii and point-by-point graphing can be used to construct their graphs.	Circulate and monitor student progress as they are working on classwork. Have students complete problems at the board. Through questioning students will be able to graph polar graphs using key features. Closure Quiz: sec 3.13- 3.14

Standards for Course Content Area and Cross Content Standards Addressed

MA.9-12.3.1.A.1	A periodic relationship can be identified between two aspects of a context if, as the input values increase, the output values demonstrate a repeating pattern over successive equal-length intervals.
MA.9-12.3.1.A.2	The graph of a periodic relationship can be constructed from the graph of a single cycle of the relationship.
MA.9-12.3.1.B.1	The period of the function is the smallest positive value k such that $f(x + k) = f(x)$ for all x in the domain. Consequently, the behavior of a periodic function is completely determined by any interval of width k.
MA.9-12.3.1.B.2	The period can be estimated by investigating successive equal-length output values and finding where the pattern begins to repeat.
MA.9-12.3.1.B.3	Periodic functions take on characteristics of other functions, such as intervals of increase and decrease, different concavities, and various rates of change. However, with periodic functions, all characteristics found in one period of the function will be in every period of the function.
MA.9-12.3.2.A.2	The radian measure of an angle in standard position is the ratio of the length of the arc of a circle centered at the origin subtended by the angle to the radius of that same circle. For a unit circle, which has radius 1, the radian measure is the same as the length of the subtended arc.
MA.9-12.3.2.A.4	Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The cosine of the angle is the ratio of the horizontal displacement of P from the y-axis to the distance between the origin and point P. Therefore, for a unit circle, the cosine of the angle is the x-coordinate of point P.
MA.9-12.3.2.A.5	Given an angle in standard position, the tangent of the angle is the slope, if it exists, of the terminal ray. Because the slope of the terminal ray is the ratio of the vertical displacement to the horizontal displacement over any interval, the tangent of the angle is the ratio of the y-coordinate to the x-coordinate of the point at which the terminal ray intersects the unit circle; alternately, it is the ratio of the angle's sine to its cosine.
MA.9-12.3.3.A.1	Given an angle of measure θ in standard position and a circle with radius r centered at the origin, there is a point, P, where the terminal ray intersects the circle. The coordinates of point P are (r cos θ, r sin θ).
MA.9-12.3.3.A.2	The geometry of isosceles right and equilateral triangles, while attending to the signs of the values based on the quadrant of the angle, can be used to find exact values for the cosine and sine of angles that are multiples of $\pi/4$ and $\pi/6$ radians and whose terminal rays do not lie on an axis.
MA.9-12.3.4.A.1	Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The sine function, f(θ) = sin θ, gives the y-coordinate, or vertical displacement from the x-axis, of point P. The domain of the sine function is all real numbers.
MA.9-12.3.4.A.2	As the input values, or angle measures, of the sine function increase, the output values oscillate between –1 and 1, taking every value in between and tracking the vertical distance of points on the unit circle from the x-axis.
MA.9-12.3.4.A.3	Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The cosine function, f(θ) = cos θ, gives the x-coordinate, or horizontal displacement from the y-axis, of point P. The domain of the cosine function is all real numbers.
MA.9-12.3.4.A.4	As the input values, or angle measures, of the cosine function increase, the output values oscillate between –1 and 1, taking every value in between and tracking the horizontal distance of points on the unit circle from the y-axis.

MA.9-12.3.5.A.1	A sinusoidal function is any function that involves additive and multiplicative transformations of $f(\theta) = \sin\theta$. The sine and cosine functions are both sinusoidal functions, with $\cos\theta = \sin(\theta + \pi/2)$.
MA.9-12.3.5.A.5	As input values increase, the graphs of sinusoidal functions oscillate between concave down and concave up.
MA.9-12.3.5.A.6	The graph of $y = \sin\theta$ has rotational symmetry about the origin and is therefore an odd function. The graph of $y = \cos\theta$ has reflective symmetry over the y-axis and is therefore an even function.
MA.9-12.3.6.A.3	The graph of the additive transformation $g(\theta) = \sin(\theta + c)$ of the sine function $f(\theta) = \sin\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units. The same transformation of the cosine function yields the same result.
MA.9-12.3.6.A.4	The graph of the multiplicative transformation $g(\theta) = a\sin\theta$ of the sine function $f(\theta) = \sin\theta$ is a vertical dilation of the graph of f and differs in amplitude by a factor of $\|a\|$. The same transformation of the cosine function yields the same result.
MA.9-12.3.6.A.5	The graph of the multiplicative transformation $g(\theta) = \sin(b\theta)$ of the sine function $f(\theta) = \sin\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\|1/b\|$. The same transformation of the cosine function yields the same result.
MA.9-12.3.6.A.6	The graph of $y = f(\theta) = a\sin(b(\theta + c)) + d$ has an amplitude of $\|a\|$ units, a period of $\|1/b\|\, 2\pi$ units, a midline vertical shift of d units from $y = 0$, and a phase shift of $-c$ units. The same transformations of the cosine function yield the same results.
MA.9-12.3.7.A.1	The smallest interval of input values over which the maximum or minimum output values start to repeat, that is, the input-value interval between consecutive maxima or consecutive minima, can be used to determine or estimate the period and frequency for a sinusoidal function model.
MA.9-12.3.7.A.3	An actual pair of input-output values can be compared to pairs of input-output values produced by a sinusoidal function model to determine or estimate a phase shift for the model.
MA.9-12.3.7.A.4	Sinusoidal function models can be constructed for a data set with technology by estimating key values or using sinusoidal regressions.
MA.9-12.3.7.A.5	Sinusoidal functions that model a data set are frequently only useful over their contextual domain and can be used to predict values of the dependent variable from values of the independent variable.
MA.9-12.3.8.B.1	Because the slope values of the terminal ray repeat every one-half revolution of the circle, the tangent function has a period of π.
MA.9-12.3.8.B.2	The tangent function demonstrates periodic asymptotic behavior at input values $\theta = \pi/2 + k\pi$, for integer values of k, because $\cos\theta = 0$ at those values.
MA.9-12.3.8.B.3	The tangent function increases and its graph changes from concave down to concave up between consecutive asymptotes.
MA.9-12.3.8.C.1	The graph of the additive transformation $g(\theta) = \tan\theta + d$ of the tangent function $f(\theta) = \tan\theta$ is a vertical translation of the graph of f and the line containing its points of inflection by d units.
MA.9-12.3.8.C.2	The graph of the additive transformation $g(\theta) = \tan(\theta + c)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units.
MA.9-12.3.8.C.3	The graph of the multiplicative transformation $g(\theta) = a\tan\theta$ of the tangent function $f(\theta) = \tan\theta$ is a vertical dilation of the graph of f by a factor of $\|a\|$. If $a < 0$, the transformation involves a reflection over the x-axis.

MA.9-12.3.9.A.1	For inverse trigonometric functions, the input and output values are switched from their corresponding trigonometric functions, so the output value of an inverse trigonometric function is often interpreted as an angle measure and the input is a value in the range of the corresponding trigonometric function.
MA.9-12.3.9.A.2	The inverse trigonometric functions are called arcsine, arccosine, and arctangent (also represented as $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$). Because the corresponding trigonometric functions are periodic, they are only invertible if they have restricted domains.
MA.9-12.3.9.A.3	In order to define their respective inverse functions, the domain of the sine function is restricted to $[-\pi/2, \pi/2]$, the cosine function to $[0, \pi]$, and the tangent function to $(-\pi/2, \pi/2)$.
MA.9-12.3.10.A.1	Inverse trigonometric functions are useful in solving equations and inequalities involving trigonometric functions, but solutions may need to be modified due to domain restrictions.
MA.9-12.3.10.A.2	Because trigonometric functions are periodic, there are often infinitely many solutions to trigonometric equations.
MA.9-12.3.10.A.3	In trigonometric equations and inequalities arising from a contextual scenario, there is often a domain restriction that can be implied from the context, which limits the number of solutions.
MA.9-12.3.11.A.1	The secant function, $f(\theta) = \sec\theta$, is the reciprocal of the cosine function, where $\cos\theta \neq 0$.
MA.9-12.3.11.A.2	The cosecant function, $f(\theta) = \csc\theta$, is the reciprocal of the sine function, where $\sin\theta \neq 0$.
MA.9-12.3.11.A.3	The graphs of the secant and cosecant functions have vertical asymptotes where cosine and sine are zero, respectively, and have a range of $(-\infty, -1] \cup [1, \infty)$.
MA.9-12.3.11.A.4	The cotangent function, $f(\theta) = \cot\theta$, is the reciprocal of the tangent function, where $\tan\theta \neq 0$. Equivalently, $\cot\theta = \cos\theta/\sin\theta$, where $\sin\theta \neq 0$.
MA.9-12.3.11.A.5	The graph of the cotangent function has vertical asymptotes for domain values where $\tan\theta = 0$ and is decreasing between consecutive asymptotes.
MA.9-12.3.12.A.1	The Pythagorean Theorem can be applied to right triangles with points on the unit circle at coordinates $(\cos\theta, \sin\theta)$, resulting in the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$.
MA.9-12.3.12.A.2	The Pythagorean identity can be algebraically manipulated into other forms involving trigonometric functions, such as $\tan^2\theta = \sec^2\theta - 1$, and can be used to establish other trigonometric relationships, such as $\arcsin x = \arccos(\sqrt{1 - x^2})$, with appropriate domain restrictions.
MA.9-12.3.12.B.1	The sum identity for sine is $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
MA.9-12.3.12.B.2	The sum identity for cosine is $\cos(\alpha + \beta) \cos\alpha\cos\beta - \sin\alpha\sin\beta$.
MA.9-12.3.12.B.3	The sum identities for sine and cosine can also be used as difference and double-angle identities.
MA.9-12.3.12.B.4	Properties of trigonometric functions, known trigonometric identities, and other algebraic properties can be used to verify additional trigonometric identities.
MA.9-12.3.12.C.1	A specific equivalent form involving trigonometric expressions can make information more accessible.
MA.9-12.3.12.C.2	Equivalent trigonometric forms may be useful in solving trigonometric equations and inequalities.
MA.9-12.3.13.A.1	The polar coordinate system is based on a grid of circles centered at the origin and on lines through the origin. Polar coordinates are defined as an ordered pair, (r, θ), such that $\|r\|$ represents the radius of the circle on which the point lies, and θ represents the measure

	of an angle in standard position whose terminal ray includes the point. In the polar coordinate system, the same point can be represented many ways.
MA.9-12.3.13.A.2	The coordinates of a point in the polar coordinate system, (r, θ), can be converted to coordinates in the rectangular coordinate system, (x, y), using $x = r \cos \theta$ and $y = r \sin \theta$.
MA.9-12.3.13.A.3	The coordinates of a point in the rectangular coordinate system, (x, y), can be converted to coordinates in the polar coordinate system, (r, θ), using $r = \sqrt{(x^2 + y^2)}$ and $\theta = \arctan (y/x)$ for $x > 0$ or $\theta = \arctan (y/x) + \pi$ for $x < 0$.
MA.9-12.3.13.A.4	A complex number can be understood as a point in the complex plane and can be determined by its corresponding rectangular or polar coordinates. When the complex number has the rectangular coordinates (a, b), it can be expressed as $a + bi$. When the complex number has polar coordinates (r, θ), it can be expressed as $(r \cos \theta) + i(r \sin \theta)$.
MA.9-12.3.14.A.1	The graph of the function $r = f(\theta)$ in polar coordinates consists of input-output pairs of values where the input values are angle measures and the output values are radii.
MA.9-12.3.14.A.2	The domain of the polar function $r = f(\theta)$, given graphically, can be restricted to a desired portion of the function by selecting endpoints corresponding to the desired angle and radius.
MA.9-12.3.14.A.3	When graphing polar functions in the form of $r = f(\theta)$, changes in input values correspond to changes in angle measure from the positive x-axis, and changes in output values correspond to changes in distance from the origin.
MA.9-12.3.15.A.1	If a polar function, $r = f(\theta)$, is positive and increasing or negative and decreasing, then the distance between $f(\theta)$ and the origin is increasing.
MA.9-12.3.15.A.2	If a polar function, $r = f(\theta)$, is positive and decreasing or negative and increasing, then the distance between $f(\theta)$ and the origin is decreasing.
MA.9-12.3.15.A.3	For a polar function, $r = f(\theta)$, if the function changes from increasing to decreasing or decreasing to increasing on an interval, then the function has a relative extremum on the interval corresponding to a point relatively closest to or farthest from the origin.

Suggested Modifications for Students with Disabilities, 504 eligible, Multilingual Learners, At Risk Students and Gifted Students

Special Education Students:

- Visual Aids and Manipulatives: Provide visual aids such as graphic organizers, anchor charts, and manipulatives like algebra tiles or graphing boards to help students visualize concepts and relationships.
- Simplified Instructions: Break down complex concepts into smaller, more manageable steps with clear and concise instructions.
- Extra Practice and Scaffolding: Offer additional practice problems with varying difficulty levels, along with scaffolding (hints, prompts, worked examples) to support students as they progress.
- Alternative Assessments: Provide alternative assessment options, such as oral presentations, projects, or portfolios, to demonstrate understanding.
- Individualized Instruction: Tailor instruction to meet individual needs, considering learning styles, strengths, and weaknesses.

Multilingual Learners:

- Visual Support: Incorporate visuals, such as diagrams, graphs, and real-world examples, to enhance understanding of vocabulary and concepts.
- Simplified Language: Use clear and concise language, avoiding jargon and idioms. Provide a glossary of key terms with definitions and translations.

- Bilingual Resources: Offer bilingual dictionaries, textbooks, or online resources to support students' understanding.
- Collaborative Learning: Encourage peer-to-peer interaction and group work to provide opportunities for language practice and collaborative problem-solving.
- Differentiated Instruction: This method tailors instruction to meet individual language proficiency levels, providing additional support for students who need it.

Gifted Students:

- Extension Activities: Provide opportunities for enrichment and extension beyond the standard curriculum, such as exploring advanced topics, independent research projects, or problem-solving challenges.
- Inquiry-Based Learning: Encourage students to ask questions, explore concepts independently, and develop their solutions.
- Higher-Order Thinking Skills: Challenge students with complex problems that require critical thinking, analysis, and synthesis.
- Acceleration: Consider accelerating gifted students to higher-level math courses if they demonstrate readiness.
- Mentorship: Pair gifted students with mentors who can guide and support their academic pursuits.

Computer Sci Design Thinking

CS.9-12.8.1.12.DA.1	Create interactive data visualizations using software tools to help others better understand real world phenomena, including climate change.
CS.9-12.8.1.12.DA.6	Create and refine computational models to better represent the relationships among different elements of data collected from a phenomenon or process.
CS.9-12.8.2.12.ED.4	Design a product or system that addresses a global problem and document decisions made based on research, constraints, trade-offs, and aesthetic and ethical considerations and share this information with an appropriate audience.
CS.9-12.8.2.12.ITH.2	Propose an innovation to meet future demands supported by an analysis of the potential costs, benefits, trade-offs, and risks related to the use of the innovation.
	Engineers use science, mathematics, and other disciplines to improve technology. Increased collaboration among engineers, scientists, and mathematicians can improve their work and designs. Technology, product, or system redesign can be more difficult than the original design.

Career Readiness, Life Literacies and Key Skills Practice

Model interdisciplinary thinking to expose students to other disciplines.

Art connection:

Name of Task: Math Music NJSLS: 1.3B.12. Cr1a

A pure tone produces a sine wave when shown on an oscilloscope. When an instrument is played, the tone is not pure. For instance, when a guitar string or piano string vibrates it does not produce a simple sine wave. It does produce other, less distinguishable harmonious waves of higher pitch called harmonic waves. For instance, y =sin (2x) is called the second harmonic, y= sin (3x) is called the third harmonic, and y= sin (4x) is called the fourth harmonic.

Science Connection:

Name of Task: Rabbits, Rabbits Everywhere NJSLS: HS-LS4-2, HS-LS4-3, HS-LS4-5

The rabbit population in a national park rises and falls throughout the year. The population is at its approximate minimum of 6000 rabbits in December. As the weather gets warmer and food becomes more available, the population grows to its approximate maximum of 16,000 rabbits in June. The function describing the rabbit population is f(x)=

5000sin 6x-2+11,000 where x is the time in months and f(x) is the rabbit population.

Name of Task: Speed of CD-RW NJSLS: HS-PS2-1, HS-PS2-2

A CD-RW has a diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute).

- Find the linear speed.
- What would the angular speed be when you reach half of the CD?
- When being burned in this writable CD-R drive, the angular speed of the CD is often much faster than when playing audio, but the angular speed still varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4800 RPM (revolutions per minute). Find the linear speed.

TECH.9.4.12.CI.1	Demonstrate the ability to reflect, analyze, and use creative skills and ideas (e.g., 1.1.12prof.CR3a).
TECH.9.4.12.CI.3	Investigate new challenges and opportunities for personal growth, advancement, and transition (e.g., 2.1.12.PGD.1).
TECH.9.4.12.CT.1	Identify problem-solving strategies used in the development of an innovative product or practice (e.g., 1.1.12acc.C1b, 2.2.12.PF.3).
TECH.9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
TECH.9.4.12.TL.1	Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specified task (e.g., W.11-12.6.).
TECH.9.4.12.IML.1	Compare search browsers and recognize features that allow for filtering of information.

Unit 4: Functions Involving Parametric, Vector and Introduction to Limits

Content Area: **Mathematics**
Course(s):
Time Period: **4th Marking Period**
Length: **8 weeks**
Status: **Published**

Summary of the Unit

Parametric equations can be used to model the path of an object. These objects can be represented by conics (ellipses, parabolas, and hyperbolas). Conics are applied in real-world situations, such as orbits of planets, flashlights, and satellites. Since conics are not always functions, they can be defined using parametric equations. The study of vectors is crucial in applied mathematics. Vectors relate to geometry, trigonometry, and physics. The unit concludes with an introduction to limits and continuity. Students will estimate limits graphically by analyzing function behavior near specific points and will determine limits algebraically by substitution, factoring, and rationalizing techniques. Students will extend their understanding to limits at infinity and use these ideas to describe end behavior and asymptotic behavior of functions. Finally, students explore the definition of continuity by verifying that the function value, limit, and left and right handed behavior all agree. Throughout the unit, students will use multiple representations - graphical, algebraic, numerical, and contextual - to analyze functions, justify reasoning, and communicate mathematical ideas clearly.

Enduring Understandings

Students will be able to

- Evaluate sets of parametric equations for given values of the parameter.
- Graph curves that are represented by sets of parametric equations with and without graphing calculators.
- Rewrite sets of parametric equations as single rectangular equations by eliminating the parameters.
- Use time as a parameter in parametric equations.
- Find parametric equations for curves defined by rectangular equations.
- Represent vectors as directed line segments.
- Write the component forms of vectors.
- Perform basic vector operations and represent vectors graphically.
- Write vectors as linear combination of unit vectors.
- Find the direction angles of vectors.
- Use vectors to model and solve real-life problems.
- Find the dot product of two vectors and use the properties of the dot product.
- Find the angle between two vectors and determine whether two vectors are orthogonal.
- Write vectors as the sums of two vector components.

- Use vectors to find the work done by a force.

Essential Questions

- How can we determine when the populations of species in an ecosystem will be relatively steady?
- How can we analyze the vertical and horizontal aspects of motion independently?
- How does high resolution computer generated imaging achieve smooth and realistic motion on screen with so many pixels?
- What is a vector in the plane?
- How do you represent and perform operations with vector quantities?
- How do you write a vector as a sum of two vector components?
- What is the dot product? How is it used to analyze vectors?
- What makes a vector orthogonal?
- What do vector quantities signify in real life situations?

- How can we use mathematical notation to formally define and interpret a limit?

- What does it mean for a limit to not exist (DNE) at a particular point, and how can we identify this graphically, numerically, and algebraically?

- How does the concept of an infinite limit relate directly to the behavior of a function near its vertical asymptotes?

- How can we estimate the limit of a function using graphical data versus numerical data (tables)?

- How do we apply the algebraic properties/theorems of limits to evaluate multi-step composite and rational functions?

- How do we algebraically handle indeterminate forms (like 0/0) using techniques like factoring or rationalizing to find a limit?

- What are the three conditions required to define and prove continuity at a specific point x = c?

- How do we classify and justify the different types of discontinuities (removable/holes, jumps, and infinite/asymptotes) both visually and analytically?

- How can the domain of a parent function family (such as logarithmic, rational, or trigonometric) be used to confirm its continuous intervals?

Unit Summative Assessment and Alternate Assessment Options

Suggested Formative/Summative Classroom Assessments

- Describe Learning Vertically
- Identify Key Building Blocks
- Make Connections (between and among crucial building blocks)
- Short/Extended Constructed Response Items
- Multiple-Choice Items (where multiple answer choices may be correct)
- Drag and Drop Items
- Use of Equation Editor
- Quizzes
- Journal Entries/Reflections/Quick-Writes
- Accountable talk
- Projects
- Portfolio
- Observation
- Graphic Organizers/ Concept Mapping
- Presentations
- Teacher-Student and Student-Student Conferencing

- WebAssign Problem Sets WebAssign Instructor Help
- Homework
- Students will take formal assessments, such as tests and quizzes, to assess knowledge of concepts learned throughout the unit.
- Students will also demonstrate mastery through various assessment criteria included in the unit such as do nows, exit slips, graded classwork activities and assignments, and/or projects.

Resources

- Classpad.net (Casio): https://classpad.net/us/
- Desmos: https://www.desmos.com/
- Geogebra: https://www.geogebra.org/?lang=en
- Math Open Reference: https://www.mathopenref.com/
- TI Education (Texas Instruments): https://education.ti.com/en
- WolframAlpha: https://www.wolframalpha.com/
- Wolfram MathWorld: https://mathworld.wolfram.com/
- Khan Academy: https://www.khanacademy.org/math/precalculus
- Digital Mathematics Word Wall: http://www.mathwords.com/index_adv_alg_precal.htm
- Extra Notes for Pre-Calculus Content: https://sites.google.com/a/evergreenps.org/ms-griffin-s-math-classes/updates
- Review Documents for Pre-Calculus: https://sites.google.com/site/dgrahamcalculus/trigpre-calculus/trig-pre-calculus-worksheets

- Pre-Calculus IXL Topics and Resources: https://www.ixl.com/math/precalculus
- Classroom Challenges to Support Teachers in Formative Assessments: http://map.mathshell.org/materials/lessons.php?gradeid=24
- Applications of Function Models: https://www.ck12.org/algebra/Applications-of-Function-Models/lesson/Applications-of-Function-Models-BSC-ALG/?referrer=featured_content
- Statistics Education Web (STEW). http://www.amstat.org/education/STEW/
- The Data and Story Library (DASL). http://lib.stat.cmu.edu/DASL/
- WebAssign Instructor Resources
- WebAssign Student Resources
- Cengage Learning: PreCalculus with Limits - A Graphing Approach, Sixth Edition

Topic/Selection Timeframe	General Objectives	Instructional Activities	Benchmarks/Assessments
4.1 Parametric Functions (2 days)	SWBAT construct a graph or table of values for a parametric function represented analytically.	Students would be able to determine when it is advantageous to define curves parametrically. Students will work in groups to determine the orientation of a curve, and thus the usefulness of parametric equations.	By the end of the lesson students will be able to: Construct a complete numerical table of values for a parametric function f(t) = (x(t), y(t)) by evaluating the independent parameter t across a specified domain. Sketch a smooth curve by plotting coordinate pairs from a data table in order of increasing t, marking distinct initial (start) and terminal (end) points resulting from a restricted domain. Indicate the direction of motion (orientation) along the graphed curve using directional arrows to show how the path is traversed as t increases. Utilize a graphing utility or/and Desmos to accurately input a parametric vector, adjust the domain restrictions, and analyze the resulting path. Give students a blank table and a parametric equation pair, such as: $x(t) = t^2 - 1$, $y(t) = 2t$ for $-2 \leq t \leq 2$ Have them fill out the table independently. Then, have them trade papers with a peer to plot the points, prompting them to verify if the coordinates map out the curve in the exact sequence of increasing t. Display a pre-drawn curve of a parametric function on the board, but draw the orientation arrows backwards. Ask students to look at the given interval, catch the directional error, and write down a 1-sentence justification for the correction.

4.2 Parametric Functions Modeling Planar Motion (2 days)	SWBAT identify key characteristics of a parametric planar motion function that are related to position.	Students will determine to write equations to describe the motion of a point in a plane; they need to introduce a third variable, or a parameter, and use parametric equations involving only x and y.	By the end of the lesson students will be able to • identify that a parametric function f(t) = (x(t), y(t)) models the specific position of a particle in a plane at any given time parameter t. • Algebraically and graphically determine the horizontal extrema (leftmost and rightmost positions) by optimizing x(t), and vertical extrema (lowest and highest positions) by optimizing y(t). Have students come to the board in small groups to calculate the horizontal and vertical bounds of a particle's path. Circulate to check for a common misconception: ensure students do not mix up finding the minimum of x(t) (horizontal constraint) with finding the minimum of y(t) (vertical constraint). Through questioning students will be able to determine how do you write equations to describe the motion of a point in a plane. Closure
4.3 Parametrically Defined Circles and Lines (2 days)	SWBAT express motion around a circle or along a line segment parametrically	Students will determine that a complete counterclockwise revolution around the unit circle, starting and ending at (1, 0) and centered at the origin, can be modeled by relating x and y to cos (t) and sin(t) with a restricted domain.	By the end of this lesson, students will be able to: • model a complete counterclockwise revolution around a circle centered at the origin with radius r using the parametric form (x(t), y(t)) = (rcos(t), rsin(t)) on the restricted domain [0,2pi]. • establish strict domain constraints for t to ensure a linear or circular path starts and terminates at the exact designated coordinate boundaries. Have students complete problems at the board. Closure

4.4 Parametrization of Implicitly Defined Functions (4 days)	SWBAT represent a curve in the plane parametrically.	Students will determine parametrization for an implicitly defined function in terms of third variable will satisfy the corresponding equation for every value in the domain.	By the end of this lesson sequence, students will be able to: • represent any standard explicit function y = f(x) parametrically as (x(t), y(t)) = (t, f(t)), and represent its invertible inverse as (x(t), y(t)) = (f(t), t) over a designated interval. • convert implicitly defined parabolas, ellipses and hyperbolas into parametric form by isolating the non-squared variable (x or y) and substituting the parameter t for the squared variable. • algebraically calculate and define the correct domain restrictions for the parameter t to ensure the parametric equations precisely match the boundaries of the original rectangular curve. Have students complete problems at the board. Through questioning students will be able to determine how to determine the domain of the function after parametrization. Closure Quiz Sec 4.1 - 4.4
4.5 Vectors (5 days)	SWBAT Identify characteristics of a vector. SWBAT Determine sums and products involving vectors. SWBAT Determine a unit vector for a given	As a class discuss the properties of vectors algebraically and geometrically. Explain and model the definition of two-dimensional vectors. Model vector addition numerically and geometrically. Multiplying a Vector by a Scalar. Explain and model the	By the end of the lesson students would be able to: • graphically represent a vector in the xy-plane as a directed line segment, clearly identifying its initial point (tail), terminal point (head), and direction. • write the component form < a, b> of a vector given its initial point and terminal point. • algebraically and geometrically compute the sum of two vectors (using the head-to-tail or parallelogram method) and find scalar products.

	vector. SWBAT Determine angle measures between vectors and magnitudes of vectors involved in vector addition.	definition of two-dimensional vectors. Geometric Interpretation of Dot Product: Applies the dot product to scenarios such as determining the angle between two vectors and using the dot product to prove if a vector is orthogonal.	• Compute the magnitude of a vector using the distance formula and calculate a corresponding unit vector in the same direction. • Calculate the dot product of two vectors to determine the angle between them and analytically justify whether the vectors are orthogonal (dot product equals 0). • Apply vector components, magnitudes, and navigation angles to model and solve real-life force, velocity, or work scenarios. Have students come to the board in small groups to model basic vector operations. This allows you to circulate and target misconceptions about vector direction and scalar stretching. Provide students with a pre-drawn, incorrect geometric vector addition graph (e.g., drawing vectors tail-to-tail instead of head-to-tail). Have students diagnose the visual error and write down the correct sequence. Closure, Exit slips. Students will be given a unit test that assesses their understanding of the concepts and skills from the material on two-dimensional vectors and will include open-ended problem-solving, including both short constructed-response and extended-response questions.
4.6 Defining Limits and Using Limit Notation 1day	SWBAT represent limits analytically using correct notation. SWBAT interpret limits expressed in analytic notation.	Explain the concept of a limit using an informal "destination point" analogy: a limit is simply the value that the outputs of a function f(x) get closer and closer to as the inputs x get closer and closer to a specific target number c, without ever needing to actually reach that target.	Give students written limit expressions and have them write out or verbally state the meaning in plain language. Present common notation mistakes—such as omitting the lim symbol during steps, writing lim = 4, or placing the x approaches to c in the wrong place and have students write the correct mathematical syntax.

		Introduce the concept of one-sided limits both graphically and analytically.	
4.7 Estimating Limit Values from Graphs. 4 days	SWBAT estimate Limits of Functions from graphs.	Use graphing software like Desmos or graphing calculators to visually demonstrate how a function approaches different values from the left versus the right (e.g., using piecewise or rational functions). Model the translation of this concept into mathematical notation. Walk students through how to write and read the expression "The limit of f(x) as x approaches c equals L." Guide students to identify each piece of the notation on a graph or within an equation, emphasizing that L must be a real number for the limit to exist	Provide a graph with distinct features (holes, clear functional values). Students must write the formal limit statements that accurately describe the behavior at those points using correct algebraic notation Provide a graph containing holes, jump discontinuities, and asymptotes. Give students a list of formal limit expressions and have them fill in the estimated y-value by visually analyzing the behavior from both sides. Students evaluate left-hand, right-hand, and overall limits at jump discontinuities, holes, and asymptotes from a complex visual graph.
4.8 Estimating Limit Values from Tables. 2 days	SWBAT estimate limits of functions from tables.	Provide students with numerical tables where the input values (x) systematically close in on a specific number from both sides. Guide them to look for the overall trend in the corresponding output values (f(x)) to estimate the limit. Teach students how to configure the table settings on a graphing utility (like a TI-calculator or Desmos) to "Ask" mode. Have them manually input values increasingly closer to a point of discontinuity to	Students are given multiple numerical tables representing different types of functions (continuous, removable holes, asymptotes). Students must state the estimated limit or write "DNE" with a brief justification. Short-response question where students are given an equation, must generate their own small table of values to estimate the limit, and then verify their guess using a graph. Quiz sec 4.6 - 4.8

		observe the behavior of indeterminate forms. Present tables where the left and right outputs approach two entirely different numbers, or grow without bound (approaching $\pm\infty$), to model how to identify when a limit does not exist (DNE) from numerical data.	
4.9 Determining Limits Using Algebraic Properties of Limits 9 days	SWBAT determine the limits of functions using limit theorems.	Model how to evaluate one-sided limits analytically using direct substitution, factoring, or rationalizing when dealing with indeterminate forms. Introduce and define the fundamental algebraic properties of limits (theorems for the limits of sums, differences, products, quotients, and composition functions). Guide students through step-by-step algebraic computations using these theorems.	Students identify and correct a conceptual error in a provided sample solution where a limit theorem was applied incorrectly. Quick, projected algebraic drills where students evaluate composite limits given basic limit values. Written items requiring students to evaluate multi-step composite limits analytically, using correct notation to show which limit theorem is applied at each step.
4.10 Connecting Infinite Limits and Vertical Asymptotes. 2 days	SWBAT interpret the behavior of functions using	Use graphing tools like Desmos to show functions with vertical asymptotes (e.g., $f(x) = 1/(x-2)$. Direct students to observe how the y-values grow without bound as x approaches the asymptote from either side. How to express unbounded behavior using formal limit notation, writing statements; Have students use a calculator table to plug in values extremely close to a vertical asymptote (e.g.,	Present a graph with a vertical asymptote. Have students write the two one-sided limit statements that describe the behavior around that asymptote. Ask students to explain in a quick sentence why writing $\lim_{x \to c} f(x)=\infty$ means the limit technically Does Not Exist (DNE), but provides specific directional information. Quiz items where students are given a rational function equation, must find the vertical asymptote analytically, and use proper limit notation to describe the behavior from the left and right.

		2.001, 2.0001) to analytically verify that the output values increase drastically.	Quiz 4.9 - 4.10
4.11 Exploring Types of Discontinuities. 2 days	SWBAT Justify conclusions about continuity at a point using the definition.	Present a single graph containing all three types of discontinuities (a hole, a jump, and a vertical asymptote). Lead a class discussion to define and classify each type based on visual behavior. Demonstrate how to identify discontinuities analytically from a function's equation (e.g., factors that cancel out create removable holes, while non-canceling denominator zeros create vertical asymptotes).	Give students a list of functions or graphs and have them quickly categorize their discontinuities as Removable, Jump, or Infinite/Asymptote.
4.12 Defining Continuity at a Point. 3 days	SWBAT justify conclusions about continuity at a point.	Connect visual features to algebraic conditions by checking the three requirements for continuity at a point x=c: 1. 1. Is f(c) defined? 2. Does limit as x approaches c, f(x) exist? 3. Does limit as x approaches c f(x) = f(c)?	Provide a graph with a discontinuity and ask students to write down exactly which part of the 3-part continuity definition fails. Students analyze a multi-part piecewise graph, state the x-values where the function is discontinuous, and classify each type. Given an algebraic function (like a rational or piecewise expression), students must find all points of discontinuity and provide a formal justification for their classification.

4.13 Confirming Continuity over an Interval. 1 day	SWBAT Determine intervals over which a function is continuous.	Guide students to identify the domain of basic function families (polynomial, rational, power, exponential, logarithmic, and trigonometric). Emphasize that these functions are automatically continuous at every point where they are defined. Show students how to write intervals of continuity using correct interval notation for polynomials or for logarithmic functions. Use graphing software like Desmos to visually demonstrate how domain restrictions (like vertical asymptotes in rational functions or endpoints in square root functions) define the boundaries of a function's continuity intervals.	Provide an equation (e.g., f(x) = ln(x-2)) and have students state its domain and state the interval where it is guaranteed to be continuous. Give statements of function which are either continuous or discontinuous at a point on its domain and have students identify the error in reasoning. Quiz items where students are given a mix of parent functions and must write the exact intervals over which each function is continuous. Short-response questions where students algebraically find the intervals of continuity for trigonometric functions, piecewise functions and rational functions. Students will be given a unit test that assesses their understanding of the concepts and skills from the material on limits an Continuity and will contain open-ended problem solving, which includes both short constructed response and extended response questions. Quiz 4.11- 4.13

Standards for Course Content Area and Cross Content Standards Addressed

MA.9-12.1.2	Defining Limits and Using Limit Notation

MA.9-12.1.2.LIM-1	Reasoning with definitions, theorems, and properties can be used to justify claims about limits.
MA.9-12.1.2.LIM-1.A	Represent limits analytically using correct notation.
MA.9-12.1.2.LIM-1.A.1	Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made arbitrarily close to R by taking x sufficiently close to c (but not equal to c). If the limit exists and is a real number, then the common notation is lim $[x \to c]$ $f(x) = R$.
MA.9-12.1.2.LIM-1.B	Interpret limits expressed in analytic notation.
MA.9-12.1.2.LIM-1.B.1	A limit can be expressed in multiple ways, including graphically, numerically, and analytically.
MA.9-12.1.3	Estimating Limit Values from Graphs
MA.9-12.1.3.LIM-1	Reasoning with definitions, theorems, and properties can be used to justify claims about limits.
MA.9-12.1.3.LIM-1.C	Estimate limits of functions.
MA.9-12.1.3.LIM-1.C.1	The concept of a limit includes one sided limits.
MA.9-12.1.3.LIM-1.C.2	Graphical information about a function can be used to estimate limits.
MA.9-12.1.3.LIM-1.C.4	A limit might not exist for some functions at particular values of x. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right.
MA.9-12.1.4	Estimating Limit Values from Tables
MA.9-12.1.4.LIM-1	Reasoning with definitions, theorems, and properties can be used to justify claims about limits.
MA.9-12.1.4.LIM-1.C	Estimate limits of functions.
MA.9-12.1.4.LIM-1.C.5	Numerical information can be used to estimate limits.
MA.9-12.1.5	Determining Limits Using Algebraic Properties of Limits
MA.9-12.1.5.LIM-1	Reasoning with definitions, theorems, and properties can be used to justify claims about limits.
MA.9-12.1.5.LIM-1.D	Determine the limits of functions using limit theorems.
MA.9-12.1.5.LIM-1.D.1	One-sided limits can be determined analytically or graphically.
MA.9-12.1.5.LIM-1.D.2	Limits of sums, differences, products, quotients, and composite functions can be found using limit theorems.
MA.9-12.4	Functions Involving Parameters, Vectors, and Matrices
MA.9-12.4.1.A	Construct a graph or table of values for a parametric function represented analytically.
MA.9-12.4.1.A.1	A parametric function in $\mathbb{R}^2$, the set of all ordered pairs of two real numbers, consists of a set of two parametric equations in which two dependent variables, x and y, are dependent on a single independent variable, t, called the parameter.
MA.9-12.4.1.A.2	Because variables x and y are dependent on the independent variable, t, the coordinates (x_i, y_i) at time t_i can be written as functions of t and can be expressed as the single parametric function $f(t) = (x(t), y(t))$, where in this case x and y are names of two functions.
MA.9-12.4.1.A.3	A numerical table of values can be generated for the parametric function $f(t) = (x(t), y(t))$ by evaluating x_i and y_i at several values of t_i within the domain.
MA.9-12.4.1.A.4	A graph of a parametric function can be sketched by connecting several points from the

	numerical table of values in order of increasing value of t.
MA.9-12.4.1.A.5	The domain of the parametric function f is often restricted, which results in start and end points on the graph of f.
MA.9-12.4.2.A.1	A parametric function given by $f(t) = (x(t), y(t))$ can be used to model particle motion in the plane. The graph of this function indicates the position of a particle at time t.
MA.9-12.4.2.A.2	The horizontal and vertical extrema of a particle's motion can be determined by identifying the maximum and minimum values of the functions $x(t)$ and $y(t)$, respectively.
MA.9-12.4.2.A.3	The real zeros of the function $x(t)$ correspond to y-intercepts, and the real zeros of $y(t)$ correspond to x-intercepts.
MA.9-12.4.4.A.1	A complete counterclockwise revolution around the unit circle that starts and ends at (1, 0) and is centered at the origin can be modeled by $(x(t), y(t)) = (\cos t, \sin t)$ with domain $0 \leq t \leq 2\pi$.
MA.9-12.4.4.A.2	Transformations of the parametric function $(x(t), y(t)) = (\cos t, \sin t)$ can model any circular path traversed in the plane.
MA.9-12.4.4.A.3	A linear path along the line segment from the point (x_1, y_1) to the point (x_2, y_2) can be parametrized many ways, including using an initial position (x_1, y_1) and rates of change for x with respect to t and y with respect to t.
MA.9-12.4.7.A.1	A parametrization $(x(t), y(t))$ for an implicitly defined function will, when $x(t)$ and $y(t)$ are substituted for x and y, respectively, satisfy the corresponding equation for every value of t in the domain.
MA.9-12.4.7.A.2	If f is a function of x, then $y = f(x)$ can be parametrized as $(x(t), y(t)) = (t, f(t))$. If f is invertible, its inverse can be parametrized as $(x(t), y(t)) = (f(t), t)$ for an appropriate interval of t.
MA.9-12.4.7.B.1	A parabola can be parametrized in the same way that any equation that can be solved for x or y can be parametrized. Equations that can be solved for x can be parametrized as $(x(t), y(t)) = (f(t), t)$ by solving for x and replacing y with t. Equations that can be solved for y can be parametrized as $(x(t), y(t)) = (t, f(t))$ by solving for y and replacing x with t.
MA.9-12.4.7.B.2	An ellipse can be parametrized using the trigonometric functions $x(t) = h + a \cos t$ and $y(t) = k + b \sin t$ for $0 \leq t \leq 2\pi$.
MA.9-12.4.7.B.3	A hyperbola can be parametrized using trigonometric functions. For a hyperbola that opens left and right, the functions are $x(t) = h + a \sec t$ and $y(t) = k + b \tan t$ for $0 \leq t \leq 2\pi$. For a hyperbola that opens up and down, the functions are $x(t) = h + a \tan t$ and $y(t) = k + b \sec t$ for $0 \leq t \leq 2\pi$.
MA.9-12.4.8.A.1	A vector is a directed line segment. When a vector is placed in the plane, the point at the beginning of the line segment is called the tail, and the point at the end of the line segment is called the head. The length of the line segment is the magnitude of the vector.
MA.9-12.4.8.A.2	A vector $\overrightarrow{P_1P_2}$ with two components can be plotted in the xy-plane from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$. The vector is identified by a and b, where $a = x_2 - x_1$ and $b = y_2 - y_1$. The vector can be expressed as a, b. A zero vector $\langle 0, 0 \rangle$ is the trivial case when $P_1 = P_2$.
MA.9-12.4.8.A.3	The direction of the vector is parallel to the line segment from the origin to the point with coordinates (a, b). The magnitude of the vector is the square root of the sum of the squares of the components.
MA.9-12.4.8.A.4	For a vector represented geometrically in the plane, the components of the vector can be found using trigonometry.

Suggested Modifications for Students with Disabilities, 504 eligible, Multilingual Learners, At Risk Students and Gifted Students

Special Education Students:

- Visual Aids and Manipulatives: Provide visual aids such as graphic organizers, anchor charts, and manipulatives like algebra tiles or graphing boards to help students visualize concepts and relationships.
- Simplified Instructions: Break down complex concepts into smaller, more manageable steps with clear and concise instructions.
- Extra Practice and Scaffolding: Offer additional practice problems with varying difficulty levels, along with scaffolding (hints, prompts, worked examples) to support students as they progress.
- Alternative Assessments: Provide alternative assessment options, such as oral presentations, projects, or portfolios, to demonstrate understanding.
- Individualized Instruction: Tailor instruction to meet individual needs, considering learning styles, strengths, and weaknesses.

Multilingual Learners:

- Visual Support: Incorporate visuals, such as diagrams, graphs, and real-world examples, to enhance understanding of vocabulary and concepts.
- Simplified Language: Use clear and concise language, avoiding jargon and idioms. Provide a glossary of key terms with definitions and translations.
- Bilingual Resources: Offer bilingual dictionaries, textbooks, or online resources to support students' understanding.
- Collaborative Learning: Encourage peer-to-peer interaction and group work to provide opportunities for language practice and collaborative problem-solving.
- Differentiated Instruction: This method tailors instruction to meet individual language proficiency levels, providing additional support for students who need it.

Gifted Students:

- Extension Activities: Provide opportunities for enrichment and extension beyond the standard curriculum, such as exploring advanced topics, independent research projects, or problem-solving challenges.
- Inquiry-Based Learning: Encourage students to ask questions, explore concepts independently, and develop their solutions.
- Higher-Order Thinking Skills: Challenge students with complex problems that require critical thinking, analysis, and synthesis.
- Acceleration: Consider accelerating gifted students to higher-level math courses if they demonstrate readiness.
- Mentorship: Pair gifted students with mentors who can guide and support their academic pursuits.

Computer Sci Design Thinking

CS.9-12.8.1.12.DA.1	Create interactive data visualizations using software tools to help others better understand real world phenomena, including climate change.
CS.9-12.8.1.12.DA.6	Create and refine computational models to better represent the relationships among different elements of data collected from a phenomenon or process.
CS.9-12.8.2.12.ED.4	Design a product or system that addresses a global problem and document decisions made based on research, constraints, trade-offs, and aesthetic and ethical considerations and share this information with an appropriate audience.
CS.9-12.8.2.12.ITH.1	Analyze a product to determine the impact that economic, political, social, and/or cultural factors have had on its design, including its design constraints.
	Engineers use science, mathematics, and other disciplines to improve technology. Increased collaboration among engineers, scientists, and mathematicians can improve their work and designs. Technology, product, or system redesign can be more difficult than the original design.

Career Readiness, Life Literacies and Key Skills Practice

Social Studies and ELA Literacy Connection:

Name of Task: Americans' spending: NJSLS: 6.1.12.HistoryCC.16.b, 6.2.12.EconGE.5.a

From July 1998 to July 1999, Americans' spending rose from 5.82 trillion dollars to 6.20 trillion dollars

a. Let x = 0 represent July 1998, x = 1 represent August 1998, ..., and x = 12 represent July 1999. Write a linear equation for Americans' spending in terms of the month x

b. Use the equation in (a) to predict Americans' spending in July 2002.

c. Based on the model created in (a) when would the aggregate expenditure exceed 10 trillion dollars?

d. What part of the US GDP is spent by the Americans in 2013?

Name of Task: Publishing Cost:

A publishing company estimates that the average cost (in dollars) for one copy of a new scenic calendar it plans to produce can be approximated by the function

$C(x) = (2.25x + 275)/x$

2.25 x+

Where x is the number of calendars printed.

a. Find the average cost per calendar when the company prints 100 calendars.

b. Identify the domain and range of this function.

c. After analyzing the function, Alex said that this company should not be allowed to publish zero calendars. As a result, the company has no option to shut down and go out of business. Write an argument to support or reject Alex's conclusion.

Science Connection:

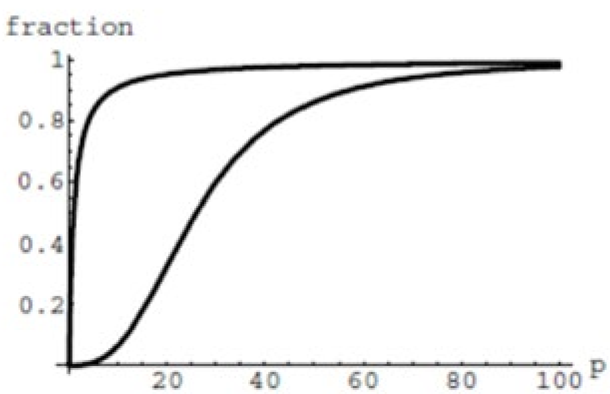
Name of Task: Myoglobin and Hemoglobin: NJSLS: HS-LS1-2; HS-LS1-4

Myoglobin and hemoglobin are oxygen-carrying molecules in the human body. Hemoglobin is found inside red blood cells, which flow from the lungs to the muscles through the

bloodstream. Myoglobin is found in muscle cells. The function y = M(p) = p/(1 + p) calculates the fraction of myoglobin saturated with oxygen at a given pressure p Torrs. For example, at a pressure of 1 Torr, M(1) = 0.5, which means half of the myoglobin (i.e. 50%) is oxygen saturated. (Note: More precisely, you need to use something called the "partial pressure", but the distinction is not important for this problem.) Likewise, the function y = H(p) = p^2.8/((26)^2.8 + p^2.8) calculates the fraction of hemoglobin saturated with oxygen at a given pressure p. [UW]

a. The graphs of M(P) and H(P) are given here on the domain

$0 \le p \le 100$



Which is which?

b. If the pressure in the lungs is 100 Torrs, what is the level of oxygen saturation of the hemoglobin in the lungs?

c. The pressure in an active muscle is 20 Torrs. What is the level of oxygen saturation of myoglobin in an active muscle? What is the level of hemoglobin in an active muscle?

d. Define the efficiency of oxygen transport at a given pressure p to be M(p) – H(p) . What is the oxygen transport efficiency at 20 Torrs? At 40 Torrs? At 60 Torrs? Sketch the graph of M(p) – H(p) ; are there conditions under which transport efficiency is maximized (explain)?

Business Connection :

Name of Task: Minimize the metal in a can: NJSLS: 9.1.12.A.4; W.11-12.1

A manufacturer wants to manufacture a metal can that holds 1000 cm3 of oil. The can is in the shape of a right cylinder with a radius r and height h. Assume the thickness of the material used to make the metal can is negligible.

For each question, include correct units of measurement and round your answers to the nearest tenth. Using your knowledge of volume and surface area of a right cylinder, write a function S(r) that represents the surface area of the cylindrical can in terms of the radius, r, of its base. Show in detail your algebraic thinking.

1. Sketch the graph of S(r) and show key features of the graph. State any restriction on the value of r so that it represents the physical model of the can.

2. What dimensions will minimize the quantity of metal needed to manufacture the cylindrical can? Show in detail your mathematical solution.

3. Calculate the minimum value of the function S(r) and interpret the result in the context of the physical model. Show the mathematical steps you used to obtain the answer.

Name of Task: Chemco Manufacturing: NJSLS: 9.1.12.A.4; W.11-12.1

Chemco Manufacturing estimates that its profit P in hundreds of dollars is $P = -4x^2 + 40x + 3$ where x is the number of units produced in thousands.

a. How many units must be produced to obtain the maximum profit?

b. Graph the profit function and identify its vertex.

c. An increase in productivity increased profit by $7 at each quantity sold. What kind of a transformation would model this situation? Show your work graphically and algebraically.

d. A decrease in marginal cost lead to a 4 units increase in the optimum level of production. What kind of a transformation would model this situation? Show your work graphically and algebraically.

TECH.9.4.12.CI.1	Demonstrate the ability to reflect, analyze, and use creative skills and ideas (e.g., 1.1.12prof.CR3a).
TECH.9.4.12.CI.3	Investigate new challenges and opportunities for personal growth, advancement, and transition (e.g., 2.1.12.PGD.1).
TECH.9.4.12.CT.1	Identify problem-solving strategies used in the development of an innovative product or practice (e.g., 1.1.12acc.C1b, 2.2.12.PF.3).
TECH.9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
TECH.9.4.12.TL.1	Assess digital tools based on features such as accessibility options, capacities, and utility for accomplishing a specified task (e.g., W.11-12.6.).
TECH.9.4.12.IML.1	Compare search browsers and recognize features that allow for filtering of information.