

AP Calculus BC

Summer Work

Unit 1 Review: Limits and Continuity

Welcome to AP Calculus BC!

This summer assignment is our Unit 1 review. Since most students entering BC have successfully completed AP Calculus AB, the concepts in this packet should be familiar. Completing this review over the summer will allow us to quickly review and assess these prerequisite skills so that we can begin new BC content earlier in the school year and spend more time on the more complex concepts.

Topics Included

This packet reviews the major Unit 1 topics from AP Calculus AB, including:

- Limits and limit notation
- Estimating limits graphically and algebraically
- Infinite limits and limits at infinity
- Continuity and discontinuities
- The Intermediate Value Theorem
- Rates of Change and Tangent Lines
- The Derivative and Differentiability
- Basic Derivative Rules
- The Chain Rule
- Velocity and Other Rates of Change

Expectations

Please complete all notes, examples, and practice problems before the first day of school. Show your work and make note of any concepts that you find challenging so that you can ask questions when we return. Some of the sections I have old video notes that I made for that topic that you can find by scanning the pink QR code at the start of the section or next to the specific topic (these are only here for extra resources and are not mandatory to watch).

Assessment

During the first week of school, we will briefly address questions from the packet and then take a **Unit 1 Test on Limits and Continuity**. This assessment will count as our first unit test of the year and will be based on the material in this packet. The stronger your understanding of these prerequisite concepts, the more successful you will be as we move into the new BC topics throughout the year. Have a great summer, and I look forward to seeing you in the fall!

Mrs. Ayala

Section 1: Everything you need to know about limits



SCAN ME

Definition of a Limit

If the y-value of a function $f(x)$ becomes close to a single value L as x gets closer and closer to a point c from both the left and the right side, then the **limit of $f(x)$ as x approaches c is L** , which can be written using mathematical notation as:

$$\lim_{x \rightarrow c} f(x) = L$$

Who remembers how to do it?!!!

$$\lim_{x \rightarrow -8} f(x) =$$

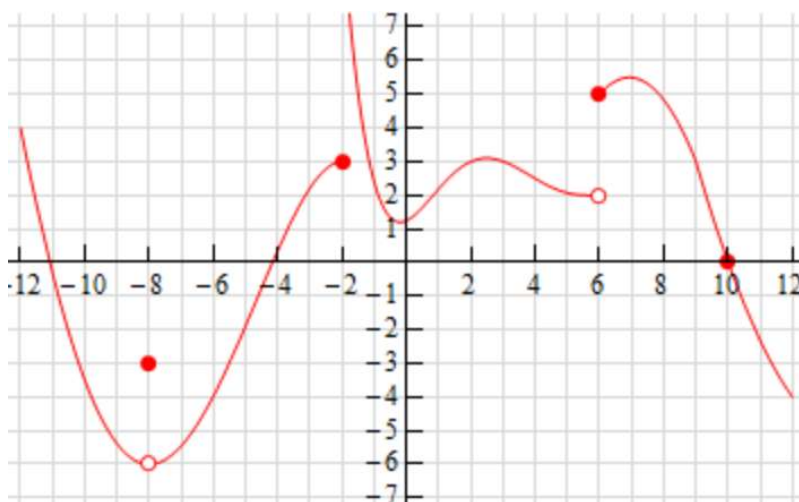
$$\lim_{x \rightarrow 10} f(x) =$$

$$\lim_{x \rightarrow 6^+} f(x) =$$

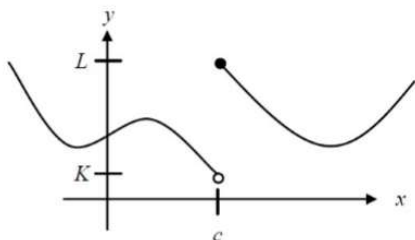
$$\lim_{x \rightarrow 6^-} f(x) =$$

$$\lim_{x \rightarrow 6} f(x) =$$

$$\lim_{x \rightarrow -2^+} f(x) =$$



One Sided Limits



$\lim_{x \rightarrow c^+} f(x) = L$... "the limit of $f(x)$ as x approaches c from the right is L ."
 $\lim_{x \rightarrow c^-} f(x) = K$... "the limit of $f(x)$ as x approaches c from the left is K ."

$$\lim_{x \rightarrow c} f(x) \text{ exists} \iff \lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x)$$

Properties of Limits

Let b and c be real numbers and let f and g be functions with the following limits:

$$\lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = K$$

Rule	Rule
$\lim_{x \rightarrow c} b = b$	$\lim_{x \rightarrow c} [b \cdot g(x)] = bL$
$\lim_{x \rightarrow c} x = c$	$\lim_{x \rightarrow c} [f(x)]^{\frac{r}{s}} = L^{\frac{r}{s}}$ provided r and s are integers and $s \neq 0$.
$\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$	$\lim_{x \rightarrow c} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{K}$ provided $K \neq 0$.
$\lim_{x \rightarrow c} [f(x) \cdot g(x)] = L \cdot K$	

Example 1: Use the given information to evaluate the following limits: $\lim_{x \rightarrow 2} f(x) = 5$ and $\lim_{x \rightarrow 2} g(x) = 4$

a) $\lim_{x \rightarrow 2} 3f(x) =$

b) $\lim_{x \rightarrow 2} 2 \cdot f(x) \cdot g(x) =$

c) $\lim_{x \rightarrow 2} f(x) + g(x) =$

d) $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)} =$

Finding a Limit when Given a Function (without a calculator!!!)



SCAN ME

Strategies

1. Direct Substitution
2. Factor
3. Simplify
4. Rationalize the Numerator
5. If it's a piecewise, plug into the function that is represented by the indicated side (If it is a 2 sided limit, you need to make sure both sides are the same!)

If direct substitution yields $\frac{0}{0}$, otherwise known as **indeterminate form**, then you cannot determine the limit in its current form. Encountering this form means you should try another technique. One way to deal with limits in this form is to use algebraic techniques like **factoring**, **simplifying**, and **rationalizing the numerator**.

Example 2: Find the value of each limit.

a) $\lim_{x \rightarrow 1} x^2 + 4x + 3$

b) $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2}$

c) $\lim_{x \rightarrow -3} \frac{x}{\left(\frac{1}{3+x}\right) - \frac{1}{3}}$

d) $\lim_{x \rightarrow 2} \frac{\sqrt{x+2} - 2}{x - 2}$

e) $\lim_{x \rightarrow 2^-} f(x), f(x) = \begin{cases} 2x + 3, & x < 2 \\ -2x - 5, & x \geq 2 \end{cases}$

f) $\lim_{x \rightarrow 3} f(x), f(x) = \begin{cases} 2x - 5, & x < 3 \\ x^2 - 6x + 10, & x \geq 3 \end{cases}$

Limits Involving Infinity

Definition: Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \text{ or } \lim_{x \rightarrow -\infty} f(x) = b$$

Complete the table below with the horizontal asymptote rules (aka rules for limits at $\pm\infty$)

Rules for Horizontal Asymptotes (in rational functions)

If....

Then the Horizontal Asymptote is....

Degree of Numerator is Larger

Degree of Numerator is Smaller

Degrees are the Same

Example 3: Evaluate each limit.

a) $\lim_{x \rightarrow -\infty} \frac{2x}{x+1}$

b) $\lim_{x \rightarrow \infty} \frac{3x^3}{2x^2+4}$

c) $\lim_{x \rightarrow \infty} -\frac{4}{x^2+1}$

d) $\lim_{x \rightarrow -\infty} \frac{4x}{3x-2}$

Limits at Infinity of Non Rational Functions

With rational functions, the graph approaches the same value on either side of the horizontal asymptote. This is not necessarily true for other functions that are in fraction form. (Remember rational functions are in fraction form and always have polynomials on top and bottom). As a general rule, if the denominator “grows” faster than the numerator (in the given direction $\pm\infty$), then the limit as x approaches $\pm\infty$, will be 0; If the numerator “grows” faster, the limit will increase/decrease without bound; if they grow at the same rate, the limit will be a constant value.

Remember to try to imagine what happens to the numerator and denominator when x gets really small or really big!

Growth Rate
Factorial
Exponential
Polynomial
Logarithmic
$\sin x / \cos x$
constants



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Example 4: Evaluate the following limits.

a) $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$

b) $\lim_{x \rightarrow \infty} e^x - 2x$

c) $\lim_{x \rightarrow \infty} \frac{5+2^x}{2-2^x}$

d) $\lim_{x \rightarrow -\infty} \frac{5+2^x}{2-2^x}$

e) $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+3x+1}}{4x}$

f) $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+3x+1}}{4x}$

Infinite Limits as $x \rightarrow a$

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty \text{ or } \lim_{x \rightarrow a^+} f(x) = \pm\infty$$

Example 5: Evaluate each limit. Try to do this without a calculator by testing values very close to the asymptote! Remember your answer will always be ∞ or $-\infty$ so you just need to determine the sign of the limit.

a) $\lim_{x \rightarrow 2^-} \frac{x^2}{2x-4}$

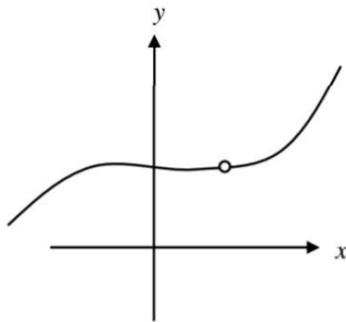
b) $\lim_{x \rightarrow 2^-} \frac{x+2}{x^2-4x+4}$

c) $\lim_{x \rightarrow -1} \frac{2x}{x+1}$

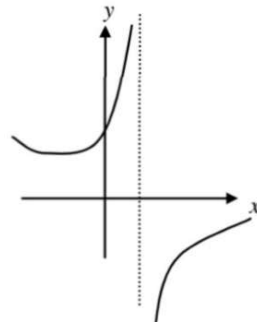
d) $\lim_{x \rightarrow 3} \frac{x+1}{x^2-6x+9}$

Section 2: Continuity

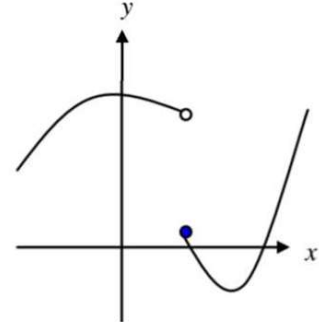
A function is continuous if you can draw the function without ever lifting your pencil. The following graphs demonstrate three types of discontinuous graphs.



Removable Discontinuity
Hole in the graph

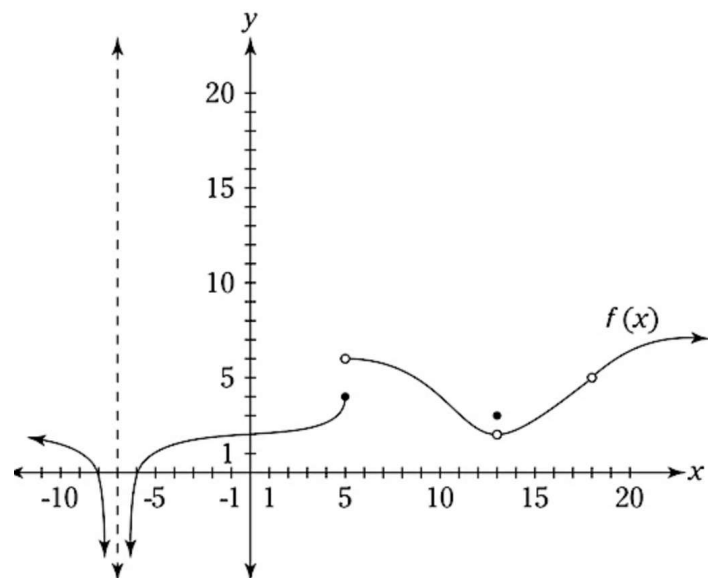


Non-Removable Discontinuity
Vertical Asymptote



Non-Removable Discontinuity
Jump

Example 1: Find all of the points of discontinuity in the graph below on the interval $[-10, 20]$. Then state the type of discontinuity.



Continuity at a point

A function $y = f(x)$ is continuous at point c if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Or in other words,

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$$

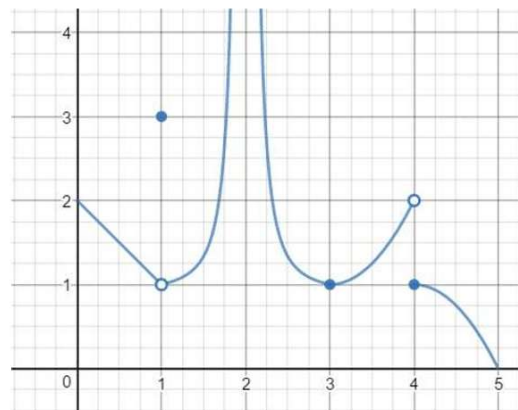
“The limit from the left, equals the limit from the right, equals the function value.”

Example 2: For $c = 1, 2$ and 4 , find $f(x)$, $\lim_{x \rightarrow c^+} f(x)$, $\lim_{x \rightarrow c^-} f(x)$, and $\lim_{x \rightarrow c} f(x)$. Take notice if how each part of the definition of continuity is important.

$$c = 1, \quad f(1) = \quad \lim_{x \rightarrow 1^+} f(x) = \quad \lim_{x \rightarrow 1^-} f(x) = \quad \lim_{x \rightarrow 1} f(x) =$$

$$c = 2, \quad f(2) = \quad \lim_{x \rightarrow 2^+} f(x) = \quad \lim_{x \rightarrow 2^-} f(x) = \quad \lim_{x \rightarrow 2} f(x) =$$

$$c = 4, \quad f(4) = \quad \lim_{x \rightarrow 4^+} f(x) = \quad \lim_{x \rightarrow 4^-} f(x) = \quad \lim_{x \rightarrow 4} f(x) =$$



Example 3: Determine if each function is continuous. If the function is not continuous, find the x value of the discontinuity and classify each one.

a) $f(x) = \frac{x}{x^2 + 4x + 4}$

b) $f(x) = \begin{cases} \frac{x}{2} + 3, & x < 0 \\ 2x + 6, & x \geq 0 \end{cases}$

c) $f(x) = \frac{x^2 + 5x + 6}{x + 2}$

d) $f(x) = \begin{cases} -2x - 2, & x < -1 \\ -\frac{x}{2} - \frac{1}{2}, & x \geq -1 \end{cases}$

Many AP multiple choice questions about continuity ask you to make a piecewise function continuous.

Example 4:

a) Find the value of k that makes $f(x)$ continuous at $x = 2$.
$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$$

b) Find the value of k that makes $f(x)$ continuous at $x = 3$.
$$f(x) = \begin{cases} -2x - 7k, & x < 3 \\ 2x - 3, & x \geq 3 \end{cases}$$

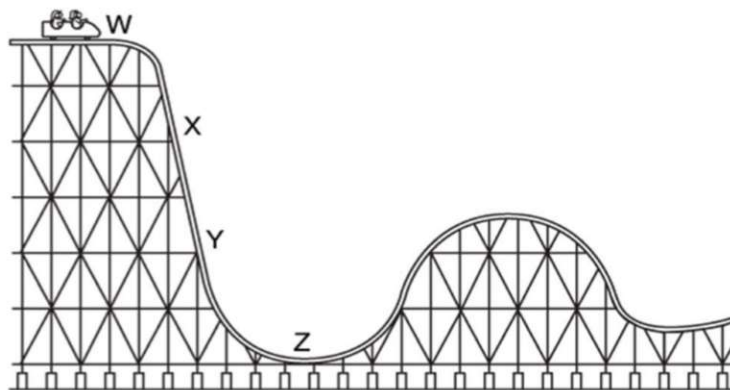
c) Find the value of k that makes $f(x)$ continuous at $x = -2$.
$$f(x) = \begin{cases} 4kx - 3, & x < -2 \\ -3x^2 - 2x, & x \geq -2 \end{cases}$$

Properties of Continuity

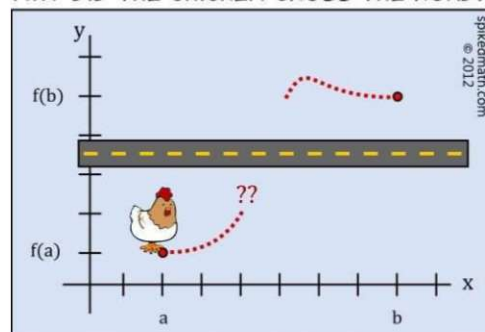
If b is a real number and f and g are continuous at $x = c$, then the following functions are continuous at c .

- | | |
|-----------------------------------|--|
| 1. Constant multiple: $b \cdot f$ | 2. Sum and Difference: $f \pm g$ |
| 3. Product: $f \cdot g$ | 4. Quotient: $\frac{f}{g}$; $g(c) \neq 0$ |

Intermediate Value Theorem (IVT)



WHY DID THE CHICKEN CROSS THE ROAD?



THE INTERMEDIATE VALUE THEOREM.

For the car to go from point W to point Z safely, do you have to go through points X and Y? Why or Why not?

Intermediate Value Theorem (IVT)

If f is continuous on the closed interval $[a, b]$ then f takes every value between $f(a)$ and $f(b)$.

Suppose k is a value between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Example 5: Show that $f(x) = x^2$ takes on the value 8 for some x between 2 and 3.

Example 6: Show that the equation $x^3 - 3x^2 + 1 = 0$ has a solution on the interval $[0, 1]$.

Example 7: Verify that the Intermediate Value Theorem applies to the following function $f(x)$ over the interval, $[-2, 2]$, and explain why IVT guarantees an x -value of c where $f(c) = 8$, and find c .

$$f(x) = 2x^2 - 5x - 4$$

***** Remember on the AP exam, these problems aren't always easily identifiable. Look for something like "must there be a value on the interval such that....." You may need to prove continuity and make sure to cite IVT!**

Section 3: Rates of Change and Tangent Lines

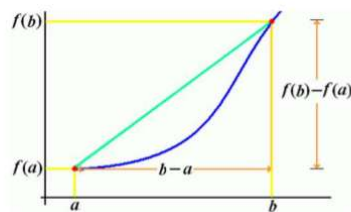
Remember from Calculus AB that calculus is all about slope! Here is our old definition for average rate of change (it's just the slope formula!) Average rate of change is the slope of the secant line.



SCAN ME

Average Rate of Change

$$\frac{f(b) - f(a)}{b - a}$$



Example 1: Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

If asked to find the instantaneous rate of change (or slope of the tangent line), we had to use a different formula because our b and a would be the same value and result in a denominator that is 0.

Slope of a Curve at a Point

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Provided the limit exists.

Example 2: Find the instantaneous rate of change at $x = 3$ for the following function. (Use the formula above, don't use any shortcuts you learned from last year!!)

$$f(x) = x^2 + 5$$

Recall that the **equation of the tangent line** can be found by plugging in the slope and the point of tangency into the point slope formula $y - y_1 = m(x - x_1)$. You may be given the point of tangency as an x value or a point but you can always find the y value of the point by plugging back into the function.

Example 3: Using the slope of the tangent line formula, write the equation of the tangent line to the function with the given point of tangency.

$$f(x) = x^2 + 2x - 9, \quad x = -1$$

Section 4: The Derivative of a Function and Differentiability

The derivative of a function is one of the two main concepts in calculus. The other is called an integral, which we will not get until later, and both are really just limits when you look under the hood. The only change from the limit definition we used before, is that we are going to treat the derivative as a function derived from f .

The Limit Definition of Derivative

The **derivative** of a function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Anywhere that the derivative exists, we say that the function is *differentiable*.

Thus, the derivative f' is a function that gives the slope of the function f at any point.

Example 1: Write down the different notations for the derivative that we learned last year.

We already practiced using this formula in the last section but remember if it asks you to find the derivative (in words or using one of the notations above), you can use the limit definition of the derivative. If you are asked to find the derivative at a point, simply plug the x value into the equation you have derived using the definition of the derivative.

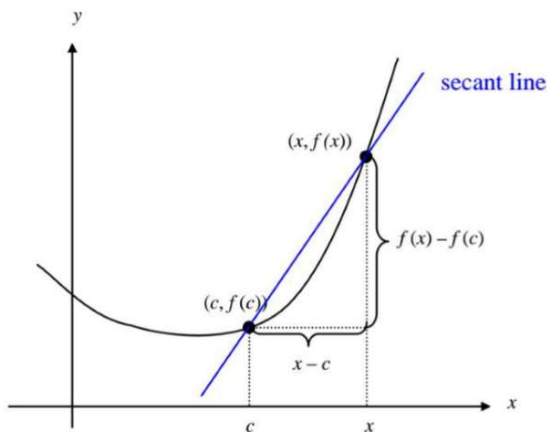
Below is an alternate version of the derivative definition and one that you will need to be able to recognize.

Alternative Definition of Derivative

An alternative definition of the derivative of f at point c is

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c},$$

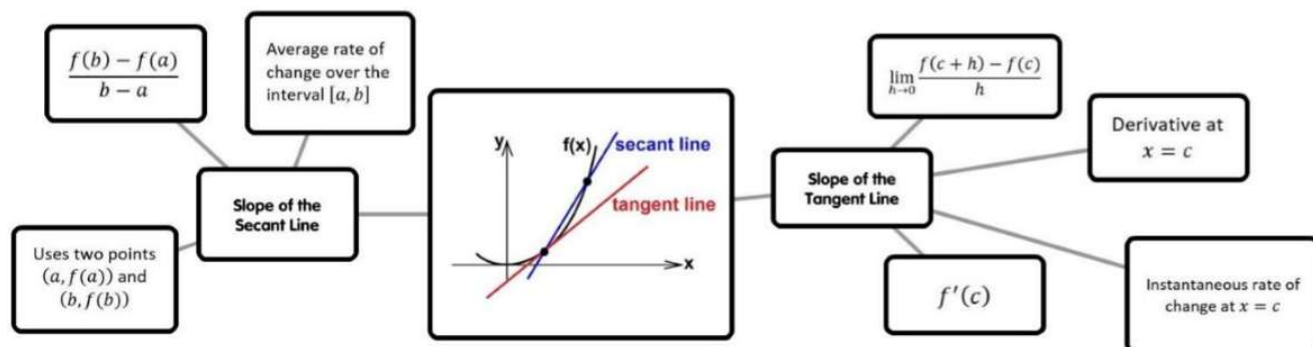
provided the limit exists.



What this alternative definition allows us to do is to examine the behavior of a function as x approaches c from the left or the right. The limit exists (and thus the derivative) as long as the left and right limits exist and are equal.

Example 2: Use the alternative definition of the derivative to find the derivative of $f(x) = x^2 - 5x$ at the point $x = -1$.

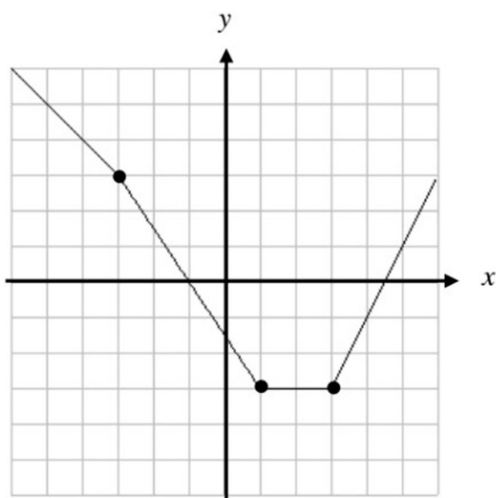
It is important to formally lay out the meanings and distinctions between secant lines and tangent lines.



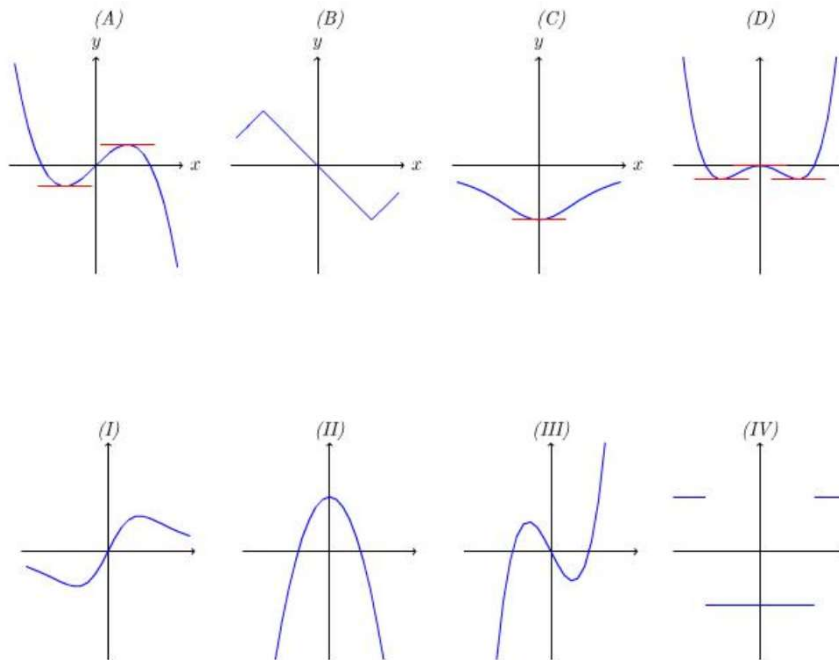
Relationships between the graphs of f and f'

Since a derivative at any point is the slope of the function at that point, you can graph the derivative given a function or estimate what an original function looks like when we are given the graph of the derivative.

Example 3: Given the graph of f , sketch the graph of the derivative on the same set of axes. Remember that it is just a sketch so pay attention to positive slopes, negative slopes and zero slopes!



Example 4: Match each function (A-D) with its derivative (I-IV).



Example 5: If $f(5) = -4$ and $f'(5) = \frac{1}{2}$, find the equation of the tangent line to $f(x)$ at $x = 5$.

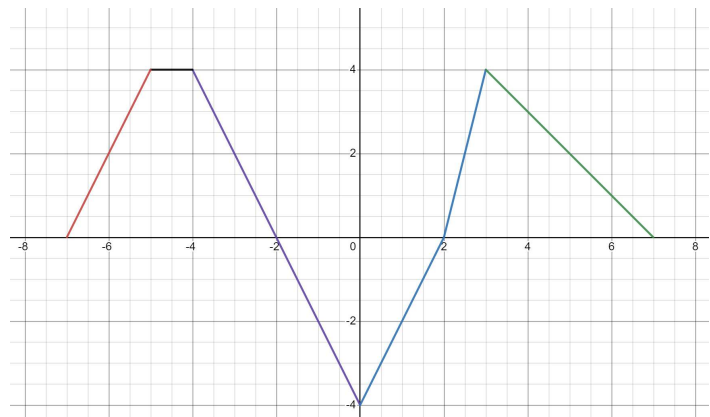
Example 6 AP Example: Given the graph to the right, evaluate the following:

a. $\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$

b. $\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$

c. $\lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x+2}$

d. $\lim_{x \rightarrow 3} f(x) - 2f(5)$



Differentiability

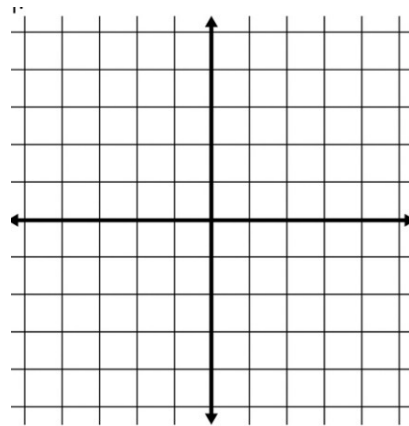
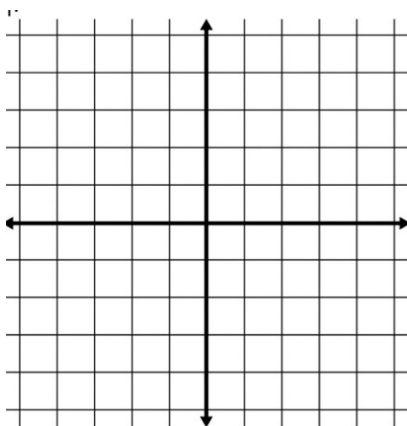
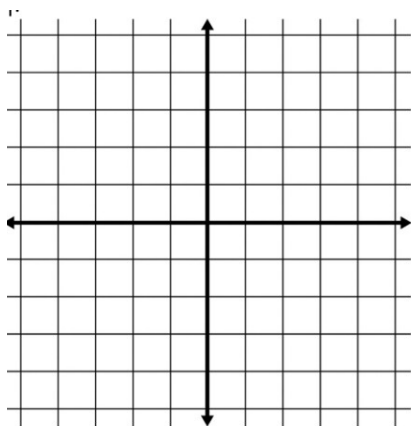
Differentiability at point $x = c$

A function $f(x)$ is differentiable at $x = c$ if and only if

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} = L = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}$$

where L is a finite value. In other words, the derivative from the left must equal the derivative from the right.

Last year we learned of a few types of graphs that were not differentiable. Sketch some examples of functions that are not differentiable on the coordinate planes below. Try to use examples of different types of non differentiable functions.



Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Example 7: If you are given that f is differentiable at $x = 2$, then explain why each statement below is true.

- a) $\lim_{x \rightarrow 2} f(x)$ exists.
- b) $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$ exists.
- c) $f(2)$ exists.
- d) $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$ exists.

Example 8: If f is a function such that $\lim_{x \rightarrow -3} \frac{f(x) - f(-3)}{x + 3} = 2$, which of the following must be true?

- A) The limit of $f(x)$ as x approaches -3 does not exist.
- B) f is not defined at $x = -3$.
- C) The derivative of f at $x = -3$ is 2.
- D) f is continuous at $x = 2$.
- E) $f(-3) = 2$

It takes more than just the slopes to be approaching the same value on either side of a value of x for a function to be differentiable at that point. You always need to be careful with piecewise functions and make sure they are also continuous.

Example 9: Determine if $f(x)$ is differentiable at $x = 1$ if $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x + 1, & x > 1 \end{cases}$

Example 10: Determine if $f(x)$ is differentiable at $x = 1$ if $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$

Rules for Finding the Derivative

Finding the Derivative on the Calculator: MATH 8!!!

The **nDeriv**(function works as follows:

$$\left. \frac{d}{dx}(\text{function}) \right|_{x=\text{value}}$$

Note: Many times it is easier to type the function into y_1 and then enter y_1 in for the function. To enter y_1 in for the function, simply press alpha , trace to find your functions.

Example 11: Use your calculator to find the derivative of $f(x) = 5x^2 + 2x - 8$ at $x = -2$. Express your answer using correct notation.

The rest of this section includes most of the basic derivative rules that we learned last year. You will need to make sure they are committed to memory. Remember that you have the derivative cheat sheet you can study as well!

Power Rule for Differentiation

If n is any number, then $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$, provided x^{n-1} exists.

**** Don't forget to change radicals into exponents to use the power rule!**

Example

$$y = 4x^3$$
$$y' = 12x^2$$

The Derivative of a Constant Function

If c is any constant value, then $\frac{d}{dx}[c] = 0$.

Example

$$y = 5$$
$$y' = 0$$

The Constant Multiple Rule for Derivatives

If u is a differentiable function of x and c is a constant value, then

$$\frac{d}{dx}[cu] = c \frac{du}{dx}$$

Example

$$y = 2 \sin x$$
$$y' = 2 \cos x$$

The Sum and Difference Rule for Derivatives

If u and v are differentiable functions of x , then wherever u and v are differentiable

$$\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$$

Example

$$y = x^2 + 5x$$
$$y' = 2x + 5$$

The Product Rule for Derivatives

If u and v are differentiable functions of x , then

$$\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \frac{du}{dx}$$

Example

$$y = x \cdot \sin x$$
$$y' = x \cos x + \sin x (1)$$

The Quotient Rule for Derivatives

If u and v are differentiable functions of x , then

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v \cdot \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Example

$$y = \frac{x}{\sin x}$$
$$y' = \frac{\sin x (1) - x \cdot \cos x}{\sin^2 x}$$

Second and Higher Order Derivatives

The first derivative of y with respect to x is denoted y' , $f'(x)$, or $\frac{dy}{dx}$. The second derivative with respect to x is denoted y'' , $f''(x)$, or $\frac{d^2y}{dx^2}$. The second derivative is an example of a higher order derivative. We can continue to take derivatives (as long as they exist) using the following notation:

First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$
Second Derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$
Third Derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$
Nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$

Derivatives of the Six Basic Trigonometric Functions

$$\frac{d}{dx} [\sin x] = \cos x$$

$$\frac{d}{dx} [\cos x] = -\sin x$$

$$\frac{d}{dx} [\tan x] = \sec^2 x$$

$$\frac{d}{dx} [\cot x] = -\csc^2 x$$

$$\frac{d}{dx} [\sec x] = \sec x \tan x$$

$$\frac{d}{dx} [\csc x] = -\csc x \cot x$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} [\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} [\tan^{-1} x] = \frac{1}{1+x^2}$$

$$\frac{d}{dx} [\cot^{-1} x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx} [\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} [\csc^{-1} x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

*Remember that arcsin and $\sin^{-1}$ are the same thing!!

Derivatives of Exponential and Logarithmic Functions

Derivative of e^x

$$\frac{dy}{dx} [e^x] = e^x$$

Derivative of $\ln x$

$$\frac{d}{dx} [\ln x] = \frac{1}{x}$$

Derivative of a Log Function

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

Derivative of a^x where a is a constant

$$\frac{d}{dx} [a^x] = \ln a \cdot a^x$$

Example 12: Find the derivatives of the following functions.

a) $5x^3 - 2x^2 + 6x + 1$	b) $\sqrt{x} - 3x^{\frac{3}{2}} + 5x^{-\frac{1}{2}}$
c) $\sin x \cos x$	d) $\frac{\tan x}{e^x}$
e) $\ln x + 4^x$	f) $\log_8 x - \csc x + \sin^{-1} x$

Example 13: AP Practice

a) Let f be the function defined below. Which of the following statements about f are true?

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

- I. f has a limit at $x = 2$
- II. f is continuous at $x = 2$
- III. f is differentiable at $x = 2$

- A) I only B) III only C) I and II D) I, II and III

b) Let $f(x) = \begin{cases} cx + d & \text{for } x \leq 2 \\ x^2 - cx & \text{for } x > 2 \end{cases}$. If c and d are constants and f is differentiable at $x = 2$, what is $c + d$?

- A) -4 B) -2 C) 0 D) 2

c) Let $f(x) = \begin{cases} 2x - 2 & \text{for } x < 3 \\ 2x - 4 & \text{for } x \geq 3 \end{cases}$. Which of the following statements is true?

- I. $\lim_{h \rightarrow 0^-} \frac{f(3+h) - f(3)}{h} = 2$
- II. $\lim_{h \rightarrow 0^+} \frac{f(3+h) - f(3)}{h} = 2$
- III. $f'(3) = 2$

- A) None B) II only C) I and II D) I, II and III

Section 5: Chain Rule



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The Chain Rule is a rule that you need to use if you are trying to take the derivative of a **composite function**.

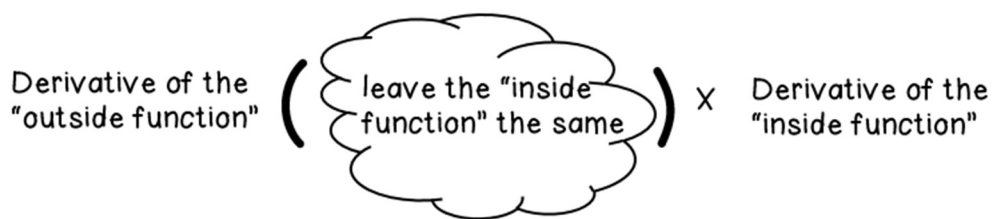
The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ OR } \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Composite functions have an “inside function” and an “outside function.”

Another way to look at this would be:



Example 1: Use the chain rule to differentiate the following functions.

1. $f(x) = \sin 6x$	2. $f(x) = (5x^2 + 3)^3$
3. $f(x) = e^{7x}$	4. $f(x) = \frac{1}{4x+7}$
5. $f(x) = \sqrt{6x^2 + 8x - 1}$	6. $f(x) = \sin^2 x$

Deriving and identifying the inside and outside functions gets tougher when you have functions with multiple inside functions. When this happens, you peel the function like an onion, taking the derivative of the outside of the remaining function that has not been derived yet and keeping the inside function the same until you work your way all the way to the innermost function.

Example 3: Use the chain rule to differentiate the following functions.

1. $f(x) = \sin^2 5x^3$	2. $f(x) = \cos e^{10x}$
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It is common on the AP exam for them to ask you to find derivatives of functions that they do not give you. They will often just give you values of the function and its derivative at specific points so it is important that you know the definition of the chain rule. (i.e. how to take the derivative of $f(g(x))$.)

Example 4: Use the given table to find the following derivatives.

x	$f(x)$	$f'(x)$	$h(x)$	$h'(x)$
4	1	7	3	2
3	4	5	2	-6
2	-2	6	0	8

a. If $g(x) = f(h(x))$, find $g'(4)$.

b. If $g(x) = h(h(x))$, find $g'(3)$

AP Practice

1. If $f(x) = (\ln x)^2$, then $f''(\sqrt{e}) =$

- (A) $\frac{1}{e}$ (B) $\frac{2}{e}$ (C) $\frac{1}{2\sqrt{e}}$ (D) $\frac{1}{\sqrt{e}}$ (E) $\frac{2}{\sqrt{e}}$

2. Let h be a differentiable function, and let f be the function defined by $f(x) = h(x^2 - 3)$. Which of the following is equal to $f'(2)$?

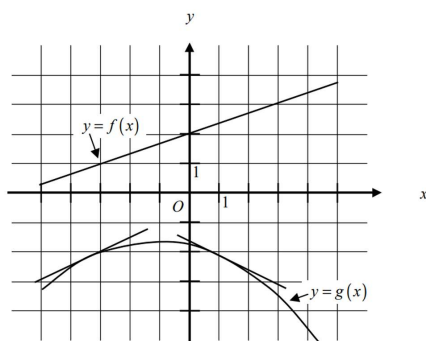
- (A) $h'(1)$ (B) $4h'(1)$ (C) $4h'(2)$ (D) $h'(4)$ (E) $4h'(4)$

3.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	-5	1	3	0
0	-2	0	1	1
1	0	-3	0	0.5
2	5	-1	5	2

The table above gives values of the differentiable functions f and g and of their derivatives f' and g' , at selected values of x . If $h(x) = f(g(x))$, what is the slope of the graph of h at $x = 2$?

- (A) -10 (B) -6 (C) 5 (D) 6 (E) 10



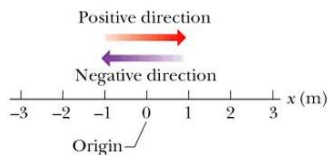
4.

The figure above shows the graphs of the functions f and g . The graphs of the lines tangent to the graph of g at $x = -3$ and $x = 1$ are also shown. If $B(x) = g(f(x))$, what is $B'(-3)$?

- (A) $-\frac{1}{2}$ (B) $-\frac{1}{6}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

Section 6: Velocity and Other Rates of Change

Motion along a Line



There is perhaps no more important type of rate of change than that of motion. After all, it was Newton's motivation for inventing calculus in the first place. We have already looked at motion briefly when we investigated the basketball shot from a World Record height. To delve into this topic further, we will need to get familiar with some terminology and relationships between position, velocity, and acceleration.



Relationships between Position, Velocity, and Acceleration

Position functions, often denoted $s(t)$, give the position of an object at any point in time.

The **displacement** of an object is the **total change in position**. If we were given a position function $s(t)$ and an interval of time $[a, b]$, then the displacement would be $s(b) - s(a)$.

The **average velocity** of the object is the **total change in position (displacement) divided by the total change in time**. It can be thought of as the slope of the line connecting two points on a position function.

$$\text{Average Velocity} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{s(b) - s(a)}{b - a}$$

The **instantaneous velocity** of an object is the derivative of the position function of the object.

$$\text{Instantaneous Velocity} = v(t) = s'(t) = \text{the slope of the tangent line at any time } t$$

Velocity is a quantity that gives both how fast something is moving as well as direction of the movement. If velocity is positive, then movement is occurring in the positive direction. If velocity is negative, then movement is occurring in the negative direction.

Speed is the **absolute value of the velocity**, therefore, speed is always positive.

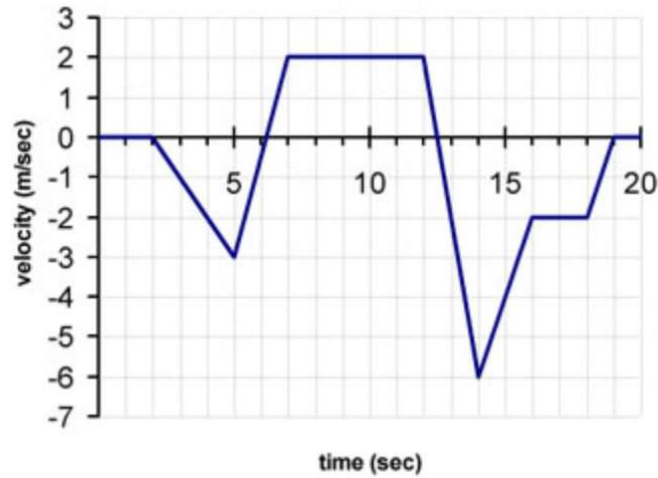
$$\text{Speed} = |v(t)| = |s'(t)|$$

Acceleration is the rate of change of velocity or **the derivative of velocity**. Since it is the derivative of velocity, it is also **the second derivative of position**.

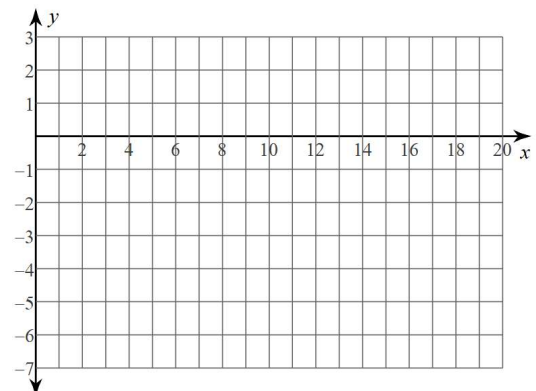
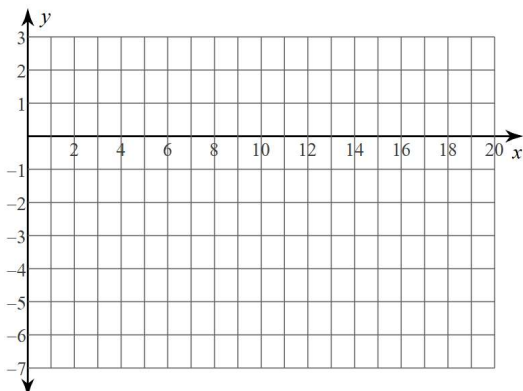
$$\text{Acceleration} = v'(t) = s''(t) = \text{the slope of the tangent line to the velocity function at any time } t$$

The **average acceleration** of an object is the **total change in velocity divided by the total change in time**. It can be thought of as the slope of the line connecting two points on a velocity function.

Example 1: The graph below shows the velocity of a particle moving along the x axis. Use the graph to answer the following questions. Justify each response.



- a. What is the velocity of the particle at 5 seconds?
- b. When is the particle moving at a constant velocity?
- c. When does the particle change direction?
- d. When is the particle moving left?
- e. When is the particle moving right?
- f. When does the particle stop?
- g. When is the particle moving the fastest?
- h. When is the particle speeding up?
- i. Graph the particle's acceleration over the interval.
- J. Graph the particle's speed over the interval.



Determining When Speed is Increasing or Decreasing

The speed of an object is increasing when it is moving away from the x-axis of a velocity-time graph. The speed is decreasing when the object is moving towards the x-axis. If you do not have a graph, Speed is increasing when velocity and acceleration have the same sign. Speed is decreasing when velocity and acceleration have different signs.

Sign of the Velocity	Sign of the Acceleration	Speed
+	+	Increasing
-	-	Increasing
+	-	Decreasing
-	+	Decreasing



Knowing how this works with a graph can help us answer motion questions when given functions.

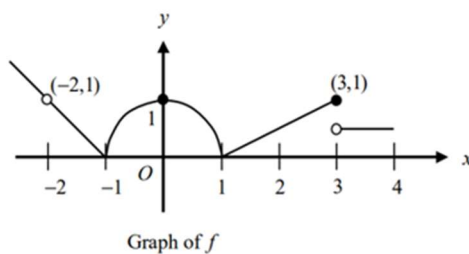
Example 2: A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^2 - 4t + 3$, where s is measured in meters and t is measured in seconds.

- Find the displacement of the particle during the first 2 seconds.
- Find the average velocity of the particle during the first 2 seconds.
- Find the instantaneous velocity of the particle when $t = 1$ and then find the speed.
- Find the acceleration of the particle when $t = 4$.
- What value of t does the particle change direction?
- At time $t = 5$, is the particle speeding up or slowing down?

You will be getting a test review packet when we get back to school but here are some practice problems in a random order to help you check for understanding as well as practice your application of the skills like AP questions will.

1. An equation of the line tangent to the graph of $f(x) = x(1 - 2x)^3$ at the point $(1, -1)$ is

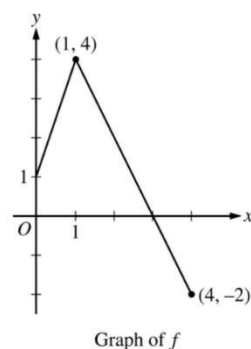
- (A) $y = -7x + 6$
 (B) $y = -2x$
 (C) $y = 2x - 3$
 (D) $y = 7x - 8$



2. The graph of the function f is shown above. For which of the following values of c does $\lim_{x \rightarrow c} f(x) = 1$?
- (A) 0 only (B) 0 and 3 only (C) -2 and 0 only (D) -2 and 3 only (E) -2, 0 and 3

3. The graph of the function f , consisting of two line segments, is shown in the figure below. Let g be the function given by $g(x) = 2x + 1$, and let h be the function given by $h(x) = f(g(x))$. What is the value of $h'(1)$?

- (A) -4
 (B) -2
 (C) 4
 (D) nonexistent

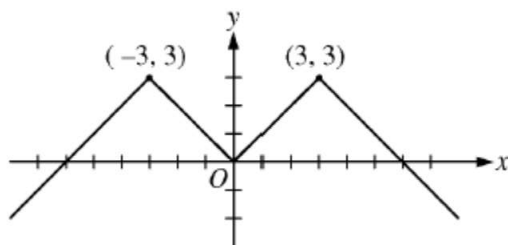


4. If $f'(x) = \frac{(x-2)^3(x^2-4)}{16}$ and $g(x) = f(x^2 - 1)$, what is $g'(2)$?

- (A) 2
 (B) $\frac{5}{4}$
 (C) $\frac{5}{8}$
 (D) $\frac{5}{16}$

5. Let f be a function that is differentiable on the open interval $(1, 10)$. If $f(2) = -5$, $f(5) = 5$, and $f(9) = -5$, which of the following must be true?
- I. f has at least 2 zeros.
 - II. The graph of f has at least one horizontal tangent.
 - III. For some c , $2 < c < 5$, $f(c) = 3$.
- (A) None
 (B) I only
 (C) I and II only
 (D) I and III only
 (E) I, II and III

6.



The graph of the even function $y = f(x)$ consists of 4 line segments, as shown above. Which of the following statements about f is false?

- (A) $\lim_{x \rightarrow 0} (f(x) - f(0)) = 0$
 (B) $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = 0$
 (C) $\lim_{x \rightarrow 0} \frac{f(x) - f(-x)}{2x} = 0$
 (D) $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = 1$
 (E) $\lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}$ does not exist.

7.

X	2.5	2.8	3.0	3.1
$f(x)$	31.25	39.20	45	48.05

The function f is differentiable and has values as shown in the table above. Both f and f' are strictly increasing on the interval $0 \leq x \leq 5$. Which of the following could be the value of $f'(3)$?

- (A) 20
 (B) 27.5
 (C) 29
 (D) 30
 (E) 30.5