

Introduction: The Logic of Mathematics

Key Perspective

Calculus is often described as the *mathematics of change*—focusing on continuous motion, velocities, and curves. In contrast, **Discrete Mathematics** is the *mathematics of distinct objects*. It deals with structures that are separate and countable, rather than continuous.

From Motion to Foundation

In the historical excerpt from *Calculus: An Intuitive and Physical Approach*, Morris Kline explains that calculus was motivated by seventeenth-century problems like planetary motion and the paths of projectiles. These problems required understanding variable velocity and acceleration. However, before we can model the physical world's complex changes, we must understand the bedrock of all mathematical reasoning: **Logic and Set Theory**.

While Calculus looks at how things move, Discrete Math looks at how things are organized and how we can prove truths about them. Logic provides the rules for valid reasoning, ensuring that our mathematical conclusions are sound. Set Theory gives us a way to group distinct objects together to study their relationships. Together, they form the language upon which all other mathematics—including Calculus—is built.

Term	Definition
Statement (Proposition)	A sentence that is either true or false, but not both. (e.g., "The earth revolves around the sun.")
Negation	The opposite of a statement. If statement <i>p</i> is true, its negation (not <i>p</i>) is false.
Conjunction	An "and" statement. It is true only if both individual parts are true.
Disjunction	An "or" statement. It is true if at least one of the parts is true.
Conditional	An "if-then" statement. It claims that if the first part is true, the second part must also be true.

Practice: Identifying Mathematical Statements

Not every sentence is a mathematical statement. Questions, commands, and opinions cannot be proven true or false in a mathematical sense. Determine if each sentence below is a **Statement** or **Not a Statement**.

Sentence	Statement? (Yes/No)
The sum of the interior angles of a triangle is 180°.	
Calculus is the most interesting branch of math.	
Solve the following equation for <i>x</i> .	

If a shape is a square, then it is a rectangle.	
$5 + 3 = 10$	
Did Newton or Leibniz create the calculus?	

Reflection: Based on the definitions above, why is the sentence " $5 + 3 = 10$ " considered a mathematical statement even though the math is incorrect?

Activity 2: Logic Puzzles and Truth Tables

Symbolic Notation Reference

In the previous activity, we defined key logical connectors. In symbolic logic, we use shorthand to represent these operations:

- **Conjunction (AND):** $p \wedge q$
- **Disjunction (OR):** $p \vee q$
- **Conditional (IF-THEN):** $p \rightarrow q$
- **Negation (NOT):** $\neg p$ or $\sim p$

Part 1: Constructing Truth Tables

A truth table shows the truth value of a compound statement for every possible combination of truth values of its individual parts. Complete the tables below.

p	q	Conjunction $p \wedge q$	Disjunction $p \vee q$	Conditional $p \rightarrow q$
T	T			
T	F			
F	T			
F	F			

Part 2: Deductive Reasoning Puzzles

Use your understanding of logical statements to solve these puzzles. Show your reasoning for each.

Puzzle 1: The Island of Knights and Knaves

You meet two inhabitants of an island where Knights always tell the truth and Knaves always lie.

Person A says: "At least one of us is a Knave."

What are Person A and Person B?

Puzzle 2: The Three Boxes

There are three boxes: one contains gold, one contains silver, and one is empty. Each box has a label, but you are told that **all labels are false**.

- **Box 1 Label:** "The gold is in here."
- **Box 2 Label:** "This box is empty."
- **Box 3 Label:** "The gold is in Box 2."

Which box contains the gold?

Puzzle 3: The Race Results

Three students (Alice, Bob, and Charlie) finished a race in 1st, 2nd, and 3rd place.

1. Alice says: "I did not come in 1st."
2. Bob says: "I came in 3rd."
3. Charlie says: "Alice came in 2nd."

If **exactly one** of these statements is true, who won the race?

Reflection: In Puzzle 3, why is it helpful to test the assumption that Alice's statement is the only true one first? How does this relate to the concept of a 'Statement' being either true or false?

Set Theory Foundations

In the previous sections, we explored the logic of statements. Now, we move to **Set Theory**, which is the study of collections of distinct objects. Just as logic provides the rules for our 'grammar,' set theory provides the 'nouns'—the objects we are actually talking about. In Discrete Math, we use sets to group numbers, symbols, or even other sets to analyze their relationships.

Key Notation Guide

- **Set:** Usually named with a capital letter (e.g., A, B, S).
- **Elements:** The individual objects inside a set, listed inside curly braces: $\{ \}$.
- **$x \in A$ (Element of):** $x \in A$ means x is an object inside set A .
- **$\subseteq$ (Subset):** $A \subseteq B$ means every element in A is also in B .
- **$\emptyset$ (Empty Set):** A set with no elements, also written as $\{ \}$.
- **U (Universal Set):** The set containing all possible elements under consideration.

Operation	Definition & Notation
Union	$A \cup B$: The set of all elements that are in A OR in B (or both).
Intersection	$A \cap B$: The set of all elements that are in A AND in B (the overlap).
Complement	A' (or A^c): The set of all elements in the Universal Set U that are NOT in A .

Part 1: Identifying Elements and Subsets

Let $S = \{2, 4, 6, 8, 10\}$. Determine if the following statements are **True** or **False**.

Statement	True or False?
$4 \in S$	
$5 \in S$	
$\{2, 6\} \subseteq S$	
$\emptyset \subseteq S$	

Part 2: Set Operations Practice

Given the following sets, find the resulting set for each operation. Show your elements inside curly braces .

Universal Set $U = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

Set $A = 1, 3, 5, 7, 9$ (Odd numbers)

Set $B = 2, 3, 5, 7$ (Prime numbers)

Set $C = 1, 2, 3, 4, 5$

1. Find $A \cup B$ (Elements in A or B)

2. Find $A \cap B$ (Elements in both A and B)

3. Find A^c (Elements in U that are not in A)

4. Find $B \cap C$

5. Find $(A \cup C)^c$ (Hint: Find the union first, then the complement)

Critical Thinking Reflection

In logic, we used the symbol $\wedge$ for 'and' and $\vee$ for 'or.' How do the set operations $\cap$ and $\cup$ relate to these logical connectors? Why do you think mathematicians use different symbols for sets versus logic statements?

Visualizing Sets with Venn Diagrams

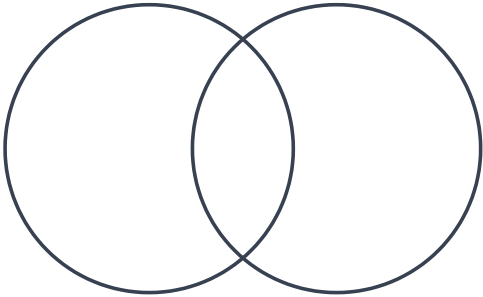
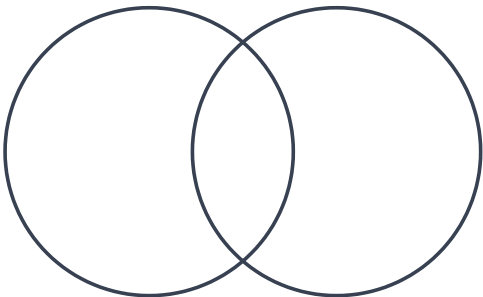
Key Concepts

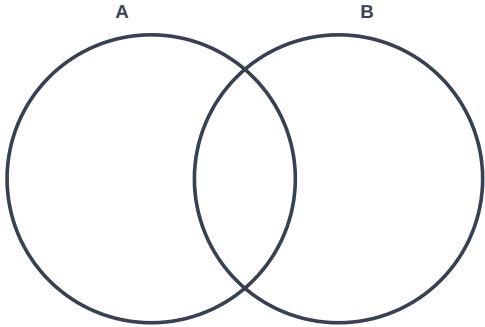
A **Venn Diagram** uses overlapping circles to represent relationships between sets. The **Universal Set (U)** is represented by the rectangle surrounding the circles.

- **Intersection ($A \cap B$):** The region where circles overlap (elements in both).
- **Union ($A \cup B$):** The entire area inside both circles (elements in either or both).
- **Complement (A'):** Everything outside circle A but inside the rectangle U .

Part 1: Shading Regions

For each diagram below, describe which specific region(s) should be shaded to represent the given set notation. Use the labels 'Only A', 'Only B', 'Both A and B', or 'Outside A and B'.

A B 	1. $A \cap B$ (Intersection) Describe the shaded region:
A B 	2. $A \cup B$ (Union) Describe the shaded region:

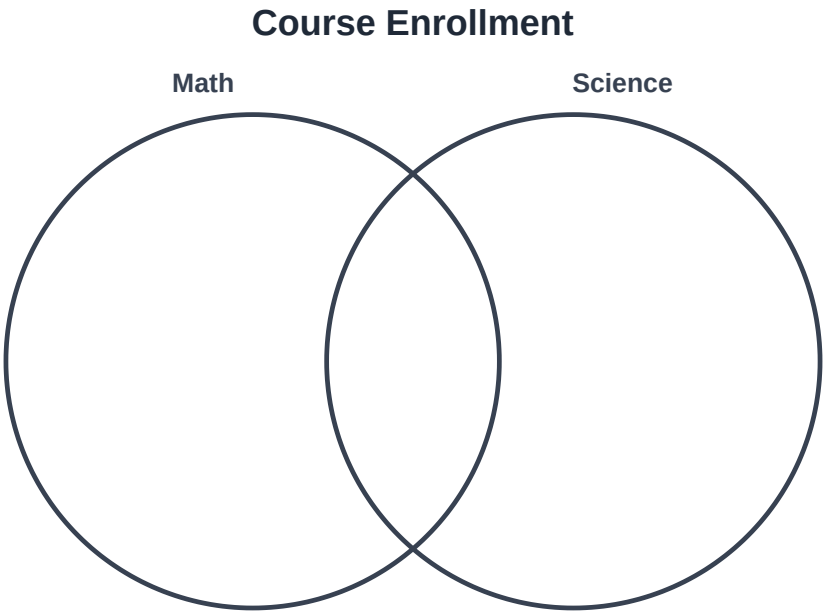
 A Venn diagram consisting of two overlapping circles. The left circle is labeled 'A' and the right circle is labeled 'B'. The circles overlap in the center.	3. $A \cap B'$ (Elements in A but NOT in B) Describe the shaded region:

Part 2: Organizing Data with Venn Diagrams

Problem 1: The Science and Math Survey

In a group of 30 high school students, 18 students are taking a Math course, 12 students are taking a Science course, and 7 students are taking both.

Step 1: Fill in the Venn Diagram below with the number of students in each region.
Hint: Start with the intersection (both) first, then subtract that number from the totals for Math and Science.



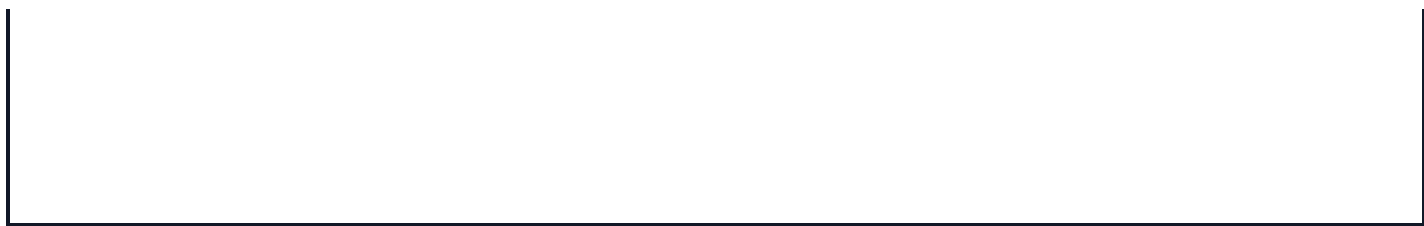
Step 2: Use your diagram to answer the following questions:

How many students are taking Math but NOT Science?
How many students are taking Science but NOT Math?
How many students are taking neither Math nor Science?

Problem 2: Programming Languages

A tech club has 25 members. 15 members know Python (*P*), 10 members know Java (*J*), and 5 members know neither language. How many members know **both** Python and Java?

Show your work below by setting up an equation or using the diagram.



Final Reflection: Why is it necessary to subtract the intersection when calculating the total number of elements in a union ($A \cup B$)? What would happen if we just added the total of Set A to the total of Set B?

Logic and Sets Cumulative Review

Part 1: Vocabulary and Foundations

Circle the best answer for each question.

1. Which of the following best describes a mathematical 'statement' (proposition)?

- A) Any sentence written in a textbook.
- B) A sentence that is either true or false, but not both.
- C) A question that requires a numerical answer.
- D) An equation that contains at least one variable.

2. In symbolic logic, what does the symbol $\vee$ represent?

- A) Negation (Not)
- B) Conjunction (And)
- C) Disjunction (Or)
- D) Conditional (If... then...)

3. If an element belongs to the intersection of Set A and Set B ($A \cap B$), where must it be located?

- A) In Set A only.
- B) In Set B only.
- C) In either Set A or Set B (or both).
- D) In both Set A and Set B simultaneously.

4. What is the definition of a 'subset' ($\subseteq$)?

- A) A set that contains no elements.
- B) A set where every element is also contained within another set.
- C) The set of all possible elements in a given discussion.
- D) Two sets that share exactly one element.

5. Which branch of mathematics is described as the 'mathematics of distinct objects' rather than the 'mathematics of change'?

- A) Calculus
- B) Trigonometry
- C) Discrete Mathematics
- D) Analytic Geometry

Part 2: Symbolic Logic and Truth Tables

6. Complete the truth table for the compound statement: $\neg p \vee q$

p	q	$\neg p$	$\neg p \vee q$
T	T		
T	F		
F	T		
F	F		

7. Under what specific condition is a conditional statement ($p \rightarrow q$) considered False?

8. **Deductive Reasoning:** On the Island of Knights (truth-tellers) and Knaves (liars), you meet a person who says: "I am a Knave OR $2 + 2 = 5$." Is this person a Knight or a Knave? Explain your logic.

9. **Translate to Symbols:** Let p be "It is raining" and q be "The ground is wet." Write the symbolic form for: "If it is not raining, then the ground is not wet."

10. Logical Consistency: If the statement "Alice won the race" is False, what is the truth value of the negation "Alice did not win the race"?

Part 3: Set Operations and Venn Diagrams

Set Reference

Use the following sets for questions 11 and 12:

Universal Set (U) = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Set A = {2, 4, 6, 8, 10}

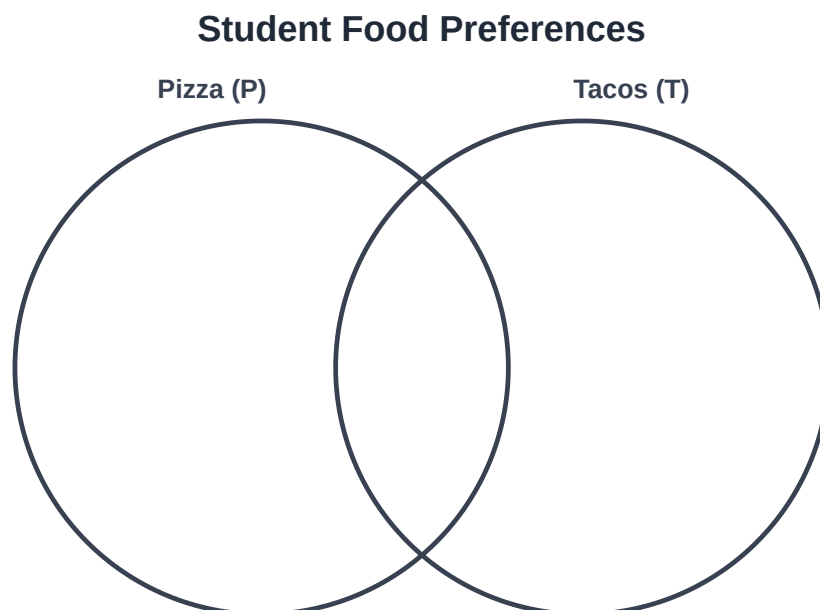
Set B = {1, 2, 3, 4, 5}

11. Find the following results based on the sets above:

- $A \cap B$ (Intersection):
- $A \cup B$ (Union):
- A' (Complement of A):

12. Is the statement $\{2, 4\} \subseteq A$ true or false? Explain why.

13. Data Organization: In a survey of 40 students, 25 like Pizza (P), 20 like Tacos (T), and 10 like both. Use the Venn Diagram below to organize this data.



14. Calculation: Based on your diagram in Question 13, how many students like **neither** Pizza nor Tacos?

15. Shading Logic: If you were asked to shade the region representing $(A \cup B)'$, which area(s) of a 2-circle Venn Diagram would you shade?

Answer Key: Logic and Sets Review Quiz**Teacher Reference**

This answer key corresponds to the 15-question cumulative review quiz. Use the explanations provided to help students understand the underlying logic of Discrete Mathematics, focusing on the distinction between continuous change (Calculus) and distinct objects (Discrete Math).

Part 1: Vocabulary and Foundations

- | | |
|----|---|
| 1. | B. A sentence that is either true or false, but not both. (A mathematical statement must have a definitive truth value.) |
| 2. | C. Disjunction (Or). The symbol $\vee$ represents the logical 'OR' operator. |
| 3. | D. In both Set A and Set B simultaneously. Intersection ($\cap$) refers to the overlap where elements must satisfy both conditions. |
| 4. | B. A set where every element is also contained within another set. This is the definition of $A \subseteq B$. |
| 5. | C. Discrete Mathematics. As noted in the introduction, Calculus is the study of change, while Discrete Math focuses on distinct, separate objects. |

Part 2: Symbolic Logic and Truth Tables

- | | |
|-----|---|
| 6. | Truth Table Results: <ul style="list-style-type: none"> Row 1 (T, T): $\neg p$ is F, final result is T. Row 2 (T, F): $\neg p$ is F, final result is F. Row 3 (F, T): $\neg p$ is T, final result is T. Row 4 (F, F): $\neg p$ is T, final result is T. |
| 7. | A conditional statement ($p \rightarrow q$) is False only when the antecedent (p) is True and the consequent (q) is False ($T \rightarrow F$). |
| 8. | The person is a Knight.
<i>Logic:</i> If the person were a Knave, the statement would have to be false. For a disjunction (OR) to be false, both parts must be false. Part 2 ($2+2=5$) is false. However, if the person is a Knave, Part 1 ("I am a Knave") would be true. Since one part is true, the whole statement is true. A Knave cannot tell a true statement, so the person must be a Knight. |
| 9. | $\neg p \rightarrow \neg q$ |
| 10. | If the statement is False, its negation is True . (Negation flips the truth value). |

Part 3: Set Operations and Venn Diagrams

11.

Results:

- $A \cap B = \{2, 4\}$ (Elements in both sets)
- $A \cup B = \{1, 2, 3, 4, 5, 6\}$ (All elements from both sets)
- $A' = \{5, 6\}$ (Elements in the Universal set not in A)

12.

True. Every element in the set $\{2, 4\}$ is also an element of Set A $\{1, 2, 3, 4\}$.

13.

Venn Diagram Values:

- Center (Both): 10
- Pizza Only: 15 (25 total - 10 both)
- Tacos Only: 10 (20 total - 10 both)
- Outside Circles: 5

14.

5 students. Calculation: Total (40) - [Pizza Only (15) + Tacos Only (10) + Both (10)] = 40 - 35 = 5.

15.

You would shade the **entire area outside of both circles** but inside the universal set rectangle. This represents everything that is NOT in the union of A and B.