Entering
Advanced Topics
Summer Homework



Name

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Advanced Topics in Math
Summer Review Packet

Post-AP Calculus BC Review

Due the first day of class

Purpose and Expectations. This packet reviews AP Calculus BC topics that are important for success in Multivariable Calculus. Complete all problems without a calculator or other technology. Show your method in an organized manner, using proper mathematical notation throughout. Unsupported answers will not receive full credit. Bring the completed packet to class on the first day.

I. Differentiation Review

1. Find the derivative of each function. Show all work and simplify where reasonable.

a) $f(x) = x^4 e^{2x}$

b) $f(x) = \left(\sin x + x^2\right)^5$

c) $f(x) = \dfrac{x^3 \ln x}{x^2 + 1}$

d) $f(x) = (3x - 1)^6 \left(x^2 + 2\right)^4$

e) $f(x) = \ln\left(\dfrac{x^2 + 1}{\sqrt{x + 3}}\right)$

2. Let $f(2) = 5$, $f'(2) = -1$, $g(2) = 3$, and $g'(2) = 4$. Find each value.

a) $(fg)'(2)$

b) $\left(\dfrac{f}{g}\right)'(2)$

c) $(f \circ g)'(2)$, if $f'(3) = 7$

II. Limits, Continuity, and Graph Behavior

3. Evaluate each limit exactly, or state that the limit does not exist. Justify your answer.

a) $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

b) $\lim\limits_{x \to \infty} \dfrac{3x^2 + \sin x}{x^2 + 5}$

c) $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3}$

d) $\lim\limits_{x \to \infty} x e^{-x}$

e) $\lim_{x \to 2} \dfrac{x^2 - 4}{x^2 - x - 2}$

4. Find all vertical and horizontal asymptotes of

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 - x - 2}.$$

State clearly whether any factors create holes rather than vertical asymptotes.

5. Show that the equation

$$x^3 + x - 1 = 0$$

has at least one solution on the interval $(0, 1)$. Name the theorem used.

III. Applications of Derivatives

6. Let $f(x) = x^2 + \ln(x + 1)$.

a) Find the equation of the tangent line to the graph of f at $x = 0$.

b) Determine the intervals on which f is concave up.

7. Find $\dfrac{dy}{dx}$ for the curve

$$x^2 + xy + y^2 = 7.$$

8. Find an expression for the slope of the curve

$$\tan y + y \sin x = 3.$$

9. A spherical balloon is being inflated so that its radius increases at a constant rate of 0.2 cm/sec. Find the rate at which its volume is increasing when $r = 5$ cm. Use exact notation before giving any decimal approximation.

IV. Integration Techniques

10. Evaluate each indefinite integral. Show your substitution or method clearly.

a) $\displaystyle\int x e^{x^2}\, dx$

b) $\displaystyle\int \frac{2x}{x^2 + 1}\, dx$

c) $\displaystyle\int x \cos(x^2)\, dx$

d) $\displaystyle\int \frac{\ln x}{x}\, dx$

e) $\displaystyle\int x^2 \sqrt{x^3 + 4}\, dx$

f) $\displaystyle\int \sin^3 x \cos x \, dx$

11. Evaluate each definite integral exactly.

a) $\displaystyle\int_0^{\pi} \sin x \, dx$

b) $\displaystyle\int_1^{e} \frac{1}{x} \, dx$

c) $\displaystyle\int_0^{2} \left(3x^2 - 4x + 1\right) \, dx$

d) $\displaystyle\int_0^{1} xe^{-x^2} \, dx$

V. The Fundamental Theorem of Calculus and Area

12. Let

$$F(x) = \int_1^{x} \sqrt{1 + t^2} \, dt.$$

a) Find $F'(x)$.

b) Evaluate $F'(2)$ exactly.

13. Find the derivative of

$$H(x) = \int_3^{x^2} \frac{t^2 + 1}{\sqrt{t + 4}} \, dt.$$

14. Sketch the region bounded by $y = x^2$ and $y = 2x$, and find its area using a definite integral.

VI. Differential Equations and Models

15. Solve the initial value problem

$$\frac{dy}{dx} = 3x^2, \qquad y(1) = 4.$$

16. A population satisfies the differential equation

$$\frac{dP}{dt} = kP(100 - P), \qquad k > 0.$$

a) Explain what happens to the growth rate as P approaches 100.

b) Explain why this model differs from unrestricted exponential growth.

VII. Parametric and Polar Review

17. A curve is given parametrically by

$$x = t^2 + 1, \qquad y = t^3 - t.$$

a) Find $\dfrac{dy}{dx}$ in terms of t.

b) Find the slope of the curve when $t = 1$.

c) Find $\dfrac{d^2y}{dx^2}$ in terms of t.

18. A polar curve is given by

$$r = 2 + 2\sin\theta.$$

a) Sketch the curve and identify its symmetry.

b) Find the exact area enclosed by the curve.

VIII. Sequences and Series

19. Determine whether each series converges or diverges. State the test or reasoning used.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

20. Find the radius and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}.$$

21. Write the first four nonzero terms of the Maclaurin series for e^{-x^2}.

IX. Vectors and Three-Dimensional Geometry

22. Let

$$\mathbf{a} = \langle 1, 2, 3\rangle, \qquad \mathbf{b} = \langle 2, -1, 4\rangle.$$

Find each of the following.

a) $\mathbf{a} + \mathbf{b}$

b) $\|\mathbf{a}\|$

c) $\mathbf{a} \cdot \mathbf{b}$

d) The angle between $\mathbf{a}$ and $\mathbf{b}$, written exactly using inverse cosine

23. Find parametric equations for the line through $(1, 2, 3)$ parallel to the vector $\langle 2, -1, 4\rangle$.

24. Find an equation of the plane through the point $(1, -2, 3)$ with normal vector $\langle 4, 0, -1\rangle$.

X. Optional Challenge Problems

25. Evaluate exactly:

$$\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx.$$

26. Suppose $\mathbf{u} = \langle 1, 0, 2\rangle$ and $\mathbf{v} = \langle 3, -1, 1\rangle$. Find a nonzero vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

27. Let

$$G(x) = \int_{\cos x}^{x^2} \ln(1+t^2)\,dt.$$

Find $G'(x)$.