

Calculus Honors Summer Assignment

Welcome to Calculus Honors! Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations. Throughout the course, we will use real problems to motivate the mathematics, and then apply the mathematics to genuine physical problems.

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The practice questions in this assignment are intended to focus on functions. My expectation for you entering Calculus is that you know what a function is **and** how to graph a function. If you feel that you are weak in the other topics mentioned above, please reach out to me directly and I can provide you additional (optional) practice. Prior to the practice problems, a brief excerpt from *Calculus: An Intuitive and Physical Approach* by Kline is available for you to read to start developing a sense of *why* we would study Calculus in the first place.

You can consult the sites under "Helpful Links" **on the last page of this assignment** to aid in your review. If you have any further questions as you work through the assignment, please feel free to email me. I will check my email throughout the summer and respond to any questions you have when I am able.

Best,

Mr. Cantlin

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CHAPTER ONE WHY CALCULUS?

1. *The Historical Motivations for the Calculus.* Each branch of mathematics has been developed to attack a class of problems that could not be solved at all or yielded to a solution only after great efforts. Thus elementary algebra was created to find answers to simple physical problems which in mathematical form called for solving first, second, and higher degree equations with one or two unknowns. Plane and solid geometry originated in the need to find perimeters, areas, and volumes of common figures and to state conditions under which two figures, say two triangles, are congruent or have the same shape—that is, are similar. Trigonometry, introduced by astronomers, enabled man to determine the sizes and distances of heavenly bodies.

In high school algebra and trigonometry we usually learn the fundamentals of another branch of mathematics, called coordinate or analytic geometry. Thus we learn to graph linear equations such as $x + 2y = 5$, to represent a circle of radius R by an equation of the form $x^2 + y^2 = R^2$, and to determine which curves correspond to such equations as $y = \sin x$ and $y = \cos x$. The primary purpose of relating equations and curves is to enable us to use the equations in the study of such important curves as the paths of projectiles, planets, and light rays. Of course, each of the above-mentioned branches of mathematics has also helped to treat problems of the physical and social sciences which arose long after the motivating questions had been disposed of.

During the seventeenth century, when modern science was founded and began to expand apace, a number of new problems were brought to the fore. Because the mathematicians of that century, like those of most great periods, were the very physicists and astronomers who raised the questions, they responded at once to the problems. Let us see what some of these problems were.

Seventeenth-century scientists were very much concerned with problems of motion. The heliocentric theory created by Nicolaus Copernicus (1473–1543) and Johannes Kepler (1571–1630) introduced the concepts of the earth rotating on its axis and revolving around the sun. The earlier theory of planetary motion, dating back to Ptolemy (c. A.D. 150), which presupposed an earth absolutely fixed in space and, indeed, in the center of the universe, was discarded. The adoption of the theory involving an earth in motion invalidated the laws and explanations of motion that had been accepted since Greek times. New insights were needed into such phenomena as the motion of a projectile shot from a cannon and an answer to the question of why objects stay with the moving earth seemed called for. Furthermore, Kepler had shown on the basis of observations that the path of each planet around the sun is an ellipse, although no theoretical explanation of why the planets move on such paths had been offered. However, the notion that all bodies in the universe attract one another in accordance with the force of gravitation became prominent, and scientists decided to investigate whether the motions of planets around the sun and of moons around planets could be deduced from the proper laws of motion and gravitation. The motion of celestial bodies became the dominant scientific study.

All of these motions—those of objects near the surface of the earth and those of the heavenly bodies—take place with variable velocity, and many involve variable acceleration. Although the difficulties in handling variable velocities and accelerations may not be apparent at the moment, the branches of mathematics that existed before the calculus was created were not adequate to treat them. We shall see later precisely what the difficulties are and how they are surmounted. In pre-calculus courses students often work on problems involving variable velocity—for example, the motion of a body falling to earth—but the intricacy is circumvented there by one dodge or another.

The second major problem of the seventeenth century was the determination of tangents to various curves (Fig. 1-1). This question is of some interest as a matter of pure geometry, but its deeper significance is that the tangent to a curve at a point represents the *direction* of the curve at the point. Thus, if a projectile moves along a curve, the direction in which the projectile is headed at any point on its path is the direction of the tangent at that point. To determine whether the projectile will strike its target head on or merely at a glancing angle, we must know in which direction the projectile is moving at that point on its path at which it strikes the target. The invention of the telescope and microscope in the seventeenth century stimulated great interest in the action of lenses. To determine the course of a light ray after it strikes the surface of a lens, we must know the angle that the light ray makes with the lens, that is, the angle (Fig. 1-2) between the light ray and the tangent to the lens. Incidentally, the study of the behavior of light was, next to the study of motion, the most active scientific field in that century. It may now be apparent why the question of finding the tangent to a curve was a major one.

Figure 1-1



Figure 1-2



A third class of problems besetting the seventeenth-century scientists may be described as maxima and minima problems. The motion of cannon balls was studied intensively from the sixteenth century onward. In fact, the mathematicians Nicolò Tartaglia (1500–1557) and Galileo Galilei (1564–1642) made significant progress in this investigation even before the calculus was applied to it. One of the important questions about the motion of cannon balls and other kinds of projectiles was the determination of the maximum range. As the angle of elevation of a cannon (angle A in Fig. 1-3) is varied, the range—that is, the horizontal distance from the cannon to the point at which the projectile again reaches the ground—also varies. The question is, at what angle of elevation is the range a maximum? Another maximum and minimum problem of considerable importance arises in planetary motion. As a planet moves about the sun, its distance from the sun varies. A basic question in this

area is, what are the maximum and minimum distances of the planet from the sun? Some simple maxima and minima problems can be solved by the methods of elementary algebra and elementary geometry, but the most important problems are beyond the power of these branches and require the calculus.

Still another class of problems in the seventeenth century concerned the lengths of curves and the areas and volumes of figures bounded by curves and surfaces. Elementary mathematics suffices to determine the areas and volumes of simple figures, principally figures bounded by line segments and by portions of planes. However, when curves or curved surfaces are involved, elementary geometry is almost helpless. For example, the shape of the earth is an oblate spheroid, that is, a sphere somewhat flattened on the top and bottom (Fig. 1-4). The calculation of the volume of this figure cannot be performed with elementary geometry; it can be done with the calculus. Euclidean geometry does have a method, called the method of exhaustion, for treating a very limited number of area and volume problems involving curves and surfaces, respectively. This method is difficult to apply and, moreover, involves concepts that can with considerable justification be regarded as belonging to the calculus, although the Greeks did not formulate them in modern terms. In any case, the method of exhaustion could not cope with the variety and difficulty of the area and volume problems that appeared in the seventeenth century. Closely related to these problems were those of finding the center of gravity of a body and the gravitational attraction exerted by, say, the earth on the moon. The relation may not be evident at the moment, but we shall see that the same method solves both types of problem.

Figure 1-3

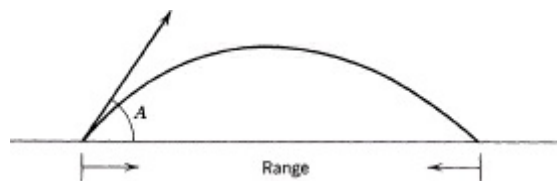


Figure 1-4

The efforts to treat the four classes of problem that we have thus far briefly described led mathematicians to methods which we now embrace under the term calculus. Of course, similar problems continue to be important in our time; otherwise the calculus would have only historical value. In fact, once a mathematical method or branch of any significance is created, many new uses are found for it that were not envisioned by its creators. For the calculus this has proved to be far more the case than for any other mathematical creation; we shall examine later a number of modern applications. Moreover, the most weighty developments in mathematics since the seventeenth century employ the calculus. Indeed, it is the basis of a number of branches of mathematics which now comprise its most extensive portion. The calculus has proved to be the richest lode that the

mathematicians have ever struck.

2. *The Creators of the Calculus.* Like almost all branches of mathematics, the calculus is the product of many men. In the seventeenth century Pierre de Fermat (1601–1665), René Descartes (1596–1650), Blaise Pascal (1623–1662), Gilles Persone de Roberval (1602–1675), Bonaventura Cavalieri (1598–1647), Isaac Barrow (1630–1677), James Gregory (1638–1675), Christian Huygens (1629–1695), John Wallis (1616–1703), and, of course, Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) all contributed to it. Newton and Leibniz are most often mentioned as the creators of the calculus. This is a half-truth. Without deprecating their contributions, it is fair to say, as Newton himself put it, that they stood on the shoulders of giants. They saw more clearly than their predecessors the generality of the methods that were gradually being developed and, in addition, added many theorems and processes to the stock built up by their predecessors.

Even Newton and Leibniz did not complete the calculus. In fact, it may be a comfort to students just beginning to work in the calculus to know that Newton and Leibniz, two of the greatest mathematicians, did not fully understand what they themselves had produced. Throughout the eighteenth century new results were obtained by, for example, James Bernoulli (1654–1705), his brother John Bernoulli (1667–1748), Michel Rolle (1652–1719), Brook Taylor (1685–1731), Colin Maclaurin (1698–1746), Leonhard Euler (1707–1783), Jean Le Rond d’Alembert (1717–1783), and Joseph-Louis Lagrange (1736–1813). However, the final clarification of the concepts of the calculus was achieved only in the nineteenth century by, among others, Bernhard Bolzano (1781–1848), Augustin-Louis Cauchy (1789–1857), and Karl Weierstrass (1815–1897). We shall find many of these great names attached to theorems that we shall be studying.*

3. *The Nature of the Calculus.* The word *calculus* comes from the Latin word for pebble, which became associated with mathematics because the early Greek mathematicians of about 600 B.C. did arithmetic with the aid of pebbles. Today a calculus can mean a procedure or set of procedures such as division in arithmetic or solving a quadratic equation in algebra. However, most often the word means the theory and procedures we are about to study in the differential and integral calculus. Usually we say *the calculus* to denote the differential and integral calculus as opposed to other calculi.

The calculus utilizes algebra, geometry, trigonometry, and some coordinate geometry (which we shall study in this book). However, it also introduces some new concepts, notably the derivative and the integral. Fundamental to both is the limit concept. We shall not attempt to describe the notion of limit and how it is used to formulate the derivative and integral, because a brief explanation may be more confusing than helpful. Nevertheless, we do wish to point out that the calculus in its introduction and utilization of the limit concept marks a new stage in the development of mathematics.

The proper study of the calculus calls for attention to several features. The first is the theory, which leads to numerous theorems about the derivative and the integral. The second feature is the technique; to use the calculus, one must learn a fair amount of technique in differentiation and integration. The third feature is application. The calculus was created in response to scientific needs, and we should study many of the applications to gain appreciation of what can be accomplished with the subject; these applications also give insight into the mathematical ideas.

The theory of the calculus, which depends primarily on the limit concept, is rather sophisticated. Complete proofs of all the theorems are difficult to grasp when one is beginning the study of the subject. Our approach to the theory attempts to surmount this hurdle. Many of our proofs are

complete. However, other proofs are made by appealing to geometric evidence; that is, we use curves or other geometric figures to substantiate our assertions. The geometric evidence does not necessarily consist of complete geometric proofs, so that we cannot say that we are *proving* geometrically. Nevertheless, the arguments are quite convincing. For example, we can be certain even without geometric proof that the bisector of angle A (Fig. 1-5) divides the isosceles triangle ABC into two congruent triangles.

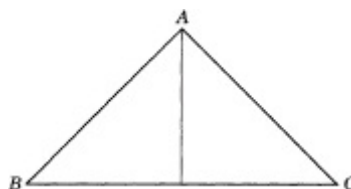


Figure 1-5

This approach, which will become clearer when we begin considering specific cases, is called the intuitive approach. It is recommended for several reasons. The first, as already suggested, is pedagogical. A thoroughly sound, deductive approach to the calculus, one which the modern mathematician would regard as logically rigorous, is meaningless before one understands the ideas and the purposes to which they are put. One should always try to understand new concepts and theorems in an intuitive manner before studying a formal and rigorous presentation of them. The logical version may dispose of any lingering doubts and may be aesthetically more satisfying to some minds, but it is not the road to understanding.

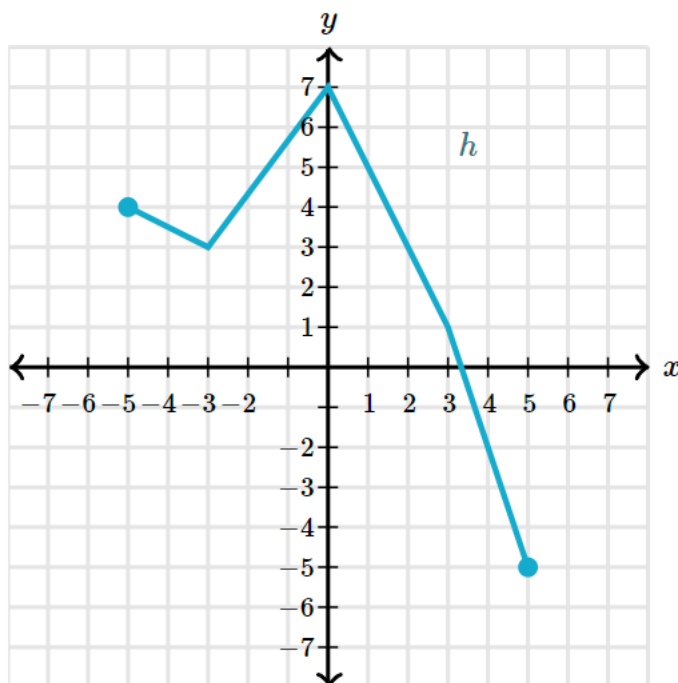
The rigorous approach, in fact, did not become available until about 150 years after the creation of the calculus. During these years mathematicians built up not only the calculus but also differential equations, differential geometry, the calculus of variations, and many other major branches of mathematics that depend on the calculus. In achieving these results the greatest mathematicians thought in intuitive and physical terms.

The second reason for adopting the intuitive approach is that we wish to have time for some techniques and for applications. Were we to study the rigorous formulations of limit, derivative, integral, and allied concepts, we would not have time for anything else. These concepts can be carefully and most profitably studied in more advanced courses in the calculus after one has some appreciation of what they mean in geometrical and physical terms and of what one wishes to do with them. As for the applications, the calculus more than any other branch of mathematics was created to solve major physical problems, and one should certainly learn what the calculus accomplishes in this connection.

After we have become reasonably familiar with the ideas, techniques, and uses of the calculus, we shall consider in the last chapter how the intuitive approach may be strengthened by a more rigorous one.

* The history of the calculus is presented very well in Carl B. Boyer, *The Concepts of the Calculus*, Dover Publications, New York, 1959.

0.1 Functions



- The graph of a function f is given above
 - State the value of $f(-3)$
 - Estimate the value of $f(1.5)$
 - For what values of x is $f(x) = 1$?
 - State the domain and range of f
- If $f(x) = x^3$, evaluate the quotient $\frac{f(2+h)-f(2)}{h}$ and simplify your answer
- If $f(x) = x^2 - 3x + 3$, simplify the quotient $\frac{f(a+h)-f(a)}{h}$
- Find the domain of the following functions
 - $f(x) = \frac{2x+1}{x^2+x-2}$
 - $g(x) = \frac{\sqrt{x}}{x^2+1}$
 - $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$
- How are the graphs of the functions obtained from the graph of f ?
 - $y = -f(x)$
 - $y = 2f(x) - 1$
 - $y = f(x-3) + 2$

6. Make a rough sketch of the graph of the following functions

a) $y = x^3$

b) $y = 4 - x^2$

c) $y = \sqrt{x}$

d) $y = -2^x$

e) $y = (x - 2)^3 + 3$

f) $y = 1 + x^{-1}$

7.

$$\text{Let } f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$$

(a) Evaluate $f(-2)$ and $f(1)$

(b) Sketch the graph of f

8. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions

a) $f \circ g$

b) $g \circ f$

c) $g \circ g \circ g$

9. Given

a) $f(x) = 3x - 2$, find $f^{-1}(x)$

b) $g(x) = \sqrt{x - 2}$, find $g^{-1}(x)$

c) $h(x) = \frac{x+4}{2x-5}$, find $h^{-1}(x)$

Helpful Links

<https://www.khanacademy.org/math/algebra>
<https://www.khanacademy.org/math/geometry>
<https://www.khanacademy.org/math/algebra2>
<https://www.khanacademy.org/math/precalculus>
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<https://www.youtube.com/@patrickjmt/playlists>
<https://www.youtube.com/@TheOrganicChemistryTutor/playlists>

Note, while these links cover the material of the review, this list is **not** exhaustive. If you are unable to find material on content you need help with, please reach out to me and I would be happy to send you additional resources.