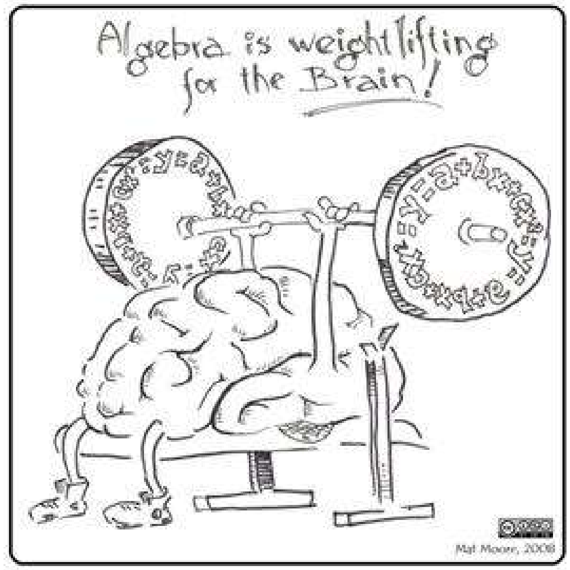
Name: ______________________________

Date: _______________________________

"Entering Geometry" Summer Math Packet

These exercises provide a thorough review of Algebra 1 topics in preparation for starting Geometry at Liberty Common Junior High School. This problem set should be completed and brought to class on the first full day of school for credit as the first homework assignment of the school year, worth five regular homework assignments. **<u>Work all problems on a separate sheet of paper and circle your answers.</u>** Answers are provided in the back so that you can check your work; you need to show your work to receive credit for the problem.



This packet also includes a reading assignment which will challenge your brain to think about the nature of logic and reasoning. There are three questions which should be answered after you complete the reading passage.

As you work through these problems, you most likely will come across topics that require a little review… you might even find some topics that you have completely forgotten! When this situation presents itself, get help. There are many resources available – websites, textbooks, friends or siblings who are ahead of you in math, previous courses' notes, etc. – that can provide information to assist you.

Have a great summer and we'll see you in the fall!

Name: ______________________

Date: ______________________

Entering Geometry Summer Math Packet

Complete all work on your own paper. Copy the original problem. No credit will be given without accompanying work.

1. a. Write an example of an expression which is a polynomial.
   b. Write an example of an expression which is not a polynomial.

Simplify. Write answers with no grouping symbols.

2. $7(3x - 14)$
3. $(2x - 3)^2$
4. $(x + 4)(2x - 5)$
5. $2x + 7 + 9(7 - 2x)$
6. $\frac{5}{6}y(-12)$
7. $2(x^2 - 3x + 4) - 5(x^2 + 2x - 3)$
8. $\frac{(6x^2+9x-1\ \ )}{2}$
9. $6 + \frac{2^3(3)}{6-4}$

Evaluate for the given values of the variable.

10. $6x - 23$ a. $x = -2$ b. $x = 12$
11. $8 - 2(3x - 5)$ a. $x = 0$ b. $x = -\frac{2}{3}$
12. $3y^2 - y - 4$ a. $x = 2$ b. $x = -6$
13. $\frac{|3x+4|}{7-x}$ a. $x = -3$ b. $x = 12$

State the axiom, property, or definition which justifies each of the following.

14. $2x + 3 = 3 + 2x$
15. $-x = -1(x)$
16. If $5 = x - 2$, the $x - 2 = 5$
17. $3(x - 12) = 3x - 36$
18. $2(3 \cdot 4) = (2 \cdot 3)4$

Solve and check.

19. $x + 3 = 7$
20. $2x - 5 = 18$
21. $0.3 - 2.5x = -0.7$
22. $4x + 3 = 8x - 5$

Name the polynomial by degree and number of terms.

23. $x^2 - 3x + 5$
24. $x + 5$
25. x^2
26. $x^3 - 4x^2 + 7x - 9$

Simplify by multiplying and adding common terms.

27. $(x + 3)(x - 2)$

28. $(3x + 1)(2x + 3)$

29. $(4x - 1)^2$

30. $(6x + 1)(6x - 1)$

31. $(3x - 5)(4x + 3)$

32. $(x + 1)^2$

33. $(x - 1)^2$

34. $(x + 1)(x - 1)$

35. $(7x - 2)(x + 4)$

36. $(3x - 2)^2$

Factor. If prime, so state.

37. $x^2 + x - 6$

38. $2x^2 - 7x - 4$

39. $25x^2 - 10x - 3$

40. $x^2 - 5x + 6$

41. $x^2 - 6x + 9$

42. $x^2 + 1$

43. $x^2 - 1$

44. $x^2 - 2x + 1$

45. $x^2 + 2x - 1$

46. $9x^2 - 16$

47. $4x^2 + 4x - 3$

48. $x^2 + 9x - 10$

49. State whether the following numbers are rational or irrational.

a. 4 b. $\sqrt{4}$ c. 14 d. $\sqrt{14}$

Solve the absolute value equations. Write solutions using set notation.

50. $|x - 6| = 9$

51. $|4x + 1| = 33$

52. $|7x - 9| = -8$

53. $|8 - x| = 7$

Solve the quadratic equations by completing the square.

54. $(x + 3)^2 = 25$

55. $(3x - 1)^2 = 64$

56. $x^2 - 4\text{x} + 4 = 36$

57. $x^2 + 2x + 1 = 5$

58. $x^2 + 8\text{x} - 9 = 0$

59. $2x^2 + 8x - 12 = 0$

Solve the quadratic equations using the quadratic formula.

60. $2x^2 - 3x + 1 = 0$

61. $3x^2 + 4x - 9 = 0$

62. $6x^2 - 3x + 1 = 0$

63. $x^2 - 8x = -12$

Find the discriminant of each of the equations in Problems 60-63 and show that the solutions are indicated by the discriminant.

60a. 61a. 62a. 63a.

Solve the quadratic equations by factoring.

64. $x^2 + 7x + 12 = 0$

65. $x^2 - 4x - 45 = 0$

66. $x^2 - x = 20$

67. $x^2 = 3x + 4$

Factor completely.

68. $3x^2 + 6x - 9$

69. $x^4 - x^3 - 2x^2$

70. $5x^2 - 125$

71. $8x^2 - 28x + 12$

72. $x(2x + 3) - 5(2x + 3)$

73. $x^2(x - 2) - 4(x - 2)$

74. $8x(3x + 1) - (3x + 1)$

75. $(x^2 + 3x)(2x - 1) + 2(2x - 1)$

76. $2x^3 + 3x^2 + 2x + 3$

77. $x^3 + 3x^2 - 9x - 27$

78. $3x^3 + 3x^2 - 4x - 12$

79. $x^3 + 3x^2 - 4x - 12$

80. $6x^2 - x - 15$

81. $15x^2 + 22x - 5$

82. A football was kicked into the air with an initial velocity of 16 meters/second.

   a. Write an equation which relates distance above where the ball was kicked with respect to time.
   b. After what time(s) will the ball be at a height of 12 meters?
   c. How high was the ball after 2 seconds? Was it on the way up or down. Justify your answer.
   d. How high was the ball after 3 seconds?
   e. How many seconds did it take the ball to fall back to the same level from which it was kicked?

For each of the following systems of equations:

a. solve each system by graphing

b. verify the solution by substitution for 83, 84, and 86

c. verify the solution by linear combination (elimination) for 85 and 87.

83. $2x - y = -3$
    $3x + y = -2$

84. $x + 3y = 6$
    $2x - 3y = 3$

85. $5x - 2y = -10$
    $x + 2y = -2$

86. $y = 3x + 8$
    $y = -\frac{2}{3}x - 3$

87. $3x - 2y = -6$
    $x + y = 3$

Simplify. Leave no powers of variables in the denominator. Evaluate all powers of numbers. Do not use decimals. If the expression can't be simplified, so state.

88. x^2x^3

89. $(x^2)^3$

90. $\frac{x^2}{x^3}$

91. $x^2 + x^3$

92. $2x^4 \cdot 3x^3$

93. $(2x)^3$

94. $(x^3y^{-3})^5$

95. $\frac{8x^6}{2x^4}$

96. 3^{-2}

97. $(6x^{-2})(-3x^4)$

98. $(2x^2y^3)^3$

99. $\left(\frac{x^3}{y^2}\right)^4$

100. $\left(\frac{2x^{-3}}{y^3}\right)^{-2}$

101. $\frac{x^4y}{xy}$

102. $4x^3 \cdot 2x^3$

103. $4x^3 - 2x^3$

104. $(x^2y^{-4})^5$

105. $(-2x^4)^3$

106. $\frac{x^{-3}y^4}{x^5y^{-2}}$

107. $(5xy^8)^0$

Write in scientific notation:

108. 30405

109. 0.00091

110. 89.5 ×103

Write in standard notation:

111. 7.8×10^{-3}

112. 7.8×10^3

Express the product or quotient in scientific notation:

113. $(2 \times 10^3)(4 \times 10^{-2})$

114. $\frac{4 \times 10^{-2}}{2 \times 10^3}$

115. Find the GCF of the two polynomials $x^2 - 4$ and $2x^2 - 10x + 12$.

Perform the indicated operation. Write the result as a reduced fraction.

116. $\frac{3}{4}+\frac{4}{9}$

117. $\frac{3}{4}\cdot\frac{4}{9}$

118. $\frac{4}{3x}+\frac{2}{x}$

119. $\frac{5}{x}+\frac{x}{x+2}$

120. $\frac{2}{x}\cdot\frac{x}{x+2}$

121. $\frac{2x}{x+2}+\frac{4}{x+2}$

122. $\frac{3}{x^2-4}+\frac{4}{2x+4}$

123. $\frac{x^2-6x-7}{x-1}\cdot\frac{x^2-1}{x^2+4x+3}$

124. $\frac{4-x^2}{x^2+5x+6}\cdot\frac{x^2-10x+9}{x^2-3x+2}$

125. $\frac{2x^2-x-1}{9x^3}\div\frac{x^2+x-2}{6x^2}$

126. $\frac{3}{x+2}-\frac{x^2+4x+3}{x^2-4}\div\frac{1+x}{2-x}$

127. Evaluate the fraction $\frac{2x-6}{x+2}$ if x is:

a. 2 b. 3 c. –2

For problems 149-151:

- **a. State the restrictions on *x*.**
- **b. State the LCD of the denominators.**
- **c. Solve the equation.**
- **d. Check for extraneous solutions.**

128. $\frac{5}{x+2}=\frac{3}{x}$

129. $\frac{x+2}{4x}-\frac{5}{8}=\frac{3}{2x}$

130. $\frac{4}{x+4}+\frac{x}{4+x}=2$

131. Find x in the proportion 3:5 = 4:x .

132. Two numbers are in a ratio of 8 to 12. If the larger number is 42, find the smaller number.

133. Two numbers are in a ratio of 12 to 17. If the sum of the numbers is 435, find the larger number.

134. It takes Frank 3 hours to do a job. If Eddie works with him, it takes him only two hours. What is Eddie's rate?

135. For the radical expression $\sqrt{x+4}$:

a. Evaluate the expression if x is 8. Write the result in simple radical form. Is the result rational or irrational?
b. Evaluate the expression if x is 12. Write the result in simple radical form. Is the result rational or irrational?
c. Find the value of x if the expression is equal to 7.
d. Find the value of x if the expression is equal to –1.

Write in simplest radical form.

136. $\sqrt{24}$

137. $\sqrt{128}$

138. $\sqrt{5} - 6\sqrt{5}$

139. $\sqrt{12} + 3\sqrt{12}$

140. $2\sqrt{24} - \sqrt{24}$

141. $3\sqrt{8} \cdot 4\sqrt{12}$

142. $\sqrt{27} + \sqrt{12}$

143. $\sqrt{6} + \sqrt{3}$

144. $(1 + \sqrt{2})^2$

145. $(1 + \sqrt{2})(1 - \sqrt{2})$

146. $\frac{6}{\sqrt{3}}$

147. $\sqrt{\frac{3}{8}}$

148. $(2 + \sqrt{3})(1 - \sqrt{6})$

149. $\sqrt{16 + 9}$

150. $\sqrt{24} \cdot \sqrt{18}$

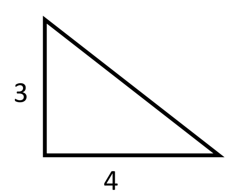
151. $\frac{2}{1-\sqrt{3}}$

152. Write each of the following repeating decimals as fractions.

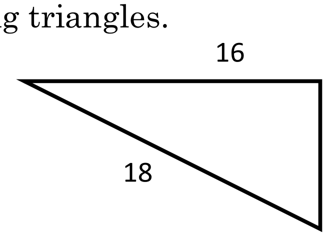
a. 0.5555555… b. 0.05050505…

153. Find the length of the third side of the following triangles.

a.



b.



154. Evaluate the radical $\sqrt[x]{81}$ if x is

a. 2 b. 3 c. 4 d. 5

e. Which of the results in *a* through *d* are rational?

Solve and graph the absolute value inequalities.

155. $|x - 5| < 9$

156. $|1 - 4x| \geq 27$

157. $|2x - 1| > -9$

158. $|5 - 2x| \leq -5$

Wow! Great work! Continue in the packet after the answers to the reading assignment. You can expect to have a quiz on the reading when you start your math class next semester.

KEY

1a. $x = 2$ b. $\sqrt{x+2}$

2. $21x - 98$

3. $4x^2 - 12x + 9$

4. $2x^2 + 3x - 20$

5. $-16x + 70$

6. $-10y$

7. $-3x^2 - 16x + 23$

8. $3x^2 + 4.5x - 6$

9. 18

10. a. -35 b. 49

11. a. 18 b. 22

12. a. 6 b. 110

13. a. $\frac{1}{2}$ b. -8

14. commutative addition

15. mult. prop. of -1

16. symmetric

17. distributive

18. associative mult.

19. $x = 4$

20. $x = 11.5$

21. $x = 0.4$

22. $x = 2$

23. quadratic trinomial

24. linear binomial

25. quadratic monomial

26. cubic polynomial with four terms

27. $x^2 + x - 6$

28. $6x^2 + 11x + 3$

29. $16x^2 - 8x + 1$

30 $36x^2 - 1$

31. $12x^2 - 11x - 15$

32. $x^2 + 2x + 1$

33. $x^2 - 2x + 1$

34. $x^2 - 1$

35. $7x^2 + 26x - 8$

36. $9x^2 - 12x + 4$

37. $(x + 3)(x - 2)$

38. $(2x + 1)(x - 4)$

39. $(5x + 1)(5x - 3)$

40. $(x - 2)(x - 3)$

41. $(x - 3)^2$

42. prime

43. $(x + 1)(x - 1)$

44. $(x - 1)^2$

45. prime

46. $(3x - 4)(3x + 4)$

47. $(2x - 1)(2x + 3)$

48. $(x + 10)(x - 1)$

49. a. rational b. rational

c. rational d. irrational

50. $\{-3, 15\}$

51. $\{-8.5, 8\}$

52. Ø

53. $\{15, 1\}$

54. $\{-8, 2\}$

55. $\{-7/3, 3\}$

56. $\{-4, 8\}$

57. $\{-3.24, 1.24\}$

58. $\{-9, 1\}$

59. $\{-5.16, 1.16\}$

60. $\{0.5, 1\}$

61. $\{-2.52, 1.19\}$

62. Ø

63. $\{6, 2\}$

60a. $D = 1$; rational;

61a. $D = 124$; irrational;

62a. $D = -15$; not real;

63a. $D = 16$; rational;

64. $\{-4, -3\}$

65. $\left\{\frac{7 \pm \sqrt{97}}{2}\right\}$

66. $\{5, -4\}$

67. $\{-1, 4\}$

68. $3(x - 1)(x + 3)$

69. $x^2(x - 2)(x + 1)$

70. $5(x + 5)(x - 5)$

71. $4(2x - 1)(x - 3)$

72. $(x - 5)(2x + 3)$

73. $(x + 2)(x - 2)^2$

74. $(8x - 1)(3x + 1)$

75. $(x + 2)(x + 1)(2x - 1)$

76. $(x^2 + 1)(2x + 3)$

77. $(x - 3)(x + 3)^2$

78. prime

79. $(x + 2)(x - 2)(x + 3)$

80. $(3x - 5)(2x + 3)$

81 $(5x - 1)(3x + 5)$

82 a. $d = 16t - 5t^2$
b. after 2 seconds & 1.2 seconds
c. $12m$, going down, also $12m$ @ 1.2s
d. $3m$
e. 3.2 seconds

83. $\{(-1, 1)\}$

84. $\{(3, 1)\}$

85. $\{(-2, 0)\}$

86. $\{(-3, -1)\}$

87. $\{(0, 3)\}$

88. x^5

89. x^6

90. x^{-1}

91. $x^2 + x^3$

92. $6x^7$

93. $8x^3$

94. $x^{15}y^{-15}$

95. $4x^2$

96. $\frac{1}{9}$

97. $-18x^2$

98. $8x^6y^9$

99. $x^{12}y^{-8}$

100. $\frac{1}{4}x^6y^6$

101. x^3

102. $8x^6$

103. $2x^3$

104. $x^{10}y^{-20}$

105. $-8x^{12}$

106. $x^{-8}y^6$

107. 1

108. 3.0405×10^4

109. 9.1×10^{-4}

110. 9.2185×10^3

111. 0.0078

112. 7800

113. 8×10^1

114. 2×10^{-5}

115. $x - 2$

116. $\frac{43}{36}$

117. $\frac{1}{3}$

118. $\frac{10}{3x}$

119. $\frac{x^2+5x+10}{x^2+2x}$

120. $\frac{2}{x+2}$

121. 2

122. $\frac{2x-1}{x^2-4}$

123. $\frac{x^2-6x-7}{x+3}$

124. $\frac{-x+9}{x+3}$

125. $\frac{4x+2}{3x^2+6x}$

126. $\frac{x+6}{x+2}$

127. a. $-\frac{1}{2}$ b. 0 c. undefined

128. a. $x \neq 0, -2$
b. LCD = x(x + 2)
c. $x = 3$

129. a. $x \neq 0$
b. LCD = $8x$
c. $x = -\frac{8}{3}$

130. a. $x \neq -4$
b. LCD = $x + 4$
c. Ø

131. $x = 6\frac{2}{3}$

132. 28

133. 255

134. 6 h per job

135. a. $2\sqrt{3}$, irrational
b. 4, rational
c. 45
d. not real

136. $2\sqrt{6}$

137. $8\sqrt{2}$

138. $-5\sqrt{5}$

139. $8\sqrt{3}$

140. $2\sqrt{6}$

141. $48\sqrt{6}$

142. $5\sqrt{3}$

143. $\sqrt{3}(\sqrt{2} + 1)$

144. $3 + 2\sqrt{2}$

145. -1

146. $2\sqrt{3}$

147. $\frac{\sqrt{6}}{4}$

148. $2 - 2\sqrt{6} + \sqrt{3} - 3\sqrt{2}$

149. 5

150. $12\sqrt{3}$

151. $-1 - \sqrt{3}$

152. a. $\frac{5}{9}$ b. $\frac{5}{99}$

153. a. 5 b. 8.25

154. a. 9 b. 4.33 c. 3
d. 2.41 e. a and c

155. $-4 < x < 14$

156. $x \leq -6.5$ or $x \geq 9$

157. All real numbers ($S = \mathbb{R}$)

179. No solution (S = Ø)

CHAPTER 4

What the Tortoise Said to Achilles

Logic and Its Loopholes

To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, or of what is not that it is not, is true.

—*Aristotle,* Metaphysics

The typical impression of mathematics is that its arguments are syllogistic—that is, they are bundles of "if, then" statements that can be linked together in long chains with a beginning that depends on other valid statements and an end that gives a conclusion. Logic ensures that the links on these chains cannot be broken. But logic can be devoid of meaning, and it alone does not persuade.

We say that an argument is *valid* if the conclusion logically follows from the premises, regardless of whether the premises are true. Examine this argument:

All startbingers are stopbingers.
All gobingers are startbingers.
Therefore, all gobingers are stopbingers.

You would probably conclude that the argument is valid without knowing anything about the truth or falsity of each of the first two statements, even if you don't know the meaning of *startbinger*, *stopbinger*, and *gobinger*. And consider the following argument:

All mammals have eight legs.
All birds are mammals.
Therefore, all birds have eight legs.

There is a distinction between the validity of an argument and the truthfulness of any one of its components. Logic is not concerned with the factual truth of the individual statements; it is more concerned with making valid deductions from collections of individual statements. We might extract meaning from the syllogism by replacing it with the following abstract model:

All x are y.
All z are x.
Therefore, all z are y.

Such a model lets one plug one's own meaning in for x, y, and z, and once again shows that the meaning of the individual words are not relevant in examining the validity of the argument. In the end, an argument is valid if and only if the truth of the conclusion is absolutely guaranteed by the truth of the premises.

An *argument* is a group of compound statements. The most interesting arguments are ones containing statements, called *premises*, which logically imply an additional statement, called the *conclusion*. The conclusion is *deduced* from the premises, and the argument is called *deduction*. But the word *deduction* is tricky. What does it mean? It seems to mean something like this: An argument (containing a group of statements) is a deduction if the statements in the argument guarantee a conclusion. A deduction guarantees that a conclusion is true whenever its premises are true. Conclusions also can validly follow from nonsense premises, just as one deduces that Euclid was human from these premises:

Euclid was a mathematician in ancient Alexandria.
All mathematicians in ancient Alexandria were human.

But one might just as well conclude that Euclid was a monkey from this:

Euclid was a mathematician in ancient Alexandria.
All mathematicians in ancient Alexandria were monkeys.

So what does it mean to say that something is proven? It must mean more than simply having a valid deduction; we can see from the last two arguments that valid deductions can be constructed from false and even nonsense premises. It means that we must show that the premises themselves must be true to make sense of the argument. For a conclusion to be accepted, it must come from a valid deduction with accepted premises, but the truth of those premises depends on the semantics of the words in the argument.

Take the following syllogism:

No kitten that loves fish is unteachable.
No kitten without a tail will play with a gorilla.
Kittens with whiskers always love fish.
No teachable kitten has green eyes.
Kittens that have no whiskers have no tails.

It came from a conversation one summer morning in 1875 between Charles Dodgson (Lewis Carroll) and his friend Lord Tennyson as they sat in an Oxford teahouse eating peach jam cakes and making logical nonsense riddles. Several such syllogisms appeared in the symbolic logic book that Lewis Carroll would eventually write.[1]

Sherlock Holmes himself would say that none of these statements is true; yet, if they were, he would agree that "no green-eyed kitten will play with a gorilla." The truth or falsity of a compound statement should be distinguished from validity. Logic does not take positions on facts or philosophical principles. It is concerned with whether a valid conclusion can be drawn from a valid argument. Apply logic to an argument, and all it does is check for conflicts between the premises and the conclusion.

In the fall of some year of my distant past, I taught a course called Plausible and Deductive Reasoning. The course explored how observations in mathematics build intuition, and how that intuition, in turn, influences ideas for conjectures that chance to be deductively proven. My joy of teaching is greatest at those moments when I feel bona fide sparks of communication between my students and me. Nichole was in the class along with five others, Susan, Scarlet, Tristan, and two other students who took the class in unplanned directions with surprising results. I began with a riddle posed by the earlier kitten syllogism.

To my astonishment, it took Nichole just a few minutes to conclude that no green-eyed kitten will play with a gorilla. Nichole seemed to have an "inner-marking" scheme for organizing syllogisms. I thought that perhaps she had seen this particular well-known syllogism before and had just memorized the answer. It was hard to believe that she could untangle such a mixed-up syllogism without a map. So I created another nonsense syllogism to present at the next meeting:

> Atoms that are not radioactive are always unexcitable.
> Heavy atoms have strong bonds.
> Uranium is tasteless.
> No radioactive atom is easy to swallow.
> No atom that is not strong is tasteless.
> All atoms are excitable, except uranium.

Again, it took her less than two minutes to conclude, "Heavy things are not easy to swallow." What extraordinary talent!

In class, we explored compound statements and rephrased them so they became conditional. For example, "No kitten that loves fish is unteachable" was first rephrased as "All kittens that love fish are teachable," which, in turn, became "If a kitten loves fish, then it is teachable." Each component of this last statement had a possibility of being true or false. "Loving fish" was being compared to "being teachable." It meant that truth values could be assigned to the components "kittens loving fish" and "teachability." A kitten could love fish or not love fish, just as it could be teachable or not.

The statements "No teachable kitten has green eyes" and "No kitten that loves fish is unteachable" were rewritten as follows:

All kittens that love fish are teachable.
All teachable kittens have green eyes.

From these, we deduced the following:

All kittens that love fish have green eyes.

This last statement is deductively inescapable from the previous two; its truth is absolutely guaranteed if each of the two previous statements is true. It is simply part of the "algebra" of human reasoning. It says that "all x are z" because "all x are y" and "all y are z."

One day I gave Nichole a number theory problem that I myself did not know how to begin. With extraordinary risk and unprofessional advice, I hinted that the problem was easy. A few hours later, Nichole came to me with an exceptionally elegant solution. I had to find out how she was able to untangle syllogisms.

"How did you get the conclusion so fast?" I asked.

"I'm not sure," she answered.

"You're not sure? Something must have been going on in your mind when you were untangling the syllogism. Can you recall how you were thinking?"

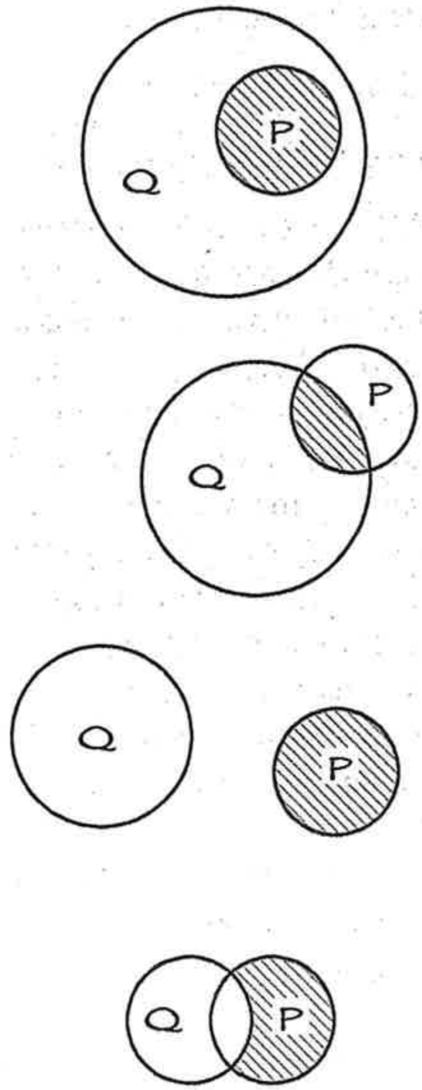
"No. The answer seemed to have just come..." she began to say. Then, after a brief pause, she said, "Wait a minute. I think that I thought of the whole syllogism as a bunch of fractions and cancelled terms."

"Of course!" I said to myself, and immediately realized how she did it.

"How would you do it?" she asked me.

"Well, you know," I said. "I was never instructed in the rules of logic or the schemes by which one can untangle syllogisms. Rarely do mathematicians have formal schooling in logic, yet they seem to do just fine in the maze of complex inference."

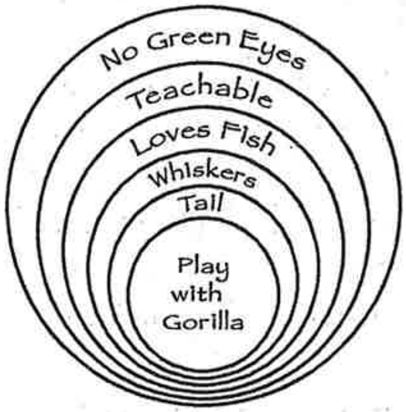
The "kitten and unteachable fish syllogism" is far too complex for the normal mind to quickly conclude what it tries to reveal. The mind needs strategies to help it do its job. Statements that involve words such as *all*, *some*, or *none* can be illustrated by intersecting circles. The four possibilities below are called *Euler diagrams.*



Henri Poincaré suggested that syllogisms can be arranged in such a way that Euler diagrams of nested statements can represent them. Perhaps logically valid arguments "feel" right because of this natural nesting.

A collection of statements connecting green eyes, kittens, tails, gorillas, whiskers, fish, and teachability could be too complex to untangle without a good method for organizing them.

Susan broke down the problem, statement by statement. Because every statement talked about kittens, she eliminated the word *kitten*



and abbreviated the attributes, and she thought of the attributes as simply names for the classes of kittens with such attributes. She put the name of the class in brackets ({ }), so {TEACHABLE} is the name of the class of *teachable kittens*; {LOVE FISH} is the name of the class of *kittens that love fish*, and so on.

With a little work, she could then construct the diagram of nested containment relations and untangle the syllogism.

Logic is used to prove theorems in mathematics. But mathematics is much more than simply logic. Perhaps we would all better understand mathematics if our arguments could be set in a clear nesting of syllogisms, such as the sort that's illustrated above. But that would surely reduce a highly intellectual branch of learning to a dull sport. We have seen that one can have a perfectly valid logical argument that produces nonsense or even untruthful facts. The principal question for us is, why do we understand anything? What gives us the very basic communication connections that guarantee understanding? How is it possible that a college freshman mathematics student who has never been taught the rules of logic or proof can unravel a syllogism and "feel" the truth of the conclusion? In other words, why is it that we know when we know?

Spoken and written languages are informal, and their ambiguities can trigger new ideas in the course of conversation. A spoken sentence might have several interpretations; and there are many ways of saying the same thing. An extreme example of language misunderstanding is found in a typically Carrollian dialog between Alice and the White Knight in *Through the Looking Glass*. The White Knight is about to sing a song to cheer up Alice, who seemed sad.

> "You are sad," the Knight said in an anxious tone: "let me sing you a song to comfort you."

"Is it very long?" Alice asked, for she had heard a good deal of poetry that day.

"It's long," said the Knight, "but it's very, *very* beautiful. Everybody that hears me sing it—either it brings the *tears* into their eyes, or else—"

"Or else what?" said Alice, for the Knight had made a sudden pause.

"Or else it doesn't, you know. The name of the song is called 'Haddocks' Eyes.'"

"Oh, that's the name of the song, is it?" Alice said, trying to feel interested.

"No, you don't understand," the Knight said, looking a little vexed. "That's what the name is *called*. The name really *is* 'The Aged Aged Man.'"

"Then I ought to have said 'That's what the *song* is called'?" Alice corrected herself.

"No you oughtn't: that's quite another thing! The *song* is called 'Ways and Means': but that's only what it's *called*, you know!"

"Well, what *is* the song, then?" said Alice, who was by this time completely bewildered.

"I was coming to that," the Knight said. "The song really *is* 'A-sitting On A Gate': and the tune's my own invention."

We are persuaded by reasoning, and reasoning involves the use of logic. But there is more to it—much more. First, it depends on the internal language we think in, which, in turn, involves words and sentences that transmit meaning from which we can gather either truth or falsehood. Take the example given by Wittgenstein[2] of different meanings of the word *is*. It could mean the copula, equality, or an expression of existence. One could say, "Green is green," ambiguously suggesting that we have either expressed a nonsense tautology, in which both the first and last words are simply names for the color green, or have meant the first word to be a proper name and the last to be the color green. Thus, Wittgenstein concluded, the only way to avoid confusion in relaying ideas is to form a logically precise language in which each word has a unique meaning. But we don't have a logically precise language, so our expressions fall prey to disagreements encouraged by environmental differences and cultural experiences. Wittgenstein observed that meaning is communicated by the sentence, not by the individual words.

You might be thinking that the language of mathematics is precise and, therefore, would qualify as a language in which each sign has a unique meaning. But mathematicians generally use an informal spoken language to communicate proof and ideas—their proofs are rarely put into a purely symbolic language form, so how is it that we assume their theorems to be universal and eternal?

Do we think in syllogisms? Many of my daily thoughts are "ordered" statements. I might say to myself, "I will wash the dishes after lunch," meaning "washing dishes follows having lunch." But thoughts can get much more complicated by webs of entangled ambiguities that cannot be systematized so easily. In 1854, George Boole wrote a book entitled *An Investigation of the Laws of Thought on Which Are Founded the Mathematical Theories of Logic and Probabilities*, in which he proposed to devise an algebraic system for modeling thought. It was, in fact, an ingenious system for mapping rules of deduction to rules of algebra. In a very naïve sense, we can metaphorically view much of common communication processing as a scheme for ordering thoughts by containment. So, the typical "if *P*, then *Q*" statement is viewed as a nest diagram with *P* contained in *Q*.[3] The project was up against strong opposition from the ambiguities of language (and, hence, of thought), but, as with many good ideas, it laid seeds for the growth of kindred ideas. Boole's project led to the development of symbolic logic, an algebra for sets, with rules and axioms that are similar to those of arithmetic. The old Aristotelian syllogistic logic was extended to a rich logic that could be manipulated by symbolic rules.

Boole stated five rules, one of which is that either a statement is true or its negation is true. His five basic rules of logic are instrumental in presenting convincing arguments and in governing clear thinking. But are they absolute? Our conventional logic dictates the use of *the law of the excluded middle*, which is the first rule on Boole's list. Could there be any logical sense to denying that something—say, "I went to the post office to buy stamps"—is either true or not true? If I did not go to the post office for no stamps, then didn't I go to buy at least one stamp? There is something so compelling about this rule of logic that it's hard to believe that there could be any other argument. But there is. In the early part of the twentieth century, several systems of logic, which included several possibilities for middle values between true and false, were developed. These are called non-Aristotelian logics.

Think of an argument you had with a highly intelligent, rational friend that ended in deadlock. You probably thought at the time that your viewpoint was totally logical, but it is likely that your friend thought the same about his. Could there have been a real conflict between the kinds of logic used? It's more likely that you had misunderstandings of the semantics. After all, you were conversing in a natural language with very flexible formal rules that might not have the strength and power to convey all feelings—one person's conception of pain, love, or sadness is quite different from another's. It is possible that you each came at the issue with a different debating strategy. Still, is it possible that you and your friend were using conflicting systems of logic? Is there more than one kind of logic? You might be surprised to hear that there is.

Are mathematical objects already there, independent of our thoughts, waiting to be discovered, or are they brought into existence by our own inventions? There is a school for each answer. Platonists believe that we discover mathematical objects and that all meaning comes from the relationship the objects have with one another. If nobody has seen the proof of a theorem in the mathematical forest, it can still be true. Such a school is forced to accept that proof of a theorem reduces to discovering its truth or falsity. On the other hand, the constructivist school sees a different kind of logic. Take the standard indirect proof that $\sqrt{2}$ is not rational. That proof assumes that $\sqrt{2}$ is rational, and that assumption leads to a contradiction.[4]

To prove that $\sqrt{2}$ is not rational, suppose that it is—that is, suppose that $\sqrt{2} = p/q$, a rational number with no common factors.

(If both p and q had common factors, they could be canceled.) Then square both sides to find that $2 = p^2/q^2$. So, $2q^2 = p^2$. This means that p^2 is an even number and, hence, that p is an even number. So, let $p = 2s$, for some s. Then $p^2 = 4s^2$ and, hence, $2q^2 = 4s^2$. Cancel the factors of 2 from both sides of the last equation to find that $q^2 = 2s^2$. This means that q^2 is an even number and, hence, that q is also an even number. This tells us that there are common factors of p and q. We therefore have a contradiction to what we supposed to be true, that $\sqrt{2}$ is a rational number.

In formal logic, we would then say that it is false to say that $\sqrt{2}$ is rational and, therefore, conclude that $\sqrt{2}$ is not rational. Constructivist logic will not accept such a method of proof. According to the constructivist school, all we can say is that the statement "$\sqrt{2}$ is rational" is not true. We cannot conclude from this that $\sqrt{2}$ is not rational.

Formal logic admits the law of the excluded middle, which, as we have seen, says that if statement A is false, then its negation must be true; likewise, if A is true, its negation is false. Formal logic has only two truth values, true and false. If the assumption of a truth value leads to a contradiction, the original truth value is switched from true to false or false to true. But what happens in a logical system with more than two truth values? Constructivists do not accept the law of the excluded middle and, therefore, do not accept the method of indirect proof, which permits a swapping of truth values whenever a contradiction is encountered.

It all comes down to the question of how your school views mathematical objects. Are they there waiting to be discovered, or are they brought into existence by invention? If they are waiting to be discovered, they must be true, independent of whether they are discovered; therefore, their negations must be false. However, if they are brought into existence by invention, the establishment of their truth depends on either an established collection of self-evident truths or not. In the latter case, truth or falsity makes no sense.

To complicate matters, Lewis Carroll playfully calls our attention to the problem of infinite regression of the hypothetical in *What the Tortoise Said to Achilles*.[5]

After Achilles and the tortoise finish a famous race—which we'll hear more about in Part 2—they rest while Achilles gives reasons for why his winning was possible. He explains that he was able to overtake his opponent because the distances in the infinite sequence of distances between them were diminishing. The tortoise thinks about this and, as a counterargument, describes a race course "that most people fancy they can get to the end of in two or three steps, though it *really* consists of an infinite number of distances, each one longer than the previous one." He takes an argument from the first proposition of Euclid while Achilles takes out an enormous notebook and pencil. "Proceed!" he says. "And speak slowly, please! *Shorthand* isn't invented yet!"

"Well, now," says the Tortoise, "let's take a little bit of the argument in that First Proposition—just two steps, and the conclusion drawn from them. Kindly enter them in your notebook. And in order to refer to them conveniently, let's call them *A*, *B*, and *Z*:

(A) Things that are equal to the same are equal to each other.
(B) The two sides of this Triangle are equal to the same.
(Z) The two sides of this Triangle are equal to each other."

While Achilles is busy writing in his notebook, the Tortoise explains that some readers, including himself, might accept *A* and *B* as true and still not accept the hypothetical proposition "*A* and *B* implies *Z*." In other words, to accept *Z* as true, one must also accept "If *A* and *B* are true, *Z* must be true."

The Tortoise asks, "What else have you got in [your notebook]?"

"Only a few memoranda," says Achilles, nervously fluttering the leaves, "a few memoranda of—of the battles in which I have distinguished myself!"

"Plenty of blank leaves, I see!" the Tortoise cheerily remarks. "We shall need them all!" (Achilles shudders.) "Now write as I dictate:

(A) Things that are equal to the same are equal to each other.
(B) The two sides of this Triangle are equal to the same.
(C) If *A* and *B* are true, then *Z* must be true.
(Z) The two sides of this Triangle are equal to each other."

Before long, Achilles is forced to enter this in his notebook:

(D) If *A* and *B* and *C* are true, then *Z* must be true.

Achilles protests that if one accepts *A* and *B* and *C* and *D*, then one must accept *Z* as true.

"Then Logic would take you by the throat, and force you to do it!' Achilles triumphantly replies. "Logic would tell you 'You can't help yourself. Now that you've accepted *A* and *B* and *C* and *D*, you must accept *Z*! So you've no choice, you see.'"

But the Tortoise does not buy it and insists that *D* would have to be granted as hypothetical, and so Achilles soon finds the leaves of his enormous notebook filling as he enters this:

(D) If *A* and *B* and *C* are true, then *Z* must be true.
(E) If *A* and *B* and *C* and *D* are true, then *Z* must be true.

Here the narrator, having pressing business at the bank, was obliged to leave the happy pair and did not again pass the spot until some months afterward. When he did so, Achilles was still seated on the bank of the much-enduring Tortoise and was writing in his notebook, which appeared to be nearly full. The Tortoise was saying, "Have you got that last step written down? Unless I've lost count, that makes a thousand and one. There are several millions more to come."

It's hard to argue that humans are programmed to be logical, when freshmen do so poorly on simple questions of logic. But the mechanics of thinking are far from the simple reasoning that declares *Z* to be true when we accept *A* and *B*, and "*A* and *B* implies *Z*." Linguists tell us that all languages have terms such as *not*, *and*, *or*, and *same*, and that in many parts of the world, children use *not, and*, and *or* before the average age of three. So there might be something to the idea that humans are quick to accept the basic building blocks of logic. In fact, it's not a far stretch to infer that humans might be programmed—or, at least, forced by their experience in communicating—to accept the *modus ponens*, the proposition of logic that tells us that *Z* is true whenever we know that *A* is true and that "*A* implies *Z*" is true. But the rules of thought extend much further than this simple force of logic.[6]

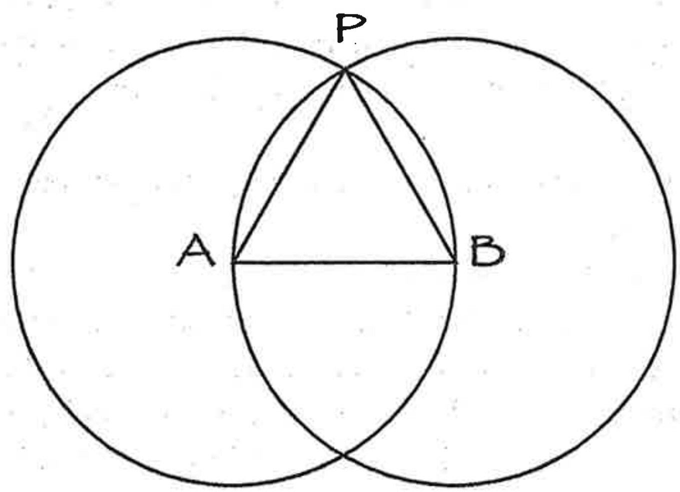
Twenty-three hundred years ago, mathematicians and philosophers struggled with the idea of defining terms such as *point*, *line*, and *circle*, mathematical objects that come from experiencing our spatial world. Euclid decided to define a point as "that which has no part." That's not a very satisfactory definition from a modern point of view, but surely one that is better than Plato's: "A point is that of which the middle covers the ends," whatever that means. (His definitions for a straight line and a circle are not much better.) Definitions of point and line are not necessary. If we postulate, as Aristotle did, that all men are mortal and that Socrates was a man, we are forced to conclude that Socrates was mortal, no matter what it means to be a man or mortal. The truthfulness of the conclusion rests on the truthfulness of the assumptions. This is the idea behind Euclid's postulates. They are true by virtue of assumption—they don't need to be proven. We can then conclude new truths, which follow logically from the assumptions.

Euclid's first proposition—that an equilateral triangle can be constructed on a finite straight line—also has a flaw, albeit one that doesn't seem to matter much. Only two postulates are used in the proof. They appear as Postulates 1 and 3 in Euclid's *Elements*.

1. *To draw a straight line from any point to any point.* By this, Euclid means that between any two points there is a unique straight line.[7]
3. *To describe a circle with any center and distance.* Here, the word *distance* can be interpreted as *radius*. Euclid was thinking that the circle consists of points that are all of the same distance from a fixed point, the center of the circle.

These are two of the five assumptions out of which Euclid built his *Elements*. They seem so self-evident and so basic to our experience with points and lines that proof is not necessary. Indeed, without some self-evident assumptions, we cannot have any notion of proof. If we accept them unconditionally, then, like the syllogism that claims Socrates mortal, these two postulates claim the truth of Euclid's first proposition. An equilateral triangle can be constructed on a finite straight line *AB*.

To prove it, Euclid had to use his third postulate to construct two circles of radius *AB*, one centered at *A* and the other at *B*. He then used his first postulate to mark one of the points of intersection as *P* and to construct two lines, one from *A* to *P* and the other from *B* to *P*.

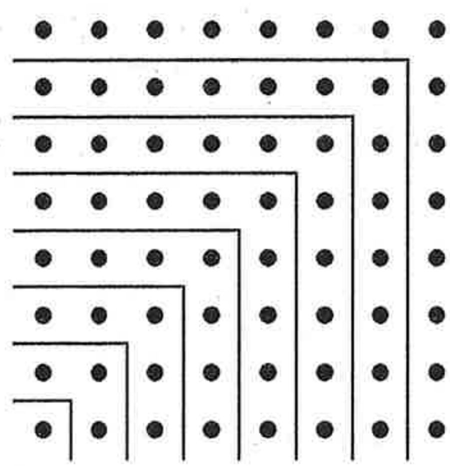


How do we know that the two circles actually intersect each other? If mathematics is axiomatic and a system of arguments that follow from each other, we should not accept Proposition 1 as valid, for there is no argument that tells us that the intersection of the two circles used in the proof exists. The problem is that we find it hard to believe that the intersection does not exist. For more than two thousand years, we felt confident that the proof of Proposition 1 was valid. It "feels" obvious that the two circles intersect. In this sense, we "feel" truth. We "know" that the proposition is true because we are so convinced, and the small detail of conceiving a possible way for the two circles to not intersect should not really get in the way of our believing the proposition.

Those circles are imagined as part of the real world of circles. However, in the nineteenth century, mathematicians realized that the real world does not matter. For example (and we'll hear more about this later), we now know that the fifth postulate of Euclid—which, in effect, says that through a given point not on a given line, there is one and only one parallel to the given line[8]—can be replaced by a postulate stating that "there are *no* straight lines parallel to a given straight line." This might sound false; but if you replace Euclid's fifth postulate with this new one, you have a new system that is logically consistent and one that gives a whole new geometry. The geometry is not that of the world we are used to, so our intuition plays a trick on us. We have a much harder time accepting statements in this new geometry and must resort to checking things not by "feelings" of truth, but by clear, logical inferences.

The symbols we use to communicate thoughts might call up the very different pictures.

"Can pictures be considered actual proofs, or are they merely used as symbolic icons?" I asked my math class for liberal arts students. "There is a well known theorem that says that the sum of the first n odd integers is equal to the square of n—that is, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$," I said and then sketched the figure below. The picture is a perfect square built from right-angled collections of odd numbers of dots, with each angled collection fitting on the previous one.



"Does the picture prove the theorem?" I asked.

"The picture is more convincing than the statement of the theorem," Thomas said as he moved his feet farther under his chair.

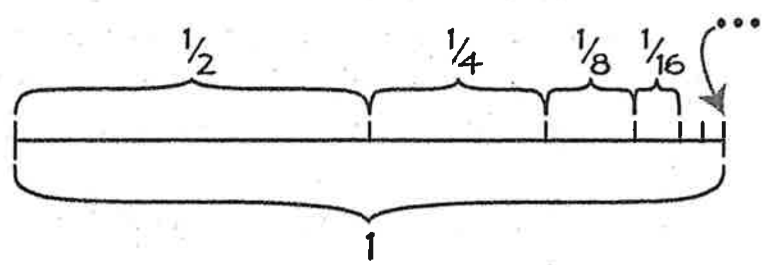
"Yes, but does it prove the theorem?" I asked again.

"I guess it doesn't really prove the theorem, but rather gives an impression that it is true because the picture is a square," Thomas said. "Maybe the picture helps us to begin to discover a real proof," he continued.

But other students disagreed. They saw the proof in the picture. So I presented another theorem. "If you add all the fractions $1/2^n$ together for every positive whole number n greater than 1, the result is just 1," I said while writing the equation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\cdots=1$$

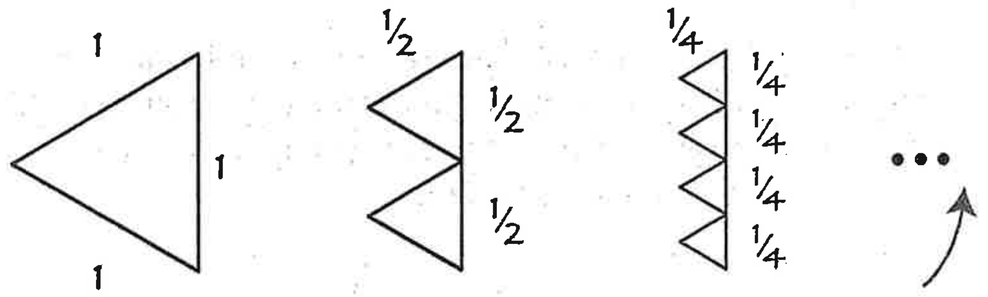
I then sketched the picture (below) of a line of length 1 with divisions in increments of $1/2^n$ and asked whether it proved the theorem.



"The space left between the end of the line and the nth marker is getting smaller as n increases," one student said. "So, of course, there's nothing left by the time n gets to infinity." Everyone but Thomas agreed.

"No!" he said. "I don't believe it is possible to know that there will be nothing left in the space when n gets to infinity. I can't tell you why I don't believe it. I just don't."

"If anyone believes that the last picture proves that the sum of the fractions $1/2^n$ equals 1, then I have a picture that proves that $1 = 2$," I said with a satirical smile and raised eyebrows. I drew a picture of an equilateral triangle with sides equal to one unit, then two equilateral triangles with sides half as long, and then four equilateral triangles with sides half as long again. The triangles in each picture were lined up, one on top of the other, so that the right side of the resulting figure was a straight line of length 1 and the left looked like the teeth of a saw.



"Notice," I ventured, "that the length of the right side—the straight line side—of every one of these figures is 1, and that the length of the left side—the saw-tooth side—is always 2. As I continue my constructions, each figure becomes narrower than the previous one, so the widths approach 0. Now doesn't that tell us that the sawtooth side will eventually meet the straight line side and, therefore, that 1 = 2?"

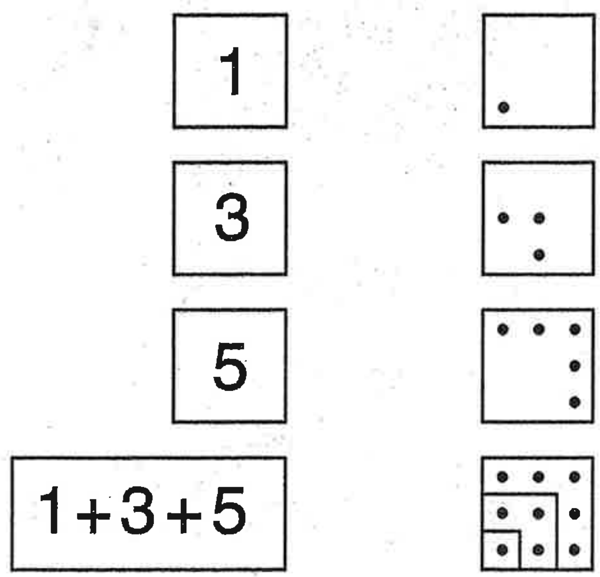
But everyone rejected my argument. The general opinion was that one could not assume that the saw tooth would actually approach the straight line, even after an infinite number of iterations of the construction. The interesting thing here is that, although the majority of students were perfectly willing to accept the picture proof of the theorem that says that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\cdots=1$$

(reasoning that the remaining space shrinks to 0 as n grows large), all students refused to accept the picture proof that 2 = 1. Of course, one reason for the heightened suspicion is that we know that 2 is not equal to 1, and that knowledge psychologically conditions us to suspect that the picture is misleading. However, if the picture on page 67 implies the theorem that says that the sum of consecutive powers of 1/2 equals 1, my students should have accepted my picture proof that 2 = 1.

The second picture is persuasive but not convincing; we cannot tell whether the infinitely many small segments will fill the space or overshoot it, unless we examine the sum of fractions itself. Perhaps it was not fair to present pictures of infinite processes to my math for liberal arts class because one inevitably will fantasize, rather than see, what happens in the infinite tail of the picture.

This leads to the question of how to define *picture*. What do we mean by *picture*? What is the difference between the pictures on the left and the corresponding pictures on the right side of the illustration below? We have an intuitive understanding of what it means to add two numbers. Therefore, I think you will agree that the pictures on the left give the same mental impressions as those on the right. In fact, for small numbers, calculating on fingers is similar to mentally joining spots—the spots either represent fingers, or fingers represent spots. But the picture on the right seems to tell us something more than the symbols on the left: It gives us an impression that the sum of 1, 3, and 5 is a perfect square.



Interpretation of a painting banks on flexible meaning, interaction, history, and culture. What you see is what you interpret. Can you say what the Marrin Robinson painting represents? The answer depends on you, the viewer. Is it a woman at the beach, or the abstract female, or simply a study in forms and shades? The title might suggest what you see: It's called *Desert Prayer*. But if this picture were a mathematical object, there would be no flexibility in interpretation—what you see must lock into a fixed universal definition.

Desert Prayer, courtesy of Marrin Robinson, American University of Cairo.