

Bluebonnet Learning

Secondary Mathematics

EDITION 1

Algebra I

Volume 1

STUDENT EDITION

Acknowledgment

Thank you to all the Texas educators and stakeholders who supported the review process and provided feedback. These materials are the result of the work of numerous individuals, and we are deeply grateful for their contributions.

Notice

These learning resources have been built for Texas students, aligned to the Texas Essential Knowledge and Skills, and are made available pursuant to Chapter 31, Subchapter B-1 of the Texas Education Code.

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Bluebonnet Learning

Secondary Mathematics

EDITION 1

Algebra I

Course Guide

Welcome to the Course Guide for Secondary Mathematics, Algebra I

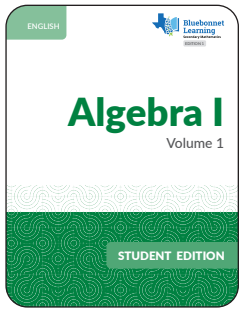
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INSTRUCTIONAL DESIGN

The instructional materials help you learn math in different ways. There are two types of resources: Learning Together and Learning Individually. These resources provide various learning experiences to develop your understanding of mathematics.

Learning Together

On **Learning Together** days, you spend time engaging in active learning to build mathematical understanding and confidence in sharing ideas, listening to one another, and learning together. The Student Edition is a consumable resource that contains the materials for each lesson.



STUDENT EDITION

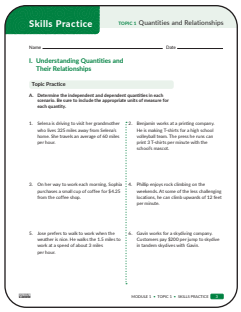
I am a record of your thinking, reasoning, and problem solving.

My lessons allow you to build new knowledge from prior knowledge and experiences, apply math to real-world situations, and learn together in a collaborative classroom.

My purpose is to create mathematical thinkers who are active learners that participate in class.

Learning Individually

On **Learning Individually** days, Skills Practice offers opportunities to engage with skills, concepts, and applications that you learn in each lesson. It also provides opportunities for interleaved practice, which encourages you to flexibly move between individual skills, enhancing connections between concepts to promote long-term learning. This resource will help you build proficiency in specific skills based on your individual academic needs as indicated by monitoring your progress throughout the course.



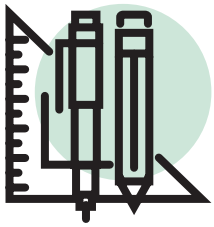
SKILLS PRACTICE

I am targeted practice of each lesson's skills, mathematical concepts, and applications for each topic in the student edition.

My purpose is to provide problem sets for additional practice, enrichment, and extension.

INSTRUCTIONAL DESIGN

The instructional materials intentionally emphasize active learning and sense-making. Deep questions ensure you thoroughly understand the mathematical concepts. The instructional materials guide you to connect related ideas holistically, supporting the integration of your evolving mathematical understanding and developing proficiency with mathematical processes.



Intentional Mathematics Design

Mathematical Coherence: The path through the mathematics develops logically, building understanding by linking ideas within and across grades so you can learn concepts more deeply and apply what you've learned to more complex problems.

TEKS Mathematical Process Standards: The instructional materials support your development of the TEKS mathematical process standards. They encourage you to experiment, think creatively, and test various strategies. These mathematical processes empower you to persevere when presented with complex real-world problems.

Multiple Representations: The instructional materials connect multiple representations of mathematical concepts. Lessons present content visually, algebraically, numerically, and verbally.

Transfer: The instructional materials focus on developing transfer. Doing A and moving on isn't the goal; being able to do A and then do B, C, and D, transferring what you know from A, is the goal.

What principles guide the design and organization of the instructional materials?

Active Learning: Studies show that learning becomes more robust as instruction moves from entirely passive to fully interactive (Chi, 2009). Classroom activities require active learning as students engage by problem solving, explaining and justifying reasoning, classifying, investigate, and analyze peer work.

Discourse Through Collaborative Learning: Collaborative problem solving encourages an interactive instructional model, and we have looked to research to provide practical guidance for making collaboration work. The collaborative activities intentionally promote active dialogue centered on structured activities.

Personalized Learning: Research has proven that problems that capture your interests are more likely to be taken seriously (Baker et. al, 2009). In each lesson, learning is linked to prior knowledge and experiences, creating valuable and authentic opportunities for you to build new understanding on the firm foundation of what you already know. You move from concrete representations and an intuitive understanding of the world to more abstract representations and procedures.

Focus on Problem Solving: Solving problems is an essential life skill that you need to develop. The problem-solving model provides a structure to support you as you analyze and solve problems. It is a strategy you can continue to use as you solve problems in everyday life.

4 Point-Slope Form of a Line

1 OBJECTIVES

- Use the slope formula to derive the point-slope form of a linear equation.
- Construct an equation in point-slope form to model a linear relationship between two quantities.
- Write equations for vertical and horizontal lines.

3 NEW KEY TERM

- point-slope form

2 You have used the slope-intercept form to represent linear relationships. Are there other forms of a linear equation that you can use? How do you write equations for horizontal and vertical lines?

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LESSON STRUCTURE

1 Objectives

Objectives are stated for each lesson to help you take ownership of the objectives.

2 Essential Question

Each lesson begins with a statement connecting what you have learned with a question to ponder.

Return to this question at the end of this lesson to gauge your understanding.

3 New Key Terms

The new mathematical terms for each lesson are identified to help you connect your everyday and mathematical language.

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4 Getting Started

Each lesson begins with a Getting Started. When working on the Getting Started, use what you know about the world, what you have learned previously, or your intuition. The goal is just to get you thinking and ready for what's to come.

4 Getting Started

PROBLEM SOLVING



Draining the Pool

Miguel and Nia are pool cleaners who have been hired to drain the community diving pools at the end of the summer. They are comparing the rate at which the three pools drain.

1. For each pool, write an equation in slope-intercept form to represent the linear relationship.

   a. Pool *A* is at a water level of 14 feet and drains at a rate of 3 feet per hour.

I wonder whether there is a way to make writing the equation of a line more efficient.

   b. Pool *B* is at a water level of 10 feet after draining for 3 hours and drains at a rate of 2 feet per hour.

   c. Pool *C* is at a water level of 15 feet after draining for 2 hours and at 12 feet after draining for 4 hours.

2. Compare your process for writing each equation. How are the processes different?

5

ACTIVITY 4.1

Writing Equations in Point-Slope Form

In the previous lesson, you used the slope, the *y*-intercept, and the slope formula to write a linear equation. You can also determine the equation of a line without knowing the *y*-intercept.

WORKED EXAMPLE

To write an equation of a line from a table of values, you can use the slope formula.

x	y
2	6
4	5
6	4

- First, calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 6}{4 - 2}$$
$$= \frac{-1}{2} = -\frac{1}{2}$$

- Next, choose any point from the table.

(2, 6)

- Then, substitute what you know into the slope formula: $m = -\frac{1}{2}$, (2, 6), and the unknown point (*x*, *y*).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$-\frac{1}{2} = \frac{y - 6}{x - 2}$$

- Finally, rewrite the equation with no variables in a denominator.

$$-\frac{1}{2} = \frac{y - 6}{x - 2}$$
$$-\frac{1}{2}(x - 2) = y - 6$$

The equation is $y - 6 = -\frac{1}{2}(x - 2)$.

This linear equation in the Worked Example is written in *point-slope form*. The **point-slope form** of a linear equation is $y - y_1 = m(x - x_1)$, where *m* is the slope of the line and (x_1, y_1) is any point on the line.

1. Solve the equation in the Worked Example for *y* so that the linear equation is in slope-intercept form. What unique information does each form of the linear equation provide? How are they similar?

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5 Activities

You are going to build a deep understanding of mathematics through a variety of activities in an environment where collaboration and conversations are important and expected.

You will learn how to solve new problems, but you will also learn why those strategies work and how they are connected to other strategies you already know.

Remember:

- It's not just about getting the answer. The process is important.
- Making mistakes is a critical part of learning, so take risks.
- There is often more than one way to solve a problem.

Activities may include real-world problems, sorting activities, Worked Examples, or analyzing sample student work.

Be prepared to share your solutions and methods with your classmates.

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6 Talk the Talk

Talk the Talk gives you an opportunity to reflect on the main ideas of the lesson.

- Be honest with yourself.
- Ask questions to clarify anything you don't understand.
- Show what you know!

Don't forget to revisit the question posed on the lesson opening page to gauge your understanding.

Talk the Talk 6

Say What?

You have learned about two forms of a linear equation: the slope-intercept form, $y = mx + b$, and the point-slope form, $y - y_1 = m(x - x_1)$.

1. What information can you determine about each line by looking at the structure of the equation?

   a. $y = \frac{3}{5}x - 4$

   b. $y - 6 = 2(x + 1)$

   c. $y + 4 = 2(x - 0)$

   d. $y = -\frac{2}{7}x$

   e. $y + 5 = -(x - 4)$

   f. $y = 19$

2. Create a context that represents a linear relationship that passes through the point (2, 56) and has an increasing slope. Then, write the equation of the line in point-slope form and slope-intercept form.

Lesson 4 Assignment

7 Write

Compare the slope-intercept and point-slope forms of a linear equation.

Remember 8

The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is a point on the line. The slope of a horizontal line is 0. The slope of a vertical line is undefined.

9 Practice

Write an equation in point-slope form.

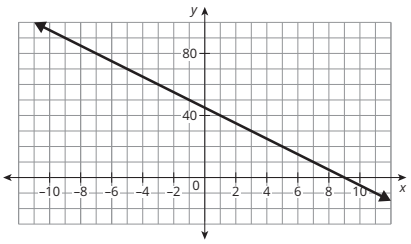
1. $m = 2$; (5, 6)
2. $m = -9.2$; (−17, 10)
3. (−2, −3) and (8, −8)
4. (79, 52) and (−87, 550)
5. A photography studio charges $50 for a sitting fee and 6 prints. Luigi increased his order to 11 prints and paid $65.
6. Lucia is taking the stairs in her building from her floor to the top of the building. After 2 minutes, she was 100 steps from the bottom floor. After 5 minutes, she was 196 steps from the bottom floor.

Write an equation in any form.

7. A newspaper charges a flat fee plus a charge per day to place a classified ad.

Number of Days	Total Charge ($)
2	8.00
4	13.00
6	18.00

8.



ASSIGNMENT

7 Write

Reflect on your work and clarify your thinking.

8 Remember

Take note of the key concepts from the lesson.

9 Practice

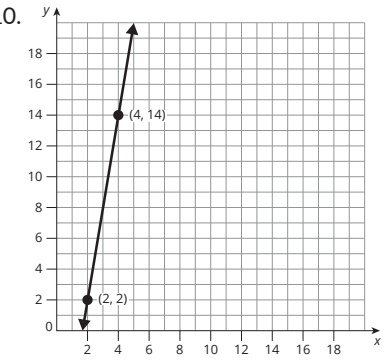
Use the concepts learned in the lesson to solve problems.

Lesson 4 Assignment

9.

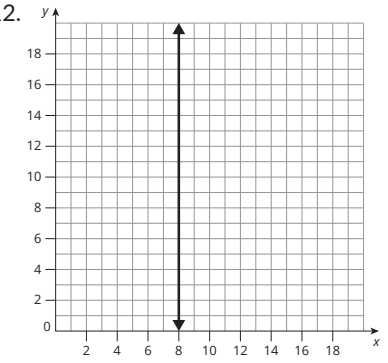
x	y
−10	50
−2	10
4	−20
14	−70

10.



11. Xavier is traveling on a toll road. He plans to exit the road 5 miles ahead and pay $1.75. He changes his plans and travels 9 miles and pays $2.75.

12.



10 Prepare

Solve each equation for *y*.

1. $-2y = -x + 7$

2. $\frac{3}{4}y = x - 6$

3. $2x + 3y = 6$

4. $\frac{1}{2}x - 4y = 8$

ASSIGNMENT

10 Prepare

Get ready for the next lesson.

WORKED EXAMPLE

Consider the sequence generated using $a_n = a_{n-1} + (-2)$ where $a_1 = 11$ and n is a whole number greater than 1.

a_1 represents the first term of the sequence and a_n represents the nth term of the sequence.

Since I know the first term of the sequence, to determine the second term a_2, I add -2 to 11.

$a_2 = a_{2-1} + -2 = a_1 + -2 = 11 + -2 = 9$

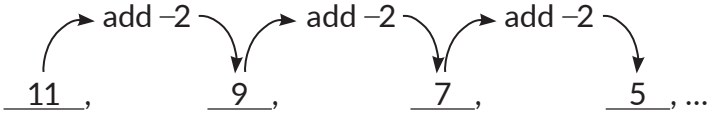
I can determine the 3rd and 4th term of the sequence by continuing the pattern.

$a_3 = a_{3-1} + -2 = a_2 + -2 = 9 + -2 = 7$

$a_4 = a_{4-1} + -2 = a_3 + -2 = 7 + -2 = 5$

The sequence is 11, 9, 7, 5. . .

The pattern is to add the same negative number, -2, to each term to determine the next term.



This sequence is arithmetic and the common difference d is -2.

Ask Yourself . . .

- What is the main idea?
- How would this work if I changed the numbers?
- Have I used these strategies before?

WORKED EXAMPLE

When you see a Worked Example:

- Take your time to read through it.
- Question your own understanding.
- Think about the connections between steps.

THUMBS UP

When you see a Thumbs Up icon:

- Take your time to read through the correct solution.
- Think about the connections between steps.

THUMBS DOWN

When you see a Thumbs Down icon:

- Take your time to read through the incorrect solution.
- Think about what error was made.

WHO'S CORRECT?

When you see a Who's Correct icon:

- Take your time to read through the situation.
- Question the strategy or reason given.
- Determine whether the work is correct or incorrect.

Maria and Abby each convert the given formula to degrees Fahrenheit.

Maria

$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}F - \frac{160}{9}$$
$$9(C) = 9\left(\frac{5}{9}F - \frac{160}{9}\right)$$
$$9C = 5F - 160$$
$$9C + 160 = 5F$$
$$\frac{9C}{5} + \frac{160}{5} = \frac{5F}{5}$$
$$\frac{9}{5}C + 32 = F$$

Abby

$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}F - 32$$
$$9(C) = 9\left(\frac{5}{9}F - 32\right)$$
$$9C = 5F - 288$$
$$9C + 288 = 5F$$
$$\frac{9C}{5} + \frac{288}{5} = \frac{5F}{5}$$
$$\frac{9}{5}C + 576 = F$$

Ask Yourself . . .

- Why is this method correct?
- Have I used this method before?

Ask Yourself . . .

- Where is the error?
- Why is it an error?
- How can I correct it?

4. Consider a sequence in which the first term is 64 and each term after that is calculated by dividing the previous term by 4. Margaret says that this sequence ends at 1 because there are no whole numbers that come after 1. Jasmine disagrees and says that the sequence continues beyond 1. Who is correct? When Margaret is correct, explain why. When Jasmine is correct, predict the next two terms of the sequence.

Ask Yourself . . .

- Does the reasoning make sense?
- If the reasoning makes sense, what is the justification?
- If the reasoning does not make sense, what error was made?

The Crew

The Crew is here to help you on your journey. Sometimes, they will remind you about things you already learned. Sometimes, they will ask you questions to help you think about different strategies. Sometimes, they will share fun facts. They are members of your group—someone you can rely on!



Teacher aides will guide you along your journey. They will help you make connections and remind you to think about the details.



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TEKS Mathematical Process Standards

Effective communication and collaboration are essential skills of a successful learner. With practice, you can develop the habits of mind of a productive mathematical thinker. The "I Can" expectations listed below align with the TEKS mathematical process standards and encourage students to develop their mathematical learning and understanding.

Apply mathematics to problems arising in everyday life, society, and the workplace.

I CAN:

- use the mathematics that I learn to solve real-world problems.
- interpret mathematical results in the contexts of a variety of problem situations.

Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying a solution, and evaluating the problem solving process and reasonableness of the solution.

I CAN:

- explain what a problem "means" in my own words.
- create a plan and change it if necessary.
- ask useful questions in an attempt to understand the problem.
- explain my reasoning and defend my solution.
- reflect on whether my results make sense.

Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate; and techniques including mental math, estimation, and number sense as appropriate, to solve problems.

I CAN:

- use a variety of different tools that I have to solve problems.
- recognize when a tool that I have to solve problems might be helpful and when it has limitations.
- look for efficient methods to solve problems.
- estimate before I begin calculations to inform my reasoning.

Communicate mathematical ideas, reasoning, and their implications using multiple representations including symbols, diagrams, graphs, and language, as appropriate.

I CAN:

- communicate and defend my own mathematical understanding using examples, models, or diagrams.
- use appropriate mathematical vocabulary in communicating mathematical ideas.
- make generalizations based on results.
- apply mathematical ideas to solve problems.
- interpret my results in terms of various problem situations.

Create and use representations to organize, record, and communicate mathematical ideas.

I CAN:

- consider the units of measure involved in a problem.
- label diagrams and figures appropriately to clarify the meaning of different representations.
- create an understandable representation of a problem situation.

Analyze mathematical relationships to connect and communicate mathematical ideas.

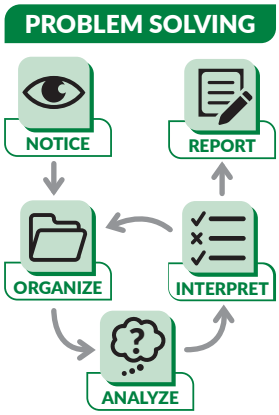
I CAN:

- identify important relationships in a problem situation.
- use what I know to solve new problems.
- analyze and organize information.
- look closely to identify patterns or structure.
- look for general methods and more efficient ways to solve problems.

Display, explain, and justify mathematical ideas and arguments using precise mathematical language in written or oral communication.

I CAN:

- work carefully and check my work.
- distinguish correct reasoning from reasoning that is flawed.
- use appropriate mathematical vocabulary when I talk with my classmates, my teacher, and others.
- specify the appropriate units of measure when I explain my reasoning.
- calculate accurately and communicate precisely to others.



Understanding the Problem-Solving Model

Productive mathematical thinkers are problem solvers. These instructional materials include a problem-solving model to help you develop proficiency with the TEKS mathematical process standards. An opportunity to use the problem-solving model is present every time you see the problem-solving model icon.

The problem-solving model represents a strategy you can use to make sense of problems you must solve. The model provides questions you can use to guide your thinking and the graphic organizer provides a place for you to show and organizer your work.

Using the Problem-Solving Model



Notice | Wonder

Understand the situation by asking these questions.

- What do I notice?
- What do I wonder?
- How do I analyze the given information to identify what is important?
- Do I have enough information to formulate a plan and determine a solution?



Organize | Mathematize

Devise a plan for your mathematical approach by asking these questions.

- What mathematical relationships exist between this problem and similar problems I have solved?
- What plan or strategy can I use to solve this problem?
- How can I efficiently solve this problem?
- How can I organize, record, and communicate my mathematics?



Predict | Analyze

Carry out your plan to determine a solution. Then, ask yourself the following questions.

- Did I display my work using multiple representations?
- Did I explain my reasoning in terms of the problem situation?
- Did I communicate the strategy used to determine the solution?
- Did I justify my mathematical argument clearly using precise mathematical language?
- Can I use my mathematical reasoning to make any predictions?



Test | Interpret

Look back at your work and ask these questions.

- Does my solution clearly and completely answer the original question/problem?
- Is my solution reasonable?
- Does my solution make sense in terms of the problem situation?
- Can I solve the problem using a different strategy? Would another strategy be more efficient?
- Can I justify my solution?



Report

As you share your mathematical reasoning with others, ask these questions.

- Did you use multiple representations to represent your mathematics?
- Did you justify your mathematical reasoning?
- Can others understand my process and solution?

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The Problem-Solving Model Graphic Organizer



Understand the Problem



Devise a Plan



Carry Out the Plan



Look Back



Report

Academic Glossary

Knowing what these terms mean and using them will help you demonstrate proficiency with the TEKS mathematical process standards as you think, reason, and communicate your ideas.

Analyze

Definition

Study or look closely for patterns. Analyzing can involve examining or breaking a concept down into smaller parts to gain a better understanding of it.

Ask Yourself

- Do I see any patterns?
- Have I seen something like this before?
- What happens if the shape, representation, or numbers change?

- Examine
- Evaluate
- Determine
- Observe
- Consider
- Investigate
- What do you notice?
- What do you think?
- Sort and match

Explain Your Reasoning

Definition

Give details or describe how to determine an answer or solution. Explaining your reasoning helps justify conclusions.

Ask Yourself

- How should I organize my thoughts?
- Is my explanation logical?
- Does my reasoning make sense?
- How can I justify my answer to others?

- Show your work
- Explain your calculation
- Justify
- Why or why not?

- Show
- Sketch
- Draw
- Create
- Plot
- Graph
- Write an equation
- Complete the table

Represent

Definition

Display information in various ways. Representing mathematics can be done using words, tables, graphs, or symbols.

Ask Yourself

- How should I organize my thoughts?
- How do I use this model to show a concept or idea?
- What does this representation tell me?
- Is my representation accurate?

- Predict
- Approximate
- Expect
- About how much?

Estimate

Definition

Make an educated guess based on the analysis of given data. Estimating first helps inform reasoning.

Ask Yourself

- Does my reasoning make sense?
- Is my solution close to my estimation?

- Demonstrate
- Label
- Display
- Compare
- Determine
- Define
- What are the advantages?
- What are the disadvantages?
- What is similar?
- What is different?

Describe

Definition

Represent or give an account of in words. Describing communicates mathematical ideas to others.

Ask Yourself

- How should I organize my thoughts?
- Is my explanation logical?
- Did I consider the context of the situation?
- Does my reasoning make sense?

What is Productive Struggle?

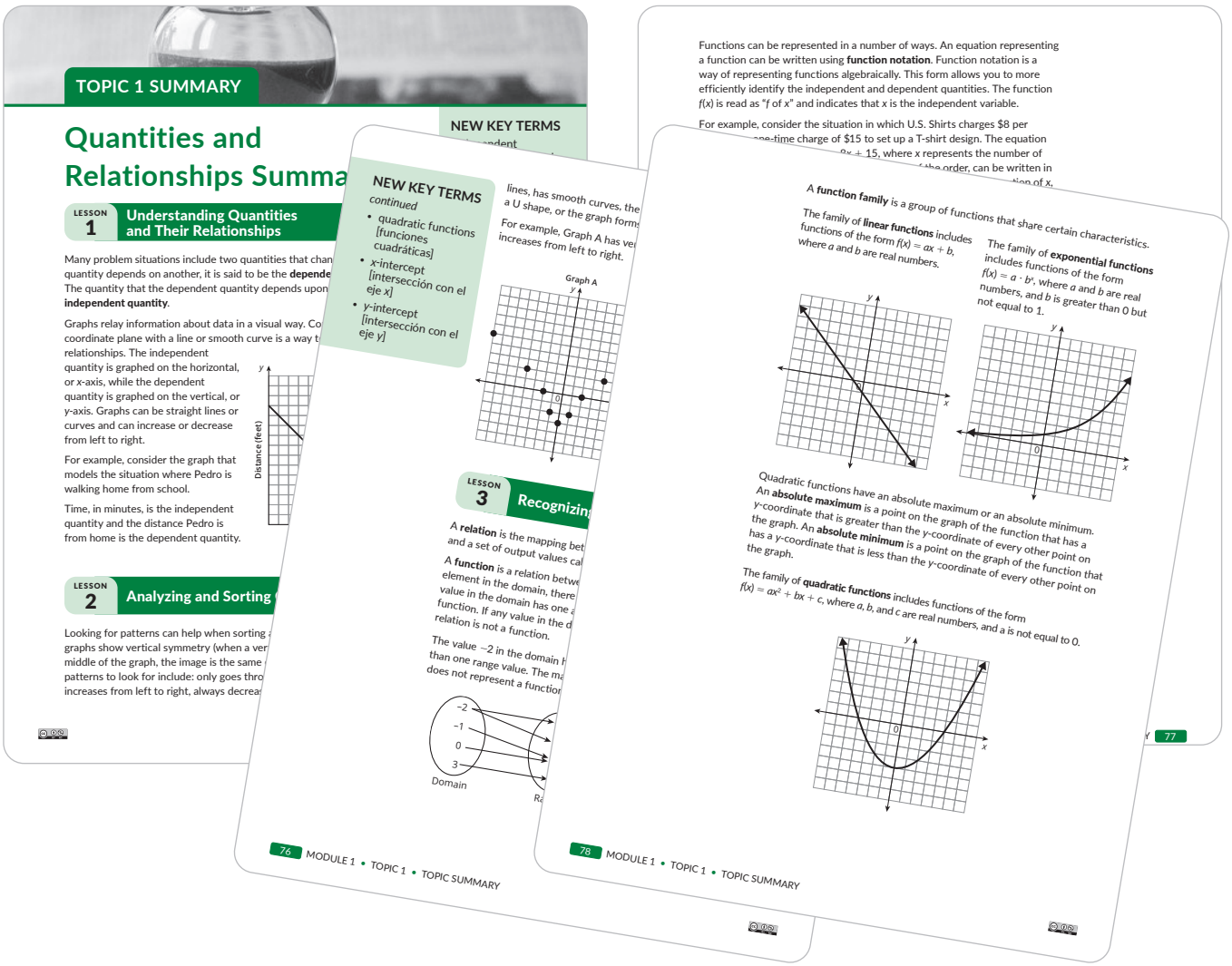
Imagine you are trying to beat a game or solve a puzzle. At some point, you are not sure what to do next. You don't give up; you keep trying repeatedly until you accomplish your goal. You call this process *productive struggle*. Productive struggle is when you choose to persist, think creatively, and try various strategies to accomplish your goal. Productive struggle supports you as you develop characteristics and habits that you will use in everyday life.

Things to do:	Things not to do:
• Persevere. • Think creatively. • Try different strategies. • Look for connections to other questions or ideas. • Ask questions that help you understand the problem. • Help your classmates without telling them the answers.	• Get discouraged. • Stop after trying your first attempt. • Focus on the final answer. • Think you have to make sense of the problem on your own.

This course will challenge you by providing you with opportunities to solve problems that apply the mathematics you know to the real-world. As you work through these problems, your first approach may not work. This is okay, even when your initial strategy does not work, you are learning. In addition, you will discover multiple ways to solve a problem. Looking at various strategies will help you evaluate the problem-solving process and identify an efficient approach. As you persist in problem-solving, you will better understand the math.

Topic Summary

Each topic includes a Topic Summary. The Topic Summary contains a list of all new key terms addressed in the topic and a summary of each lesson, including worked examples and new key term definitions. Use the Topic Summary to review each lesson's major concepts and strategies as you complete assignments and/or share your learning outside of class.



Topic Self-Reflection

The Topic Self-Reflection, provided at the end of each topic, empowers you to develop confidence in your mathematical understanding and monitor your own learning processes. Taking the time for self-reflection helps you identity your strengths and where you want to focus efforts to improve.

Use the Topic Self Reflection throughout the topic to monitor your progress toward the mathematical goals for the topic.

TOPIC 1 SELF-REFLECTION Name: _____

Quantities and Relationships

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Quantities and Relationships* topic by:

TOPIC 1: *Quantities and Relationships*	Beginning of Topic	Middle of Topic	End of Topic
choosing appropriate scale and origin for graphs.	○	○	○
identifying the appropriate unit of measure for each variable or quantity.	○	○	○
analyzing a graph and stating the key characteristics of the graph.	○	○	○
using a problem situation to explain what the key features of a graph mean in real-world context.	○	○	○
deciding whether relations represented verbally, tabularly, graphically, and symbolically define a function.	○	○	○
recognizing a linear, exponential, or quadratic function by its equation or graph.	○	○	○
evaluating functions, expressed in function notation, given one or more elements in their domain.	○	○	○
determining the domain and range and the independent and dependent quantities in a relationship.	○	○	○

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

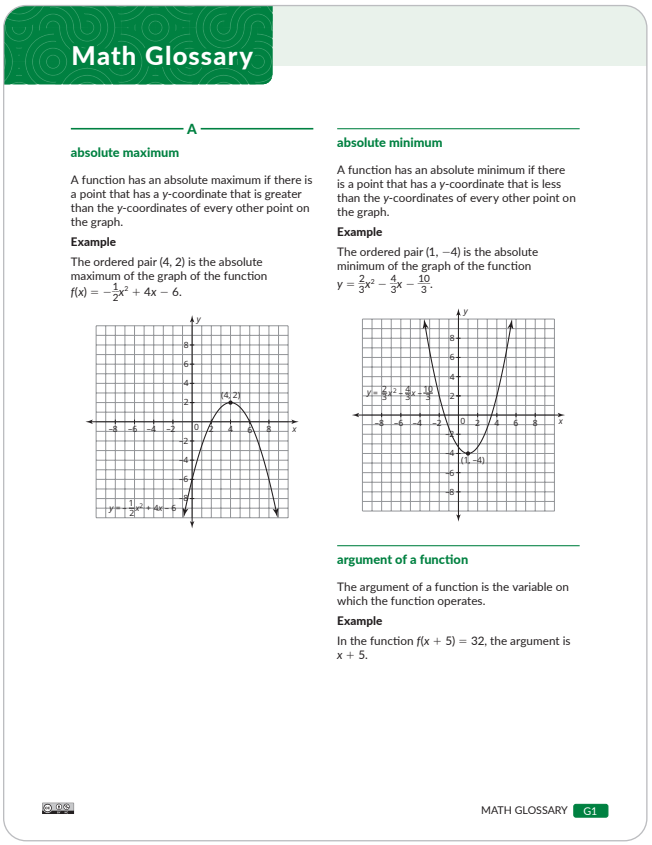
1. Describe a new strategy you learned in the *Quantities and Relationships* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

Math Glossary

A course-specific math glossary is available to utilize and reference while you are learning. Use the glossary to locate definitions and examples of math key terms.



Math Glossary

A

absolute maximum

A function has an absolute maximum if there is a point that has a *y*-coordinate that is greater than the *y*-coordinates of every other point on the graph.

Example

The ordered pair (4, 2) is the absolute maximum of the graph of the function $f(x) = -\frac{1}{2}x^2 + 4x - 6$.

absolute minimum

A function has an absolute minimum if there is a point that has a *y*-coordinate that is less than the *y*-coordinates of every other point on the graph.

Example

The ordered pair (1, −4) is the absolute minimum of the graph of the function $y = \frac{2}{3}x^2 - \frac{4}{3}x - \frac{10}{3}$.

argument of a function

The argument of a function is the variable on which the function operates.

Example

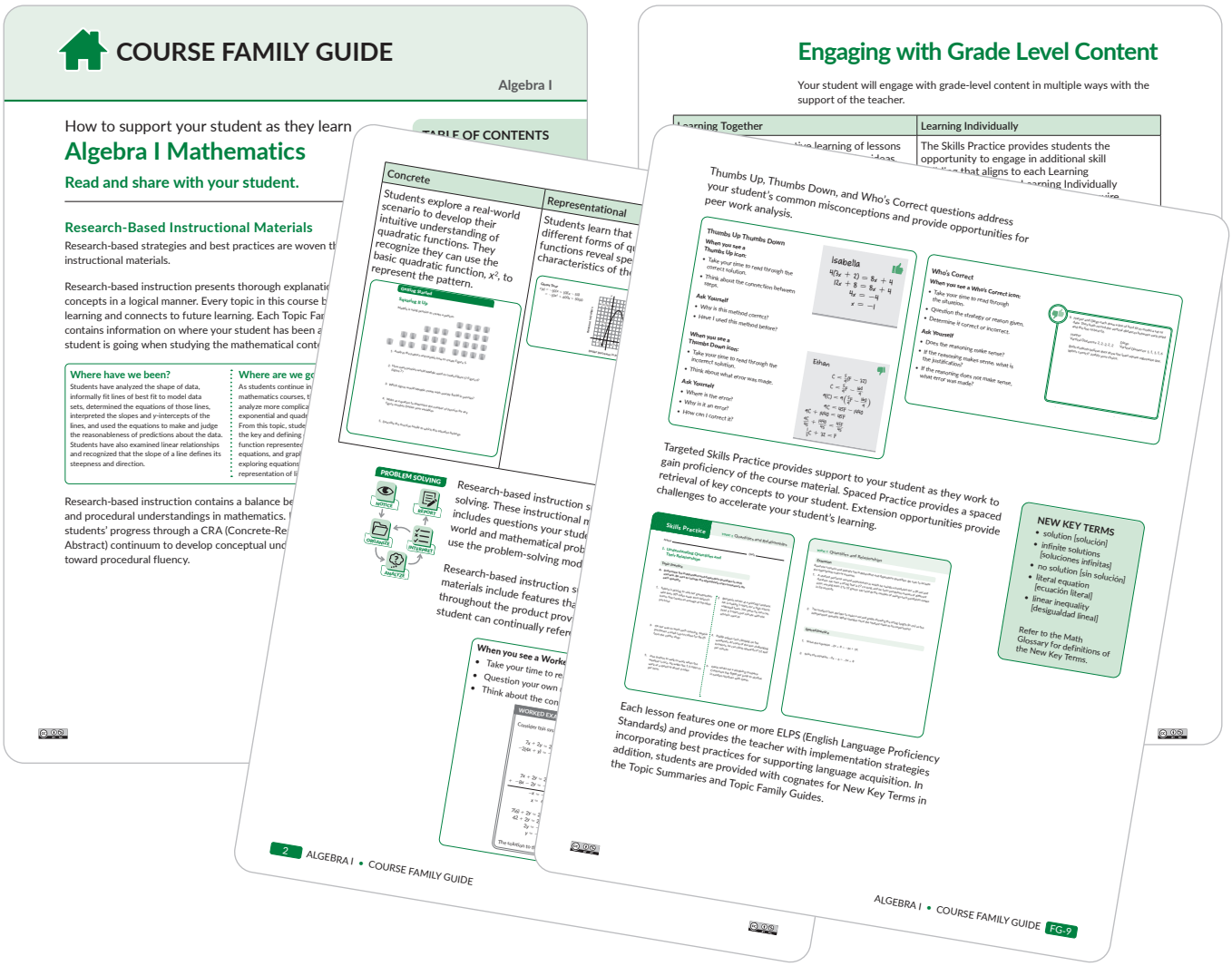
In the function $f(x + 5) = 32$, the argument is $x + 5$.

MATH GLOSSARY G1

Course Family Guide

The Course Family Guide provides you and your family an overview of the course design. The guide details the resources available to support your learning, such as the Math Glossary, the Topic Family Guide, the Topic Self-Reflection, and the Topic Summaries.

The purpose of the Course Family Guide is to bridge your learning in the classroom to your learning at home. The goal is to empower you and your family to understand the concepts and skills learned in the classroom so that you can review, discuss, and solidify the understanding of these key concepts together.



Topic Family Guides

Each topic contains a Topic Family Guide that provides an overview of the math of the topic and answers the questions, “Where have we been?” and “Where are we going?” Additional components of the Topic Family Guide are an example of a math model or strategy taught in the topic, definitions of key terms, busting of a math myth, and questions family members can ask you to support your learning.

Learning outside of the classroom is crucial to student success at school. The Topic Family Guide serves to assist families in talking to students about the learning that is happening in the classroom.

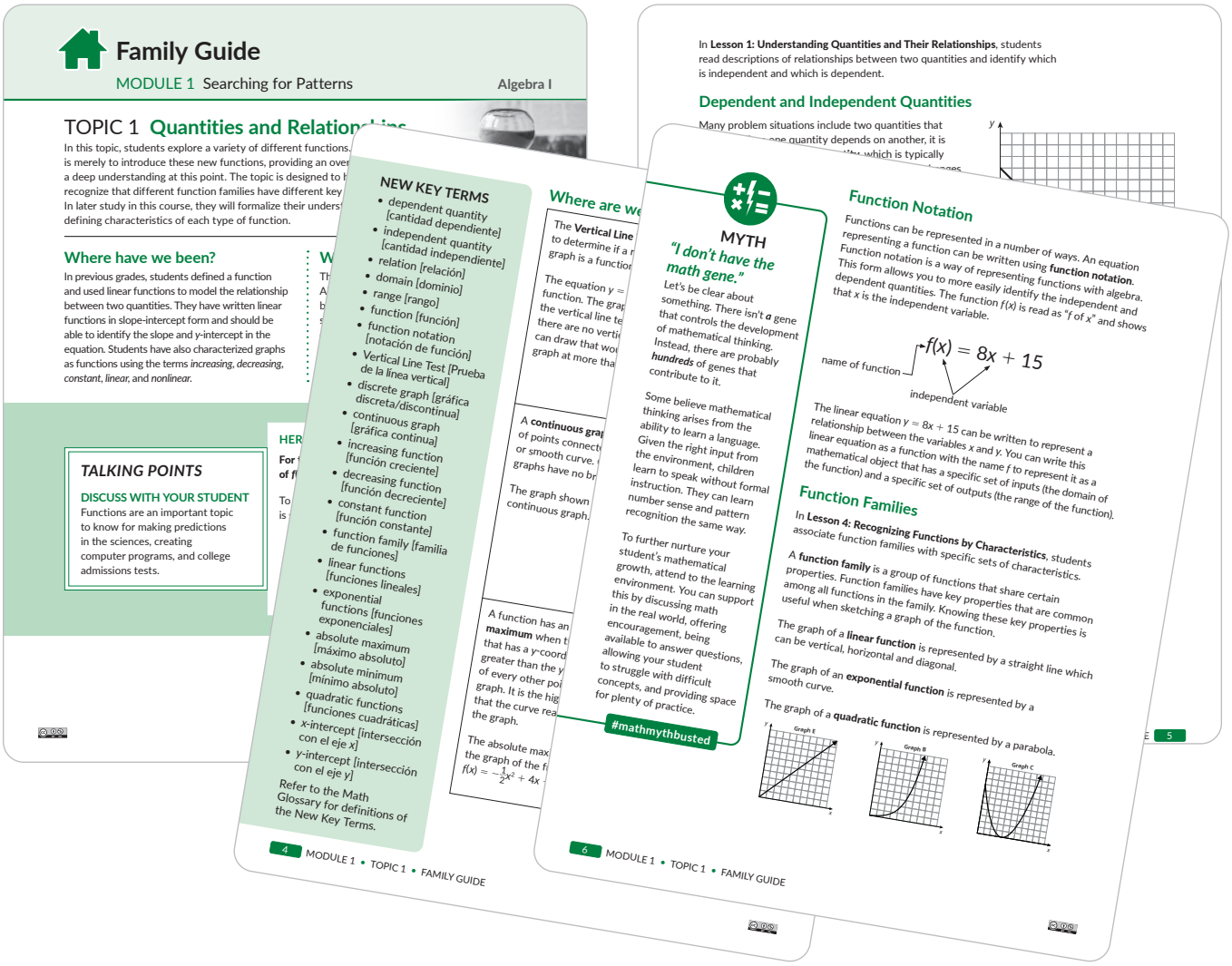


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Math Glossary **G-1**

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MODULE
1

Searching for Patterns

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TOPIC 1



The amount of sand in the lower bulb of an hourglass is directly proportional to the time since the glass was turned over.

Quantities and Relationships

INTRODUCTION LESSON

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INTRODUCTION LESSON

Introduction to the Problem-Solving Model and Learning Resources

OBJECTIVES

- Establish a community of learners.
- Discover learning resources available for this course.
- Apply the problem-solving model to a real-life situation.

In previous math classes, you have analyzed patterns and relationships, learned about numbers and operations in base ten and fractions, measurement and data, and geometry.

What resources are available in this course to help you extend your mathematical thinking?

Getting Started

You Already Know A Lot

Each lesson in this book begins with a Getting Started that gives you the opportunity to use what you know about the world and what you have learned in previous math classes. You know a lot from a variety of learning experiences.

Think back to how you learn something new.

1. List three different skills that you recently learned. Then, describe why you wanted to learn that skill and the strategies that you used.

New Skill	Motivation to Learn the New Skill	Strategies I Used to Learn This Skill

Ask Yourself . . .
How do your strategies change based on what you are learning and what you already know about it?

One learning strategy is to talk with your peers. In this course, you will work with your classmates to solve problems, discuss strategies, and learn together.

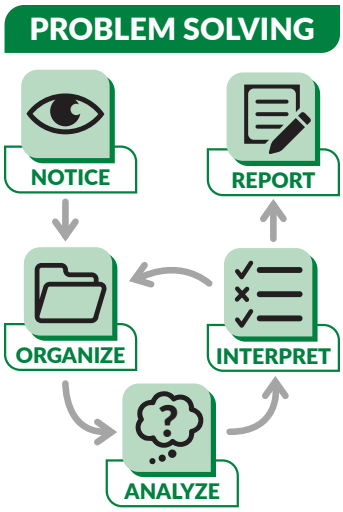
Compare and discuss your list with a classmate.

2. Which strategies do you have in common? Which strategies does your classmate have that you did not think of on your own?

Think About . . .
Listening well, cooperating with others, and appreciating different perspectives are essential life skills.

Be prepared to share your list of learning strategies with the class.

ACTIVITY
1.1 Learning Resources



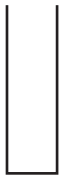
In this course, you will learn new math concepts by exploring and investigating ideas, reading, writing, and talking to your classmates. You will even learn by making mistakes with concepts you haven't mastered yet.

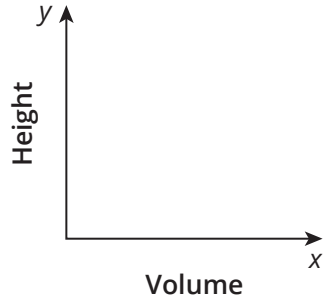
Let's practice exploring and investigating. You do not need to answer the question yet. You will solve the question as you work through the problem-solving model.

Students at East High School are designing ceramic drinking cups for an art project. The students can choose from a variety of different shapes for their cups. To test the cups, hot water is poured into each cup at a constant rate. Mario chooses the cup shown.

Create a graph to represent the height of the liquid in the cup as the volume changes in Mario's cup. **Explain your reasoning**.

Mario's Cup





The Academic Glossary provides definitions of terms you will see throughout the course as you think, reason, and communicate your ideas. The Math Glossary provides the definitions of new key terms in each lesson.

The Academic Glossary is your guide as you engage with the kind of thinking you do as you are learning the content.

1. Locate the phrase **Explain your reasoning** in the Academic Glossary in the Course Guide. What questions should you ask yourself as you describe the relationship between the height of the liquid and the volume in Mario's cup?

2. What is a related word or phrase for **explain your reasoning**?

The problem-solving model provides a structure to help you become a better problem-solver.



Notice and Wonder

The first step in modeling a situation mathematically is to understand the problem, gather information, notice patterns, and formulate mathematical questions about what you notice.

Read through the questions to ask yourself for the first step of the problem-solving model. Use the *Problem-Solving Model Graphic Organizer* to answer the questions and show your thinking.

3. What do you notice about Mario's cup?

4. Why do you think the first step of the problem-solving model is important? How will it help you when you solve problems?



Organize and Mathematize

The second step in the problem-solving model is to devise a plan. When devising a plan, you will organize your information and begin to represent it using mathematical notation.

Read through the questions to ask yourself for the second step of the problem-solving model. Use the *Problem-Solving Model Graphic Organizer* to answer the questions and show your thinking.

5. Describe your strategy to sketch a graph that represents the height of the liquid in the cup as the volume changes in Mario's cup.

6. Why do you think the second step of the problem-solving model process is important? How can you use the questions to ask yourself to help develop a strategy to solve the problem?

Predict and Analyze



The third step in the modeling process involves carrying out your plan. As you carry out the plan you will complete operations, analyze mathematical results, make predictions, and extend patterns.

Read through the questions to ask yourself for the third step of the problem-solving model. Use the *Problem-Solving Model Graphic Organizer* to answer the questions and show your thinking.

7. Use your strategy to create a graph that represents the height of the liquid in the cup as the volume changes in Blake's cup.

8. How can the questions to ask for the third step help you communicate your mathematical thinking?

Test and Interpret



The fourth step in the modeling process is to look back, interpret your results, and test your mathematical predictions in the real world. When your predictions are incorrect, or your results are not reasonable, you can revisit your mathematical work and make adjustments—or start all over!

Read through the questions to ask yourself for the fourth step of the problem-solving model. Use the *Problem-Solving Model Graphic Organizer* to answer the questions and show your thinking.

9. Alyssa choses a different design. Can you apply the same reasoning to create a graph to represent the height of the liquid in the cup as the volume changes in Jacquelyn's cup.

10. How do the questions for the fourth step of the problem-solving model help you evaluate the reasonableness of your solution?



Report

The final step in the modeling process is to share your results.

11. How does listening to others help you learn mathematics?

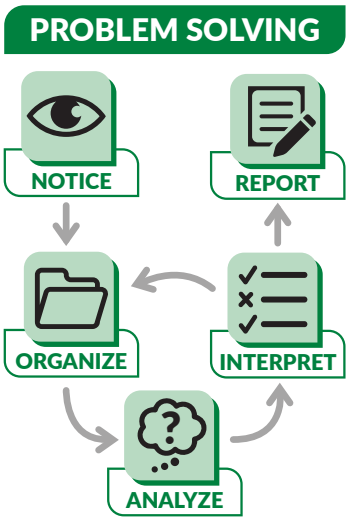
Locate the TEKS mathematical process standards in the Course Guide.

12. Which TEKS mathematical process standard(s) did you use to create a graph that represents the height of the liquid in the cup as the volume changes in Mario's cup?

Talk the Talk

The Problem-Solving Model

In this lesson, you used a problem-solving model to solve a real-world problem. The basic steps of the problem-solving model are summarized in the diagram.



Summarize what is involved in each phase of this problem-solving model.

Notice and Wonder

Organize and Mathematize

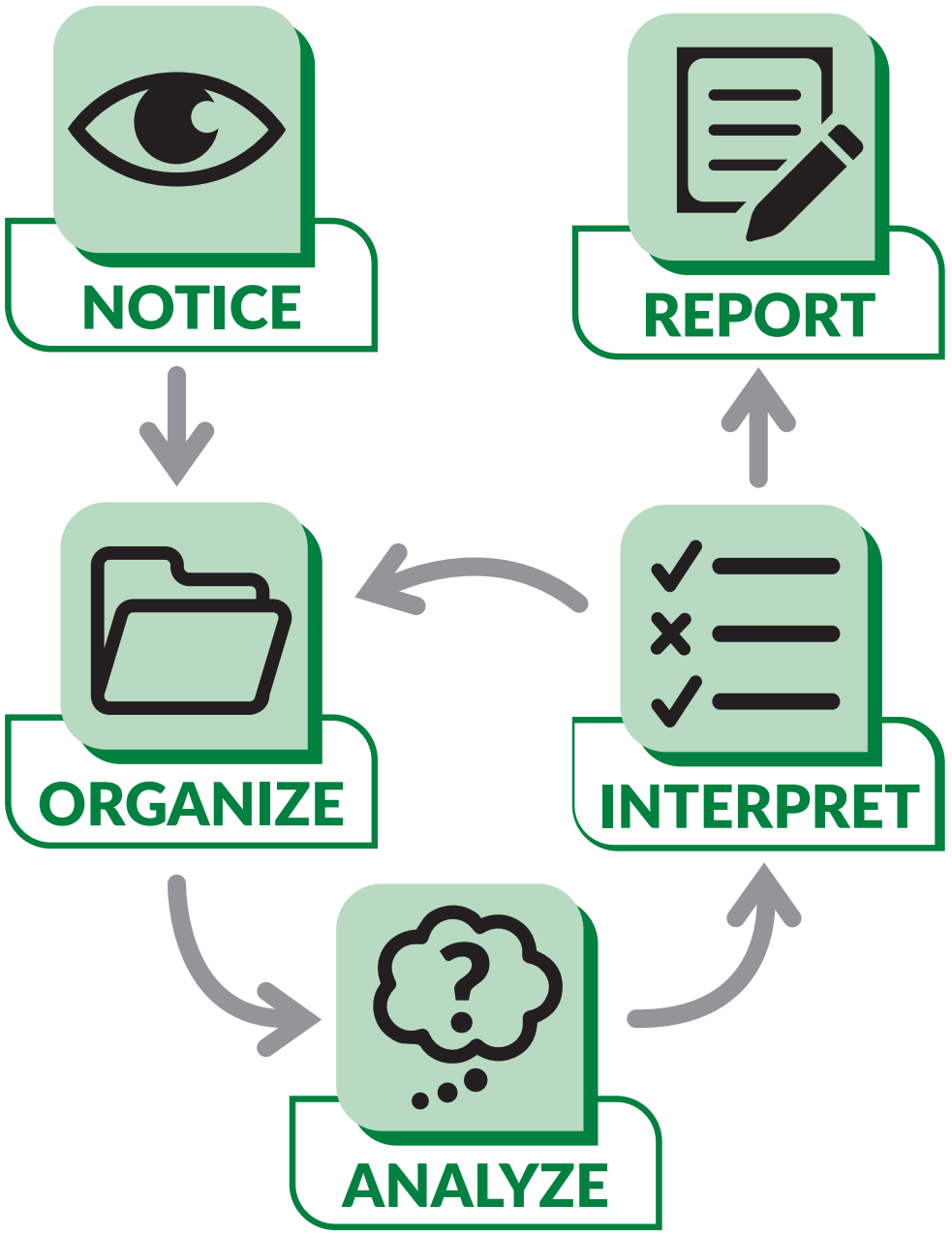
Predict and Analyze

Test and Interpret

Report

You will see this symbol throughout the course to remind you to use the problem-solving model.

The Problem-Solving Model



1 Understanding Quantities and Their Relationships

OBJECTIVES

- Understand quantities and their relationships with each other.
- Identify the independent and dependent quantities for a scenario.
- Match a graph with an appropriate scenario.
- Use a reasonable scale for a graph modeling a scenario.
- Identify key characteristics of graphs.
- Describe similarities and differences between pairs of graphs and scenarios.

NEW KEY TERMS

- dependent quantity
- independent quantity

You have analyzed graphs of relationships and identified important features, such as intercepts and slopes. How can the key characteristics of a graph tell a story?

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Getting Started

What Comes First?

Have you ever planned a birthday party? You may have purchased ice, gone grocery shopping, selected music, made food, or even cleaned in preparation. Many times, these tasks depend on another task being done first. For instance, you wouldn't make food before grocery shopping, now would you?

Consider the two quantities that are changing in each relationship.

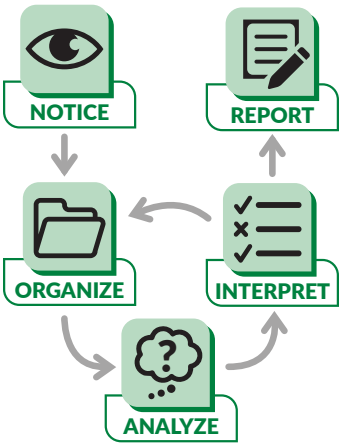
- The number of movie tickets purchased and the total cost
- The number of eggs used and the number of cakes baked
- The number of students in attendance at school and the number of lunches served
- The number of hours driven and the number of miles to a vacation destination
- The number of minutes a swimming pool is filled with water and the number of gallons of water in the swimming pool

When one quantity is determined by another in a problem situation, it is called the **dependent quantity**. The quantity it is determined from is called the **independent quantity**.

1. Circle the independent quantity and underline the dependent quantity in each relationship.
2. Describe how you can determine which quantity is independent and which quantity is dependent in any problem situation.

ACTIVITY
1.1 Connecting Scenarios and Their Graphs

PROBLEM SOLVING



While a person can describe the monthly cost to operate a business or talk about a marathon pace a runner ran to break a world record, graphs on a coordinate plane enable people to see the data. Graphs relay information about data in a visual way.

You can use lines or smooth curves to represent relationships between points on a graph. In some problem situations, all the points on the line will make sense. In other problem situations, not all the points will make sense. So, when you model a relationship with a line or a curve, it is up to you to consider the situation and interpret the meaning of the data values shown.

This activity includes six scenarios and six graphs that are located at the end of the lesson.

Ask Yourself . . .
How can graphs be used to tell a story in everyday life?

1. Read each scenario. Determine the independent and dependent quantities. Then, match each scenario to its corresponding graph. Glue the graph next to the scenario. For each graph, label the *x*- and *y*-axes with the appropriate quantity and a reasonable scale and then interpret the meaning of the origin.

   Be sure to include the appropriate units of measure for each quantity.

   Music Club
   Natalia loves music. She can lip sync almost any song at a moment's notice. She joined Songs When I Want Them, an online music store. By becoming a member, Natalia can purchase just about any song she wants. Natalia pays $1 per song.

   - Independent quantity:

   - Dependent quantity:

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Ask Yourself . . .
What strategies will you use to match each graph with one of the six scenarios?

Something's Fishy

Kaya is the building manager for an office building. One of her responsibilities is cleaning the office building's 200-gallon aquarium. For cleaning, she must remove the fish from the aquarium and drain the water. The water drains at a constant rate of 10 gallons per minute.

- Independent quantity:

- Dependent quantity:

Smart Phone, but Is It a Smart Deal?

You have your eye on an upgraded smart phone. However, you currently do not have the money to purchase it. Your cousin will provide the funding, as long as you pay him back with interest. He tells you that you only need to pay $1 in interest initially and then the interest will double each week after that. You consider his offer and wonder if this *really* is a good deal.

- Independent quantity:

- Dependent quantity:

It's Magic

The Amazing Alejandro is practicing one of his tricks. As part of this trick, he cuts a rope into many pieces and then magically puts the pieces of rope back together. He begins the trick with a 20-foot rope and then cuts it in half. He then takes one of the halves and cuts that piece in half. He repeats this process until he is left with a piece so small he can no longer cut it.

- Independent quantity:

- Dependent quantity:

Baton Twirling

Samantha is a drum major for the high school marching band. For the finale of the halftime performance, Samantha tosses her baton in the air so that it reaches a maximum height of 22 feet. This gives her 2 seconds to twirl around twice and catch the baton when it comes back down.

- Independent quantity:

- Dependent quantity:

Skateboarding

Andrew is skateboarding on a 4-foot-tall half-pipe at the local skate park. He wants to know how far his skateboard's wheels are from the ground as he skates from the top of one side of the half-pipe to the top of the other side.

- Independent quantity:

- Dependent quantity:

ACTIVITY
1.2 Comparing and Contrasting Graphs

Now that you have matched a graph with the appropriate problem situation, let's go back and examine all the graphs.

1. What similarities do you notice in the graphs?

Think About . . .
Look closely when analyzing the graphs. What do you see?

2. What differences do you notice in the graphs?

3. How did you label the independent and dependent quantities in each graph?

4. **Analyze** each graph from left to right. Describe any graphical characteristics you notice.

Review the definition of analyze in the Academic Glossary.

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Think About . . .

What do the points on each graph represent?

5. Compare the graphs for each pair of scenarios given and describe any similarities and differences you notice.

   a. *Smart Phone, but Is It a Smart Deal?* and *Music Club*

   b. *Something's Fishy* and *It's Magic*

   c. *Baton Twirling* and *Skateboarding*

Talk the Talk

A Writer and a Mathematician

1. Write a scenario and sketch a graph to describe a possible trip to school.

Scenario

_____

_____

_____

_____

_____

_____

_____

Graph

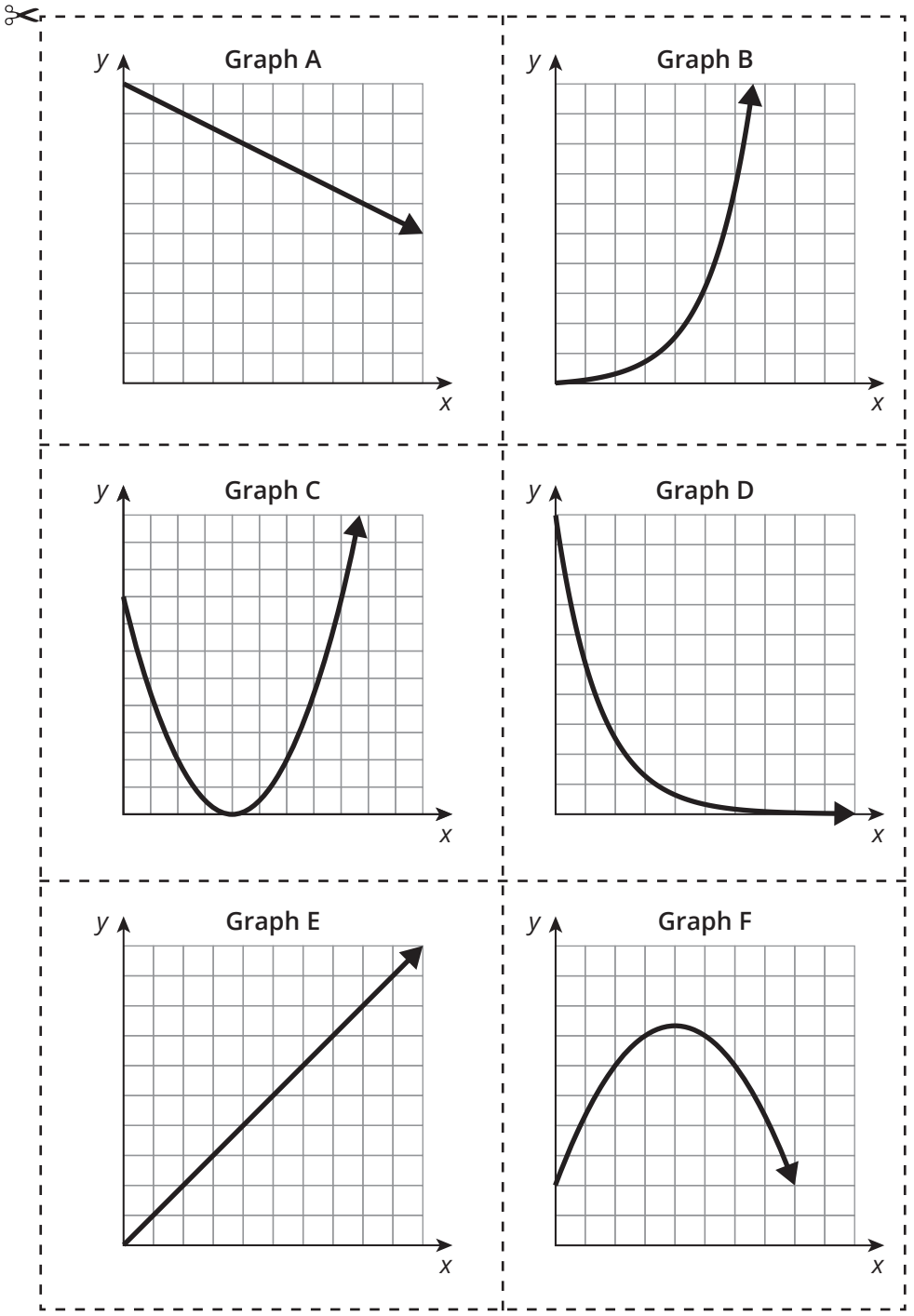


2. Describe the meaning of the points, or smooth curve, represented by your graph.

3. Compare your scenario and sketch with your classmates' scenarios and sketches. What similarities do you notice? What differences do you notice?

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Graph Cutouts



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Why is this page blank?

So you can cut out the graphs on the other side

Lesson 1 Assignment

Write

Describe how you can distinguish between an independent quantity and a dependent quantity. Use an example in your description.

Remember

When one quantity is determined by another in a problem situation, it is called the *dependent quantity*. The quantity it is determined from is called the *independent quantity*. The independent quantity is represented on the *x*-axis and the dependent quantity is represented on the *y*-axis.

Practice

1. Read each scenario and identify the independent and dependent quantities. Be sure to include the appropriate units of measure. Then, analyze each graph and determine which of the provided scenarios it models. For each graph, label the *x*- and *y*-axes with the appropriate quantity and unit of measure.

   a. Endangered Species

   A local animal conservation organization is working with various reptile species to increase their populations. The initial population of 450 endangered turtles tripled each year for the past five years.

   b. Video Games

   Trung is playing video games at an arcade. Trung starts with $40 and is playing games that cost 50 cents per game.

   c. Sales Commission

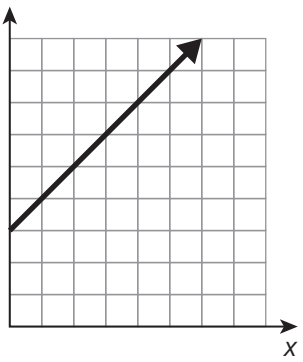
   Eduardo works as a salesman. He receives a monthly salary of $3000 as well as a 10% commission on the amount of sales.

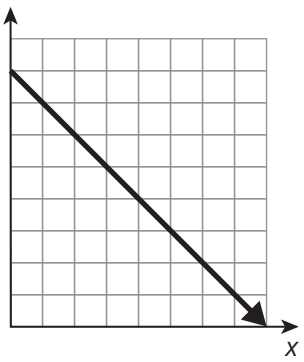
   d. Cooling Tea

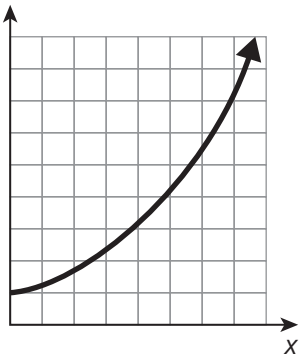
   A freshly made cup of tea is served at a temperature of about 180°F. The tea cools rapidly at first and then slows down gradually as it approaches room temperature.

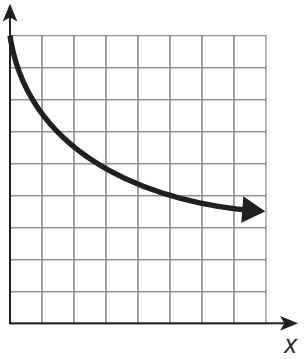
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Lesson 1 Assignment

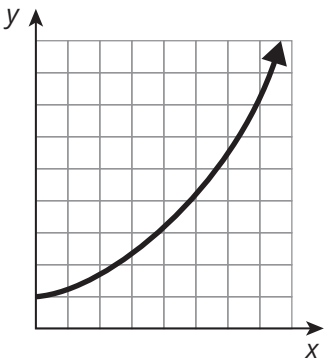
A. 

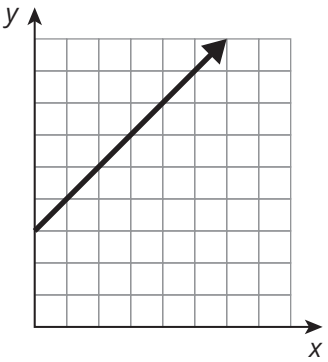
B. 

C. 

D. 

2. Compare the pair of graphs and describe any similarities and differences you notice.

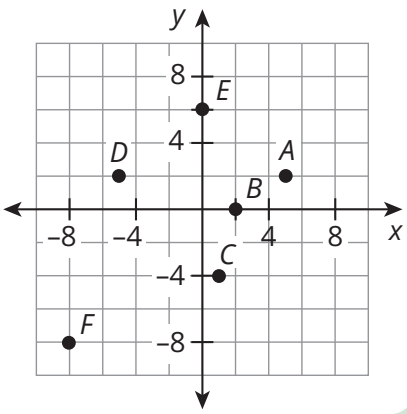




Lesson 1 Assignment

Prepare

1. Write the coordinates of each point and name the quadrant or axis where the point is located.



2 Analyzing and Sorting Graphs

OBJECTIVES

- Review and analyze graphs and graphical behavior.
- Determine similarities and differences among various graphs.
- Sort graphs and give reasons for the similarities and differences between the groups of graphs.

You have used graphs to analyze the relationship between independent and dependent quantities. Do the graphs of certain types of relationships share any characteristics?

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Keep your graphs. You will need them in the next lesson.

Let's Sort Some Graphs

Mathematics is the science of patterns and relationships. Looking for patterns and sorting patterns into different groups based on similarities and differences can provide valuable insights. In this lesson, you will analyze many different graphs and sort them into various groups.

1. Cut out the 13 graphs at the end of the lesson. Then, analyze and sort the graphs into at least two different groups. You may group the graphs in any way you feel is appropriate.

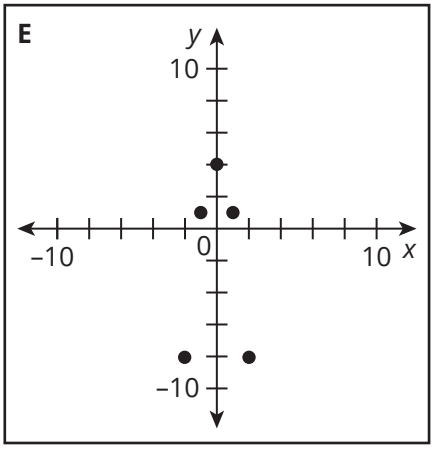
   Record the following information for each of your groups.

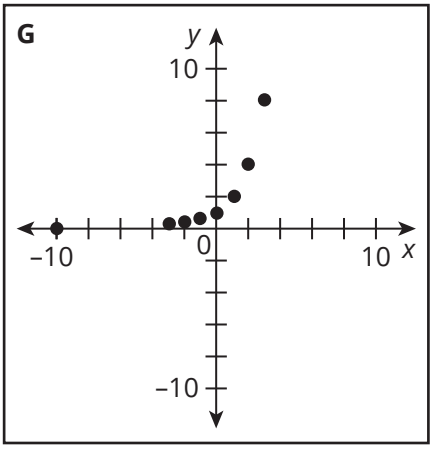
   - Name each group of graphs.
   - List the letters of the graphs in each group.
   - Provide a rationale for why you created each group.

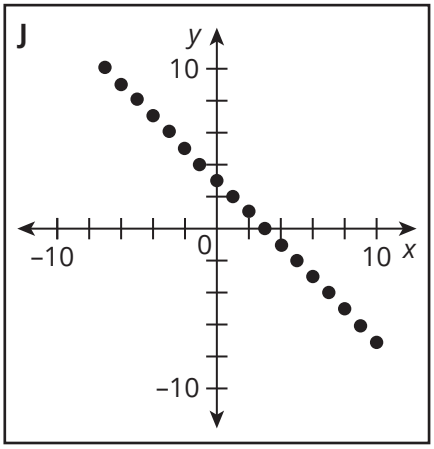
ACTIVITY
2.1 Identifying Graphical Behaviors

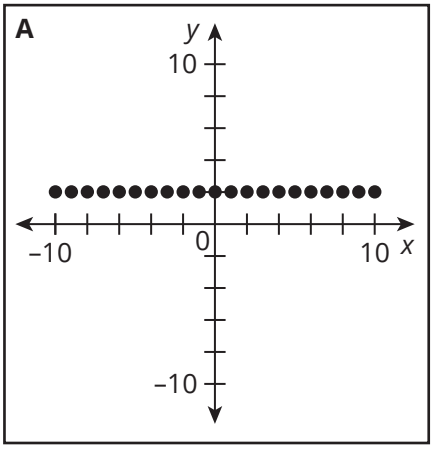
In this activity, consider the different ways the graphs are grouped.

1. Chris grouped these graphs together. Why do you think Chris put these graphs in the same group?







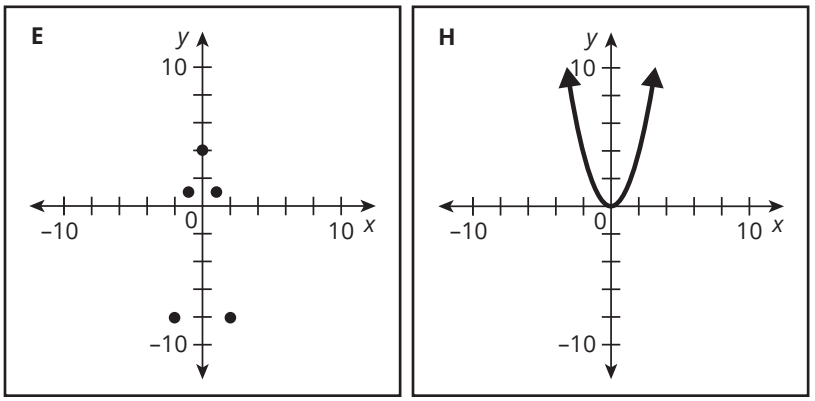


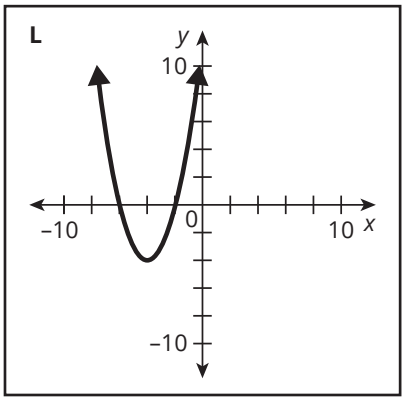
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2. Consider Paola's correct grouping.

Paola

I grouped these graphs together because they all have a vertical axis of symmetry. If I draw a vertical line through the middle of the graph, the image is the same on both sides.





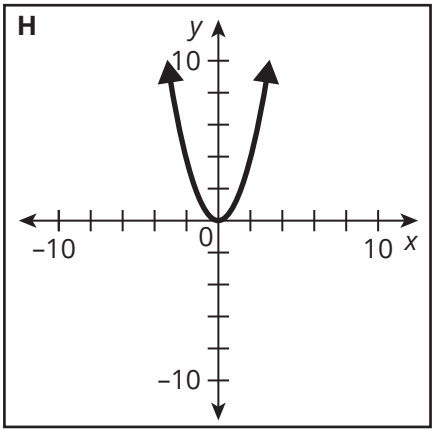
a. Show why Paola's reasoning is correct.

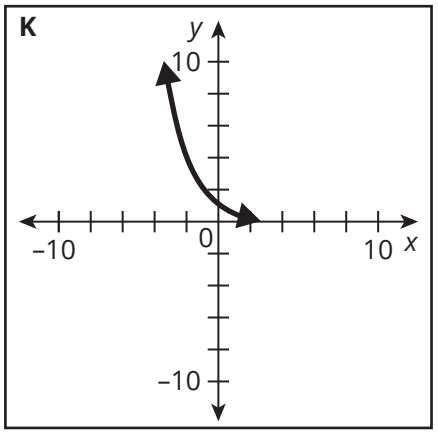
b. When possible, identify other graphs in this activity that have a vertical axis of symmetry.

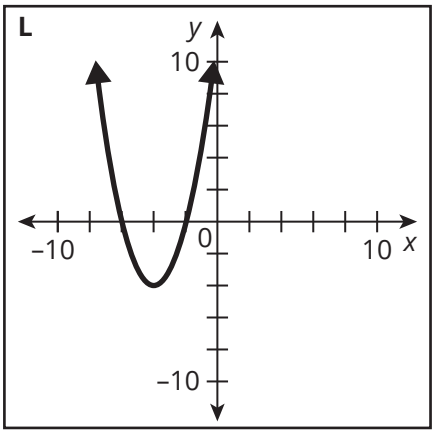
3. Consider Jorge's incorrect grouping.

Jorge

I grouped these graphs together because each graph goes through only two quadrants.



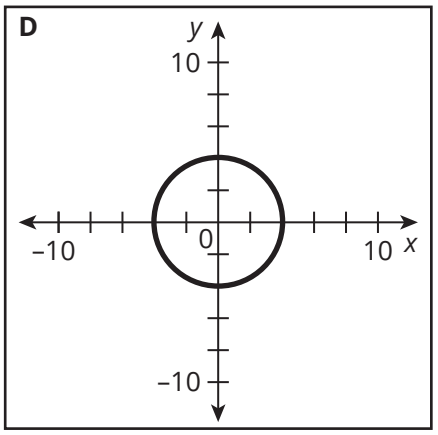


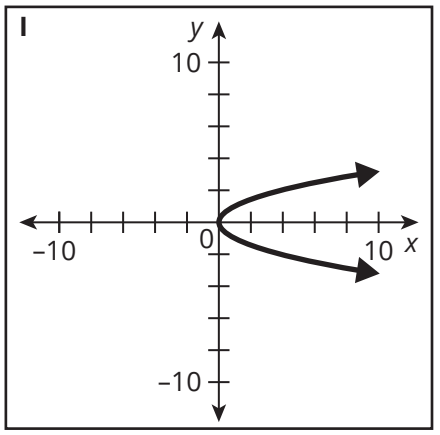


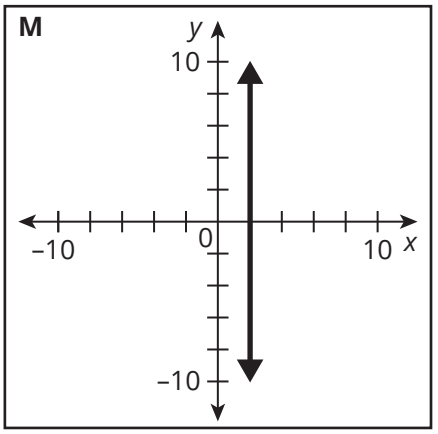
a. Explain why Jorge's reasoning is not correct.

b. When possible, identify other graphs in this activity that go through only two quadrants.

4. Elena grouped these graphs together but did not provide any rationale.







a. What do you notice about the graphs?

b. What rationale could Elena have provided?

Talk the Talk

Compare and Contrast

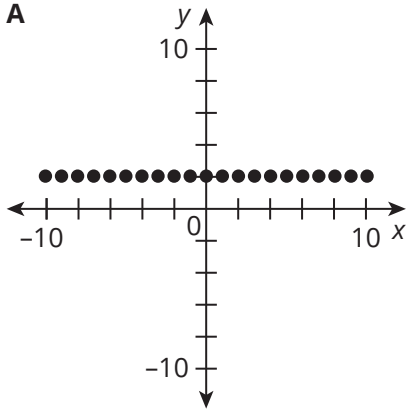
1. Compare your groups with your classmates' groups. Create a list of the different graphical behaviors you noticed.

Ask Yourself . . .
Are any of the graphical behaviors shared among your groups? Or are they unique to each group?

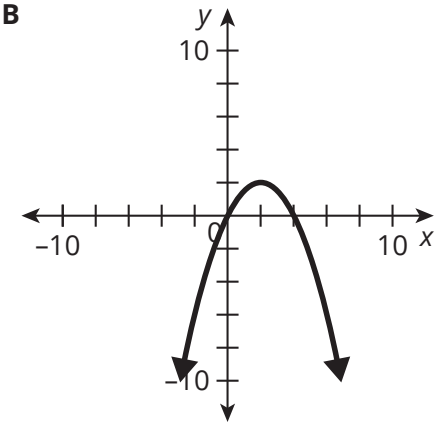
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Graph Cards

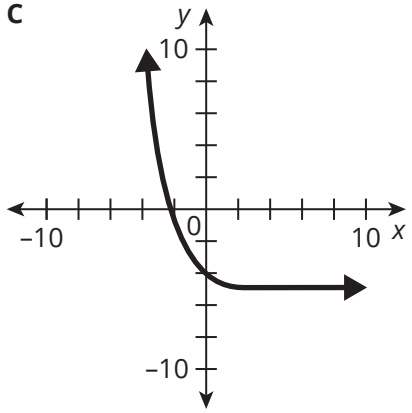
A



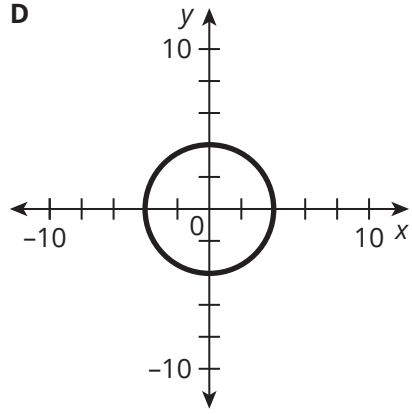
B



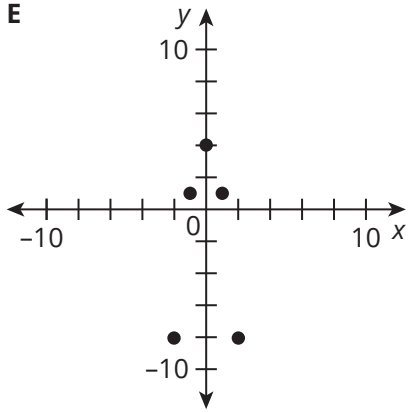
C



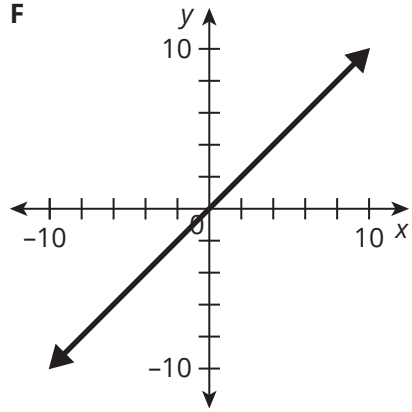
D



E



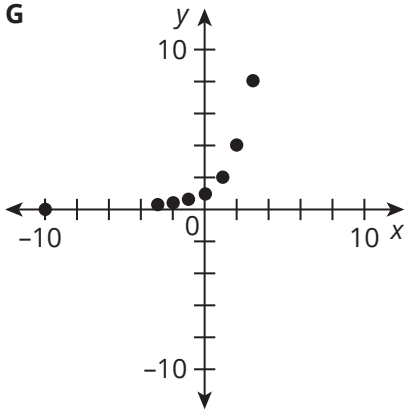
F



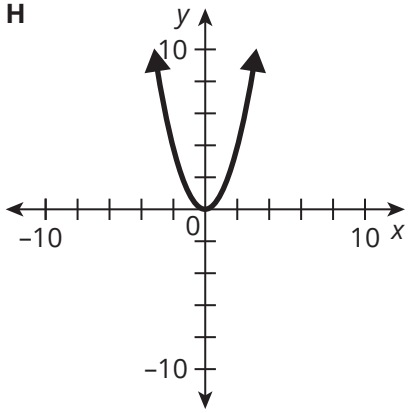
Why is this page blank?

So you can cut out the Graph Cards on the other side

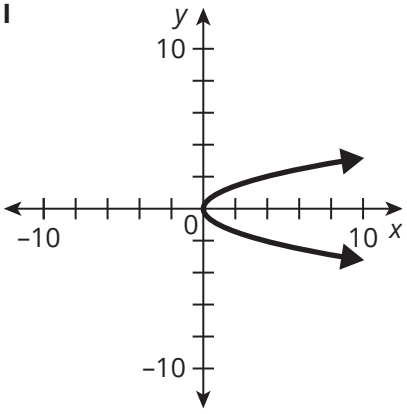
G



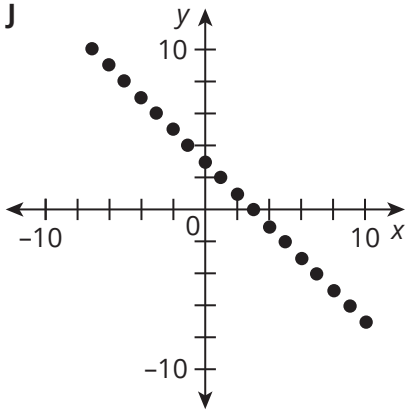
H



I



J

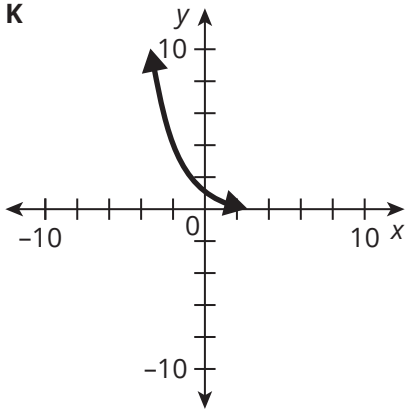


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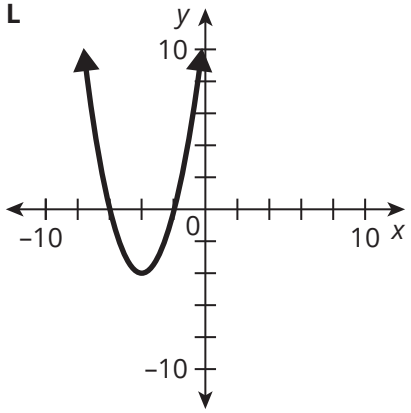
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So you can cut out the Graph Cards on the other side

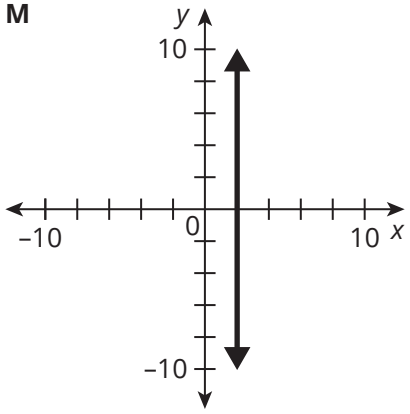
K



L



M



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Why is this page blank?

So you can cut out the Graph Cards on the other side

Lesson 2 Assignment

Write

Describe the importance of graphical representations.

Remember

Graphs of relationships between quantities have characteristics that can give you important information about the relationship. For example, a graph can be increasing, decreasing, neither increasing nor decreasing, or both increasing and decreasing. A graph can have straight lines or smooth curves, a maximum or a minimum, or no maximum or minimum, and so on.

Practice

1. Record the letter of each graph with the given characteristic.

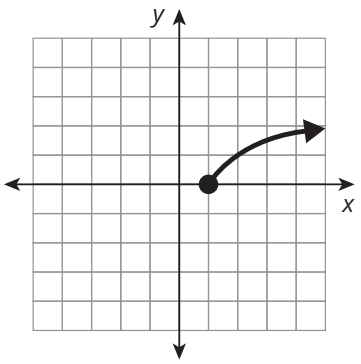
   a. Has a vertical axis of symmetry

   b. Passes through exactly 1 quadrant

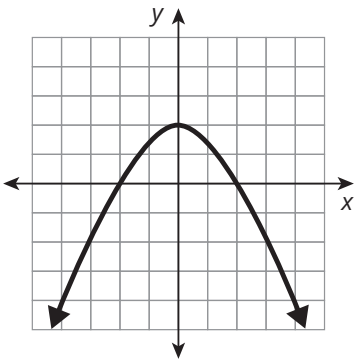
   c. Has a horizontal axis of symmetry

   d. Passes through all 4 quadrants

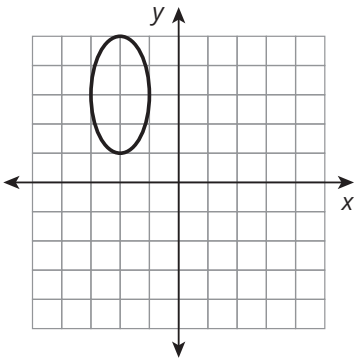
A.



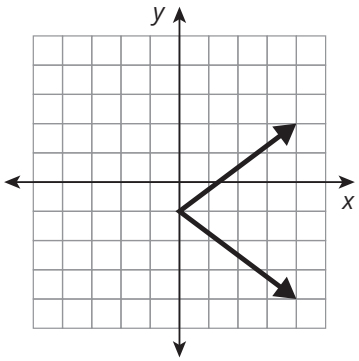
B.



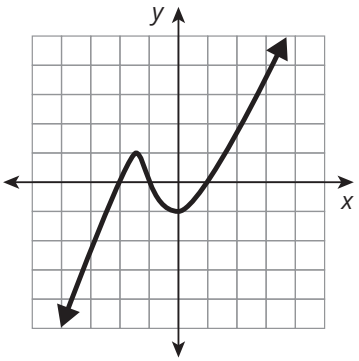
C.



D.



E.



Lesson 2 Assignment

Prepare

Create a list of the *x*- and *y*-values of the relation described by each set of ordered pairs. Write an equation using the variables *x* and *y* that could map the *x*-value to the *y*-value.

1. {(−3, −6), (−2, −4), (−1, −2), (0, 0), (1, 2)}

2. {(−3, 9), (−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4)}

3. {(−3, 3), (−2, 2), (−1, 1), (0, 0), (1, −1), (2, −2)}

3 Recognizing Functions and Function Families

OBJECTIVES

- Define a *function* as a relation that assigns each element of the domain to exactly one element of the range.
- Write equations using function notation.
- Recognize multiple representations of functions.
- Determine and recognize characteristics of functions.
- Determine and recognize characteristics of function families.

You have sorted graphs by their graphical behaviors.
How can you describe the common characteristics of the graphs of the functions?

NEW KEY TERMS

- relation
- domain
- range
- function
- function notation
- Vertical Line Test
- discrete graph
- continuous graph
- increasing function
- decreasing function
- constant function
- function family
- linear functions
- exponential functions
- absolute minimum
- absolute maximum
- quadratic functions
- *x*-intercept
- *y*-intercept

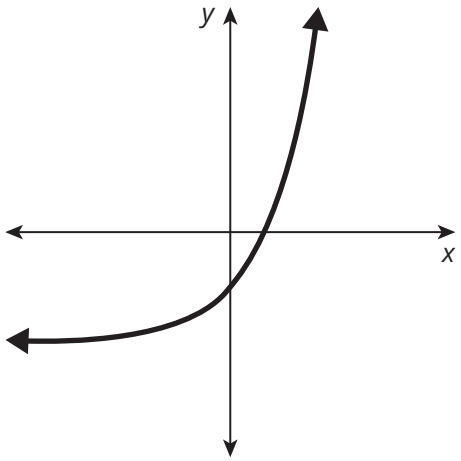
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Getting Started

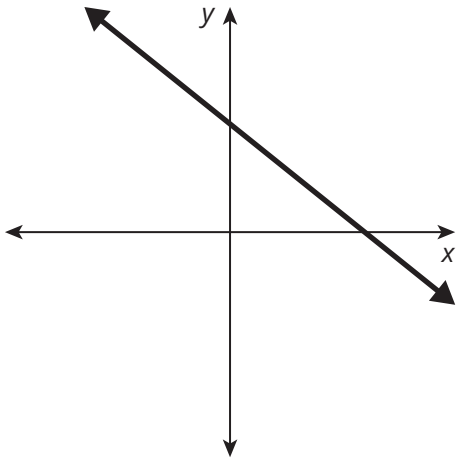
Odd One Out

1. Which of the graphs does not belong with the others? Explain your reasoning.

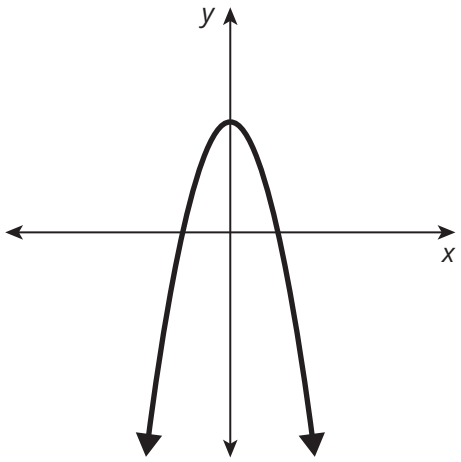
Graph A



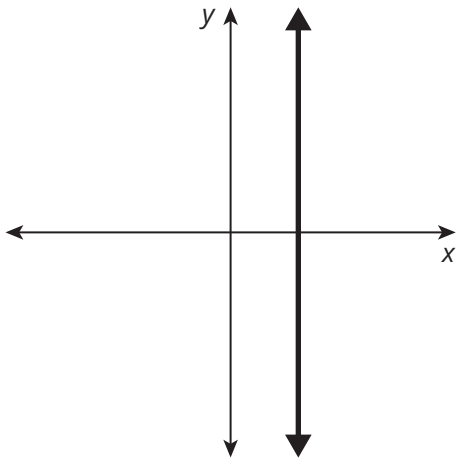
Graph B



Graph C



Graph D



ACTIVITY
3.1 Functions and Non-Functions

A relation can be represented in the following ways.

Ordered Pairs

$\{(-2, 2), (0, 2), (3, -4), (3, 5)\}$

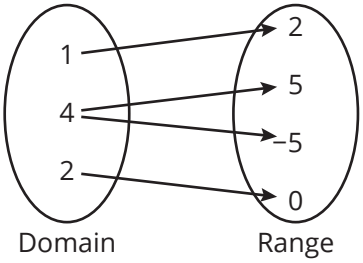
Equation

$y = \frac{2}{3}x - 1$

Verbal

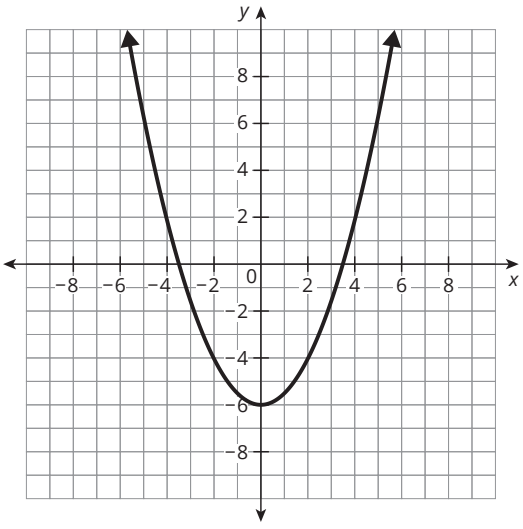
The relation between students in your school and each student's birthday

Mapping



A **relation** is the mapping between a set of input values called the **domain** and a set of output values called the **range**.

Graph



Table

Domain	Range
−1	1
2	0
5	−5
6	−5
7	−8

Think About . . .

So, all functions are relations, but only some relations are functions.

A **function** is a relation that assigns to each element of the domain exactly one element of the range. Functions can be represented in a number of ways.

1. Analyze the relation represented as a table. Is the relation a function? Explain your reasoning.

2. Analyze the relation represented as a mapping. Is the relation a function? Explain your reasoning.

3. Analyze the relation represented verbally. Is the relation a function? Explain your reasoning.

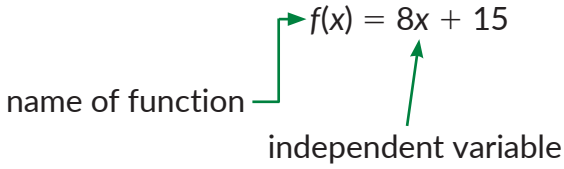
You can write an equation representing a function using *function notation*. Let's look at the relationship between an equation and function notation.

Function notation is a way of representing functions algebraically. The function notation $f(x)$ is read as "f of x" and indicates that x is the independent variable.

Consider this scenario. A shirt company charges \$8 per shirt plus a one-time charge of \$15 to set up a T-shirt design.

The equation $y = 8x + 15$ can be written to model this situation. The independent variable x represents the number of shirts ordered, and the dependent variable y represents the total cost of the order, in dollars.

This is a function because for each number of shirts ordered (independent value) there is exactly one total cost (dependent value) associated with it. Because this relationship is a function, you can write $y = 8x + 15$ in function notation.



The cost, defined by f, is a function of x, defined as the number of shirts ordered.

When f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x.

You can write a function in a number of different ways. You could write the T-shirt cost function as $C(s) = 8s + 15$, where the cost, defined as C, is a function of s, the number of shirts ordered.

4. Consider the function, $C(s) = 8s + 15$. What expression in the function equation represents:

   a. the domain of the function?

   b. the range of the function?

5. Describe the possible domain and range for this situation.

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WORKED EXAMPLE

You can write domain and range using set notation.

A teacher is restocking his pencil supply for this semester's tutoring group. All new students to the tutoring group receive 12 pencils. The teacher expects to have no more than 6 new students attending his group.

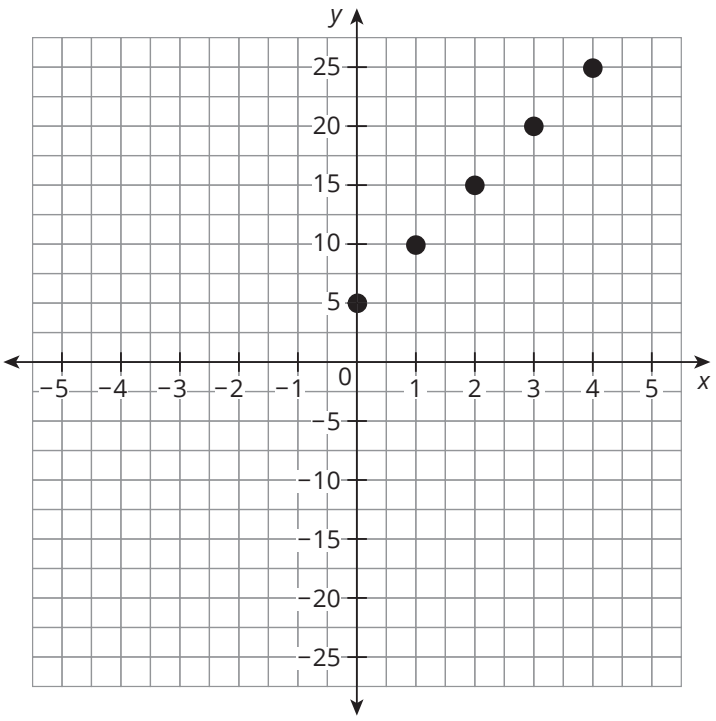
The domain for this situation is {0, 1, 2, 3, 4, 5, 6} for the possibility of having 0 to 6 new students.

The range for this situation is {0, 12, 24, 36, 48, 60, 72} for the possibility of needing between 0 and 72 pencils.

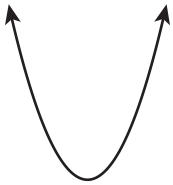
6. Write the domain and range using set notation.

   a. A neighborhood bakery makes and sells bagels every morning. The bagels are sold in sets of 6 and only the first 9 customers can purchase them.

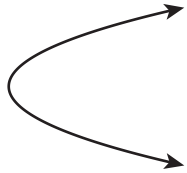
   b.



The **Vertical Line Test** is a visual method used to determine whether a relation represented as a graph is a function. To apply the Vertical Line Test, consider all of the vertical lines that could be drawn on the graph of a relation. If any of the vertical lines intersect the graph of the relation at more than one point, then the relation is not a function.



Function



Not a function

The Vertical Line Test applies for both *discrete* and *continuous graphs*.

A **discrete graph** is a graph of isolated points. A **continuous graph** is a graph of points that are connected by a line or smooth curve on the graph. Continuous graphs have no breaks.

7. How can you determine if a relation represented as a set of ordered pairs is a function? Explain your reasoning.

8. How can you determine if a relation represented as an equation is a function? Explain your reasoning.

9. Determine which relations represent functions. If the relation is not a function, state why not.

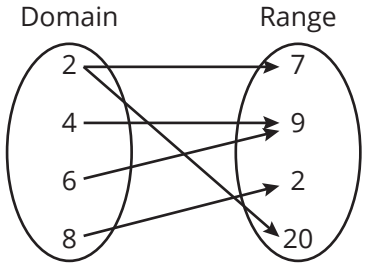
a. $y = 3^x$

b. For every house, there is one and only one street address.

c.

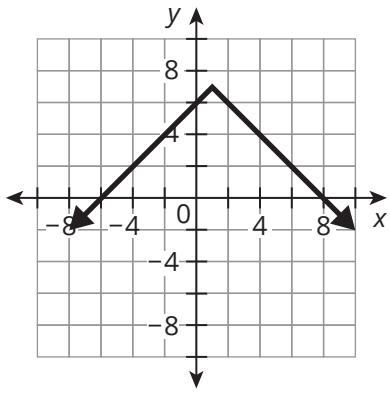
Domain	Range
−1	4
0	0
3	−2
0	4

d.



e. {(−7, 5), (−5, 5), (2, −2), (3, 5)}

f.



10. Analyze the three graphs Elena grouped together in the previous lesson, graphs *D*, *I*, and *M*. Are the graphs she grouped functions? Explain your conclusion.

11. Use the Vertical Line Test to sort the graphs in the previous lesson into two groups: functions and non-functions. Record your results by writing the letter of each graph in the appropriate column in the table shown.

Functions	Non-Function

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ACTIVITY
3.2 Domain and Range of a Function

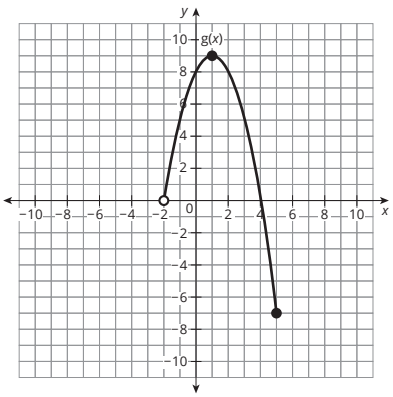
You have identified the domain and range of a function given its equation.

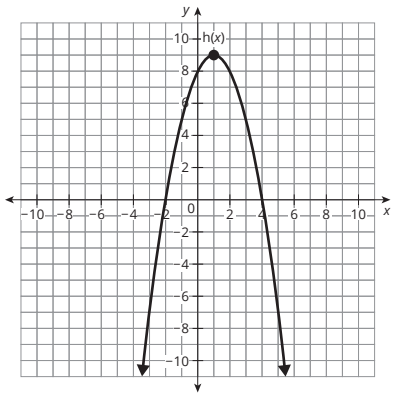
Ask Yourself . . .
How does an open or closed circle on a graph reflect the domain? The range?

1. Explain how you can identify the domain and range of a function given:
   a. a verbal statement.
   b. a graph.

WORKED EXAMPLE

There are different ways to write the domain and range of a function given its graph.





	Domain		Range	
	$g(x)$	$h(x)$	$g(x)$	$h(x)$
In Words	The domain is all real numbers greater than -2 and less than or equal to 5.	The domain is the set of all real numbers.	The range is all real numbers greater than or equal to -8 and less than or equal to 8.	The range is all real numbers less than or equal to 8.
Using Notation	$-2 < x \leq 5$	$-\infty < x < \infty$	$-8 \leq x \leq 8$	$y \leq 8$

2. Label each of the Graph Cards from the previous activity with the appropriate domain and range.

ACTIVITY
3.3 Linear, Constant, and Exponential Functions

Gather all of the graphs that you identified as functions.

A function is described as increasing when the value of the dependent variable increases as the value of the independent variable increases. If a function increases across the entire domain, then the function is called an **increasing function**.

A function is described as decreasing when the value of the dependent variable decreases as the value of the independent variable increases. If a function decreases across the entire domain, then the function is called a **decreasing function**.

If the value of the dependent variable of a function remains constant over the entire domain, then the function is called a **constant function**.

When determining whether a graph is increasing or decreasing, read the graph from left to right.

1. Analyze each graph from left to right. Sort all the graphs into one of the four groups listed.
   - Increasing function
   - Decreasing function
   - Constant function
   - A combination of increasing and decreasing

Record the function letter in the appropriate column of the table shown.

Increasing Function	Decreasing Function	Constant Function	Combination of Increasing and Decreasing

2. Each function shown represents one of the graphs in the increasing function, decreasing function, or constant function categories. Use graphing technology to determine the shape of its graph. Then, match the function to its corresponding graph by writing the function directly on the graph that it represents.

Think About . . .

Be sure to correctly interpret the domain of each function. Also, remember to use parentheses when entering fractions into your graphing technology.

- $f(x) = x$
- $f(x) = \left(\frac{1}{2}\right)^x$
- $f(x) = \left(\frac{1}{2}\right)^x - 5$, where x is an integer
- $f(x) = 2$, where x is an integer
- $f(x) = 2^x$, where x is an integer
- $f(x) = -x + 3$, where x is an integer

3. Consider the six graphs and functions that are increasing functions, decreasing functions, or constant functions.

   a. Sort the graphs into two groups based on the equations representing the functions and record the function letter in the table.

Group 1	Group 2

   b. What is the same about all the functions in each group?

You have just sorted the graphs into their own *function families*. A **function family** is a group of functions that share certain characteristics.

The family of **linear functions** includes functions of the form $f(x) = ax + b$, where a and b are real numbers.

The family of **exponential functions** includes functions of the form $f(x) = a \cdot b^x$, where a and b are real numbers, and b is greater than 0 but not equal to 1.

Ask Yourself . . .
What other variables have you used to represent a linear function?

4. Go back to your table in Question 3 and identify which group represents linear and constant functions and which group represents exponential functions.

5. When $f(x) = ax + b$ represents a linear function, describe the a and b values that produce a constant function.

Place these two groups of graphs off to the side. You will need them again.

ACTIVITY
3.4 Quadratic Functions

A function has an **absolute minimum** if there is a point on the graph of the function that has a *y*-coordinate that is less than the *y*-coordinate of every other point on the graph. A function has an **absolute maximum** if there is a point on the graph of the function that has a *y*-coordinate that is greater than the *y*-coordinate of every other point on the graph.

1. Sort the graphs from the combination of increasing and decreasing category in the previous activity into one of the two groups listed.
   - Those that have an absolute minimum value
   - Those that have an absolute maximum value

   Then, record the function letter in the appropriate column of the table shown.

Absolute Minimum	Absolute Maximum

2. Each function shown represents one of the graphs with an absolute maximum or an absolute minimum value. Use graphing technology to determine the shape of its graph. Then, match the function to its corresponding graph by writing the function directly on the graph that it represents. Identify the absolute maximum or minimum of each graph.

   - $f(x) = x^2 + 8x + 12$
   - $f(x) = -3x^2 + 4$, where x is integer
   - $f(x) = x^2$

The family of **quadratic functions** includes functions of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, and a is not equal to 0.

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ACTIVITY
3.5 Function Families

You have now sorted each of the graphs and equations representing functions into one of three function families: linear, exponential, and quadratic.

1. Glue your sorted graphs and functions to the appropriate function family graphic organizer located at the end of the lesson. Write a description of the graphical behavior for each function family.

Hang on to your graphic organizers. They will be a great resource moving forward!

You've done a lot of work up to this point! You've been introduced to linear, exponential, and quadratic functions. Don't worry—you don't need to know everything there is to know about these function families right now. As you progress through this course, you will learn more about each function family.

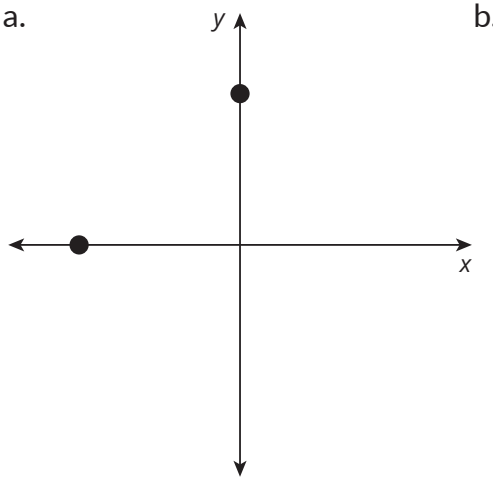
Talk the Talk

Interception!

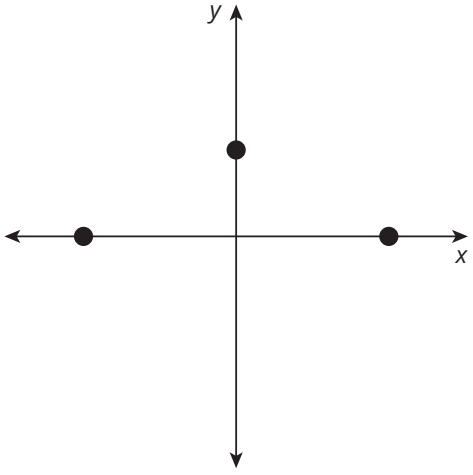
Recall that the ***x*-intercept** is the point where a graph crosses the *x*-axis. The ***y*-intercept** is the point where a graph crosses the *y*-axis.

1. The graphs shown represent relations with just the *x*- and *y*-intercepts plotted. If possible, draw a function that has the given intercepts. If it is not possible, explain why not.

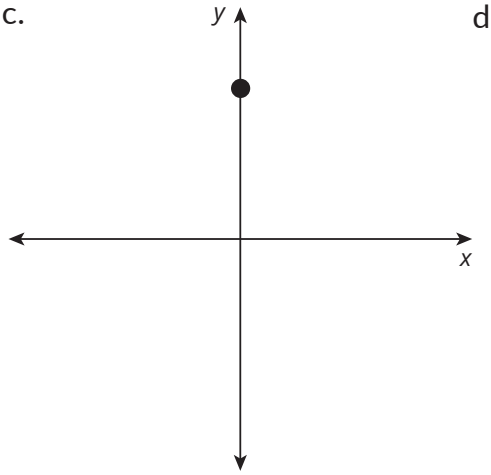
a.



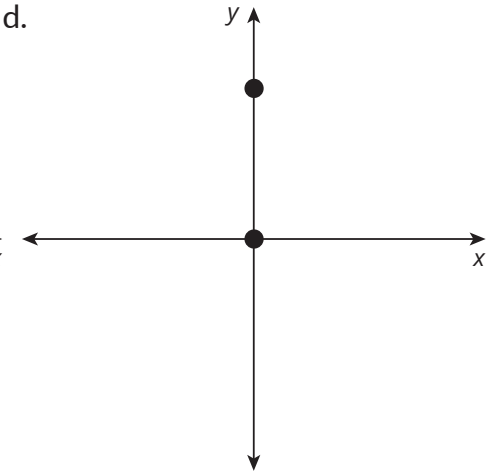
b.



c.



d.



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The family of **linear functions** includes functions of the form $f(x) = ax + b$, where a and b are real numbers.

Graphs

Linear Functions

Increasing/Decreasing/Constant:

Domain and Range:

Maximum/Minimum:

Curve/Line:

Graphical Behaviors

The family of **exponential functions** includes functions of the form $f(x) = a \cdot b^x$, where a and b are real numbers, and b is greater than 0 but not equal to 1.

Graphs

Exponential Functions

Increasing/Decreasing/Constant:

Domain and Range:

Maximum/Minimum:

Curve/Line:

Graphical Behaviors

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The family of **quadratic functions** includes functions of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, and a is not equal to 0.

Graphs

Quadratic Functions

Increasing/Decreasing/Constant:

Domain and Range:

Maximum/Minimum:

Curve/Line:

Graphical Behaviors

Lesson 3 Assignment

Write

function notation	increasing function	constant function
absolute maximum	decreasing function	absolute minimum

Choose the term that best completes each statement.

1. A way to represent equations algebraically that makes it more efficient to recognize the independent and dependent variables is called _____.
2. When both the independent and dependent variables of a function increase across the entire domain, the function is called a(n) _____.
3. A function has a(n) _____ if there is a point on its graph that has a *y*-coordinate that is greater than the *y*-coordinates of every other point on the graph.
4. When the dependent variable of a function decreases as the independent variable increases across the entire domain, the function is called a(n) _____.
5. If the dependent variable of a function does not change or remains constant over the entire domain, then the function is called a(n) _____.
6. A function has a(n) _____ if there is a point on its graph that has a *y*-coordinate that is less than the *y*-coordinate of every other point on the graph.

Remember

A function is a relation that assigns to each element of the domain exactly one element of the range. The different function families include linear functions, exponential functions, and quadratic functions.

Lesson 3 Assignment

Practice

For each scenario, use graphing technology to determine the shape of its graph. Then, identify the function family, whether it is increasing, decreasing, or a combination of both, and whether it is a smooth curve or straight line. Finally, identify whether the function has an absolute minimum or absolute maximum, if there is one.

1. A fitness company is selling DVDs for one of its new cardio routines. Each DVD will sell for \$15. Due to fixed and variable costs, the profit that the company will see after selling x DVDs can be represented by the function $P(x) = 11.5x - 0.1x^2 - 150$.

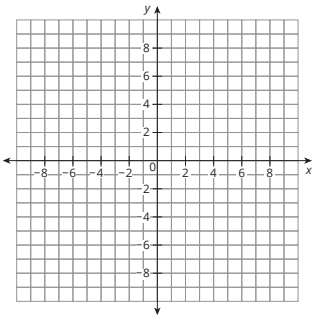
2. Mariana is going to put \$500 into an account with her bank. The bank is offering a 3% interest rate compounded annually. The amount of money that Mariana will have after x years can be represented by the function $A(x) = 500(1.03)^x$.

3. A calendar company is going to buy a new 3D printer for \$20,000. In order to plan for the future, the owners are interested in the salvage value of the printer each year. The salvage value after x years can be represented by the function $S(x) = 20{,}000 - 2000x$.

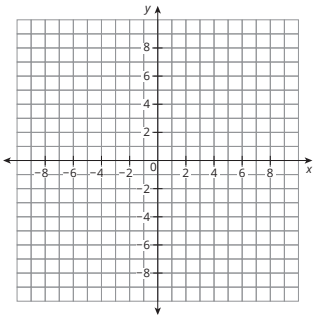
Prepare

1. Sketch a graph and write an equation for each function.

   a. Decreasing linear function



   b. Increasing exponential function



4 Recognizing Functions by Characteristics

OBJECTIVES

- Recognize similar characteristics among function families.
- Recognize different characteristics among function families.
- Determine function types given certain characteristics.

You have identified key characteristics of graphs.

How can the key characteristics help you sketch the graph of a function?

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Getting Started

Name That Function!

Function Families
linear
exponential
quadratic

You have sorted graphs according to their function family. Now, consider which function families have the given characteristics.

1. Which function families can be described by the characteristic provided? Choose from the given list.

   a. The graph is a smooth curve.

   b. The graph is made up of a straight line.

   c. The graph increases or decreases over the entire domain.

   d. The graph has an absolute maximum or minimum.

2. One or more characteristics have been added to the graphical description of each function. Name the possible function families.

   a. The graph has an absolute minimum or absolute maximum and is a smooth curve.

   b. The graph either increases or decreases over the entire domain and is a straight line.

   c. The graph is a smooth curve, and either increases or decreases over the entire domain.

Each function family has certain graphical behaviors, with some behaviors common among different function families. Notice, the more specific characteristics that are given, the more specifically you can name that function!

ACTIVITY
4.1 Categorizing Scenarios into Their Function Families

You have been introduced to several function families: linear, exponential, and quadratic. Let's revisit the first lesson: *Understanding Quantities and Their Relationships*. Each of the scenarios in that lesson represents one of these function families.

1. Describe how each scenario represents a function.

2. Complete the table on the following pages to describe each scenario.

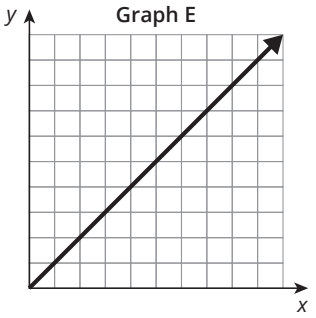
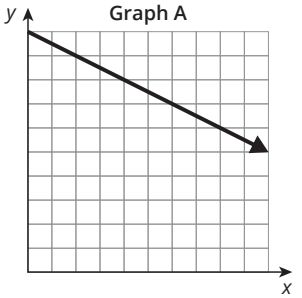
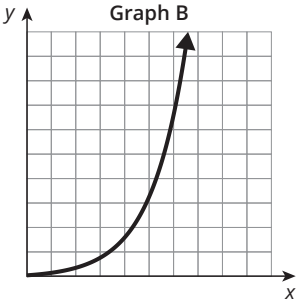
   a. Identify the appropriate function family under the scenario name.

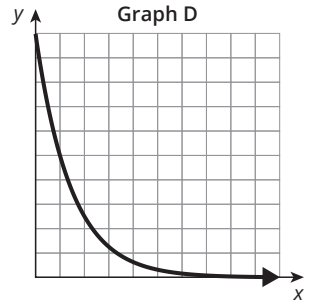
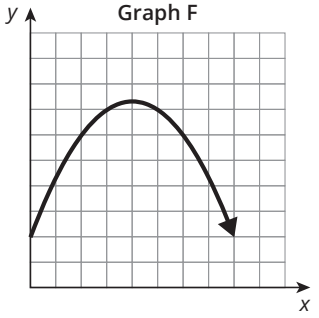
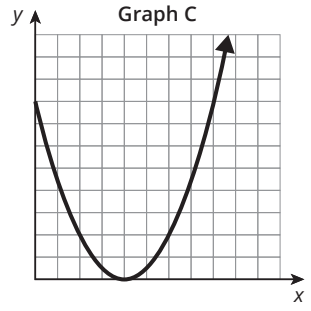
   b. Based on the context, identify the domain as continuous or discrete.

   c. Describe the graphical behavior as increasing, decreasing, constant, or a combination.

Remember . . .
Each of the graphs representing the scenarios was drawn with either a continuous line or a continuous smooth curve to model the problem situation.

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Scenario	Domain of the Real-World Situation	Graph of the Mathematical Model	Graphical Behavior
Music Club			
Something's Fishy			
Smart Phone, but Is It a Smart Deal?			

Scenario	Domain of the Real-World Situation	Graph of the Mathematical Model	Graphical Behavior
It's Magic			
Baton Twirling			
Skateboarding			

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PROBLEM SOLVING



ACTIVITY
4.2 Building Graphs from Characteristics

In this activity, you will write equations and sketch a graph based on given characteristics for the three main function families we will study this year in Algebra I: Linear Functions, Exponential Functions, and Quadratic Functions.

Use the problem-solving model whenever you see this icon.

1. Use the given characteristics to create an equation and sketch a graph. Use the equations given in the box as a guide. When creating your equation, use a, b, and c values that are any real numbers between -3 and 3. Do not use any functions that were used previously in this topic.

Linear Function

$f(x) = ax + b$

Exponential Function

$f(x) = a \cdot b^x$

Quadratic Function

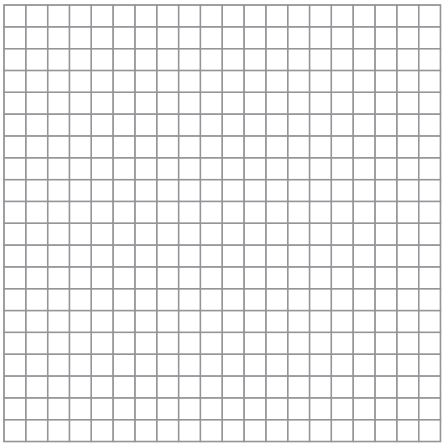
$f(x) = ax^2 + bx + c$

   a. Create an equation and sketch a graph that is:
      - a function,
      - exponential,
      - continuous, and
      - decreasing.

Think About . . .

Don't forget about the function family graphic organizers you created if you need some help.

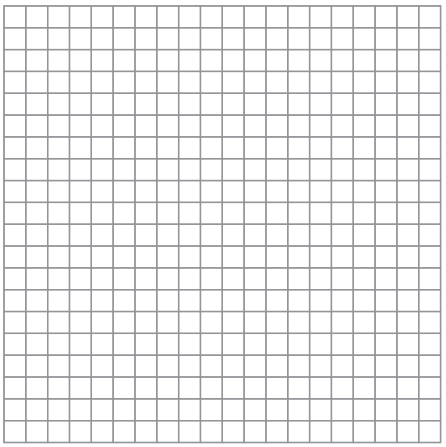
Equation: _____



b. Create an equation and sketch a graph that:

- has a minimum,
- is discrete, and
- is a quadratic function.

Equation: _____

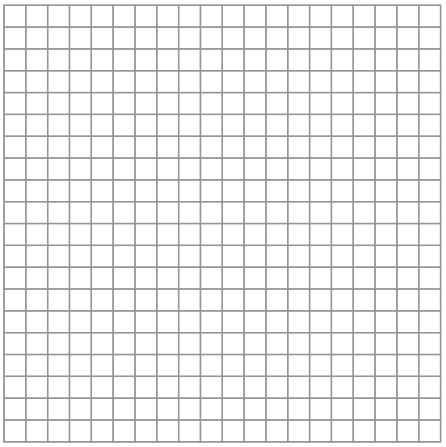


Ask Yourself . . .
Is the domain the same or different for each function?

c. Create an equation and sketch a graph that is:

- linear,
- discrete,
- increasing, and
- a function.

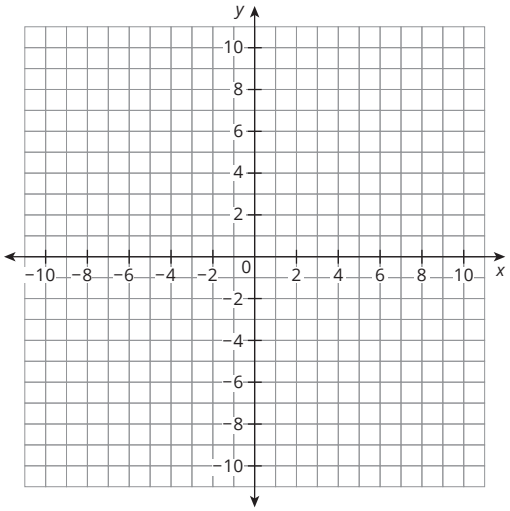
Equation: _____



d. Create an equation and sketch a graph that:

- is continuous,
- has a maximum,
- is a function, and
- is quadratic.

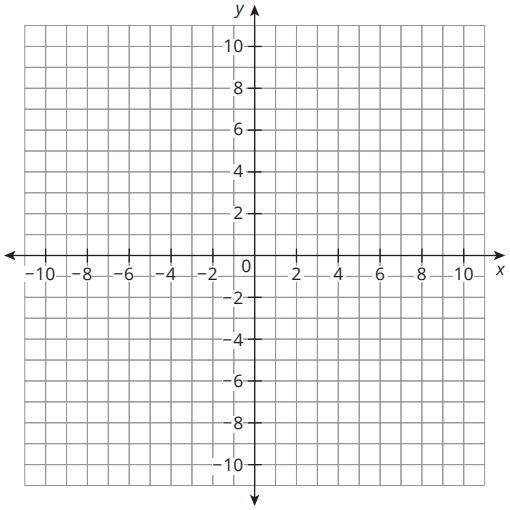
Equation: _____



e. Create an equation and sketch a graph that is:

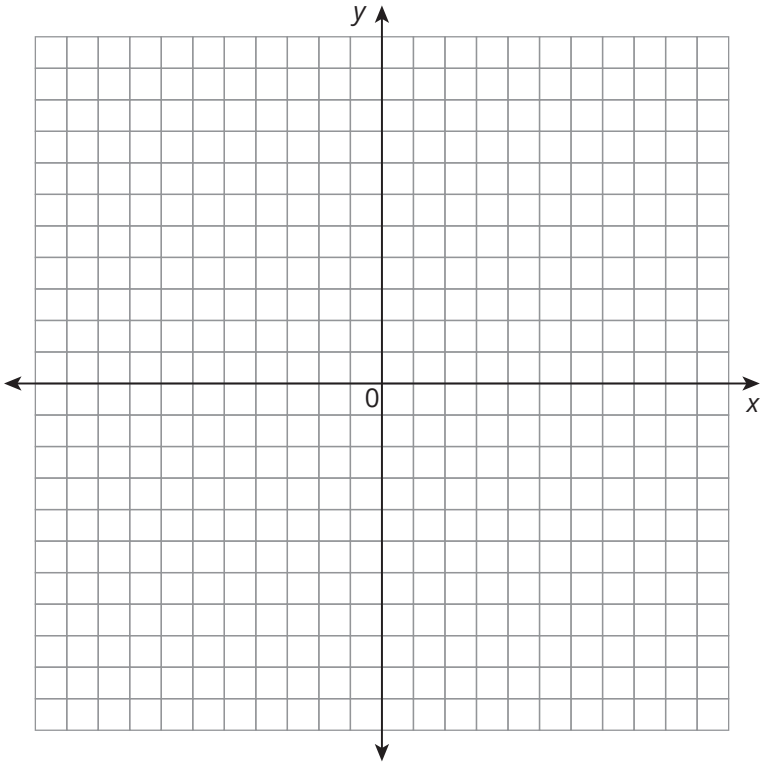
- not a function,
- continuous, and
- a straight line.

Equation: _____

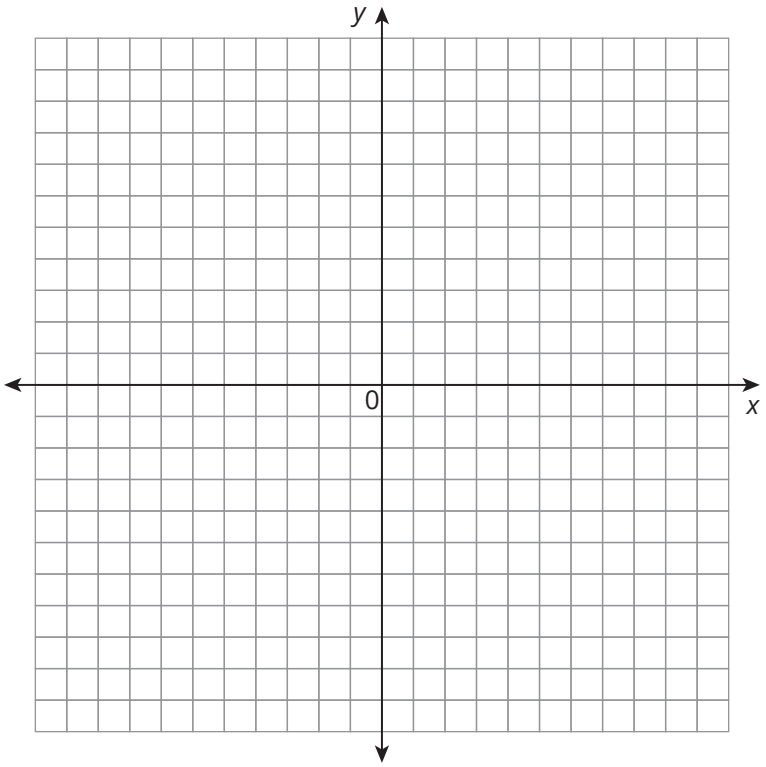


2. Create your own function. Describe certain characteristics of the function and see if your partner can sketch it. Then, sketch your partner's function based on characteristics provided.

Your Function:



Your Partner's Function:



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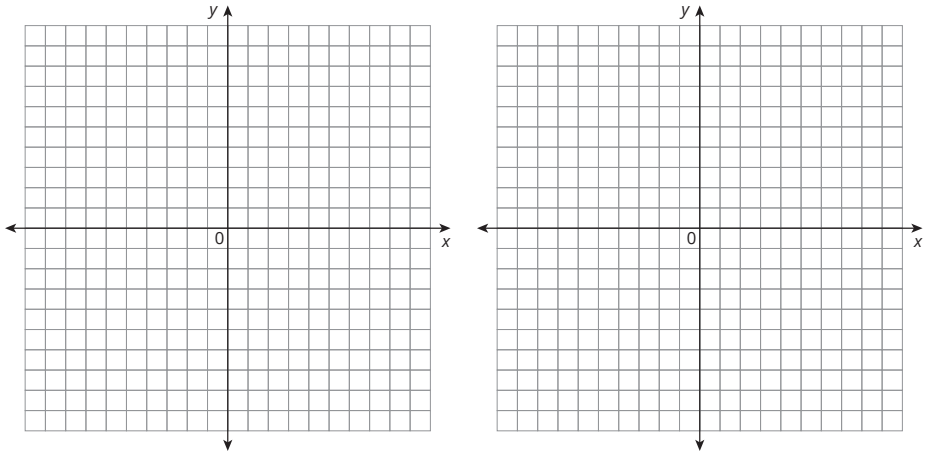
Talk the Talk

Trying to Be Unique

Throughout this lesson, you used characteristics to describe graphs.

1. Write a list of four characteristics to describe a graph.
   - _____
   - _____
   - _____
   - _____

2. Sketch two possible graphs based on your characteristics.



3. How could you modify your list of characteristics to describe a unique graph?

Lesson 4 Assignment

Write

Identify the function family or families that are described by the given characteristic(s). Choose from linear, exponential, and quadratic functions.

1. The graph of this function family has an absolute minimum.
2. The graph of this function family is decreasing over the entire domain.
3. The graph of this function family has an increasing interval and a decreasing interval and forms a U shape.
4. The graph of this function family does not have an absolute maximum or absolute minimum and is a smooth curve.
5. The graph of this function family contains straight lines and does not have an absolute maximum or absolute minimum.

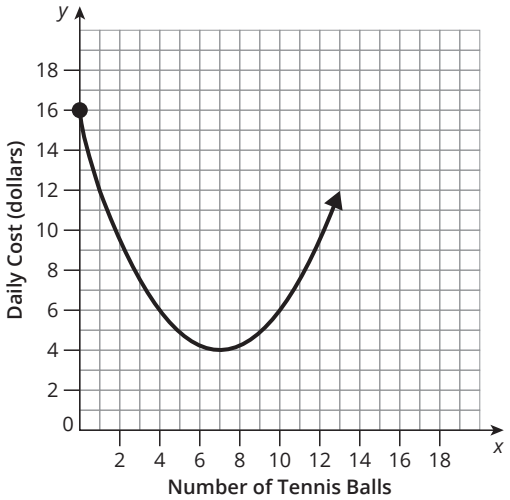
Remember

Function families have key characteristics that are common among all functions in the family. Knowing these key characteristics is useful when sketching a graph of the function.

Practice

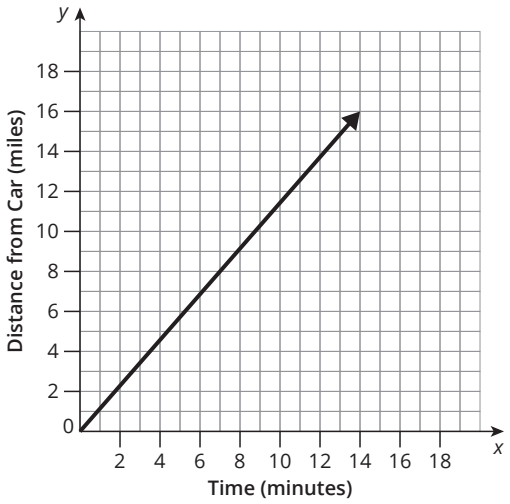
For each scenario and its graph, identify the appropriate function family. Then, based on the problem situation, identify whether the data values represented in the graph are discrete or continuous. Finally, identify the graphical behavior of the function that models the scenario based on the characteristics of its function family.

1. A manufacturing company finds that the daily costs associated with making tennis balls is high if they don't make enough balls and then becomes high again if they make too many balls. The function graphed models the daily costs of making x tennis balls.

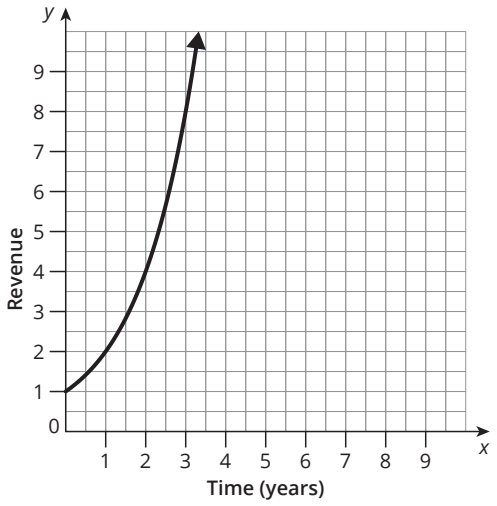


Lesson 4 Assignment

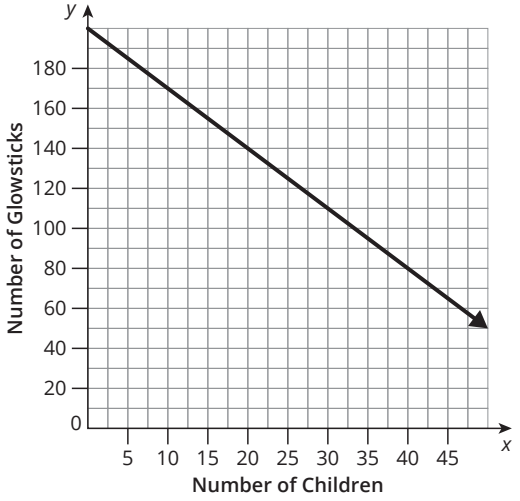
2. Jamal is training for a mountain bike race. He bikes 14 miles. If he bikes at a constant rate, the function graphed models the distance he bikes after x minutes.



3. A local television company determines that the revenue it gets from running ads doubles each year. The function graphed models the revenue from advertising after x years.



4. Mr. Patel's neighborhood is hosting a summer nighttime party in the park. They are handing out glow sticks to all the children who attend. They start with 200 glow sticks and each child receives 3 glow sticks. The function graphed models the number of glow sticks they have left after x children have entered.



Lesson 4 Assignment

Prepare

Write the next three terms in each pattern and explain how you generated each term.

1. J, F, M, A, M, J, J, A, S, ...

2. S, M, T, W, ...

3. 5, 10, 15, 20, ...

4. 100, 81, 64, 49, ...

TOPIC 1 SELF-REFLECTION

Name: _____

Quantities and Relationships

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Quantities and Relationships* topic by:

TOPIC 1: *Quantities and Relationships*	Beginning of Topic	Middle of Topic	End of Topic
choosing appropriate scale and origin for graphs.	○	○	○
identifying the appropriate unit of measure for each variable or quantity.	○	○	○
analyzing a graph and stating the key characteristics of the graph.	○	○	○
using a problem situation to explain what the key features of a graph mean in real-world context.	○	○	○
deciding whether relations represented verbally, tabularly, graphically, and symbolically define a function.	○	○	○
recognizing a linear, exponential, or quadratic function by its equation or graph.	○	○	○
evaluating functions, expressed in function notation, given one or more elements in their domain.	○	○	○
determining the domain and range and the independent and dependent quantities in a relationship.	○	○	○

continued on the next page

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TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Quantities and Relationships* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

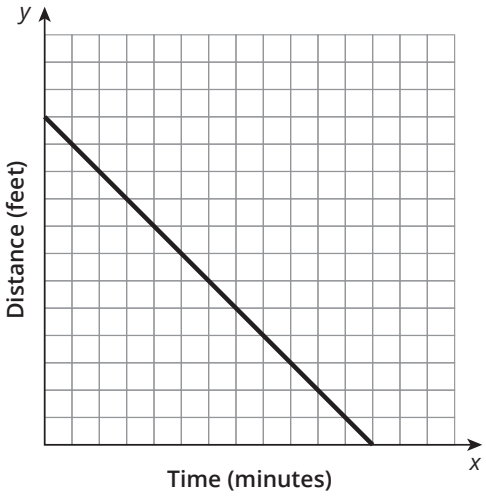


Quantities and Relationships Summary

LESSON 1 Understanding Quantities and Their Relationships

Many problem situations include two quantities that change. When one quantity depends on another, it is said to be the **dependent quantity**. The quantity that the dependent quantity depends upon is called the **independent quantity**.

Graphs relay information about data in a visual way. Connecting points on a coordinate plane with a line or smooth curve is a way to model or represent relationships. The independent quantity is graphed on the horizontal, or *x*-axis, while the dependent quantity is graphed on the vertical, or *y*-axis. Graphs can be straight lines or curves and can increase or decrease from left to right.



For example, consider the graph that models the situation where Pedro is walking home from school.

Time, in minutes, is the independent quantity and the distance Pedro is from home is the dependent quantity.

LESSON 2 Analyzing and Sorting Graphs

Looking for patterns can help when sorting and comparing graphs. Some graphs show vertical symmetry (when a vertical line is drawn through the middle of the graph, the image is the same on both sides). Other possible patterns to look for include: only goes through two quadrants, always increases from left to right, always decreases from left to right, has straight

NEW KEY TERMS

- dependent quantity [cantidad dependiente]
- independent quantity [cantidad independiente]
- relation [relación]
- domain [dominio]
- range [rango]
- function [función]
- function notation [notación de función]
- Vertical Line Test [Prueba de la línea vertical]
- discrete graph [gráfica discreta/ discontinua]
- continuous graph [gráfica continua]
- increasing function [función creciente]
- decreasing function [función decreciente]
- constant function [función constante]
- function family [familia de funciones]
- linear functions [funciones lineales]
- exponential functions [funciones exponenciales]
- absolute maximum [máximo absoluto]
- absolute minimum [mínimo absoluto]

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NEW KEY TERMS

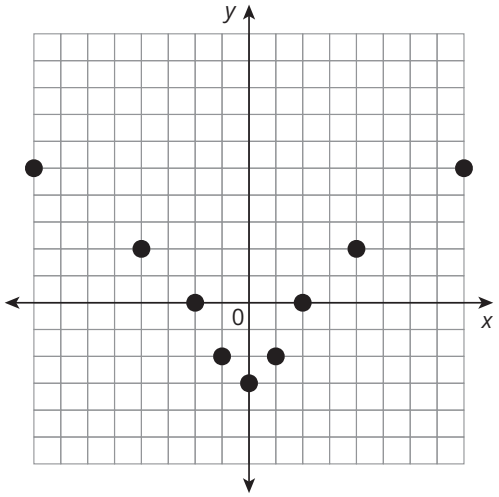
continued

- quadratic functions [funciones cuadráticas]
- *x*-intercept [intersección con el eje *x*]
- *y*-intercept [intersección con el eje *y*]

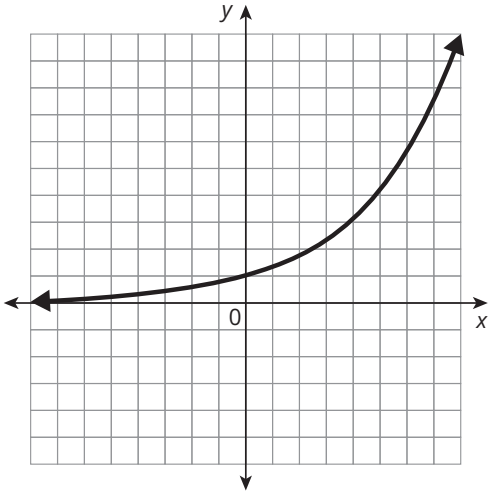
lines, has smooth curves, the graph goes through the origin, the graph forms a U shape, or the graph forms a V shape.

For example, Graph A has vertical symmetry. Graph B is a smooth curve that increases from left to right.

Graph A



Graph B



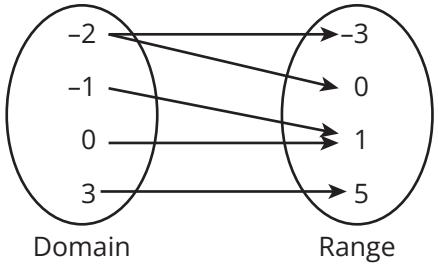
LESSON 3

Recognizing Functions and Function Families

A **relation** is the mapping between a set of input values called the **domain** and a set of output values called the **range**.

A **function** is a relation between a given set of elements such that for each element in the domain, there exists exactly one element in the range. If each value in the domain has one and only one range value, then the relation is a function. If any value in the domain has more than one range value, then the relation is not a function.

The value −2 in the domain has more than one range value. The mapping does not represent a function.



Each element in the domain has exactly one element in the range. The table represents a function.

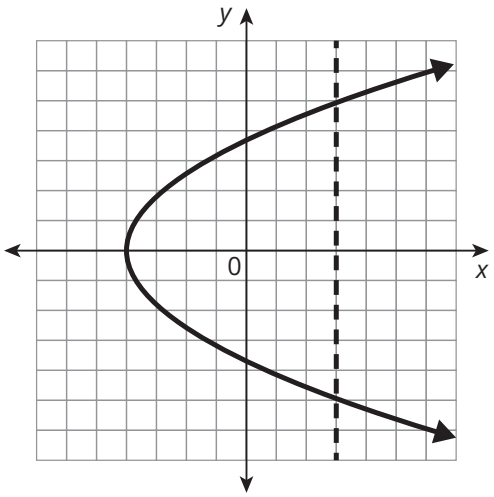
Domain	Range
2	1
6	3
10	5
14	7

Functions can be represented in a number of ways. An equation representing a function can be written using **function notation**. Function notation is a way of representing functions algebraically. This form allows you to more efficiently identify the independent and dependent quantities. The function $f(x)$ is read as "f of x" and indicates that x is the independent variable.

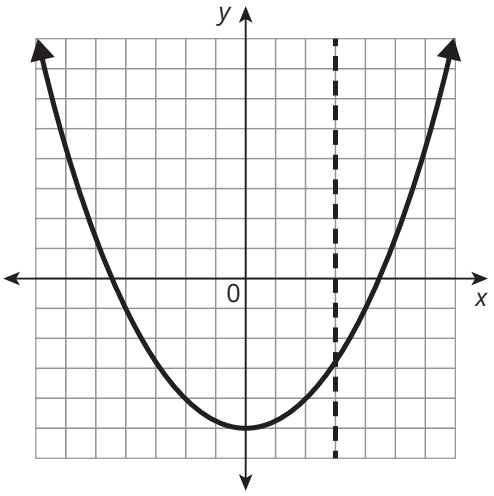
For example, consider the situation in which U.S. Shirts charges $8 per shirt plus a one-time charge of $15 to set up a T-shirt design. The equation that models the situation, $y = 8x + 15$, where x represents the number of shirts ordered and y represents the total cost of the order, can be written in function notation as $f(x) = 8x + 15$. The cost, defined by f, is a function of x, defined as the number of shirts ordered.

The **Vertical Line Test** is a visual method used to determine whether a relation represented as a graph is a function. To apply the Vertical Line Test, consider all of the vertical lines that could be drawn on the graph of a relation. If any of the vertical lines intersect the graph of the relation at more than one point, then the relation is not a function. The Vertical Line Test applies to both discrete and continuous graphs. A **discrete graph** is a graph of isolated points. A **continuous graph** is a graph of points that are connected by a line or smooth curve with no breaks in the graph.

A line drawn vertically through the graph touches more than one point. The graph does not represent a function.



A line drawn vertically through the graph only touches one point. The graph represents a function.



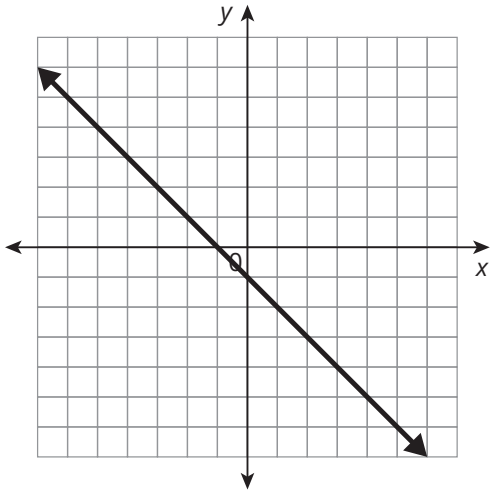
A function is described as increasing when both the independent and dependent variables are increasing. If a function increases across the entire domain, then the function is called an **increasing function**. A function is described as decreasing when the dependent variable decreases as the independent variable increases. If a function decreases across the entire domain, then the function is called a **decreasing function**. If the dependent variable of a function does not change or remains constant over the entire domain, then the function is called a **constant function**.

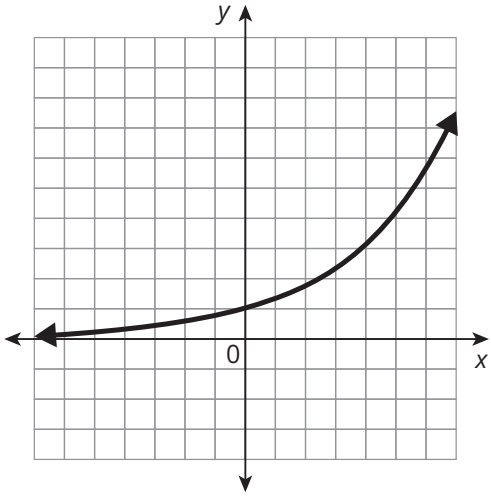
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A **function family** is a group of functions that share certain characteristics.

The family of **linear functions** includes functions of the form $f(x) = ax + b$, where a and b are real numbers.

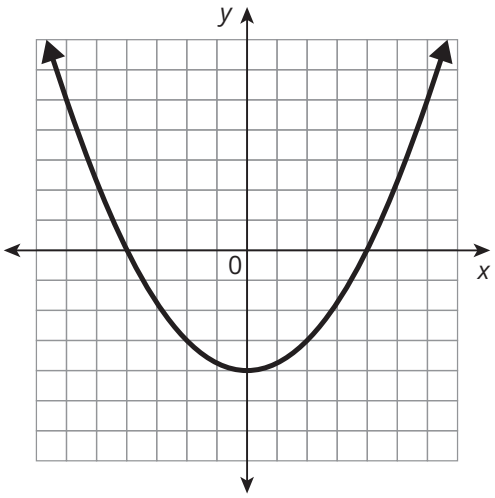
The family of **exponential functions** includes functions of the form $f(x) = a \cdot b^x$, where a and b are real numbers, and b is greater than 0 but not equal to 1.





Quadratic functions have an absolute maximum or an absolute minimum. An **absolute maximum** is a point on the graph of the function that has a y-coordinate that is greater than the y-coordinate of every other point on the graph. An **absolute minimum** is a point on the graph of the function that has a y-coordinate that is less than the y-coordinate of every other point on the graph.

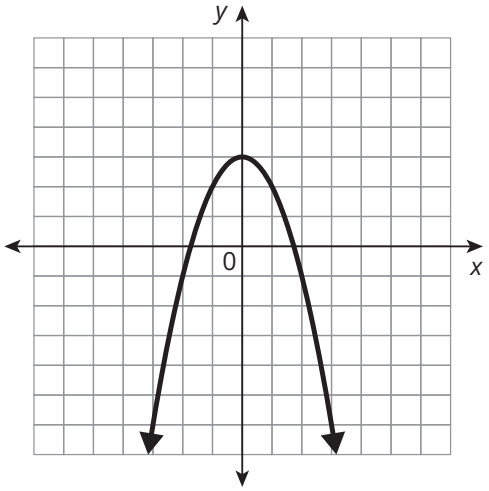
The family of **quadratic functions** includes functions of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, and a is not equal to 0.



LESSON 4

Recognizing Functions by Characteristics

Function families have key characteristics that are common among all functions in the family. Knowing these key characteristics is useful when sketching a graph of the function. For example, to sketch a graph of a continuous quadratic function with a maximum, the graph should be a smooth, U-shaped curve that increases to point and then decreases again.



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TOPIC 2



Everyday sequences might include sizes or ages. What other sequences have you noticed today?

Sequences

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LESSON 2 Arithmetic and Geometric Sequences 95

LESSON 3 Determining Recursive and Explicit Expressions from Contexts 119

1 Recognizing Patterns and Sequences

OBJECTIVES

- Recognize and describe patterns.
- Represent patterns as sequences.
- Predict the next term in a sequence.
- Represent a sequence as a table of values.

NEW KEY TERMS

- sequence
- term of a sequence
- infinite sequence
- finite sequence

Since early elementary school, you have been recognizing and writing patterns involving shapes, colors, letters, and numbers.

How are patterns related to sequences and how can sequences be represented using a table of values?

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Getting Started

A Pyramid of Patterns

Pascal's Triangle is a famous pattern named after the French mathematician and philosopher Blaise Pascal. A portion of the pattern is shown.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1. List at least two patterns that you notice.

2. Describe the pattern for the number of terms in each row.

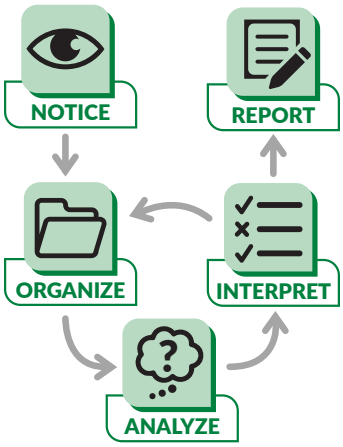
3. Describe the pattern within each row.

4. Describe the pattern that results from determining the sum of each row.

5. Determine the next two rows in Pascal's Triangle. Explain your reasoning.

ACTIVITY
1.1 Patterns to Sequences to Tables

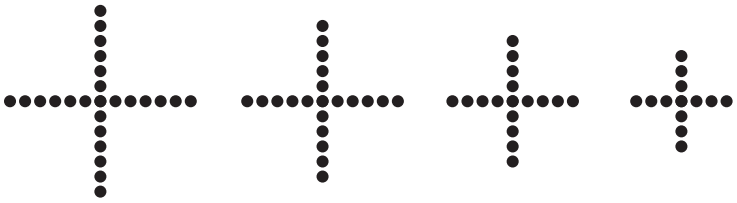
PROBLEM SOLVING



A **sequence** is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects. A **term of a sequence** is an individual number, figure, or letter in the sequence.

Four examples of sequences are given in this activity. For each sequence, describe the pattern, draw or describe the next terms, and represent each sequence numerically.

1. **Positive Thinking**



a. Analyze the number of dots. Describe the pattern.

b. Draw the next three figures of the pattern.

c. Represent the number of dots in each of the seven figures as a numeric sequence.

d. Represent the number of dots in each of the first seven figures as a function using a table of values.

Term Number	1	2	3	4	5	6	7
Term Value							

All numeric sequences can be represented as functions. The independent variable is the term number beginning with 1, and the dependent variable is the term value of the sequence.

Ask Yourself . . .
What tools or strategies can you use to solve this problem?

2. **Family Tree**
Trung is investigating her family tree by researching each generation, or set, of parents. She learns all she can about the first four generations, which include her two parents, her grandparents, her great-grandparents, and her great-great-grandparents.

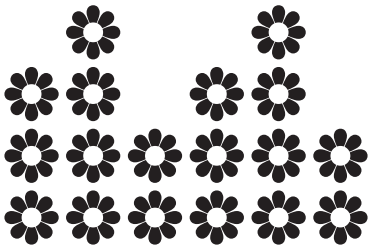
a. Think about the number of parents. Describe the pattern.

b. Determine the number of parents in the fifth and sixth generations.

c. Represent the number of parents in each of the 6 generations as a numeric sequence. Then, represent the sequence using a table of values.

Term Number	1	2	3	4	5	6
Term Value						

3. **Paola's Pattern**
Paola is decorating the top border of her bedroom walls with a flower pattern. She is applying decals with each column having a specific number of flowers.



a. Think about the number of flowers in each column. Describe the pattern.

b. Determine the number of flowers in each of the next two columns.

c. Represent the number of flowers in each of the first 8 columns as a numeric sequence. Then, represent the sequence using a table of values.

Term Number	1	2	3	4	5	6	7	8
Term Value								

4. **Gamer Guru**
Eduardo is trying to beat his high score on his favorite video game. He unlocks some special mini-games where he earns points for each one he completes. Before he begins playing the mini-games, Eduardo has 500 points. After completing 1 mini-game, he has a total of 550 points; after completing 2 mini-games, he has a total of 600 points; and after completing 3 mini-games, he has a total of 650 points.

a. Think about the total number of points Eduardo gains from mini-games. Describe the pattern.

Ask Yourself . . .
How else can you represent this information?

b. Determine Eduardo's total points after he plays the next two mini-games.

c. Represent Eduardo's total points after completing each of the first 5 mini-games as a numeric sequence. Be sure to include the number of points he started with. Then, represent the sequence using a table of values.

Term Number	1	2	3	4	5	6
Term Value						

ACTIVITY
1.2 Looking at Sequences More Closely

There are many different patterns that can generate a sequence of numbers. For example, you may have noticed that some of the sequences in the previous activity were generated by performing the same operation using a constant number. In other sequences, you may have noticed a different pattern.

The next term in a sequence is calculated by determining the pattern of the sequence and then using that pattern on the last known term of the sequence.

1. For each sequence in the previous activity, write the numeric sequence, record whether the sequence increases or decreases, and describe the sequence by stating the first term and the operation(s) used to create the sequence. The first one has been completed for you.

Problem Name	Numeric Sequence	Increases or Decreases	Sequence Description
Positive Thinking	25, 21, 17, 13, 9, 5, 1	Decreases	Begin at 25. Subtract 4 from each term.
Family Tree			
Paola's Pattern			
Gamer Guru			

2. Which sequences are similar? Explain your reasoning.

3. What do all sequences have in common?



4. Consider a sequence in which the first term is 64 and each term after that is calculated by dividing the previous term by 4. Elena says that this sequence ends at 1 because there are no whole numbers that come after 1. Chris disagrees and says that the sequence continues beyond 1. Who is correct? If Elena is correct, explain why. If Chris is correct, predict the next two terms of the sequence.

5. What is the domain of a sequence? What is the range?

When a sequence continues on forever, it is called an **infinite sequence**. If a sequence terminates, it is called a **finite sequence**.

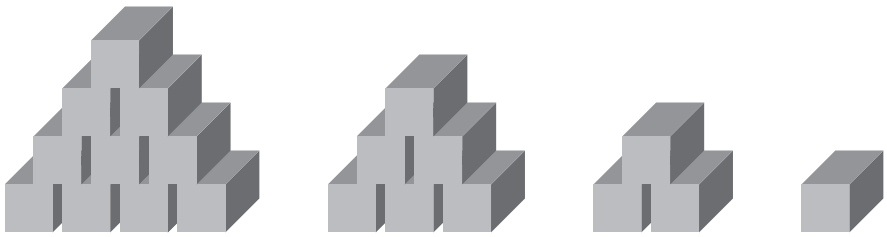
An ellipsis is three periods, which means "and so on." An infinite sequence can be represented using an ellipsis.

For example, consider an auditorium where the seats are arranged according to a specific pattern. There are 22 seats in the first row, 26 seats in the second row, 30 seats in the third row, and so on. Numerically, the sequence is 22, 26, 30, ... , which continues infinitely. However, in the context of the problem, it does not make sense for the number of seats in each row to increase infinitely. Eventually, the auditorium would run out of space! Suppose that this auditorium can hold a total of 10 rows of seats. The correct sequence for this problem situation is:

22, 26, 30, 34, 38, 42, 46, 50, 54, 58.

Therefore, because of the problem situation, the sequence is a finite sequence.

6. Does the pattern shown represent an infinite or finite sequence? Explain your reasoning.



7. For each sequence in the previous activity, write the domain and range using set notation.

   a. Positive Thinking

   b. Family Tree (6 generations):

   c. Paola's Pattern (8 columns)

   d. Gamer Guru (completes 5 mini games)

8. For each sequence in the previous activity, determine whether the sequence is a function.

   a. Positive Thinking

   b. Family Tree

   c. Paola's Pattern

   d. Gamer Guru

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Talk the Talk

Searching for a Sequence

In this lesson, you have seen that many different patterns can generate a sequence of numbers.

1. Explain why the definition of a function applies to all sequences.

2. Create a sequence to fit the given criteria. Describe your sequence using figures, words, or numbers. Provide the first four terms of the sequence. Explain how you know that it is a sequence.

   a. Create a sequence that begins with a positive integer, is decreasing by multiplication, and is finite.

   b. Create a sequence that begins with a negative rational number, is increasing by addition, and is infinite.

Lesson 1 Assignment

Write

Explain why all sequences can be described as functions.

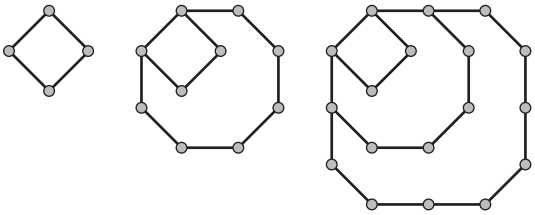
Remember

A sequence is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects. All numeric sequences can be represented as functions.

Practice

Consider the three sequences given. For each sequence, describe the pattern. Then, represent the sequence as a numeric sequence and as a table of values, including the first six terms.

1. **Stellar Segments**



Term Number						
Term Value						

2. **Welcome Home**

A local construction company specializes in building and selling homes. When the company first started, they sold 1 home the first month, 3 homes the second month, 9 homes the third month, and 27 homes the fourth month.

Term Number						
Term Value						

Lesson 1 Assignment

3. **Samantha's Streaming**
   Samantha is a yoga instructor who regularly streams exercise videos on a website for her clients. One week after launching the website, she had posted a total of 6 videos. At the end of week 2, she had a total of 10 videos. At the end of week 3, she had a total of 14 videos. At the end of week 4, she had a total of 18 videos.

Term Number						
Term Value						

Prepare

Write the next three terms in each sequence. Explain how you determined each term and whether it is a function.

1. $-2, 4, -8, 16, \ldots$

2. $60, 53, 46, 39, 32, \ldots$

3. $1, 5, 17, 53, 161, 485, \ldots$

4. $4, 10, 16, 22, \ldots$

2 Arithmetic and Geometric Sequences

OBJECTIVES

- Determine the next term in a sequence.
- Recognize arithmetic sequences and geometric sequences.
- Determine the common difference or common ratio for a sequence.
- Graph arithmetic and geometric sequences.
- Recognize graphical behavior of sequences.
- Sort sequences that are represented graphically.
- Generate terms of a sequence when the sequence is given in function form using the recursive process.

NEW KEY TERMS

- arithmetic sequence
- common difference
- geometric sequence
- common ratio

You have represented patterns as sequences of numbers—a relationship between term numbers and term values.

What patterns appear when sequences are represented as graphs?

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Getting Started

What Comes Next, and How Do You Know?

Review the definition of **describe** in the Academic Glossary.

Cut out Sequences *A* through *H* located at the end of the lesson.

1. Determine the unknown terms of each sequence. Then, describe the pattern under each sequence.

2. Sort the sequences into groups based on common characteristics. In the space provided, record the following information for each of your groups.
   - List the letters of the sequences in each group.
   - Provide a rationale as to why you created each group.

3. What mathematical operation(s) did you perform in order to determine the next terms of each sequence?

ACTIVITY 2.1

Defining Arithmetic and Geometric Sequences

For some sequences, you can describe the pattern as adding a constant to each term to determine the next term. For other sequences, you can describe the pattern as multiplying each term by a constant to determine the next term. Still other sequences cannot be described either way.

An **arithmetic sequence** is a sequence of numbers in which the difference between any two consecutive terms is a constant. In other words, it is a sequence of numbers in which a constant is added to each term to produce the next term. This constant is called the **common difference**. The common difference is typically represented by the variable d.

The common difference of a sequence is positive when the same positive number is added to each term to produce the next term. The common difference of a sequence is negative when the same negative number is added to each term to produce the next term.

Remember:
When you add a negative number, it is the same as subtracting a positive number.

WORKED EXAMPLE

Consider the sequence generated using $a_n = a_{n-1} + (-2)$, where $a_1 = 11$ and n is a whole number greater than 1.

a_1 represents the first term of the sequence and a_n represents the nth term of the sequence.

Since I know the first term of the sequence, to determine the second term a_2, I add -2 to 11.

$a_2 = a_{2-1} + -2 = a_1 + -2 = 11 + -2 = 9$

I can determine the 3rd and 4th term of the sequence by continuing the pattern.

$a_3 = a_{3-1} + -2 = a_2 + -2 = 9 + -2 = 7$

$a_4 = a_{4-1} + -2 = a_3 + -2 = 7 + -2 = 5$

The sequence is 11, 9, 7, 5, ...

The pattern is to add the same negative number, -2, to each term to determine the next term.

Sequence: 11 (add -2) 9 (add -2) 7 (add -2) 5, ...

This sequence is arithmetic and the common difference d is -2.

a_{n-1} is the term before a_n. So, when $n = 2$, a_{n-1} is a_1.

Ask Yourself . . .
How does the function notation relate to the notation in the Worked Example?

You can also write the sequence from the Worked Example in function form. Given that $f(1) = 11$ and n is a whole number greater than 1, the function $f(n) = f(n-1) + -2$ represents the sequence.

1. Suppose a sequence has the same starting number as the sequence in the Worked Example, but is generated by $a_n = a_{n-1} + 4$.

   a. Write the first four terms of the sequence.

   b. How does the pattern change?

   c. Is the sequence still arithmetic? Why or why not?

2. Analyze the sequences you cut out in the Getting Started.

   a. List the sequences that are arithmetic.

   b. Write the common difference of each arithmetic sequence you identified.

A **geometric sequence** is a sequence of numbers in which the ratio between any two consecutive terms is a constant. In other words, it is a sequence of numbers in which you multiply each term by a constant to determine the next term. This integer or fraction constant is called the *common ratio*. The **common ratio** is represented by the variable *r*.

WORKED EXAMPLE

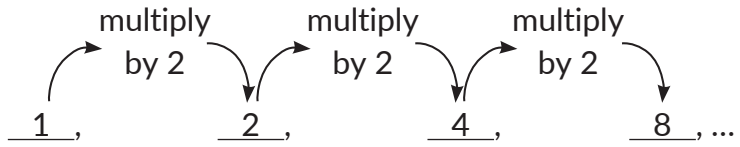
Consider the sequence generated by the function $g_n = g_{n-1} \cdot 2$ where $g_1 = 1$ and n is a whole number greater than 1.

Since I know the first term of the sequence, to determine the second term g_2, I multiply 1 by 2.

$g_2 = g_1 \cdot 2 = 1 \cdot 2 = 1$
$g_3 = g_2 \cdot 2 = 2 \cdot 2 = 4$
$g_4 = g_3 \cdot 2 = 4 \cdot 2 = 8$

The sequence is 1, 2, 4, 8, ...

The pattern is to multiply each term by the same number, 2, to determine the next term.



This sequence is geometric and the common ratio *r* is 2.

You can also write the sequence from the Worked Example in function form. Given that $f(1) = 1$ and n is a whole number greater than 1, the function $f(n) = 2f(n-1)$ describes the sequence.

Ask Yourself . . .
How does the function notation relate to the notation in the Worked Example?

3. Suppose a sequence has the same starting number as the sequence in the Worked Example, but is generated by $a_n = a_{n-1} \cdot 3$.

   a. Write the first four terms of the sequence.

   b. How does the pattern change?

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c. Is the sequence still geometric? Explain your reasoning.

4. Suppose a sequence has the same starting number as the sequence in the Worked Example, but its common ratio is $\frac{1}{3}$.

   a. How would the pattern change?

   b. Is the sequence still geometric? Why or why not?

   c. Write the first four terms of the new sequence.

5. Suppose a sequence has the same starting number as the sequence in the Worked Example, but it is generated by $g_n = 1 \cdot (-2)^{n-1}$.

   a. How would the pattern change?

   b. Is the sequence still geometric? Explain your reasoning.

   c. Write the first four terms of the new sequence.

6. Consider the sequence shown.

270, 90, 30, 10, ...

Jorge says that he can determine each term of this sequence by multiplying each term by $\frac{1}{3}$, so the common ratio is $\frac{1}{3}$. Jaylen says that he can determine each term of this sequence by dividing each term by 3, so the common ratio is 3. Who is correct? Explain your reasoning.

7. Consider the sequences you cut out in the Getting Started. List the sequences that are geometric. Then, write the common ratio on each Sequence Card.

8. Consider the sequences that are neither arithmetic nor geometric. List these sequences. Explain why these sequences are neither arithmetic nor geometric.



9. Consider the first two terms of the sequence 3, 6, ...

   Mariana says, "This is how I wrote the sequence for the given terms."

   3, 6, 9, 12, ...

   Ashley says, "This is the sequence I wrote."

   3, 6, 12, 24, ...

   Who is correct? Explain your reasoning.

10. Consider the sequence 2, 2, 2, 2, 2 ... Identify the type of sequence it is and describe the pattern.

11. Begin to complete the graphic organizers located at the end of the lesson to identify arithmetic and geometric sequences. Glue each arithmetic sequence and each geometric sequence to a separate graphic organizer according to its type. Discard all other sequences.

ACTIVITY
2.2 Matching Graphs and Sequences

As you have already discovered when studying functions, graphs can help you see trends of a sequence—and at times can help you predict the next term in a sequence.

1. The graphs representing the arithmetic and geometric sequences from the previous activity are located at the end of this lesson. Cut out these graphs. Match each graph to its appropriate sequence and glue it into the Graph section of its graphic organizer.

2. What strategies did you use to match the graphs to their corresponding sequences?

3. How can you use the graphs to verify that all sequences are functions?

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Talk the Talk

Name That Sequence!

Write the first five terms of each sequence described and identify the sequence as arithmetic or geometric.

1. The first term of the sequence is 0 and the common difference is -6.

2. The first term of the sequence is -3 and the common ratio is $-\frac{1}{4}$.

Sequence Cards

A $-2, -6, -18, -54,$ _____, _____, …	**B** $4, \frac{7}{4}, -\frac{1}{2}, -\frac{11}{4},$ _____, _____, …
C 1, –2, 3, –4, _____, _____, …	**D** –20, –16, –12, –8, –4, _____, _____, …
E $-5, -\frac{5}{2}, -\frac{5}{4}, -\frac{5}{8},$ _____, _____, …	**F** 86, 85, 83, 80, 76, _____, _____, …
G $-16, 4, -1, \frac{1}{4},$ _____, _____, …	**H** 1473.2, 1452.7, 1432.2, 1411.7, _____, _____, …

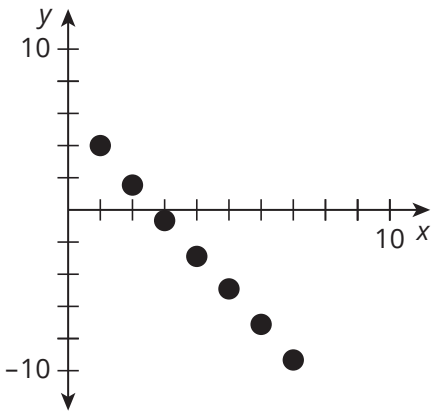
CC BY NC

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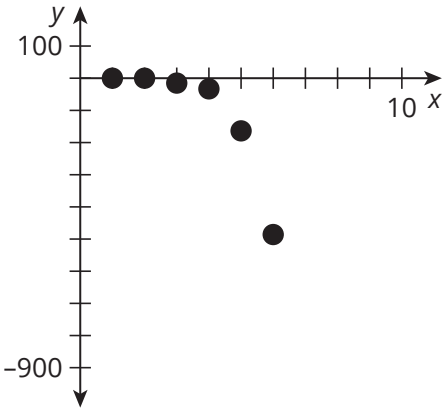
So you can cut out the Sequence Cards on the other side

Graph Cards

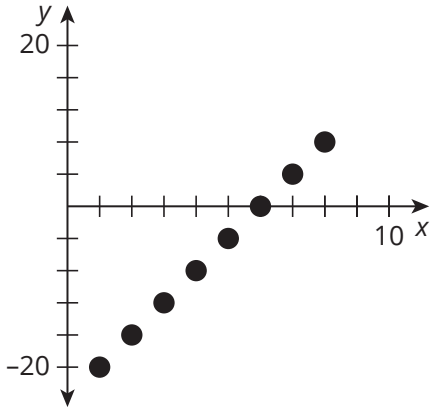
Graph 1



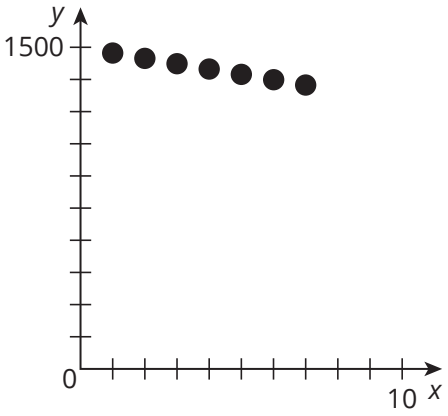
Graph 2



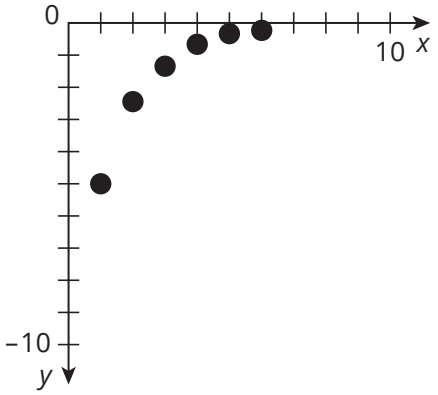
Graph 3



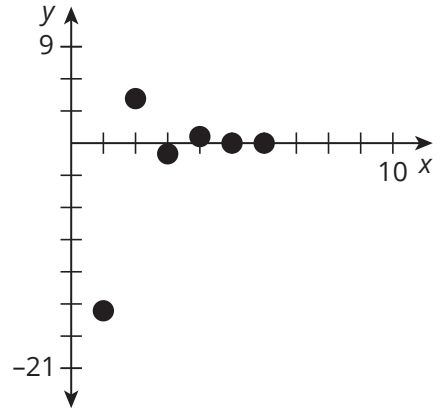
Graph 4



Graph 5



Graph 6



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Sequence

Graph

Arithmetic Sequence

Recursive Formula

Explicit Formula

CC BY NC

Sequence

Graph

Arithmetic Sequence

Recursive Formula

Explicit Formula

Sequence

Graph

Arithmetic Sequence

Recursive Formula

Explicit Formula

CC BY NC

Sequence

Graph

Geometric Sequence

Recursive Formula

Explicit Formula

Sequence

Graph

Geometric Sequence

Recursive Formula

Explicit Formula

CC BY NC

Sequence

Graph

Geometric Sequence

Recursive Formula

Explicit Formula

Lesson 2 Assignment

Write

Complete each sentence.

1. A sequence which terminates is called a(n) _____ .
2. A(n) _____ is an individual number, figure, or letter in a sequence.
3. A(n) _____ is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects.
4. A sequence which continues forever is called a(n) _____ .

Remember

An arithmetic sequence is a sequence of numbers in which the difference between any two consecutive terms is a constant.

A geometric sequence is a sequence of numbers in which the ratio between any two consecutive terms is a constant.

Practice

Consider the first two terms of the sequence 28, 14, ...

1. Determine whether the sequence is arithmetic or geometric. Explain your reasoning.

2. Suppose the sequence 28, 14, . . . is arithmetic.

   a. Determine the common difference.

   b. List the next 3 terms in the sequence. Explain your reasoning.

   c. Determine whether the sequence is finite or infinite. Explain your reasoning.

Lesson 2 Assignment

3. Suppose the sequence 28, 14, ... is geometric.

   a. Determine the common ratio.

   b. List the next 3 terms in the sequence. Explain your reasoning.

   c. Determine whether the sequence is finite or infinite. Explain your reasoning.

4. Using the first two terms 28 and 14, write the next three terms of a sequence that is neither arithmetic nor geometric.

5. A sequence is generated by the function $a_n = a_{(n-1)} + 5$ where $a_1 = 4\frac{1}{2}$ and n is a whole number greater than 1.

   a. Determine the first four terms of the sequence.

   b. Identify the sequence type and the common difference or common ratio.

Lesson 2 Assignment

6. A sequence is generated by the function $a_n = a_{n-1} \cdot \frac{1}{4}$ where $a_1 = 12$ and n is a whole number greater than 1.

   a. Determine the first four terms of the sequence.

   b. Identify the sequence type and the common difference or common ratio.

Prepare

The local bank has agreed to donate $250 to the annual turkey fund to help feed families in need. In addition, for every bank customer that donates $50, the bank will donate $25.

1. A sequence describes the relationship between the number of $50 donations and the amount of the bank's donation. Is the sequence arithmetic or geometric?

2. How can you calculate the 10th term based on the 9th term?

3. What is the 20th term?

3 Determining Recursive and Explicit Expressions from Contexts

OBJECTIVES

- Write recursive formulas for arithmetic and geometric sequences from contexts.
- Write explicit expressions for arithmetic and geometric sequences from contexts.
- Use formulas to determine unknown terms of a sequence.

NEW KEY TERMS

- recursive formula
- explicit formula

You have learned that arithmetic and geometric sequences always describe functions. How can you write equations to represent these functions?

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Getting Started

Can I Get a Formula?

Notice that the first term in this sequence is the amount Jamal donates if the team hits 0 home runs.

While a common ratio or a common difference can help you determine the next term in a sequence, how can they help you determine the thousandth term of a sequence? The ten-thousandth term of a sequence?

Consider the sequence represented in this situation.

Jamal owns a sporting goods store. He has agreed to donate $125 to his old high school's baseball team for their equipment fund. In addition, he will donate $18 for every home run the team hits during the season. The sequence shown represents the possible dollar amounts that Jamal could donate for the season.

125, 143, 161, 179, ...

Number of Home Runs	Term Number (n)	Donation Amount (dollars)
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

1. Identify the sequence type. Describe how you know.

2. Determine the common difference or common ratio for the sequence.

3. Complete the table.

4. Explain how you can calculate the tenth term based on the ninth term.

ACTIVITY
3.1 Writing Formulas for Arithmetic Sequences

In the previous lesson you used the **recursive formula** to determine term values in a sequence. The recursive formula expresses each new term of a sequence based on the preceding term in the sequence. The recursive formula to determine the *n*th term of an arithmetic sequence is:

$$\underbrace{a_n}_{n\text{th term}} = \underbrace{a_{n-1}}_{\text{previous term}} + \underbrace{d}_{\text{common difference}}$$

WORKED EXAMPLE

Consider the sequence −2, −9, −16, −23, ...
You can use the recursive formula to determine the 5th term.

$$a_n = a_{n-1} + d$$
$$a_5 = a_{5-1} + -7$$

The expression a_5 represents the 5th term. The previous term is −23, and the common difference is −7.

$$a_5 = a_4 + -7$$
$$a_5 = -23 + -7$$
$$a_5 = -30$$

The 5th term of the sequence is −30.

You only need to know the previous term and the common difference to use the recursive formula.

Ask Yourself . . .
What observations can you make?

Consider the sequence showing Jamal's contribution to his old high school's baseball team in terms of the number of home runs hit.

1. Use a recursive formula to determine the 11th term in the sequence. Explain what this value means in terms of this problem situation.

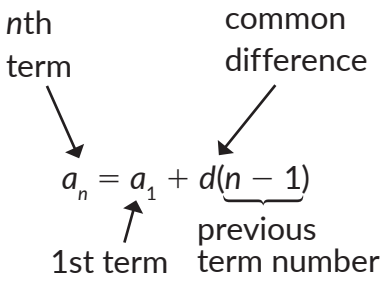
2. Is there a way to calculate the 20th term without first calculating the 19th term? If so, describe the strategy.

Ask Yourself . . .
Did you justify your mathematical reasoning?

You can determine the 93rd term of the sequence by calculating each term before it and then adding 18 to the 92nd term, but this will probably take a while! A more efficient way to calculate any term of a sequence is to use an *explicit formula*.

Remember . . .
The first term in this sequence is the amount Jamal donates when the team hits 0 home runs. So, the 93rd term represents the amount Jamal donates if the team hits 92 home runs.

An **explicit formula** of a sequence is a formula to calculate the *n*th term of a sequence using the term's position in the sequence. The explicit formula for determining the *n*th term of an arithmetic sequence is:



WORKED EXAMPLE

You can use the explicit formula to determine the 93rd term in this problem situation.

$$a_n = a_1 + d(n - 1)$$
$$a_{93} = 125 + 18(93 - 1)$$

The expression a_{93} represents the 93rd term. The first term is 125, and the common difference is 18.

$$a_{93} = 125 + 18(92)$$
$$a_{93} = 125 + 1656$$
$$a_{93} = 1781$$

The 93rd term of the sequence is 1781.

This means Jamal will contribute a total of $1781 when his old high school's baseball team hits 92 home runs.

3. Use the explicit formula to determine the amount of money Jamal will contribute for each number of home runs hit.

   a. 35 home runs

   b. 48 home runs

   c. 86 home runs

   d. 214 home runs

Jamal decides to increase his initial contribution and amount donated per home run hit. He decides to contribute $500 and will donate $75 for every home run the team hits.

4. Write the first five terms of the sequence representing the new contribution Jamal will donate to the the baseball team.

5. Determine Jamal's contribution for each number of home runs hit.

   a. 39 home runs

   b. 50 home runs

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WORKED EXAMPLE

Use the distributive property to rewrite the explicit formula for the problem situation.

The explicit formula for the problem situation is $a_n = 125 + 18(n + 1)$.

Step 1: Use the distributive property.

$a_n = 125 + 18n - 18$.

Step 2: Combine like terms.

$a_n = 18n + 107$

6. For each arithmetic sequence, identify the first term and the common difference. Then, use the distributive property to rewrite the explicit formula.

a. $p_n = 16 - \frac{1}{2}(n - 1)$

b. $k_n = 0.6 + 0.2(n - 1)$

c. $r_n = -7(n - 1) - 2$

ACTIVITY 3.2

Writing Formulas for Geometric Sequences

When it comes to bugs, bats, spiders, and—ugh, any other creepy crawlers—finding one in your house is finding one too many! Then again, when it comes to cells, the more the better. Animals, plants, fungi, slime, molds, and other living creatures are composed of eukaryotic cells. During growth, generally there is a cell called a "mother cell" that divides itself into two "daughter cells." Each of those daughter cells then divides into two more daughter cells, and so on.

Notice that the 1st term in this sequence is the total number of cells after 0 divisions (that is, the mother cell).

1. The sequence shown represents the growth of eukaryotic cells.

   1, 2, 4, 8, 16, ...

   a. Describe why this sequence is geometric and identify the common ratio.

   b. Complete the table of values. Use the number of cell divisions to identify the term number and the total number of cells after each division.

Number of Cell Divisions	Term Number (n)	Total Number of Cells
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

   c. Explain how you can calculate the tenth term based on the ninth term.

In the previous lesson, you used the recursive formula for a geometric sequence. The recursive formula to determine the nth term of a geometric sequence is:

$$a_n = a_{n-1} \cdot r$$

(a_n: nth term; r: common ratio; a_{n-1}: previous term)

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WORKED EXAMPLE

Consider the sequence shown.

$$4, 12, 36, 108, \ldots$$

You can use the recursive formula to determine the 5th term.

$$b_n = b_{n-1} \cdot r$$
$$b_5 = b_{5-1} \cdot (3)$$

The expression a_5 represents the 5th term. The previous term is 108, and the common ratio is 3.

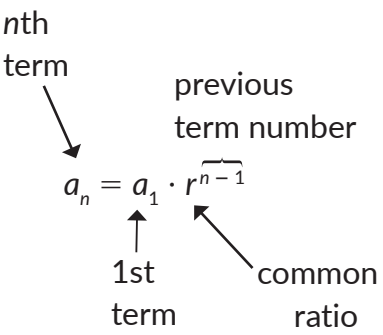
$$b_5 = b_4 \cdot (3)$$
$$b_5 = 108 \cdot (3)$$
$$b_5 = 324$$

The 5th term of the sequence is 324.

Consider the sequence of cell divisions and the total number of resulting cells.

2. Write a recursive formula for the sequence and use the formula to determine the 12th term in the sequence. Explain what your result means in terms of this problem situation.

The explicit formula to determine the nth term of a geometric sequence is:



WORKED EXAMPLE

You can use the explicit formula to determine the 20th term in this problem situation.

$$a_n = a_1 \cdot r^{n-1}$$
$$a_{20} = 1 \cdot 2^{20-1}$$

The expression a_{20} represents the 20th term. The first term is 1, and the common ratio is 2.

$$a_{20} = 1 \cdot 2^{19}$$
$$a_{20} = 1 \cdot 524{,}288$$
$$a_{20} = 524{,}288$$

The 20th term of the sequence is 524,288.

This means that after 19 cell divisions, there are a total of 524,288 cells.

Remember . . .

The first term in this sequence is the total number of cells after 0 divisions. So, the 20th term represents the total number of cells after 19 divisions.

3. Use the explicit formula to determine the total number of cells for each number of divisions.

   a. 11 divisions

   b. 14 divisions

   c. 18 divisions

   d. 22 divisions

Remember . . .

You can use parentheses or a · to represent multiplication.

Suppose that a scientist has 5 eukaryotic cells in a petri dish. She wonders how the growth pattern would change when each mother cell divides into 3 daughter cells.

4. Write the first five terms of the sequence for the scientist's hypothesis.

5. Determine the total number of cells in the petri dish for each number of divisions.

   a. 13 divisions

   b. 16 divisions

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ACTIVITY
3.3

Writing Recursive and Explicit Formulas

In the previous lesson, you identified sequences as either arithmetic or geometric and then matched a corresponding graph.

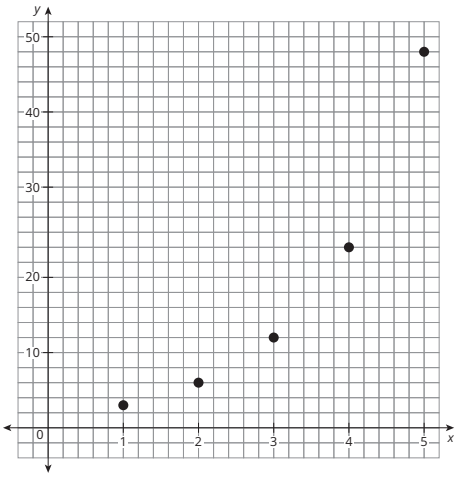
1. Go back to the graphic organizers from the previous lesson. Write the recursive and explicit formulas for each sequence.

ACTIVITY
3.4 Arithmetic and Geometric Sequences

1. For each sequence, identify the sequence type and the common difference or ratio. Write an explicit formula and a recursive formula to represent the sequence.

   a. $a_1 = 2.3, a_2 = 0.3, a_3 = -1.7, a_4 = -3.7$

   b.

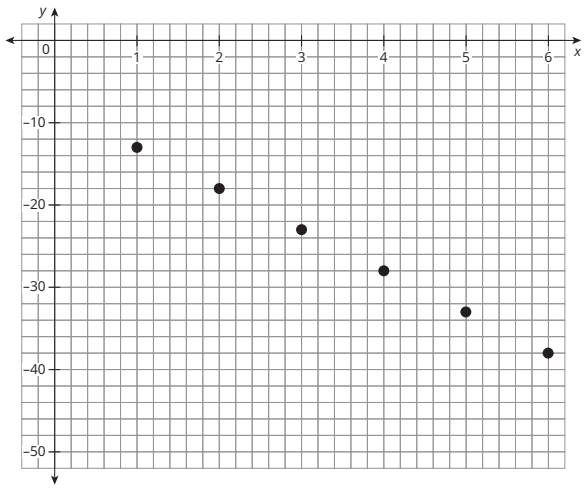


   c.

n	c_n
4	−567
5	189
6	−63
7	21
8	−7

d. $a_5 = -5.9,\ a_6 = -5.3,\ a_7 = -4.7,\ a_8 = -4.1$

e.



2. The local diner makes the best omelets in town! The chef begins each day with 150 eggs to make his famous omelets. After making 1 omelet, he has 144 eggs. After making 2 omelets, he has 138 eggs left. After making 3 omelets, he has 132 eggs left. Write a recursive and explicit formula to represent the sequence. Then, determine how many eggs the chef will have left after he makes 20 omelets.

3. Tiara is participating in a pizza-making contest. Each contestant has to bake the largest and most delicious pizza they can. Tiara's pizza has a 6-foot diameter! After the contest, she plans to cut the pizza so that she can pass the slices out to share. She begins with 1 whole pizza. Then, she cuts it in half. After that, she cuts each of those slices in half. Then she cuts each of those slices in half and so on. Write a recursive and an explicit formula for the sequence. Then, determine the size of the slice compared to the whole pizza after 10 cuts.

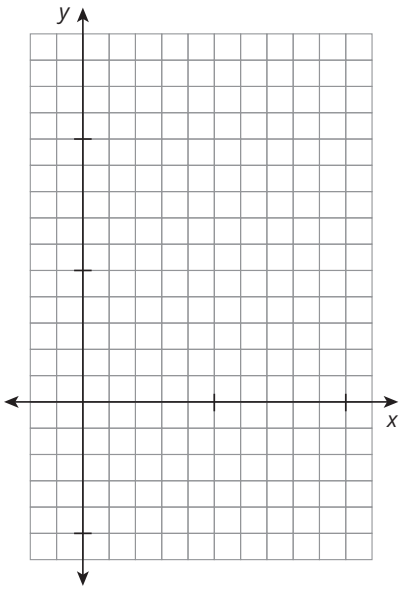
4. Determine the 9th term of the sequence $a_n = a_{n-1} \cdot 8$ and $a_3 = 2$.

5. Determine the 12th term of the sequence $a_n = a_{n-1} + \frac{3}{4}$ and $a_9 = 6$.

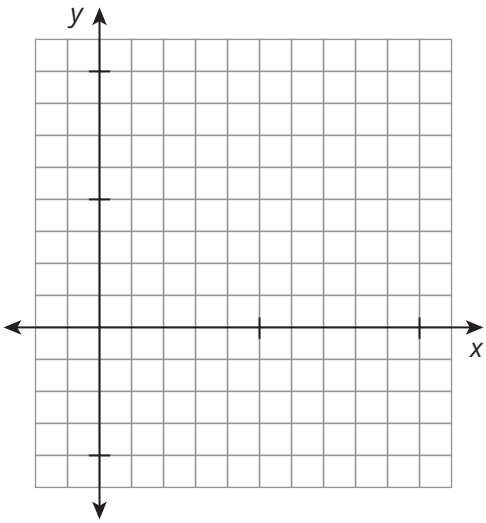
ACTIVITY
3.5 Graphing Sequences

Identify the sequence type and the common difference or ratio for each sequence. Write the explicit formula you can use to represent the terms of the sequence, then represent the sequence on the coordinate plane.

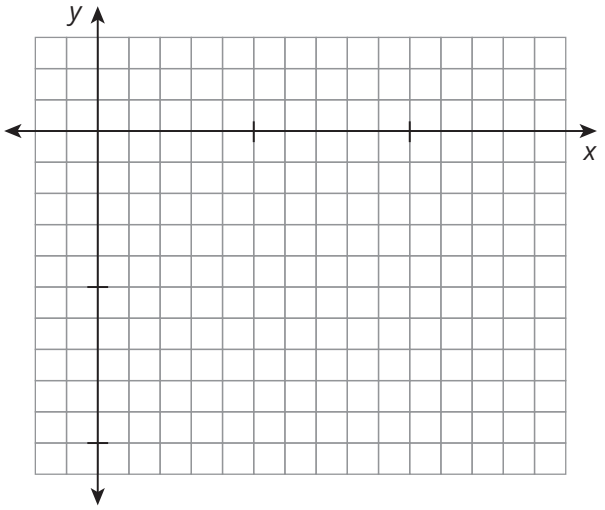
1. $a_1 = \frac{1}{3}, a_2 = -1, a_3 = 3, a_4 = -9, a_5 = 27$



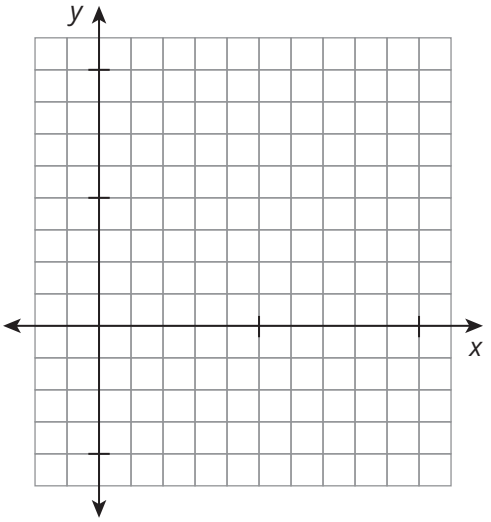
2. $h_1 = -1.2, h_2 = -1.1, h_3 = -1.0, h_4 = -0.9, h_5 = -0.8$



3. $a_3 = -12, a_4 = -14, a_5 = -16, a_6 = -18$



4. $k_3 = \frac{1}{8}, k_4 = \frac{1}{32,} k_5 = \frac{1}{128}, k_6 = \frac{1}{512}$



Talk the Talk

Pros and Cons

1. Explain the advantages and disadvantages of using a recursive formula.

2. Explain the advantages and disadvantages of using an explicit formula.

Lesson 3 Assignment

Write

Explain the difference between a recursive formula and an explicit formula in your own words.

Remember

All sequences describe functions.

The explicit formula for an arithmetic sequence is $a_n = a_1 + d(n - 1)$, where n is the term number, a_1 is the first term in the sequence, a_n is the nth term in the sequence, and d is the common difference.

The explicit formula for a geometric sequence is $a_n = a_1 \cdot r^{(n-1)}$, where n is the term number, a_1 is the first term in the sequence, a_n is the nth term in the sequence, and r is the common ratio.

Practice

1. Lizzie must volunteer 225 hours for a community service project. She plans to volunteer for 6 hours each week. The sequence shown represents the number of volunteer hours she has left after three weeks have passed.

   225, 219, 213, 207, ...

   a. Describe this sequence.

   b. Write and simplify the explicit formula for the problem situation.

Lesson 3 Assignment

c. Use your explicit formula from part (b) to determine how many volunteer hours Lizzie has left to fulfill her requirement after 33 weeks have passed. Show your work.

d. Which formula should you use to determine how many volunteer hours Lizzie has left to fulfill her requirement after 40 weeks have passed? Explain your reasoning.

e. Calculate the number of volunteer hours Lizzie has left to fulfill her requirement after 40 weeks have passed. Explain what your answer means in terms of the problem situation.

2. The half-life of a substance is defined as the period of time it takes for the amount of the substance to decay by half. The sequence below shows the amount of a substance that will be left after a certain number of half-lives have elapsed.

   $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

   a. Describe this sequence.

Lesson 3 Assignment

b. Calculate how much of the substance will be left after 21 half-lives have elapsed. Show your work. Does your answer make sense in this problem context? Why or why not?

Prepare

Consider the first two terms of this sequence $\frac{1}{16}, -\frac{3}{16}, \ldots$

- Determine the 63rd term if this is an arithmetic sequence. Write your answer as an improper fraction in lowest terms.

- Determine the 63rd term if this is a geometric sequence. Write your answer in scientific notation.

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TOPIC 2 SELF-REFLECTION

Name: _____

Sequences

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill.**

I can demonstrate an understanding of the standards in the *Sequences* topic by:

TOPIC 2: *Sequences*	Beginning of Topic	Middle of Topic	End of Topic
understanding that a sequence represents a relationship between term numbers (inputs) and term values (outputs).	○	○	○
stating the appropriate domain for a sequence.	○	○	○
distinguishing between arithmetic and geometric sequences.	○	○	○
recognizing that an arithmetic sequence has a common difference between terms and a geometric sequence has a common ratio between terms.	○	○	○
determining the common difference between two terms in an arithmetic sequence and the common ratio between two terms in a geometric sequence represented in tables and graphs.	○	○	○
describing the graph of an arithmetic and geometric sequence.	○	○	○
explaining that a recursive formula tells you how to determine the next value of a sequence from the previous value.	○	○	○
explaining that an explicit formula tells you how to determine any value given the term number.	○	○	○

continued on the next page

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TOPIC 2 SELF-REFLECTION *continued*

TOPIC 2: *Sequences*	Beginning of Topic	Middle of Topic	End of Topic
distinguishing between explicit and recursive formulas.	○	○	○
writing recursive and explicit formulas for any sequence, including those presented as real-world scenarios.	○	○	○
rewriting explicit formulas of arithmetic sequences using the distributive property.	○	○	○
translating between explicit and recursive formulas.	○	○	○
deciding when real-world problems model an arithmetic or geometric sequence.	○	○	○

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Sequences* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?



Sequences Summary

LESSON 1 Recognizing Patterns and Sequences

A **sequence** is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects. A **term in a sequence** is an individual number, figure, or letter in the sequence. Many different patterns can generate a sequence of numbers.

A sequence that continues on forever is called an **infinite sequence**. A sequence that terminates is called a **finite sequence**.

For example, consider the situation in which an album that can hold 275 baseball cards is filled with 15 baseball cards at the end of each week. A sequence to represent how many baseball cards can fit into the album after 6 weeks is 275 cards, 260 cards, 245 cards, 230 cards, 215 cards, and 200 cards. This sequence begins at 275 and decreases by 15 with each term. The pattern cannot continue forever since you cannot have a negative number of cards, so this is a finite sequence.

LESSON 2 Arithmetic and Geometric Sequences

An **arithmetic sequence** is a sequence of numbers in which the difference between any two consecutive terms is a constant. This constant is called the **common difference** and is typically represented by the variable *d*. The common difference of a sequence is positive when the same positive number is added to each term to produce the next term. The common difference of a sequence is negative when the same negative number is added to each term to produce the next term.

For example, consider the sequence 14, $16\frac{1}{2}$, 19, $21\frac{1}{2}$, ... The pattern of this sequence is to add $2\frac{1}{2}$ to each term to produce the next term. This is an arithmetic sequence, and the common difference *d* is $2\frac{1}{2}$.

A **geometric sequence** is a sequence of numbers in which the ratio between any two consecutive terms is a constant. The constant, which is either an integer or a fraction, is called the **common ratio** and is typically represented by the variable *r*.

NEW KEY TERMS

- sequence [secuencia/sucesión]
- term of a sequence [término de una secuencia]
- infinite sequence [secuencia infinita]
- finite sequence [secuencia finita]
- arithmetic sequence [secuencia aritmética]
- common difference [diferencia común]
- geometric sequence [secuencia geométrica]
- common ratio [razón común]
- recursive formula [fórmula recursiva]
- explicit formula [fórmula explícita]
- mathematical modeling [modelado matemático]

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For example, consider the sequence 27, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$. The pattern is to multiply each term by the same number, $\frac{1}{3}$, to determine the next term. Therefore, this sequence is geometric and the common ratio, r, is $\frac{1}{3}$.

LESSON 3 Determining Recursive and Explicit Expressions from Context

A **recursive formula** expresses each new term of a sequence based on a preceding term of the sequence. The recursive formula to determine the nth term of an arithmetic sequence is $a_n = a_{n-1} + d$. The recursive formula to determine the nth term of a geometric sequence is $a_n = a_{n-1} \cdot r$. When using the recursive formula, it is not necessary to know the first term of the sequence.

For example, consider the geometric sequence 32, 8, 2, $\frac{1}{2}$, ... with a common ratio of $\frac{1}{4}$. The 5th term of the sequence can be determined using the recursive formula.

The 5th term of the sequence is $\frac{1}{8}$.

$a_n = a_{n-1} \cdot r$
$a_5 = a_4 \cdot r$
$a_5 = \frac{1}{2} \cdot \frac{1}{4}$
$a_5 = \frac{1}{8}$

An **explicit formula** for a sequence is a formula for calculating each term of the sequence using the index, which is a term's position in the sequence. The explicit formula to determine the nth term of an arithmetic sequence is $a_n = a_1 + d(n - 1)$. You can use the distributive property to rewrite the formula for an arithmetic sequence. The explicit formula to determine the nth term of a geometric sequence is $g_n = g_1 \cdot r^{n-1}$.

For example, consider the situation of a cactus that is currently 3 inches tall and will grow $\frac{1}{4}$ inch every month. The explicit formula for arithmetic sequences can be used to determine how tall the cactus will be in 12 months.

In 12 months, the cactus will be $5\frac{3}{4}$ inches tall.

$a_n = 3 + \frac{1}{4}(n - 1)$
$a_n = 3 + \frac{1}{4}n - \frac{1}{4}$
$a_n = \frac{1}{4}n + 2.75$
$a_{12} = \frac{1}{4}(12) + 2.75$
$a_{12} = 3 + 2.75$
$a_{12} = 5.75$ or $5\frac{3}{4}$

MODULE 2

Exploring Constant Change

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TOPIC 1



A set of points in a straight line can be modeled by a linear function.

Linear Functions

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LESSON 4 Point-Slope Form of a Line 201

LESSON 5 Using Linear Equations 217

LESSON 6 Making Sense of Different Representations of a Linear Function 239

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1 Least Squares Regressions

OBJECTIVES

- Create a graph of data points with and without technology.
- Determine an equation for a line of best fit by visual approximation of a hand-drawn line.
- Determine a linear regression function using technology.
- Make predictions about data using a linear regression function.
- Explain the calculations involved in the Least Squares Method.
- Choose a level of accuracy appropriate when reporting quantities.

NEW KEY TERMS

- Least Squares Method
- centroid
- linear regression function
- interpolation
- extrapolation

You have searched for patterns in graphs and sequences of numbers.

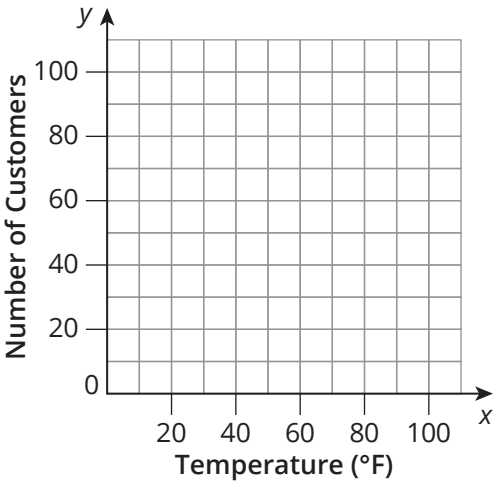
How can you use what you know to identify patterns in sets of data?

Getting Started

Frozen Yogurt . . . When It's Freezing?

A school's club for aspiring business leaders is helping a local frozen yogurt shop analyze how the business is affected by the weather. The owner is wondering whether there is a relationship between the temperature and the number of customers that buy yogurt during the 2 hours immediately after school. The school's club collected these data.

Temperature (°F)	Number of Customers
45	97
25	55
60	85
15	37
100	100



1. Construct a scatterplot of the collected data.

   a. Plot the first data point. Is there a pattern? Use a piece of spaghetti to approximate a line that models the data.

   b. Add the second data point to the graph. Is there a pattern? Adjust the piece of spaghetti to approximate a line that models the data with the additional point.

c. Add the third data point to the graph. Describe the pattern that you see. Approximate the line using the spaghetti.

d. Continue this process until all five data points are plotted and recorded in the table.

2. Use your linear model to describe the relationship between the temperature outside and the number of customers at the frozen yogurt shop.

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ACTIVITY
1.1 A Line Of Best Fit

You have approximated the line that best represents the data with each additional data point.

1. Use the full data set and the line that you approximated to write an equation that you think best represents the data.

2. Based on your equation, predict the number of customers to visit the frozen yogurt shop in the two hours after school for each given temperature.

   a. 85°F b. 115°F c. 10°F

3. Compare your predictions with your classmates. Did your predictions differ from the other groups? Explain why or why not.

You have noticed that estimating a line of best fit can give different predictions. Fortunately, with technology you can create prediction equations as well as a scatterplots from tables of data. You just need to build a data table that has an independent variable and a dependent variable.

4. Identify the independent and dependent variable. What is the significance of those designations?

5. Use the data table and graphing technology to generate a line of best fit. What is the slope and *y*-intercept of the line and what do they represent?

6. Use the new line of best fit to predict the number of customers at the frozen yogurt shop immediately after school for each given temperature.

   a. 85°F

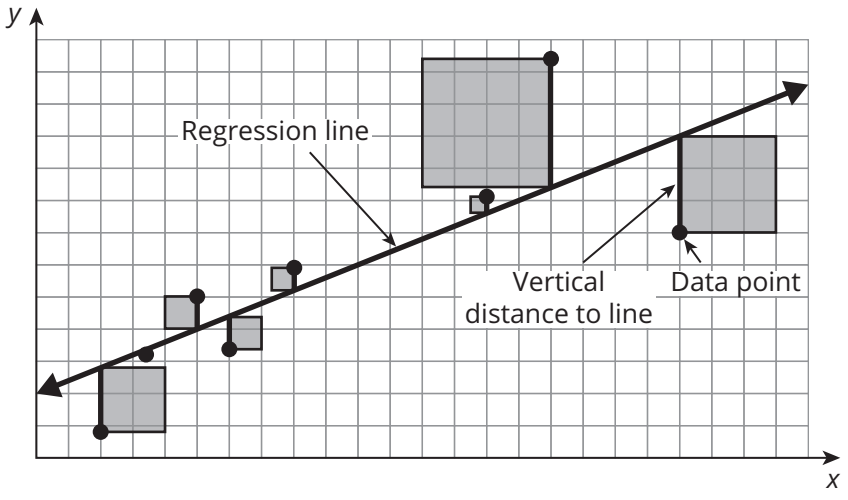
   b. 115°F

   c. 10°F

7. How do your predictions compare to the predictions from the other groups?

The equation that your graphing technology uses to give you the line of best fit is called the **Least Squares Method**. This is a method that creates a line of best fit for a scatterplot that has two basic requirements:

- The line must contain the *centroid* of the data set. The **centroid** is a point whose *x*-value is the mean of all the *x*-values of the points on the scatterplot and its *y*-value is the mean of all the *y*-values of the points on the scatterplot.
- Even though infinitely many lines can pass through the centroid, the **linear regression function** has the smallest possible vertical distances from each given data point to the regression line. The sum of the squares of those distances are at a minimum with this line.



8. Consider the graph of the sample linear regression function.

   a. What do the vertical lines in bold represent?

   b. What do the shaded squares represent? How do they relate to the Least Squares Method?



9. Linh and James each draw a line of best fit to model a set of data. They both record the vertical distances between each point and the line of best fit.

Linh	James
Vertical Distances: 2, 2, 2, 2, 2	Vertical Distances: 1, 1, 1, 1, 6

   Both students believe they drew the least square regression line. Who's correct? Justify your choice.

10. How does your decision in Question 9 inform you about the placement of a line of best fit using the Least Squares Method?

ACTIVITY
1.2 Making Predictions

The table shows the population of a small town for each year from 2010 to 2017.

Years	2010	2011	2012	2013	2014	2015	2016	2017
Population	21,359	22,906	22,542	23,048	23,562	23,609	24,008	24,716

1. What is the range of the data set?

2. Determine a linear regression model for the data. Round the slope and *y*-intercept to the nearest whole number.

3. Identify the independent and dependent variables and their units of measure.

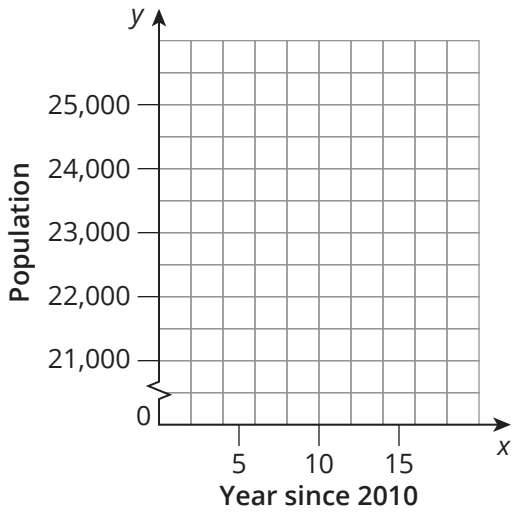
Remember . . .

Data comparing two variables can show a positive association, negative association, or no association.

4. Does the data represent a function? Does it appear that there is a specific function that could model this data set? When so, describe the function. When not, state why not.

5. Use technology to graph a scatterplot demonstrating the relationship between the year and population. What association do you notice?

6. Between which consecutive years was there a decrease in the population?



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7. Use graphing technology to determine the linear regression function for the population. Then, sketch the data points and the line of best fit that you see.

8. What is the relationship between the equation for the line of best fit and any association you notice in the graph? Do you think that this line fits the data well?

9. For each expression from your linear regression function about populations, write an appropriate unit of measure and describe the contextual meaning. Then, choose a term from the word box to describe the mathematical meaning of each part.

Word Box

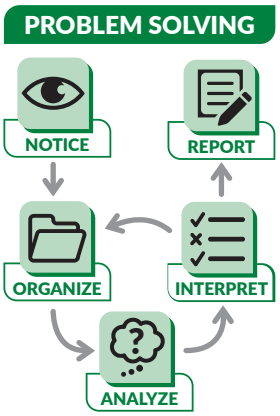
- input value
- output value
- rate of change
- *y*-intercept

		What it Means	
Expression	**Unit**	**Contextual Meaning**	**Mathematical Meaning**
$f(x)$			
390			
x			
21,855			

10. Use your linear regression function to predict the population for the year 2024.

ACTIVITY 1.3

Making Predictions Within and Outside a Data Set



The music industry is constantly changing how it delivers music to its listeners. The table shows the percent of total U.S. music sales revenues from streaming.

Year	2010	2011	2012	2013	2014	2015
Percent of Total U.S. Music Sales Revenue From Streaming	7	9	15	21	27	34

1. Use graphing technology to determine the linear regression function for the data.

2. Interpret the equation of the line in terms of this problem situation.

Ask Yourself . . .
What is an appropriate level of accuracy needed throughout this situation?

When there is a linear association between the independent and dependent variables of a data set, you can use a linear regression function to make predictions within the data set. Using a linear regression function to make predictions within the data set is called **interpolation**.

3. Use your equation to predict the percent of streaming revenues in 2013. Compare the predicted value percent in 2013 with the actual value.

4. Compute the predicted value percent for 2011 and compare it with the actual value.

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5. Do you think a prediction made using interpolation will always be close to the actual value? Explain your reasoning.

To make predictions for values of x that are outside of the data set is called **extrapolation**.

6. Use the equation to predict the percent of streaming revenues:

   a. in 2040.

   b. in 2004.

7. Are these predictions reasonable? Explain your reasoning.

Talk the Talk

Tell Me Ev-ery-thing

You have used technology to determine linear regression functions. You have then used those linear regression functions to predict unknown values within and without a data set.

1. Why is the linear regression function generated using technology more accurate than the line of best fit that you can write using two points?

2. Why are predictions made by extrapolation more likely to be less accurate than predictions made by interpolation?

Lesson 1 Assignment

Write

Complete each sentence with the appropriate vocabulary term.

1. A _____ can be used to organize and display the values of two variables in a data set.
2. A _____ models the relationship between two variables in a data set by producing a line of best fit.
3. A(n) _____ is a line that best approximates the linear relationship between two variables in a data set.
4. The _____ is used to approximate a line of best fit by minimizing the squares of the distances of the points from the line.
5. _____ is using a linear regression function to make predictions within the data set.
6. Using a linear regression function to make predictions outside of the data set is _____.
7. After a scatterplot is created, the _____ is a point with an x-value that is the mean of all the x-values of the points on the plot and a y-value that is the mean of all the y-values of the points on the plot.

Remember

Patterns in data can be modeled with lines of best fit. The Least Squares Method is one way to create a linear regression function, and it is the method that graphing technology tends to use.

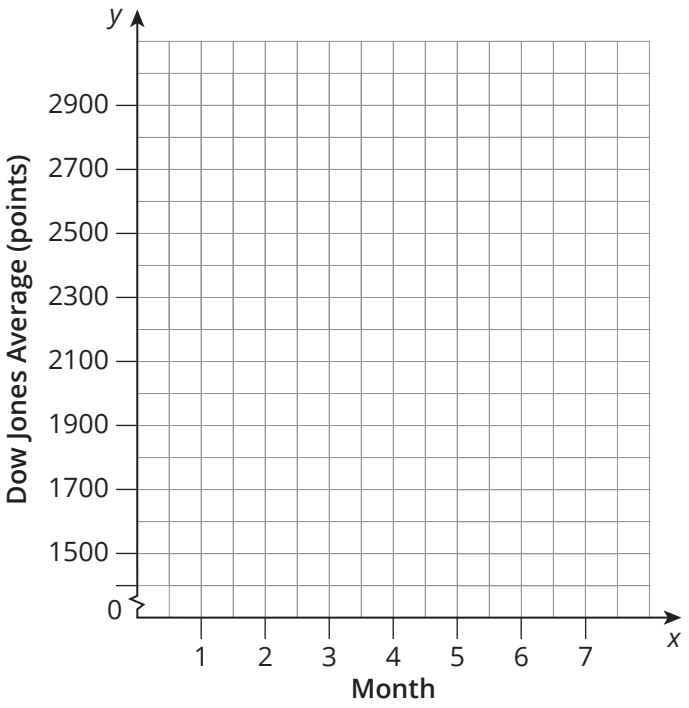
Lesson 1 Assignment

Practice

1. The table shows the Dow Jones Industrial Average on the NYC Stock Exchange over a period of six consecutive months (August through January) in 1987. On Monday, October 16, 1987, stock markets around the world crashed. This decline was the greatest one-day percentage decline in stock market history to date.

Time (months)	Dow Jones Average (points)
1	2700
2	2600
3	1750
4	1950
5	1900
6	2000

   a. Create a scatterplot of the NYC Stock Exchange data. What information can you gather about the stock market from the scatterplot?



   b. Generate a linear regression function that would model the Dow Jones Average using the data for months 1–6.

Lesson 1 Assignment

c. Generate a linear regression function that would model the Dow Jones Average using only the data for months 3–6.

d. When will the model for the data from months 3–6 provide a better prediction?

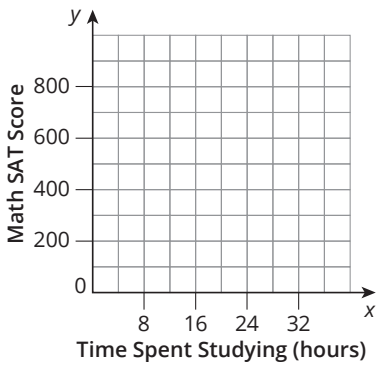
e. Interpret the slope and *y*-intercept of the linear regression function you wrote in part (b). What do these values represent in terms of the problem situation?

f. Use the linear regression function from part (b) to determine the Dow Jones Average in October 1987. Is this the same as the Dow Jones Average recorded in the table? If not, explain the difference.

2. Mr. Flores is a high school teacher and is preparing his students to take the SAT test. He collected data from 10 students who took the test last year and presented this information to the students in a table. The highest math SAT score a student can achieve is 800. Analyze the data in the table.

Time Spent Studying (hours)	Math SAT Score
1	350
22	780
12	600
14	700
4	380
10	650
9	580
3	400
7	530
4	410

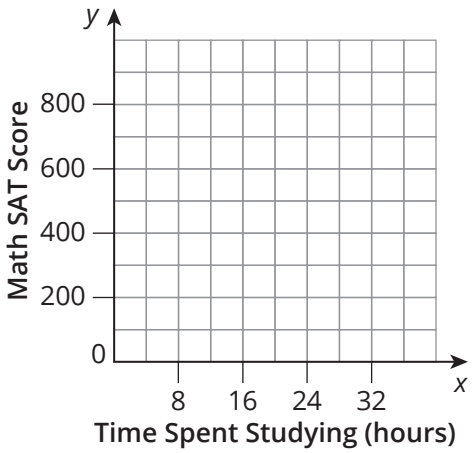
a. Construct a scatterplot of the data and describe any patterns you see in the data.



Lesson 1 Assignment

b. Determine the equation of a line passing through (14, 700) and (7, 530). Then, determine the equation of a line passing through (22, 780) and (4, 380). Graph both lines on the same graph as the scatterplot.

c. Which line seems to best fit the data? Would you use either one of these lines to make predictions about a student's math SAT score based on the amount of studying they do? Why or why not?



d. Use graphing technology to determine the linear regression function.

e. Interpret the linear regression function in terms of the problem situation.

f. Use the linear regression function to predict the math SAT score for a student who studies for 17 hours. Did you use interpolation or extrapolation to make this prediction? Is this prediction reasonable for this problem situation? Explain your reasoning.

Lesson 1 Assignment

g. Use the linear regression function to predict the math SAT score for a student who studies for 40 hours. Did you use interpolation or extrapolation to make this prediction? Is this prediction reasonable for this problem situation? Explain your reasoning.

h. One of Mr. Flores's students comes back to him the following year and says that he studied for 15 hours for the math SAT and got a score of 610. He argues that the linear regression function predicted that he would have scored a 682. What do you think explains the discrepancy?

Prepare

Describe a possible flaw in the reasoning for each situation.

1. When I wash my hands regularly, I will not get sick.

2. When I practice my guitar every day, I will be a rock star.

3. When I wear my favorite football jersey to support the team, they will win the game.

4. When I am a good driver, I will not have an accident.

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2 Correlation

OBJECTIVES

- Determine the correlation coefficient using technology.
- Interpret the correlation coefficient for a set of data.
- Understand the difference between r and r^2.
- Understand the difference between correlation and causation.
- Understand necessary conditions.
- Understand sufficient conditions.
- Choose a level of accuracy appropriate when reporting quantities.

NEW KEY TERMS

- correlation
- correlation coefficient
- coefficient of determination
- causation
- necessary condition
- sufficient condition
- common response
- confounding variable

You have learned how to write a line of best fit relating two variables using the Least Squares Method.

Is there a way to measure the strength of the relationship between the variables?

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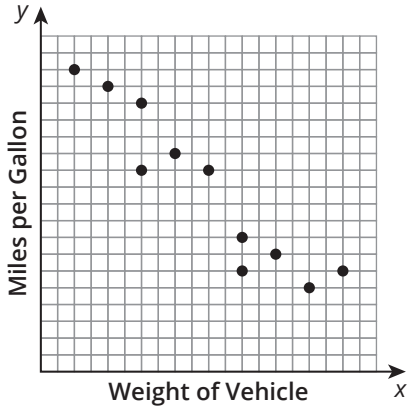
Getting Started

Associate, Formulate, Correlate!

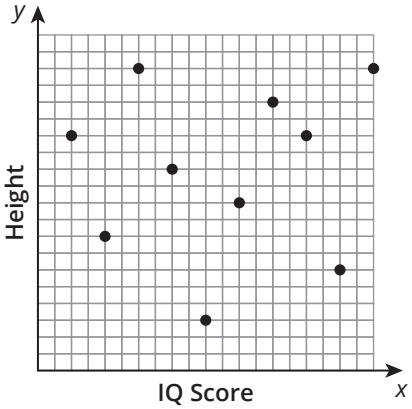
Consider each relationship shown.

1. Describe any associations between the independent and dependent variables, and then draw a line of best fit, if possible.

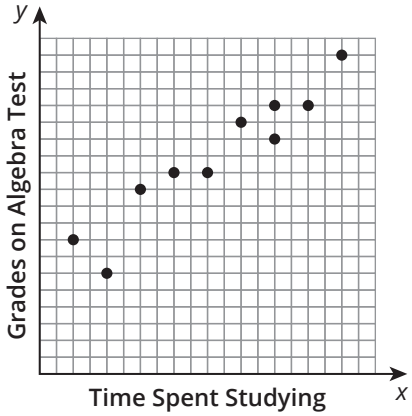
a.



b.



c.



ACTIVITY
2.1 The Correlation Coefficient

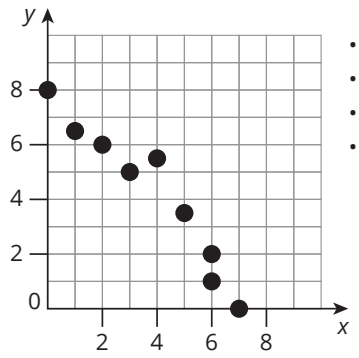
A measure of how well a regression model fits a set of data is called **correlation**. The **correlation coefficient** is a value between −1 and 1, which indicates how close the data are to the graph of the regression function. The closer the correlation coefficient is to 1 or −1, the stronger the relationship is between the two variables. The variable *r* is used to represent the correlation coefficient.

The correlation coefficient falls between −1 and 0 when the data show a negative association or between 0 and 1 when the data show a positive association.

1. Determine whether the points in each scatterplot have a positive correlation, a negative correlation, or no correlation. Four possible *r*-values are given. Circle the *r*-value you think is most appropriate. Explain your reasoning for each.

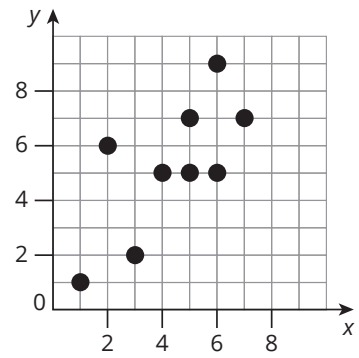
The closer the *r*-value gets to 0, the less of a linear relationship there is in the data.

a.



- $r = 0.9$
- $r = -0.9$
- $r = 0.09$
- $r = -0.09$

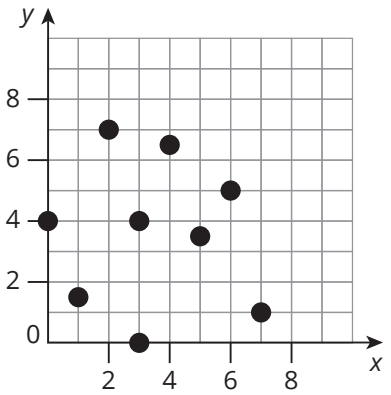
b.



- $r = 0.7$
- $r = -0.7$
- $r = -0.7$
- $r = -0.07$

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c.



- $r = 1$
- $r = 0.5$
- $r = 0.01$
- $r = -0.5$

You can calculate the correlation coefficient of a data set using the formula:

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

Fortunately, graphing technology can do this arithmetic. Previously you used graphing technology to determine the linear regression function using the Least Squares Method. Along with calculating the equation for the line, graphing technology also calculated the value *r*, the correlation coefficient.

Let's use technology to compute the value of the correlation coefficient.

2. Consider the data set $(-3, -3)$, $(1, 2)$, and $(3, 4)$.

   a. Use technology to compute the correlation coefficient.

   b. Interpret the correlation coefficient of the data set.

ACTIVITY
2.2 Is It Linear?

A group of friends completed a survey about their monthly income and how much they pay for rent each month. The table shows the results.

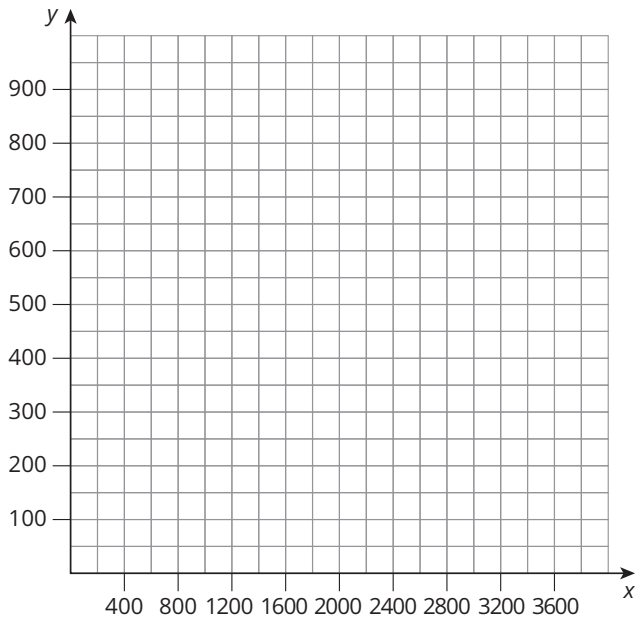
Monthly Net Income (dollars)	Monthly Rent (dollars)
1400	450
1550	505
2000	545
2600	715
3000	930
3400	1000

Ask Yourself . . .
What do you notice as you read through the data?

1. Identify the independent and dependent variables in this problem situation.

2. Construct a scatterplot of the data using technology.

   a. Sketch and label the scatterplot.



   b. Do you think a linear regression function would best describe this situation? Explain your reasoning.

Ask Yourself . . .
What is the appropriate level of accuracy needed for this linear regression function?

3. Use technology to determine whether a line of best fit is appropriate for these data.

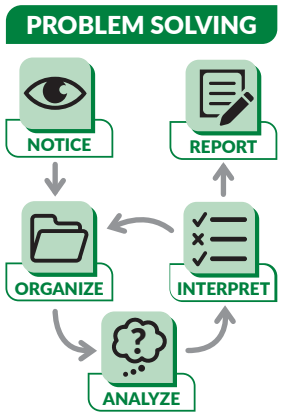
   a. Determine and interpret the linear regression function.

   b. Compute the correlation coefficient.

4. Would a line of best fit be appropriate for this data set? Explain your reasoning.

The correlation coefficient, r, indicates the type (positive or negative) and strength of the relationship that may exist for a given set of data points. The **coefficient of determination**, r^2, measures how well the graph of the regression fits the data. It represents the percentage of variation of the observed values of the data points from their predicted values.

ACTIVITY
2.3 Using the Correlation Coefficient to Assess a Line of Best Fit



The amount of antibiotic that remains in your body over a period of time varies from one drug to the next. The table given shows the amount of Antibiotic X that remains in your body over a period of two days.

Time (hours)	Amount of Antibiotic X in Body (mg)
0	60
6	36
12	22
18	13
24	7.8
30	4.7
36	2.8
42	1.7
48	1

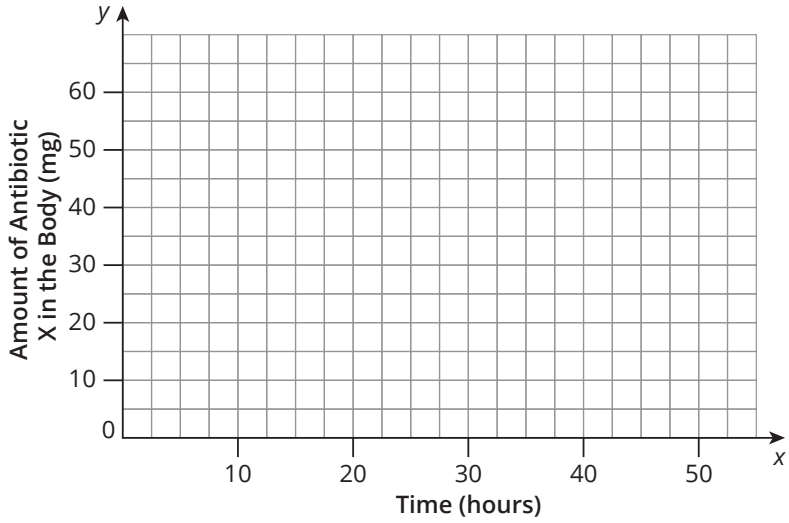
1. Determine and interpret a linear regression function for this data set.

2. Compute and interpret both the correlation coefficient and coefficient of determination of this data set.

3. Does it seem appropriate to use a line of best fit? If no, explain your reasoning.

4. Sketch a scatterplot of the data.

5. Look at the graph of the data. Do you still agree with your answer to Question 3? Explain your reasoning.



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ACTIVITY
2.4 Correlation Vs. Causation

Does correlation mean causation? What do you think causation means? That is a question that statisticians are always trying to determine.



Read the three true statements that Jackson and Ricardo are given by their Algebra I teacher. She asks them to decide what conclusions they can draw from the data. Do you agree with them? Why or why not?

1. The number of smartphones sold in the United States has increased every year since 2005. The number of flat screen televisions sold in the United States has also increased during the same period of time.

   Jackson and Ricardo reached the conclusion that owning a cell phone causes a person to buy a flat screen television.

2. Since 2004, the average salary of an NFL football player has increased every year. The average weight of an NFL player has also increased yearly since 2004.

   After much discussion, Jackson and Ricardo reached the conclusion that higher salaries cause the players to gain weight.

3. Worldwide, the number of automobiles sold annually has steadily increased since 1920. Gasoline production has also steadily increased since 1920.

   Jackson and Ricardo concluded that the increase in the number of automobiles sold caused an increase in the amount of gasoline produced.

Proving causation is challenging. The scenarios Jackson and Ricardo analyzed demonstrate that even though two quantities are correlated, this does not mean that one quantity caused the other. This is one of the most misunderstood and misapplied uses of statistics.

Causation is when one event effects the outcome of a second event. A correlation is a **necessary condition** for causation, but a correlation is not a **sufficient condition** for causation. While determining a correlation is straightforward, using statistics to establish causation is very difficult.

4. Many medical studies have tried to prove that not washing your hands can cause you to get sick.

   a. Is not washing your hands a necessary condition for getting sick? Why or why not?

   b. Is not washing your hands a sufficient condition for getting sick?

   c. Is there a correlation between people who don't wash their hands and people who get sick? Explain your reasoning.

   d. Is it true that not washing your hands causes you to get sick? If so, how was it proven?

5. It is often said that being outside for an extended amount of time can cause you to get a sunburn.

   a. Is going outside for a extended amount of time a necessary condition for a sunburn? Why or why not?

   b. Is going outside for an extended amount of time a sufficient condition for a sunburn? Why or why not?

   c. Is there a correlation between being outside for an extended amount of time and getting a sunburn? Explain your reasoning.

   d. Is it true that being outside for an extended amount of time causes sunburns? Explain your reasoning.

Does school absenteeism cause poor performance in school?
A correlation between the independent variable of days absent to the dependent variable of grades makes sense. However, this alone does not prove causation.

6. To prove that the number of days that a student is absent causes the student to get poor grades, we would need to conduct more controlled experiments.

   a. List several ways that you could design experiments to attempt to prove this assertion.

b. Will any of these experiments prove the assertion? Explain your reasoning.

There are two relationships that are often mistaken for causation. A **common response** is when some other reason may cause the same result. A **confounding variable** is when there are other variables that are unknown or unobserved.

7. Consider each relationship. List two or more common responses that could also cause this result.

   a. In North Carolina, the number of shark attacks increases when the temperature increases. Therefore, a temperature increase appears to cause sharks to attack.

   b. A company claims that their weight loss pill caused people to lose 20 pounds when following the accompanying exercise program.

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Talk the Talk

Correlations R Us

Consider the given data sets.

Set A

x	*y*
0	24
2	19
5	12
10	6
20	0

Set B

x	*y*
8	13
10	4
14	15
15	14
19	73

1. Determine the linear regression function for each set.

2. Compare the correlation coefficient and the coefficient of determination of each data set. Describe which linear regression function is the better fit and why.

Lesson 2 Assignment

Write

Complete each sentence.

1. A correlation is a _____ for causation, but a correlation is not a _____ for causation.
2. A _____ is when some other reason may cause the same result.
3. _____ is when one event causes a second event.
4. A _____ is when there are other variables that are unknown or unobserved.
5. The _____ is a value between −1 and 1 that indicates how close the data are to forming a straight line.
6. The percentage of variation of the observed values of the data points from their predicted values is represented by the _____.

Remember

Sets of data can frequently be modeled by using a linear function called a *linear regression function*. A value called the *correlation coefficient* can also be calculated to assist in determining how well the regression model fits the data.

Practice

1. The table shows the percent of the United States population who did not receive needed dental care services.

Year	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Percent	7.9	8.1	8.7	8.6	9.2	10.7	10.7	10.8	10.5	12.6	13.3

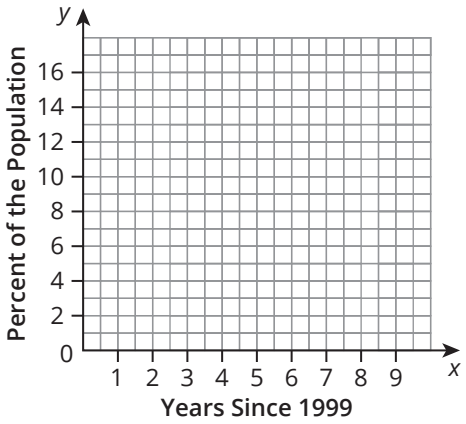
   a. Do you think a linear regression function would best describe this situation? Why or why not?

   b. Determine the linear regression function for these data. Interpret the equation in terms of this problem situation.

Lesson 2 Assignment

c. Compute and interpret the correlation coefficient of this data set. Does it seem appropriate to use a line of best fit? Explain your reasoning.

d. Sketch a scatterplot of the data. Then, plot the equation of the linear regression function on the same grid. Do you still think a linear regression model is appropriate? Explain your answer.



2. A teacher claims that students who study will receive good grades.

a. Do you think that studying is a necessary condition for a student to receive good grades?

b. Do you think that studying is a sufficient condition for a student to receive good grades?

c. Do you think that there is a correlation between students who study and students who receive good grades?

Lesson 2 Assignment

d. Do you think that it is true that studying will cause a student to receive good grades?

e. List two or more confounding variables that could have an effect on this claim.

3. For each situation, decide whether the correlation implies causation. List reasons why or why not.

   a. The number of newspapers sold in a city is highly correlated to the number of runs scored by the city's professional baseball team.

   b. The number of mouse traps found in a person's house is highly correlated to the number of mice found in their house.

Prepare

Use what you know about arithmetic sequences to complete each task.

1. Rewrite the sequence $a_n = 10 - 3(n - 1)$ in simplest terms. Then, write the first 5 terms of the sequences generated by a_n.

2. Given the function $f(x) = -3x + 10$, calculate $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$.

3 Making Connections Between Arithmetic Sequences and Linear Functions

OBJECTIVES

- Use algebraic properties to prove the explicit formula for an arithmetic sequence is equivalent to the equation of a linear function.
- Relate the defining characteristics of an arithmetic sequence, the first term and common difference, and the defining characteristics of a linear function, the *y*-intercept and slope.
- Connect the slope of a line to the average rate of change of a function.

NEW KEY TERMS

- conjecture
- first differences
- average rate of change

You know that all sequences are functions. What type of function is an arithmetic sequence?

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Getting Started

Line Up in Sequential Order

Ask Yourself . . .
What is the difference between an arithmetic and geometric sequence?

Lauren counts the number of new flowers that are blooming in her garden each day in the spring. Sequence A represents the number of new flowers on day 1, day 2, etc.

Sequence A: 3, 6, 12, 24, 48

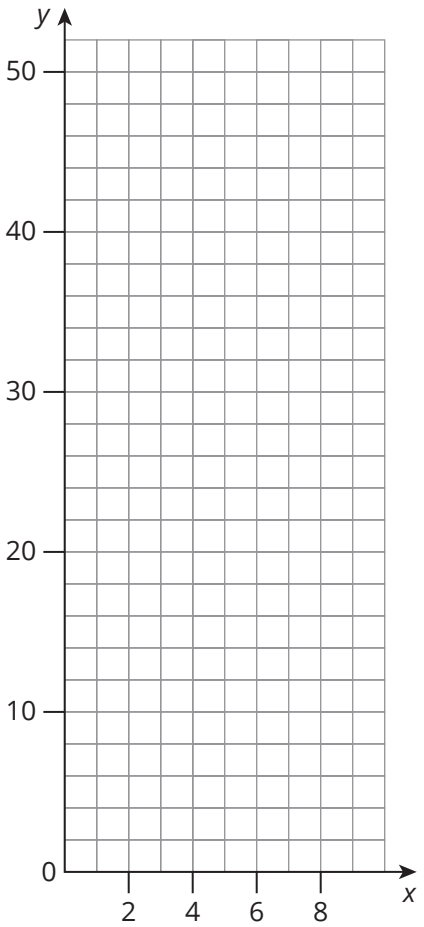
She also measures the height of the first sunflower that has started blooming. Sequence B represents the height in centimeters of the sunflower on day 1, day 2, etc.

Sequence B: 3, 6, 9, 12, 15

1. For each sequence, determine whether it is arithmetic or geometric and write the explicit formula that generates the sequence. Then, graph and label each on the coordinate plane.

   Sequence A:

   Sequence B:



2. List at least three common characteristics of the graphs. How do the two sequences compare?

ACTIVITY
3.1 Connecting Forms

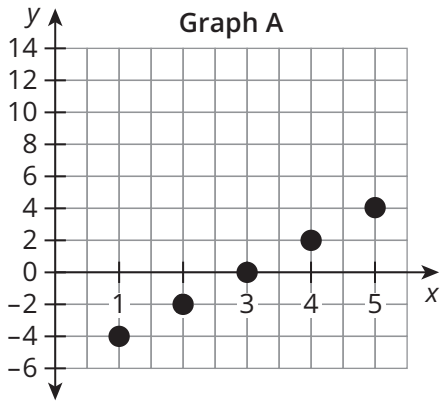
Consider the four explicit formulas, each representing a different arithmetic sequence.

- $a_n = 2 - 4(n - 1)$
- $a_n = 2 + 4(n - 1)$
- $a_n = -4 + 2(n - 1)$
- $a_n = 4 + 2(n - 1)$

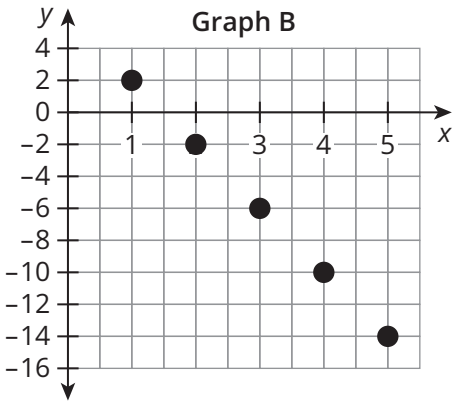
Ask Yourself . . .
How do you know by the form of the explicit formula that it represents an arithmetic sequence?

1. Match each explicit formula with its graph. Describe the strategies you used.

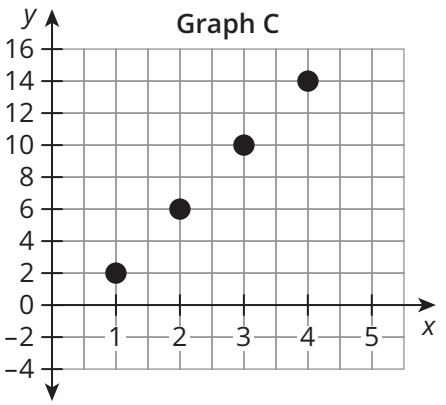
Graph A



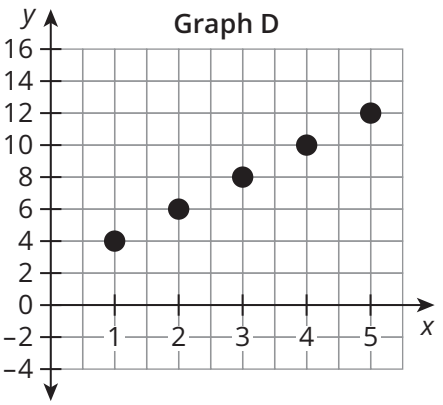
Graph B



Graph C



Graph D



2. Consider the set of graphs and identify the function family represented. Based on these formulas and graphs, do you think that all arithmetic sequences belong to this function family? Explain your *conjecture*.

A **conjecture** is a mathematical statement that appears to be true, but has not been formally proven.

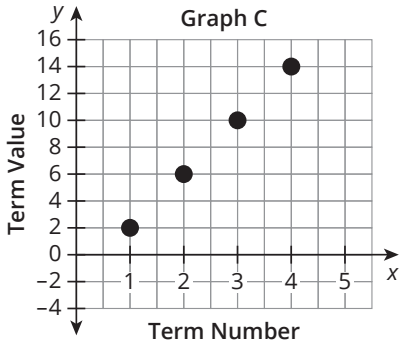
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Remember . . .

An arithmetic sequence is a sequence of numbers in which the differences between any two consecutive terms is constant. The explicit formula is of the form $a_n = a_1 + d(n - 1)$.

Let's take a closer look at the relationship between arithmetic sequences and the family of linear functions. You know a lot about each relationship.

The explicit formula and the table of values that represents Graph C are shown.



$a_n = 2 + 4(n - 1)$

Term Number n	Term Value a_n	
1	a_1	2
2	a_2	6
3	a_3	10
4	a_4	14

3. Describe the domain of the sequence.

4. Identify the common difference in each representation.

A linear function written in general form is $f(x) = ax + b$, where a and b are real numbers. In this form, a represents the slope and b represents the y-intercept.

5. Draw a line to model the linear relationship seen in the graph. Then, write the equation to represent your line. Describe your strategy.

6. Describe the domain of the graph of the linear model.

An explicit formula is one equation that you can use to model an arithmetic sequence. You can rewrite the explicit formula for the arithmetic sequence $a_n = 2 + 4(n - 1)$ in function notation.

WORKED EXAMPLE

You can represent a_n using function notation.

$$a_n = 2 + 4(n - 1)$$
$$f(n) = 2 + 4(n - 1)$$

Next, rewrite the expression $2 + 4(n - 1)$.

$f(n) = 2 + 4n - 4$	Distributive property
$= 4n + 2 - 4$	Commutative property
$= 4n - 2$	Combine like terms

So, $a_n = 2 + 4(n - 1)$ written in function notation is $f(n) = 4n - 2$.

7. Compare the equation you wrote to model the graph of the arithmetic sequence to the explicit formula written in function form. What do you notice?

8. Compare the common difference with the slope. What do you notice?

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9. Explain why Nakota's reasoning is not correct.

Nakota

The y-intercept of a linear function is the same as the first term of an arithmetic sequence.

ACTIVITY 3.2

Connecting Constant Difference and Slope



At the end of the last show, stagehands at the community theater stack the audience chairs and place them in storage. The height of one chair is 34 inches, and as each additional chair is stacked, the height increases by 8 inches.

1. Write an equation using the explicit formula to represent this scenario.

2. Rewrite the explicit formula in function form and define your variables.

3. Compare the two algebraic representations of this scenario.
   a. How is the value of d in the explicit formula related to the value of a in function form? How are d and a represented in the scenario? Be sure to include units of measure.

   b. How is the value of a_1 from the explicit formula related to b in function form? How are a_1 and b represented in the scenario?

c. If a_1 represents the first term of a sequence, what does a_0 represent? How can you rewrite the arithmetic sequence using a_0?

$a_n =$ _____

In this stacked chair scenario, both d and a represent the additional 8 inches of height per one chair, or the unit rate of change. The common difference, d, of the explicit formula is the same as the slope, a, of a general linear function. They both represent constant change.

In a sequence, the common, or constant, difference is the difference in term values between consecutive terms.	In a linear function, the slope describes the direction and steepness of the line. It is the difference in output values divided by the difference in corresponding input values.

When you see a graph or rewrite an explicit formula for an arithmetic sequence, it is apparent that it represents a linear function. However, the structure of a table requires other strategies to determine whether it represents a linear function.

One strategy is to examine *first differences*. **First differences** are the values determined by subtracting consecutive output values when the input values have an interval of 1. If the first differences of a table of values are constant, the relationship is linear.

The tables that represent the explicit formula and function form of the stacking chairs scenario are shown.

n	a_n
1	34
2	42
3	50
4	58

x	$y = f(x)$
0	26
1	34
2	42
3	50

The expression $y = f(x)$ means that the value of y depends on the value of x. That is, for different values of x, there is a function f which determines the value of y.

4. Determine the first differences in each table to verify they both represent a linear relationship.

5. Joey and Alejandra agree that the table shown represents a linear relationship because there is a constant difference between consecutive points. Joey claims the slope is 5 and Alejandra claims the slope is −5.

Who's correct? Explain your reasoning.

x	y
1	22
2	17
3	12
4	7

22 − 17 = 5

17 − 12 = 5

12 − 7 = 5

Ask Yourself . . .
What does slope describe?

6. Use first differences to determine whether each table represents a linear function. Then, create your own table to represent a linear function. Describe your strategy.

a.

x	y
5	12
6	15
7	21
8	30

b.

x	y
−2	18
−1	14
0	10
1	6

c.

x	y
10	1
11	4
12	9
13	16

d.

x	y

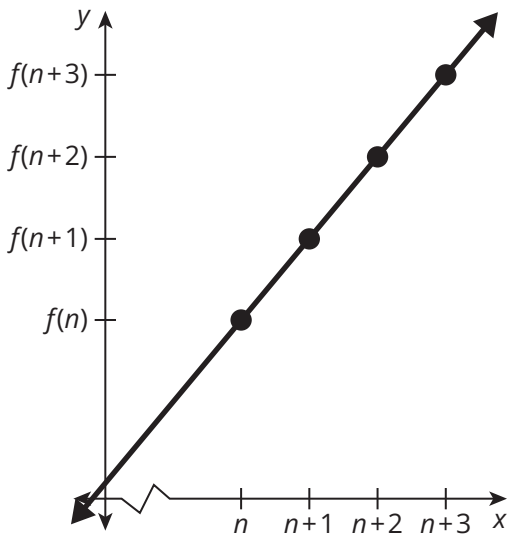
ACTIVITY
3.3 Connecting Constant Difference, Slope, and Average Rate of Change

In the previous activity, you determined that the constant difference of the chair heights and the slope of the line are both equal to 8 inches per chair. Is the slope of a linear function always equal to the constant difference of the corresponding arithmetic sequence?

Consider the graph of the arithmetic sequence represented by the general linear function $f(x) = ax + b$.

1. Identify the constant difference of the sequence in the graph.

2. Complete the table for consecutive values of the input.



Term Number	Term Value	
n	$f(n)$	$a(n) + b$
$n + 1$	$f(n + 1)$	$a(n + 1) + b$
$n + 2$		
$n + 3$		

3. Select any two consecutive input values in the sequence. Use the expressions for the term values to determine the constant difference of the sequence.

Recall that the slope of a line is constant between any two points on the line, not just consecutive points.

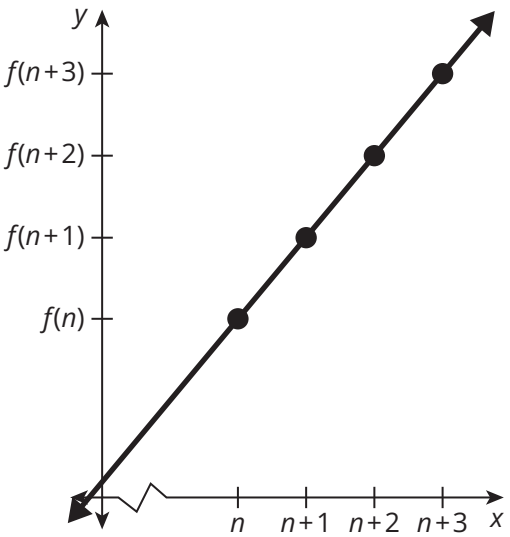
Remember . . .

The slope formula is $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{\Delta y}{\Delta x}$. Delta ($\Delta$) is a Greek letter used in mathematics to represent "change in". So, $\frac{\Delta y}{\Delta x}$ means "change in y" divided by "change in x".

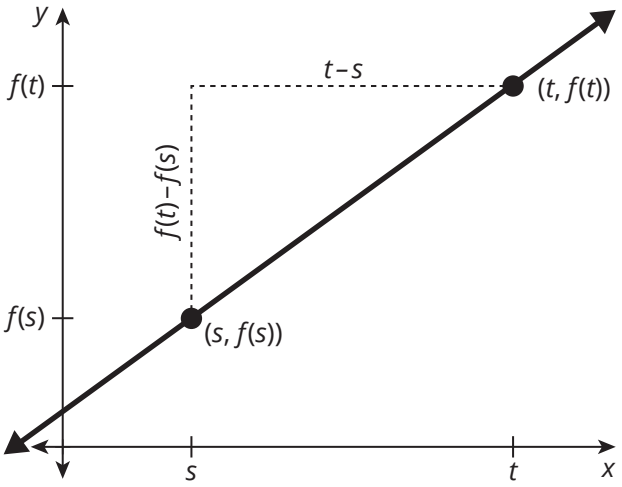
4. Use the table and graph from Question 2 to complete each task.

   a. Identify the slope of the function on the graph.

   b. Select two non-consecutive points in the table and determine the slope between those two points.



The slope, a, is equal to the constant difference. Another name for the slope of a linear function is **average rate of change**. The formula for the average rate of change is $\frac{f(t)-f(s)}{t-s}$. This represents the change in the output as the input changes from s to t.



5. Show that the slope formula and the average rate of change formula represent the same ratio.

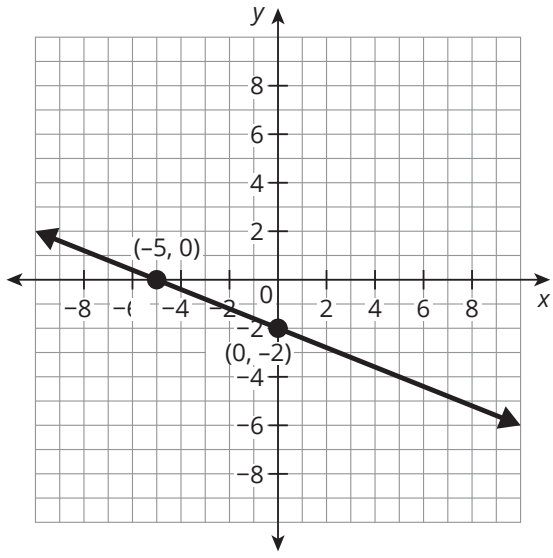
6. Determine the average rate of change for each graph, table, or function.

   a. $g(x) = 3x + 2$

   b.

x	y
4	1
6	−1
8	−3
10	−5

   c.



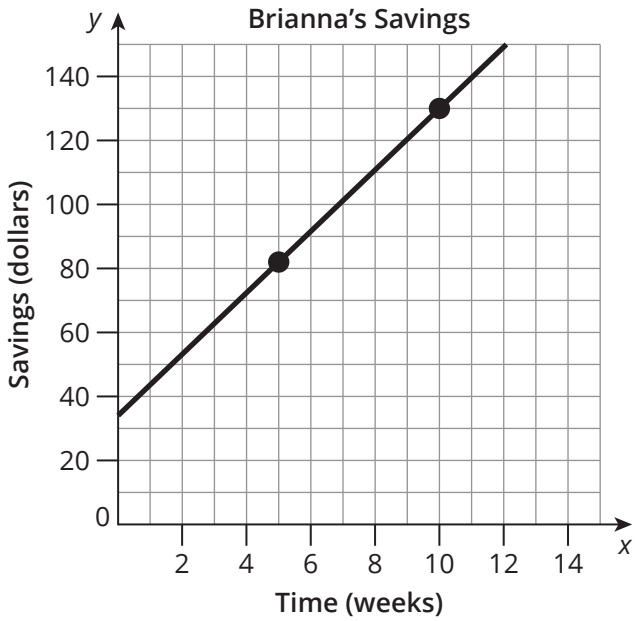
   d. The table shows the linear relationships between the distance Hailey runs and the time he spends running.

Average Distance Traveled

Time (minutes)	4	6	8	12
Distance (miles)	0.5	0.75	1	1.5

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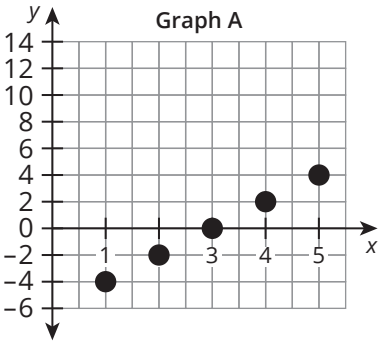
e. The graph shows the relationship between the balance of Brianna's bank account and the number of weeks she has been saving.

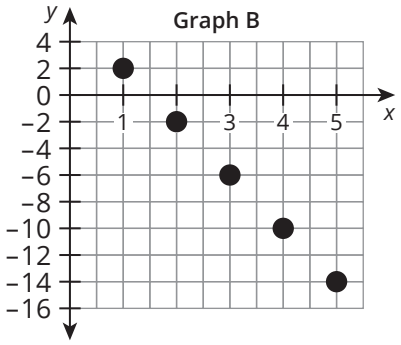


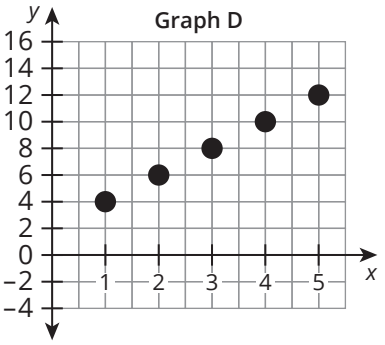
f. The function $y = 2.5x - 15.25$ models the amount of money the student council earns from selling bagels after spending money on supplies for the sale.

ACTIVITY
3.4 Verifying that Common Differences and Slopes are the Same

The remaining explicit formulas and graphs from Activity 1 are shown.







$a_n = -4 + 2(n - 1)$ $a_n = 2 - 4(n - 1)$ $a_n = 4 + 2(n - 1)$

1. For each graph and arithmetic sequence, complete each task.

   a. Identify the constant difference in the explicit formula and on the graph.

   b. Rewrite each explicit formula in function notation.

   c. Verify that the constant difference is the same as the slope of the linear function.

   d. Describe how the first term in the explicit formula is related to the *y*-intercept of the function.

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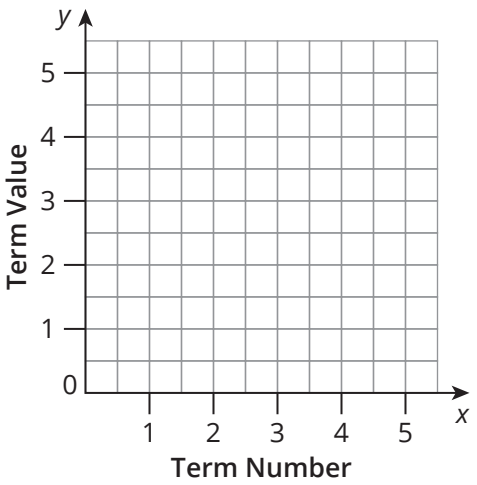
ACTIVITY
3.5 Describing Constant Sequences

Lauren's cat loves to knock over flowerpots. Each morning, she counts the number of flowerpots her cat knocked over during the night before she uprights them again. Sequence C represents the number of flowerpots knocked over on day 1, day, 2, etc.

Sequence C: 3, 3, 3, 3, 3

1. Determine the constant difference and write the explicit formula to represent Sequence C. Then, create a table of values and graph it.

Term Number (n)	Term Value (a_n)



2. Use function notation to write an equation representing the relationship between the number of days, x, and the number of knocked over flowerpots, C. Interpret the function in terms of the context.

3. Prove that the slope of $C(x)$ is equal to the common difference of Sequence C.

Consider a similar sequence.

Sequence D: −5, −5, −5, −5, −5

4. Write the function, $D(x)$, to model this sequence.

5. Prove $D(x)$ is a constant function.

Remember . . .

When the the values of the dependent variable of a function remain constant over the entire domain, then the function is called a constant function.

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Talk the Talk

Making It Plain and Clear

You have proven that all arithmetic sequences can be represented by linear functions.

1. Complete the graphic organizer to summarize the connections between arithmetic sequences and linear functions. Then, describe how you can tell a linear relationship exists given a table of values or a graph.

Equation

Arithmetic Sequence	Linear Function	Mathematical Meaning
$a_n = a_1 + d(n - 1)$	$f(x) = ax + b$	
a_n		
d		
n		
$a_1 - d$		

Characteristics and Representations of Linear Functions

Table of Values

Graph

Lesson 3 Assignment

Write

Describe how the terms *constant difference*, *slope*, and *average rate of change* are related.

Remember

The explicit formula of an arithmetic sequence can be rewritten as a linear function in the general form $f(x) = ax + b$, where a and b are real numbers, using algebraic properties. The constant difference of an arithmetic sequence is always equal to the slope of the corresponding linear function.

Practice

1. Gracie claims that the equation $f(n) = 5n - 7$ is the function notation for the sequence that is represented by the explicit formula $a_n = -2 + 5(n - 1)$. Nicky doesn't understand how this can be the case.

   a. Help Nicky by listing the steps to write the explicit formula of the given sequence in function notation. Provide a rationale for each step.

   b. Graph the function. Label the first 5 values of the sequence on the graph.

Lesson 3 Assignment

2. Determine whether each table of values represents a linear function. For those that represent linear functions, write the function. For those that do not, explain why not.

a.

x	$f(x)$
3	14
4	18
5	23
6	29

b.

x	$g(x)$
0	2
1	−1
2	−4
3	−7

c.

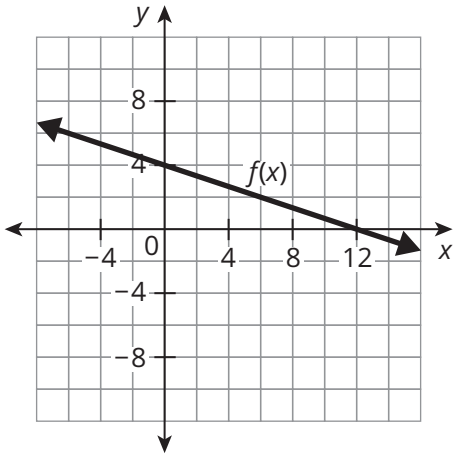
x	$h(x)$
1	11
2	16
3	21
4	26

3. Calculate the average rate of change for each linear function using the formula. Show your work.

a.

x	$f(x)$
3	−4
7	4
9	8
12	14

b.

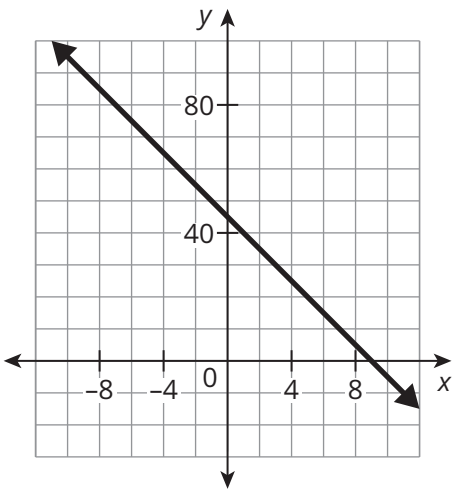


Lesson 3 Assignment

Prepare

Write an equation for each linear relationship.

1. The contestant on a game show has already won a total of $2750 from a previous episode when the game show continued today. He earns an additional $250 for each question he answers correctly today.

2. 

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4 Point-Slope Form of a Line

OBJECTIVES

- Use the slope formula to derive the point-slope form of a linear equation.
- Construct an equation in point-slope form to model a linear relationship between two quantities.
- Write equations for vertical and horizontal lines.

NEW KEY TERM

- point-slope form

You have used the slope-intercept form to represent linear relationships. Are there other forms of a linear equation that you can use? How do you write equations for horizontal and vertical lines?

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PROBLEM SOLVING



Draining the Pool

Miguel and Nia are pool cleaners who have been hired to drain the community diving pools at the end of the summer. They are comparing the rate at which the three pools drain.

1. For each pool, write an equation in slope-intercept form to represent the linear relationship.

   a. Pool *A* is at a water level of 14 feet and drains at a rate of 3 feet per hour.

I wonder whether there is a way to make writing the equation of a line more efficient.

   b. Pool *B* is at a water level of 10 feet after draining for 3 hours and drains at a rate of 2 feet per hour.

   c. Pool *C* is at a water level of 15 feet after draining for 2 hours and at 12 feet after draining for 4 hours.

2. Compare your process for writing each equation. How are the processes different?

ACTIVITY 4.1

Writing Equations in Point-Slope Form

In the previous lesson, you used the slope, the *y*-intercept, and the slope formula to write a linear equation. You can also determine the equation of a line without knowing the *y*-intercept.

WORKED EXAMPLE

To write an equation of a line from a table of values, you can use the slope formula.

x	y
2	6
4	5
6	4

- First, calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 6}{4 - 2}$$
$$= \frac{-1}{2} = -\frac{1}{2}$$

- Next, choose any point from the table.

(2, 6)

- Then, substitute what you know into the slope formula: $m = -\frac{1}{2}$, (2, 6), and the unknown point (x, y).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$-\frac{1}{2} = \frac{y - 6}{x - 2}$$

- Finally, rewrite the equation with no variables in a denominator.

$$-\frac{1}{2} = \frac{y - 6}{x - 2}$$
$$-\frac{1}{2}(x - 2) = y - 6$$

The equation is $y - 6 = -\frac{1}{2}(x - 2)$.

This linear equation in the Worked Example is written in *point-slope form*. The **point-slope form** of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is any point on the line.

1. Solve the equation in the Worked Example for y so that the linear equation is in slope-intercept form. What unique information does each form of the linear equation provide? How are they similar?

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Write the equation for each linear relationship in point-slope form.

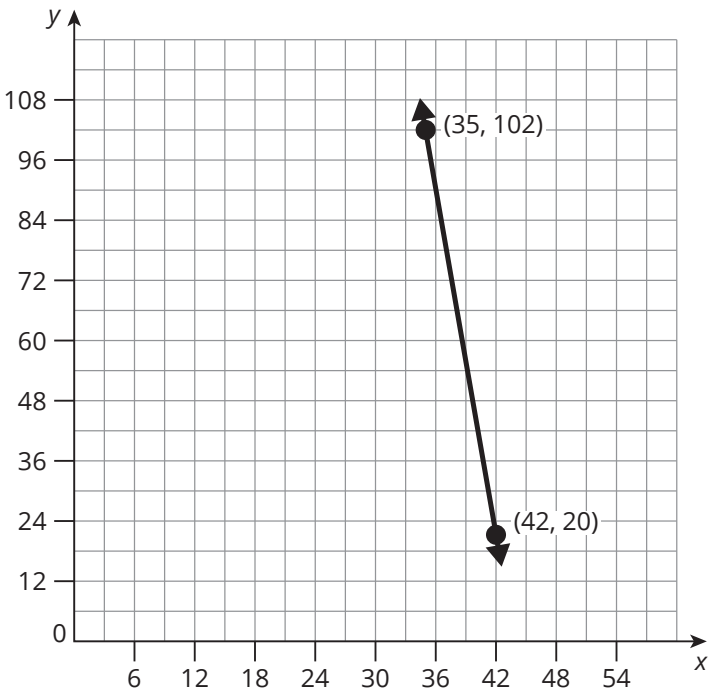
2. The slope is -8. The point (3, 12) lies on the line.

3. The slope is $\frac{2}{3}$. The point (−4, 5) lies on the line.

4. It costs \$1.10 per pound plus a basic shipping charge to mail a package. A 10-pound package costs \$14 to mail.

5. $m = -\frac{3}{8}$ and point (50, 7)

6.



7.

x	y
−5	−6
1	−6
2	−6

8. Rewrite the equation you wrote for Question 7 in slope-intercept form. What do you notice about this equation?

9. Consider the information you may have about a linear relationship. Which form of the equation do you prefer to use in each case? Explain your reasoning.

   a. Given slope and y-intercept

   b. Given two points

   c. Given slope and a point other than the y-intercept

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ACTIVITY
4.2 Horizontal and Vertical Lines

Horizontal and vertical lines represent linear relationships, but their equations are different from the equations of lines that are not horizontal or vertical.

1. Consider the equation, $y = -6$, that you wrote for the table shown in the previous activity.

   a. How is this equation different from the other equations? What is its slope?

   b. Describe the graph of the coordinate pairs in this table. Why does the value of its slope make sense?

x	y
−5	−6
1	−6
2	−6

What is the *y*-intercept?

   c. Explain why the equation makes sense in terms of the graph and the table.

2. Write an equation for each linear relationship. Describe the graph of the linear relationship. State the slope and *y*-intercept.

   a.

x	y
−7	11
−2	11
0	11

   b. A line that passes through (−15, −3.75) and (89, −3.75)

3. Consider a new table of values representing a linear relationship.

x	y
−2	5
−2	14
−2	29

   a. Explain how this table is similar to and different from the tables in Questions 1 and 2.

   b. Write an equation for the linear relationship in the table.

   c. Describe the graph of this linear relationship.

   d. Use the slope formula to calculate the slope between two points in the table. What do you notice?

   e. What is the y-intercept of this linear relationship? Explain why this makes sense.

4. Write an equation for each linear relationship. Describe the graph of the linear relationship.

   a.

x	y
$\frac{17}{2}$	−18
$\frac{17}{2}$	23
$\frac{17}{2}$	267

   b. A line that passes through (−7, −973) and (−7, 542)

   c. Create an additional table of values and write the equation for a vertical line.

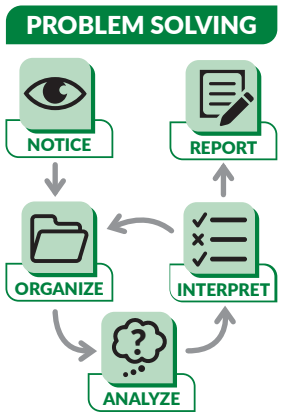
In a horizontal line there is no change in the y-values as the x-values change. Therefore, the slope is 0. A horizontal line has zero steepness. In a vertical line there is no change in the x-values as the y-values change. Therefore, the slope is undefined. A vertical line has an undefined steepness.

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ACTIVITY
4.3 Matching Representations

Take out your scissors. It's time to cut and sort!

1. Carefully cut out the graphs, tables, contexts, and equations located the end of the lesson. Match each equation with its correct graph, table, or context. Explain how you matched the equations with the representations.



2. Compare the graphs.

   a. How are they different? How are these differences reflected in the slope-intercept form of their equation?

   b. Identify the *y*-intercept for each graph. How can you determine this point in the slope-intercept form of the equation for each graph?

   c. Identify the slope for each graph. How is the slope represented in the slope-intercept form of each equation?

3. Analyze the equation for each table.

   a. Determine the coefficient of x for each linear relationship using the slope formula.

Can you remember the ways to determine the rate of change from a table?

   b. How can the number that is added in each equation written in slope-intercept form be determined from the table?

4. Analyze the equation for each context. Explain what each term of the equation means in each context.

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Talk the Talk

Say What?

You have learned about two forms of a linear equation: the slope-intercept form, $y = mx + b$, and the point-slope form, $y - y_1 = m(x - x_1)$.

1. What information can you determine about each line by looking at the structure of the equation?

   a. $y = \frac{3}{5}x - 4$

   b. $y - 6 = 2(x + 1)$

   c. $y + 4 = 2(x - 0)$

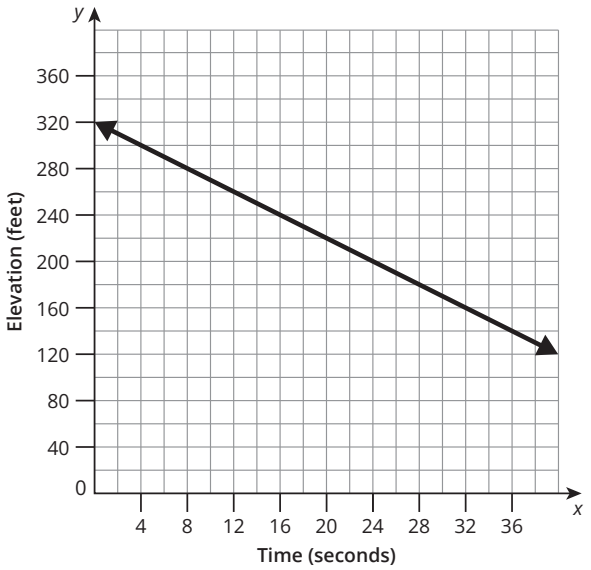
   d. $y = -\frac{2}{7}x$

   e. $y + 5 = -(x - 4)$

   f. $y = 19$

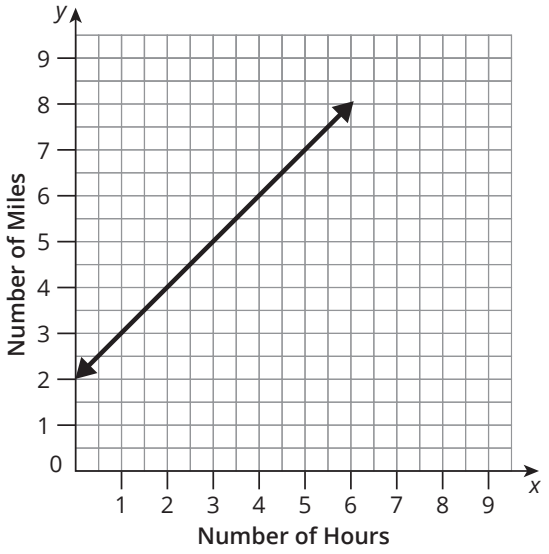
2. Create a context that represents a linear relationship that passes through the point (2, 56) and has an increasing slope. Then, write the equation of the line in point-slope form and slope-intercept form.

a.



b. Juliana read the first 40 pages of a mystery novel before she fell asleep. The next day, she read one page every two minutes until she finished the book, which was a total of 325 pages.

c.



d.

Time (hours)	Water Level (feet)
x	*y*
2	15
4	13.5
8	10.5
10	9

e.

Number of Games Ben Won Today	Number of Credits on Ben's Player's Card
x	*y*
12	216
18	264
25	320
40	440

f. A pizza shop charges $4.50 for a small pizza, $7.00 for a medium pizza, and $9.00 for a large pizza. Additional toppings cost extra depending on the size of the pizza ordered. Antonio ordered a large pizza with three toppings that cost a total of $12.60.

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So you can cut out the cards on the other side.

$y = 1.2x + 9$	$y = -\frac{3}{4x} + \frac{33}{2}$
$y = \frac{1}{2}x + 40$	$y - x = 2$
$y - 200 = -5(x - 24)$	$y = 8x + 120$

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So you can cut out the cards on the other side.

Lesson 4 Assignment

Write

Compare the slope-intercept and point-slope forms of a linear equation.

Remember

The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is a point on the line. The slope of a horizontal line is 0. The slope of a vertical line is undefined.

Practice

Write an equation in point-slope form.

1. $m = 2$; (5, 6)

2. $m = -9.2$; (−17, 10)

3. (−2, −3) and (8, −8)

4. (79, 52) and (−87, 550)

5. A photography studio charges $50 for a sitting fee and 6 prints. Luigi increased his order to 11 prints and paid $65.

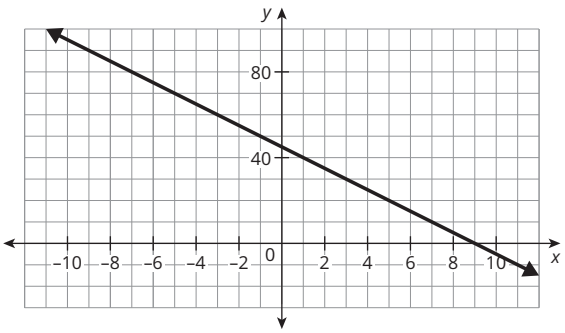
6. Lucia is taking the stairs in her building from her floor to the top of the building. After 2 minutes, she was 100 steps from the bottom floor. After 5 minutes, she was 196 steps from the bottom floor.

Write an equation in any form.

7. A newspaper charges a flat fee plus a charge per day to place a classified ad.

Number of Days	Total Charge ($)
2	8.00
4	13.00
6	18.00

8.



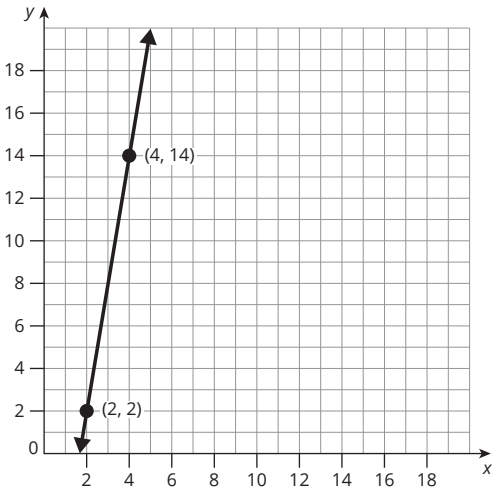
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Lesson 4 Assignment

9.

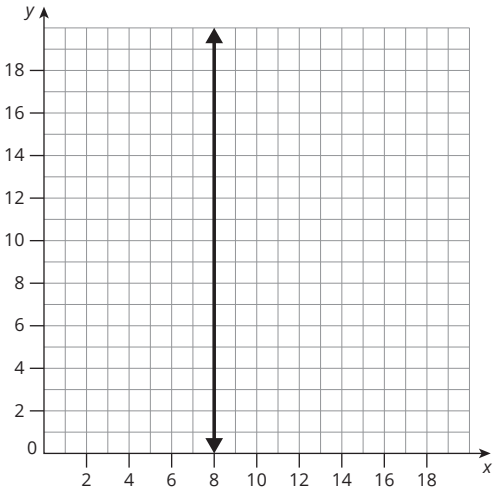
x	y
−10	50
−2	10
4	−20
14	−70

10.



11. Xavier is traveling on a toll road. He plans to exit the road 5 miles ahead and pay $1.75. He changes his plans and travels 9 miles and pays $2.75.

12.



Prepare

Solve each equation for y.

1. $-2y = -x + 7$

2. $\frac{3}{4}y = x - 6$

3. $2x + 3y = 6$

4. $\frac{1}{2}x - 4y = 8$

5 Using Linear Equations

OBJECTIVES

- Construct linear equations to model relationships between two quantities.
- Graph lines using the slope-intercept form of a linear equation.
- Graph lines using the point-slope form of a linear equation.
- Graph lines using the standard form of a linear equation.
- Convert equations from point-slope or standard form to slope-intercept form.
- Discuss the advantages and disadvantages of slope-intercept, point-slope, and standard form.

NEW KEY TERMS

- standard form
- *x*-intercept

You have graphed equations using tables of values.

Is there a more efficient method to graphing a linear relationship? Can you use the equation of a linear relationship to create a graphical representation?

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Getting Started

Jump In the Line

Describe what you know about the graph of each relationship by analyzing each equation. Then, explain how you might graph each line given its equation.

1. $y = \frac{2}{3}x + 7$

2. $y - 3 = 5(x + 1)$

3. $x = -4$

4. $-3x + 8y = 10$

ACTIVITY 5.1

Using Slope-Intercept Form to Graph a Line

As you learned previously, the slope-intercept form of a linear equation is $y = mx + b$, where m is the slope of the line and $(0, b)$ is the y-intercept. You can use the equation to graph the relationship without first creating a table of values using the y-intercept and the slope.

Douglas is giving away tickets to a concert that he won from a radio station contest. Currently, he has 10 tickets remaining. He gives a pair of tickets to each person who asks for them.

This situation can be modeled by the equation $y = -2x + 10$, where x represents the number of people who request tickets and y represents the number of tickets available.

To graph the equation $y = -2x + 10$, you will first plot the y-intercept, (0, 10), and then use the slope, -2, to plot two more points. Remember, slope describes the steepness and direction of a line. Slope is the ratio of the change in y-values to the change in x-values, commonly referred to as rise over run. In this equation, you can think of $m = -2$ as two different ratios: $\frac{-2}{1}$ or $\frac{2}{-1}$. The sign of the number tells you the direction to go to plot a new point. The ratio $\frac{-2}{1}$ has a negative rise and a positive run. It is interpreted as down 2 units and to the right 1 unit. The ratio $\frac{2}{-1}$ has a positive rise and a negative run. It is interpreted as up 2 units and to the left 1 unit.

A rule of thumb when graphing a line is to plot at least three points.

WORKED EXAMPLE

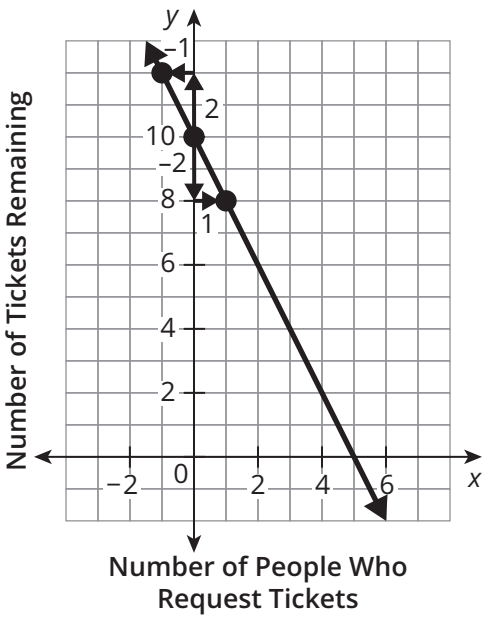
Graph $y = -2x + 10$.

Begin by plotting the y-intercept, (0, 10).

Use the slope and count from the y-intercept to graph two more points on the line.

- For $m = \frac{-2}{1}$, go down 2 units and to the right 1 unit.
- For $m = \frac{2}{-1}$, go up 2 units and to the left 1 unit.

Connect the points to form a straight line.

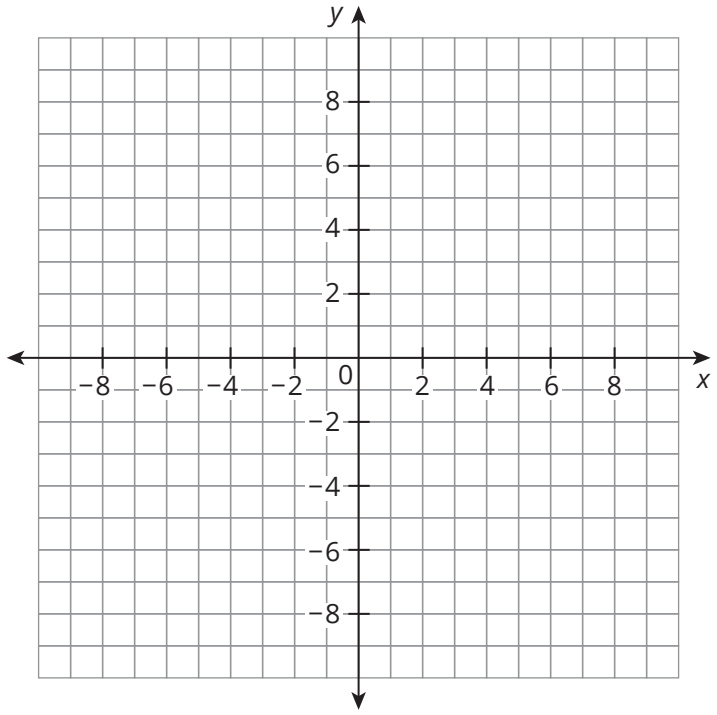


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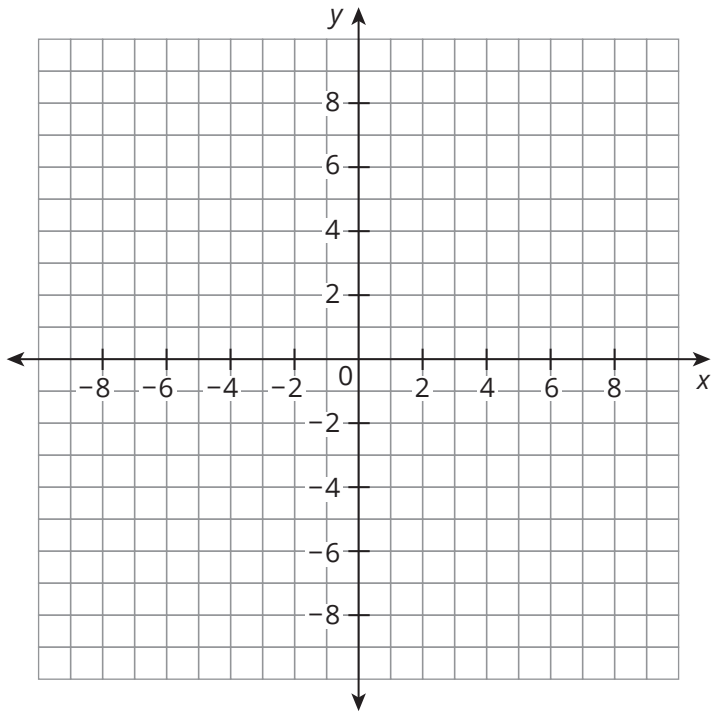
Use the equations to graph each line.

1. $y = \frac{3}{2}x - 1$

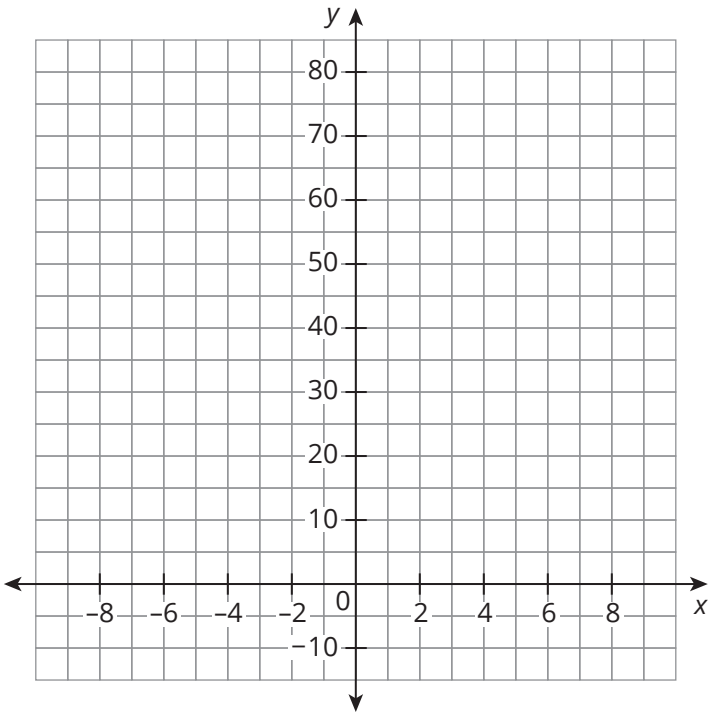
Think about the direction of the line before you start graphing.



2. $y = -\frac{5}{2}x + 3$



3. $y = 10x + 25$

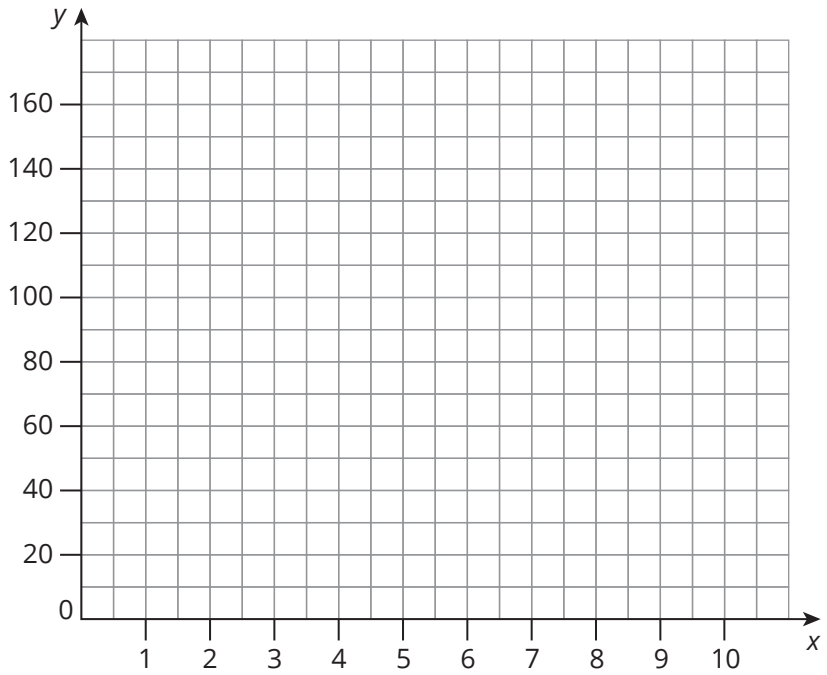


Use a straightedge to draw your lines.

Ask Yourself . . .
Should each graph have arrows or points on each end? When using a point, how do you know whether to use an open circle or a closed circle?

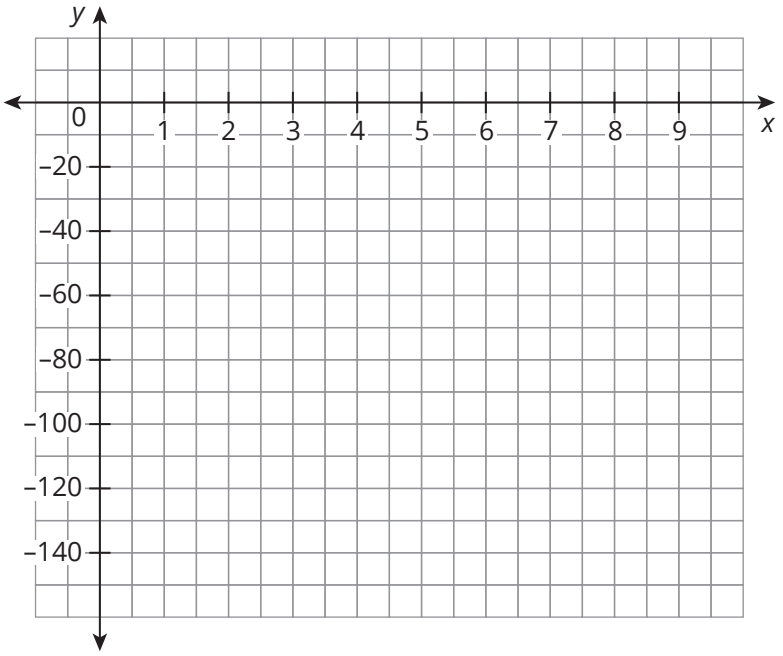
Write an equation for each problem situation. Then, graph each equation.

4. Avery wants to buy a virtual reality headset. He already has $30 saved and plans to mow lawns for $15 each.



Remember to label the axes with the appropriate variable quantities.

5. A scuba diver is 130 feet below sea level. The diver ascends at a constant rate of 20 feet per minute.



ACTIVITY
5.2 Using Point-Slope Form to Graph a Line

In the previous lesson, you learned that the point-slope form of a linear equation is $y - y_1 = m(x - x_1)$ where m is the slope of the line and (x_1, y_1) is a point on the line. Use this form of a linear equation for the next problem situation.

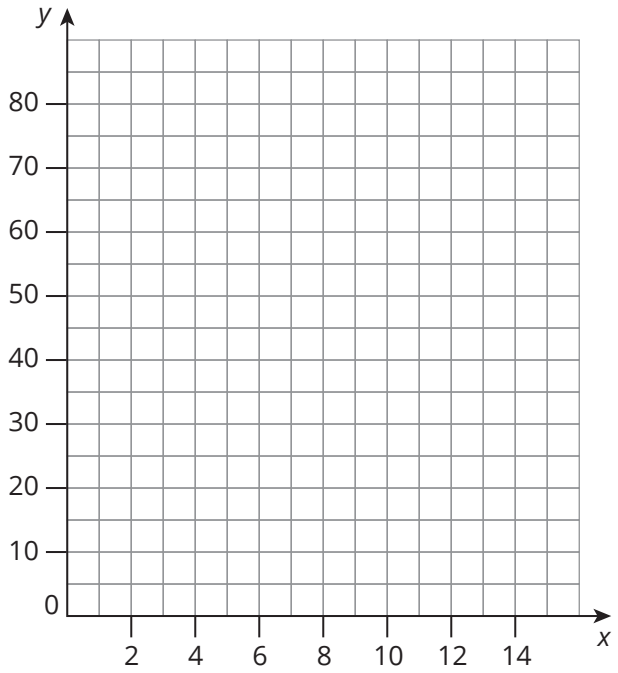
The jazz band is selling tickets to raise money for new music stands. They already had some money in their account when they started selling tickets at $5.00 each. After selling 3 tickets, the band had a total of $50 in their account.

1. Let x = the number of tickets sold, and let y = the total amount of money in the jazz band's account.

   a. Write an equation in point-slope form.

   b. Use the point-slope form to graph the equation.

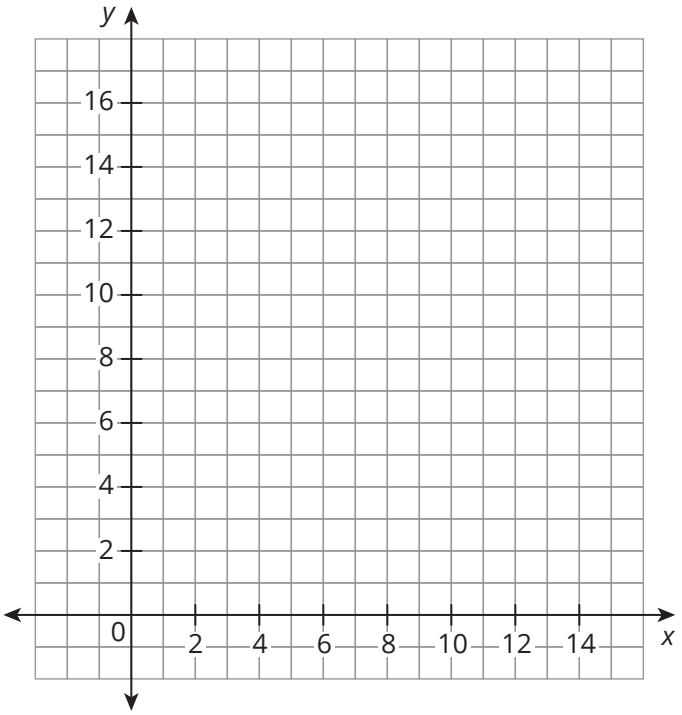
      - Write the coordinates for the known point. Plot the point on the coordinate plane.
      - Write the slope as a ratio. Then, use the slope and count from the point. To identify another point on the graph, start at the point and count either down (negative) or up (positive) for the rise. Then, count either left (negative) or right (positive) for the run.
      - Continue the counting process to plot at least two more points.
      - Connect the points to form a straight line.



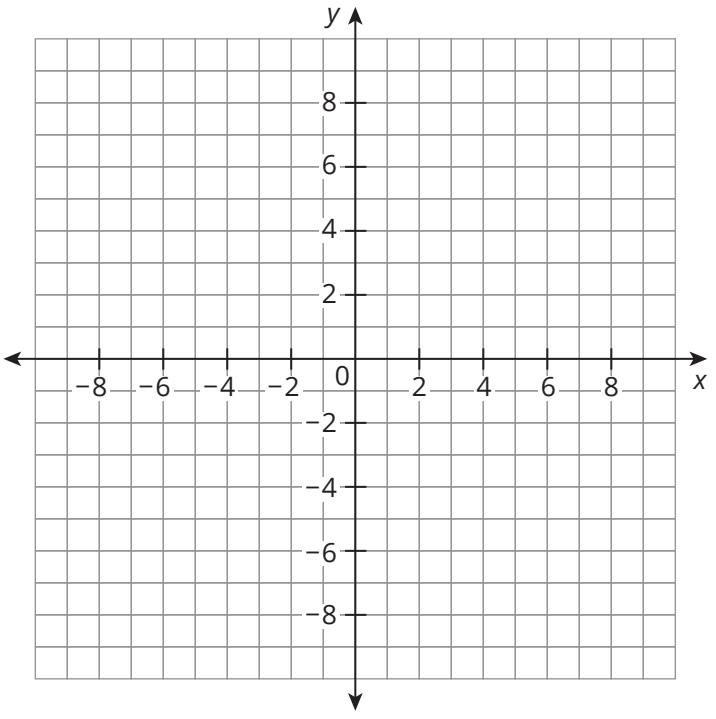
Remember, start with the given point and then use the slope to plot two more points.

2. Identify the slope and a point on the line from the given equation. Then, graph each line. Be careful to take into account the scales on the axes and the signs of the points given in the equation.

a. $y - 5 = -\frac{3}{4}(x - 14)$

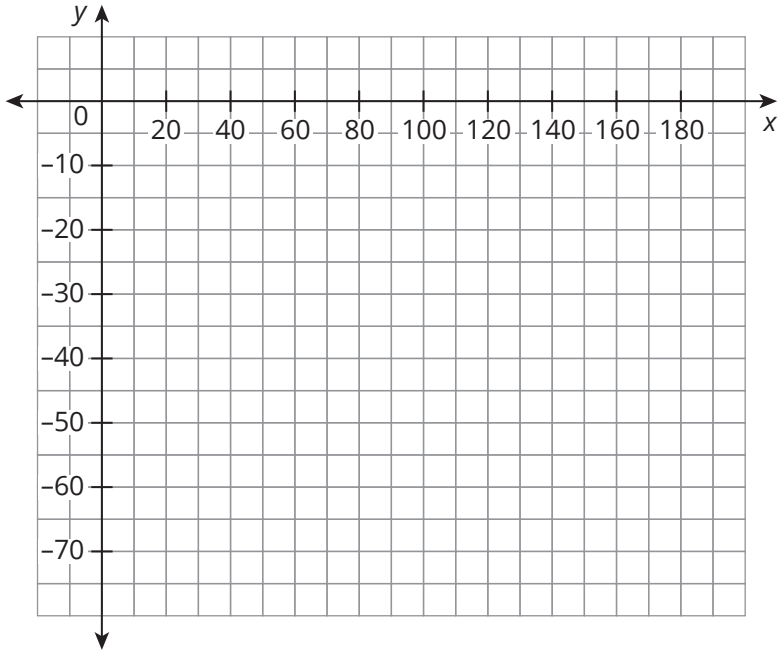


b. $y + 8 = \frac{1}{2}(x + 6)$



3. An underwater remote operated vehicle (ROV) is lowered from a boat at a constant rate of 0.3 meters per second. After 90 seconds, the ROV is at a depth of −25.8 meters.

   Define your variables and units. Then, write and graph an equation for the depth of the submarine over time.



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ACTIVITY
5.3 Standard Form of a Linear Equation

Tickets for a school play cost $5.00 for students and $8.00 for adults. On opening night, $1600 is collected in ticket sales.

This situation can be modeled by the equation $5x + 8y = 1600$. You can define the variables as shown.

x = number of student tickets sold
y = number of adult tickets sold

This equation is not written in slope-intercept form or in point-slope form. It is written in *standard form*. The **standard form** of a linear equation is $Ax + By = C$, where A, B, and C are integers and A and B are not both zero.

1. Explain what each term of the equation represents in the problem situation.

   a. $5x$

   b. $8y$

   c. 1600

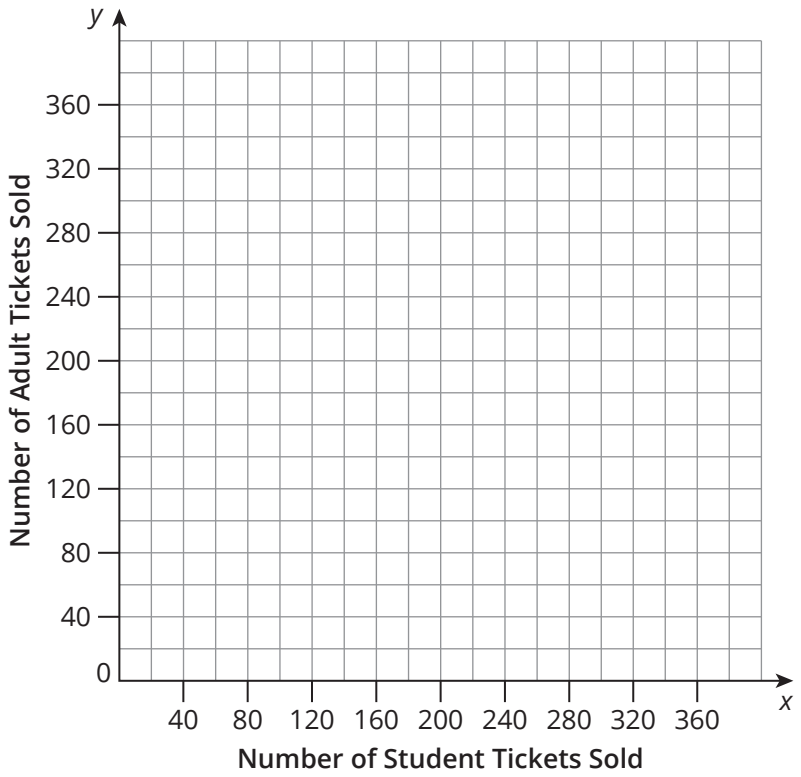
2. What is the independent variable? What is the dependent variable? Explain your reasoning.

Remember, the *y*-intercept, (0, *b*) is where a line crosses the *y*-axis, so the value of *x* is 0. To calculate a *y*-intercept, substitute 0 for *x* and solve the equation for *b*.

The ***x*-intercept**, (*x*, 0), is where the line crosses the *x*-axis, so the value of *y* is 0. To calculate an *x*-intercept, substitute 0 for *y* and solve the equation for *x*.

3. Calculate and interpret the meanings of the *x*-intercept and *y*-intercept for this equation.

4. Use the *x*-intercept and *y*-intercept to graph the equation of the line.



5. Determine the slope of this line. Interpret the meaning of the slope in this problem situation.

6. Complete the table.

Standard Form	x-Intercept	y-Intercept	Slope
$5x + 2y = 6$			
	$\left(\frac{7}{3}, 0\right)$		$m = -\frac{3}{4}$
		$(0, 23)$	$m = \frac{2}{3}$
$-5x + 7y = 11$			

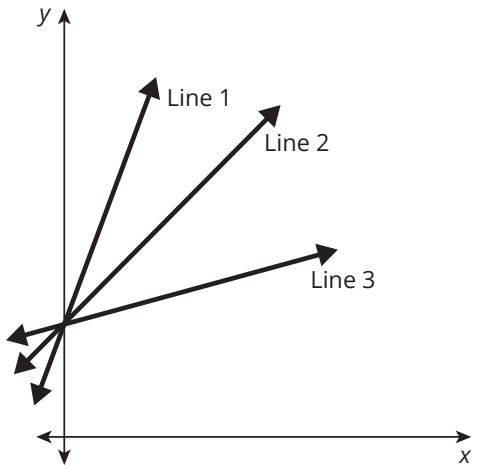
7. What do you notice about the relationship between the integers A, B, and C from the standard form and

   a. the x-intercepts?

   b. the y-intercepts?

   c. the slope?

8. Match each graph with the correct equation written in standard form. Explain your reasoning.



a. $3x - 12y = -60$

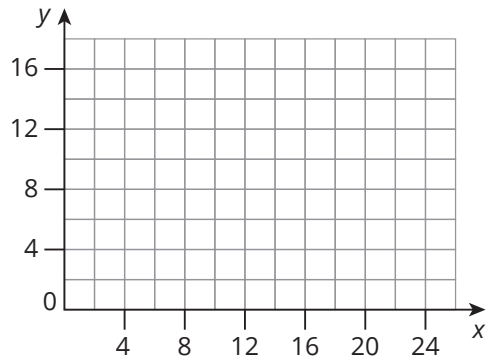
b. $6x - 2y = -10$

c. $9x - 9y = -45$

Notice that there are no values on the x- and y-axis. What strategies can you use to determine which graph goes with which equation?

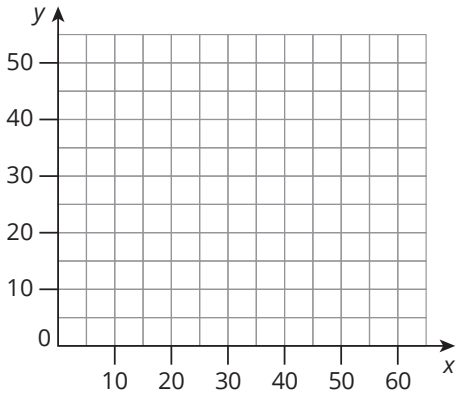
Define variables for each problem situation. Then, write an equation in standard form and use the intercepts to graph the linear relationship.

9. Nia burns 20 calories for every 5 minutes she jumps rope and 50 calories for every 5 minutes she runs. On Tuesday, Nia burned a total of 500 calories.



Ask Yourself . . .
Why is the standard form of a linear equation appropriate to represent this situation?

10. For the show choir's holiday performance, they are selling tickets for $5 per student and $6.00 per adult. On the night of their final performance, they collect $270 in ticket sales.



WORKED EXAMPLE

Write a linear equation in standard form, $Ax + By = C$, that goes through the points $(-1, 3)$ and $(-4, -3)$.

Begin by determining the slope of the line.

$$m = \frac{(-5 - 3)}{(-4 - (-1))} = \frac{(-6)}{(-3)} = 2$$

Pick one of the points and write an equation in point-slope form for the line.

$y - 3 = 2(x - (-1))$

$y - 3 = 2(x + 1)$

Rewrite the equation in standard form.

$y - 3 = 2x + 2$

$y = 2x + 5$

$-2x + y = 5$

In standard form, A, B, and C are integers.

11. Write a linear equation in standard form given two points on the line.

a. (0, −4) and (2, 2)

b. (1, −5) and (−2, 7)

c. (12, 10) and (16, 12)

ACTIVITY
5.4 Identifying Slope and *y*-Intercept

Each equation represents a linear relationship. Examine each and determine the slope and the *y*-intercept. Write the *y*-intercept as an ordered pair.

1. $y = 11x - 9$

2. $4x + 6y = 270$

3. $y = 5(2x - 9)$

4. $8y = -6x + 24$

5. $y + 9 = 6(x - 3)$

6. $y = 9$

7. $4x - 12y = 48$

8. $x = 10$

9. $y - 4 = 3(x + 1)$

10. $y = 9 - \frac{1}{2}x$

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Talk the Talk

Choose Your Medium

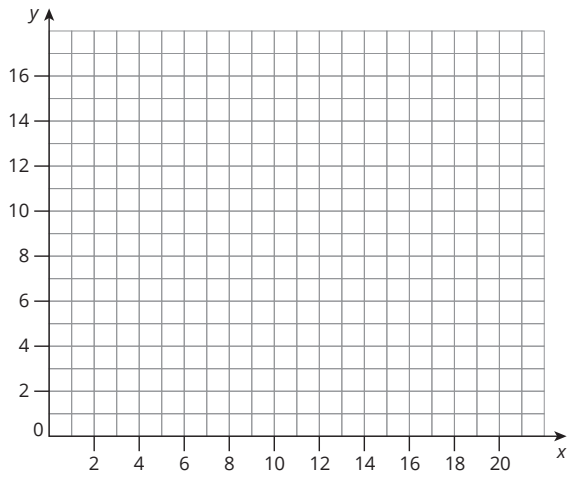
Ask Yourself . . .
What tools or strategies can you use to solve this problem?

For each context, complete each task:

- Write an equation in slope-intercept, point-slope, or standard form.
- State the form of the equation you used and your reason for using it.
- Graph the line using any method.
- Explain the graphing method you used and your reason for using it.
- Match each transformed graph to the equation in the table, written in terms of $f(x)$.

1. On a math quiz, students earned 2 points for every correct multiple-choice question and 3 points for every correct short answer question. Joey earned a total of 36 points on the quiz.

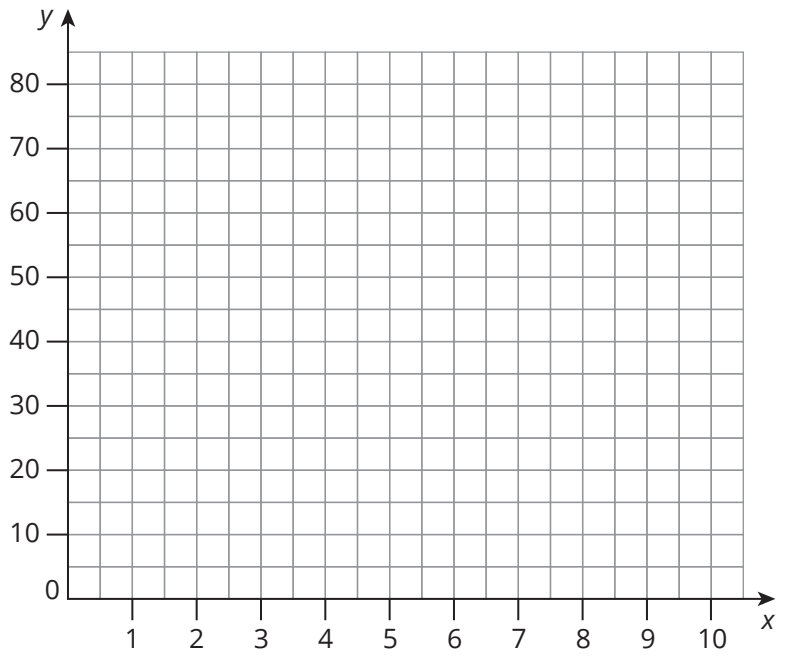
   Let x = number of correct multiple-choice questions
   Let y = number of correct short answer questions
   Equation: Reasoning:



Reasoning:

2. Nicky has $20, and he plans to save an additional $10 every two weeks.

Let x = number of weeks
Let y = Nicky's total savings
Equation: Reasoning:



Reasoning:

3. What are the advantages and disadvantages of using each form of a linear equation?

   a. Slope-intercept form

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b. Point-slope form

c. Standard form

Lesson 5 Assignment

Write

Explain how to graph a line when the equation is written in slope-intercept form, point-slope form, or standard form.

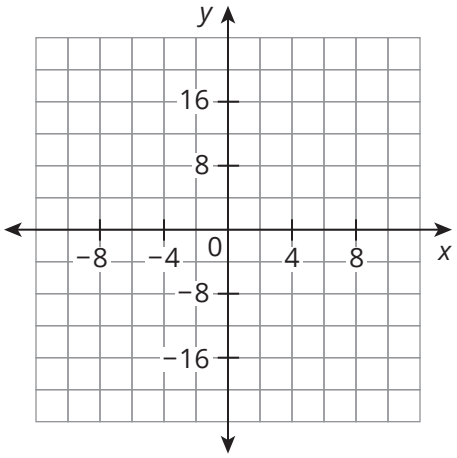
Remember

The standard form of a linear equation is $Ax + By = C$, where A, B, and C are integers and A and B are not both zero.

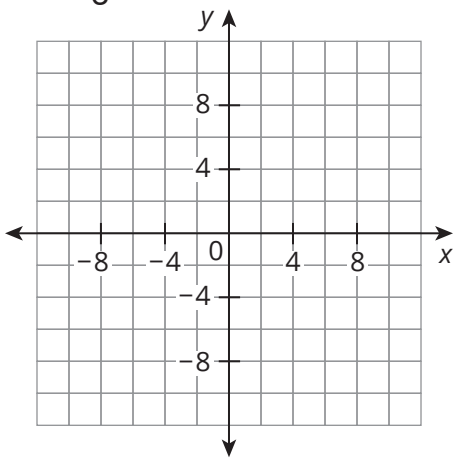
Practice

1. Graph each equation using its given form.

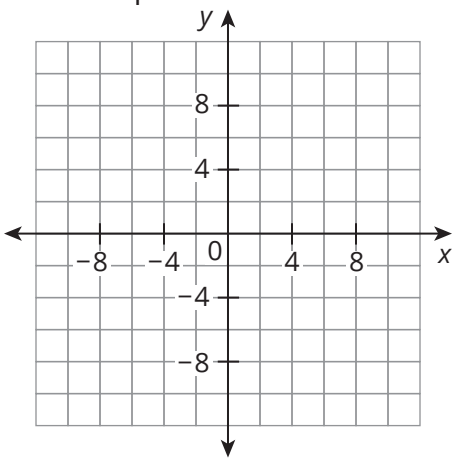
a. $y = 4x + 2$



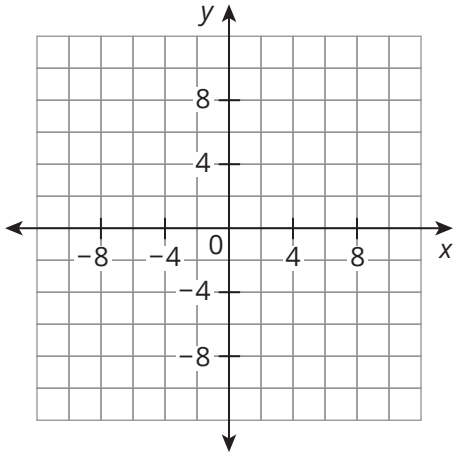
b. $y = -\frac{1}{3}x - 5$



c. $y + 1 = \frac{3}{4}(x - 8)$



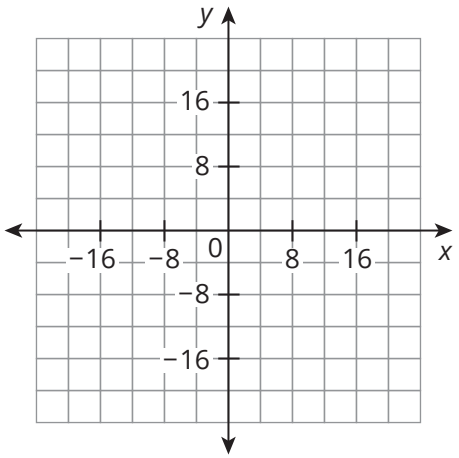
d. $y - 4 = -\frac{2}{3}(x - 6)$



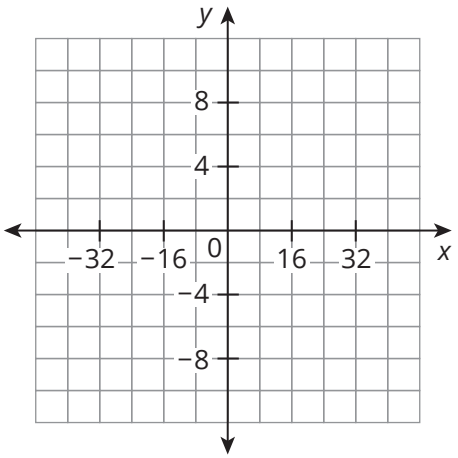
Lesson 5 Assignment

2. Graph each equation using its intercepts.

   a. $4x + 6y = 48$



   b. $-2x + 8y = 56$



3. Write a linear equation in standard form given two points on the line.

   a. (0, −9) and (−3, 6)

   b. (−4, 2) and (2, 5)

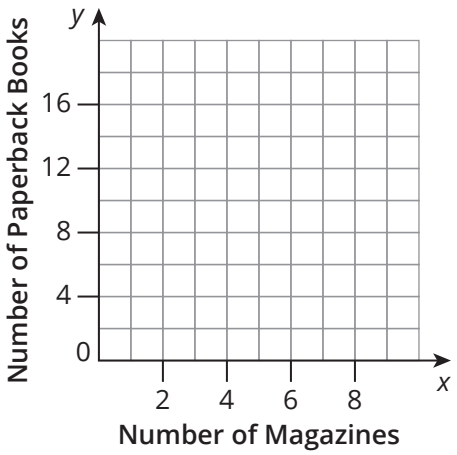
4. Brianna bought magazines for $6 each and paperback books for $3 each for a total of $54.

   a. Define your variables and write an equation in standard form to represent the situation.

   b. Calculate and interpret the *x*-intercept and the *y*-intercept for this equation.

Lesson 5 Assignment

c. Graph the equation of the line using the intercepts.



d. Calculate and interpret the slope of this line.

5. Each equation represents a linear relation. State the slope and y-intercept for each.

a. $-9x + 2y = -36$

b. $y + 5 = -7(x + 3)$

c. $y = 2$

d. $y = \frac{5}{2}x - 9$

Prepare

Determine the slope of the line between each pair of points.

1. (0, 10) and (3, 12)

2. (−1, 4.5) and (1, −4.5)

3. (21, 0) and (0, 12)

6 Making Sense of Different Representations of a Linear Function

LEARNING OBJECTIVES

- Determine whether a scenario, equation, table, or graph represents a linear relationship.
- Calculate the average rate of change from a table.
- Write functions given a table of values.
- Interpret expressions that represent different quantities in terms of a context and a graph.
- Compare different equation representations of linear functions.

NEW KEY TERMS

- polynomial
- degree
- leading coefficient
- zero of a function

You know how to determine whether a relationship represents a linear function, and you know how to write an equation for the function. How can you use the structure of the equation to identify characteristics of the function?

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Getting Started

Well, Are Ya or Aren't Ya?

1. Determine whether each representation models a linear or nonlinear function. Explain your reasoning.

Scenario A
A tree grows 3.5 inches each year.

Scenario B
The strength of a medication decreases by 50% each hour it is in the patient's system.

Scenario C
The area of a square depends on its side length.

Equation A
$y = 14 - 9x$

Equation B
$y = 2^x + 1$

Equation C
$y = \frac{1}{4}(x + 7) - 1$

Table A

x	y
1	3
1	4
1	5

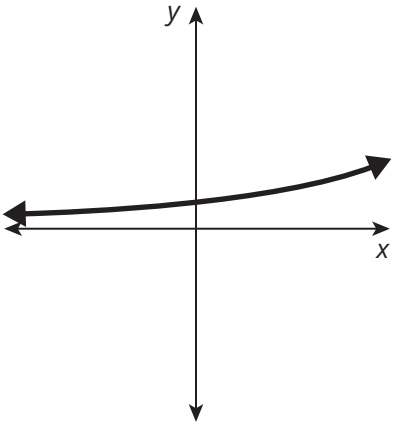
Table B

x	y
3	1
4	1
5	1

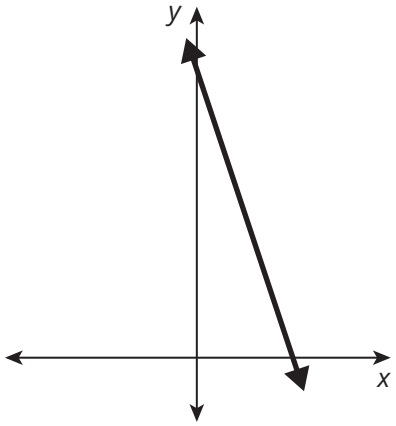
Table C

x	y
−9	45
−8	30
−7	15

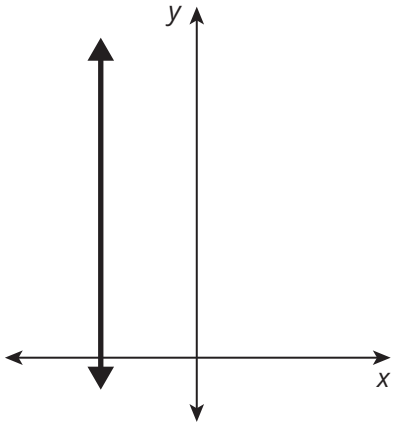
Graph A



Graph B



Graph C



ACTIVITY
6.1 Interpreting Linear Functions

Each table of values in the previous activity had consecutive input values. Tables in this format allow you to use differences to determine whether the representation is linear. Often, input values are in intervals other than 1, and sometimes the input values are in random order. To determine whether these tables represent linear functions, you need to make sure the slope, or average rate of change, is constant between all given points.

Ask Yourself . . .
Is there a pattern in the input values?

1. Determine whether each table represents a linear function. If so, write the function.

a.

x	y
−2	5.5
1	4.75
4	4
7	3.25

b.

x	y
0	5
2	13
4	21
8	29

Remember . . .
In a linear relationship, when the value of the dependent variable is directly proportional to the value of the independent variable, the relationship is a direct variation. Another way to say this is that in a direct variation the value of the dependent variable varies directly with the independent value. You can use these phrases interchangeably.

Analyze each situation represented as a table of values.

2. Nakota sells pretzels at festivals on weekends. The table shown is a record of past sales.

Number of Pretzels Sold	Amount of Money Earned (dollars)
15	37.5
42	105
58	145
29	72.5

   a. What does this table tell you about his sales?

   b. Determine whether the amount of money Nakota earns is directly proportional to the number of pretzels sold. Justify your reasoning.

   c. Write an equation to represent the situation.

   d. How much money does Nakota earn when he sells 75 pretzels?

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3. Alejandra and her friends recently went to the community fair. They had to pay an entrance fee and then purchase 1 ticket for each ride. Lauren is going to the fair tomorrow and wants to know the cost of each ride ticket. Alejandra and her friends help Lauren by writing down how much money they spent and the number of tickets they purchased.

Number of Ride Tickets	Amount of Money Spent (dollars)
2	7.5
4	9
6	10.5
11	14.25

a. What does this table tell you about the cost to go to the fair and ride the rides?

b. Determine whether the amount of money Alejandra and and her friends spent varies directly with the number of ride tickets they purchased. Justify your reasoning.

c. Write an equation to represent the situation.

d. If Lauren has $20 to spend, how many ride tickets can she buy?

4. The local pet store has a fish tank on display at the community fair. Darren is responsible for draining the tank at the end of the fair. The pet store manager provides him with this information from when they drained the tank at the end of the fair last year.

Time (hours)	Amount of Water Remaining (gallons)
$\frac{1}{4}$	169
$\frac{1}{2}$	163
$\frac{3}{4}$	157
1	151

a. Determine whether the amount of water remaining is directly proportional to the time the drain spent draining the tank. Justify your reasoning.

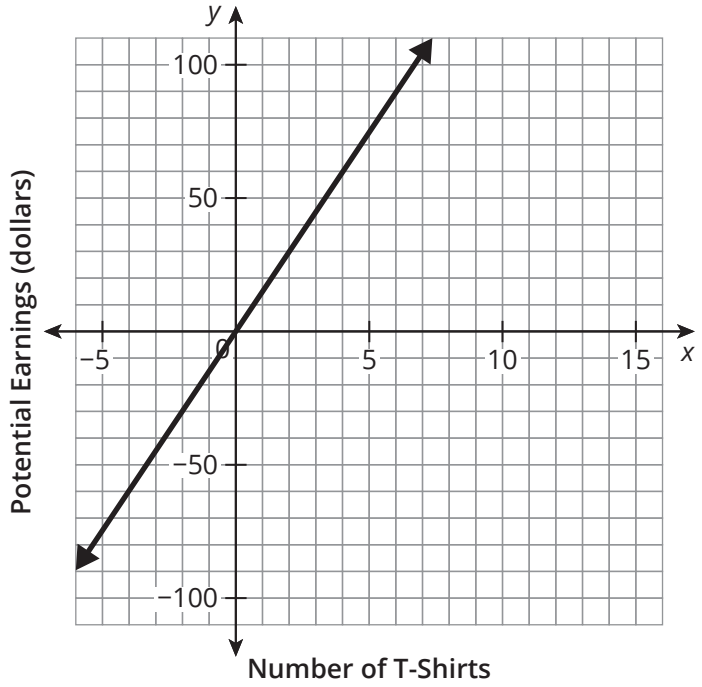
b. How many gallons did the fish tank hold?

c. Write an equation to represent the situation.

d. When will the tank be empty?

ACTIVITY
6.2 Interpreting Graphs of Linear Functions

Gracie sells silk screened T-shirts for her mom at local festivals. After each festival, she returns whatever money she earns to her mom. The graph shown represents her potential earnings based on the number of T-shirts she sells.



Ask Yourself . . .
What is the meaning of the slope, *x*- and *y*-intercepts, domain, and range in terms of this situation?

1. Analyze and interpret the graph. List as many facts as you can about the scenario based on what you see in the graph and describe how they relate to the scenario.

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Ask Yourself . . .
Does the function $E(t)$ represent a direct variation? Explain your reasoning.

2. Interpret the meaning of the x- and y-intercept.

3. Write a function, $E(t)$, to model Gracie's potential earnings given the number of T-shirts she sells.

4. What does $(t, E(t))$ represent in terms of the function and the graph?

5. Evaluate each and interpret the meaning in terms of the equation, the graph, and the scenario.

   a. $E(2)$

   b. $E(5)$

   c. $E(2.75)$

Gracie has a goal to earn \$100 at the festival. Let's consider how to determine the number of T-shirts she needs to sell to meet her goal.

WORKED EXAMPLE

To determine the number of T-shirt sales it takes to earn \$100 using the function, $E(t) = 15t$, substitute 100 for $E(t)$ and solve.

$$E(t) = 15t$$
$$100 = 15t$$
$$\frac{100}{15} = t$$
$$6.67 = t$$

6. Consider the Worked Example.

   a. Interpret the meaning of $t = 6.67$.

b. Why can you substitute 100 for $E(t)$?

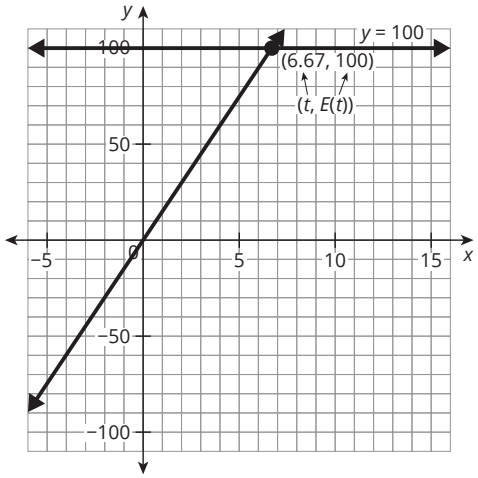
Remember . . .
The graph of an equation plotted on the coordinate plane represents the set of all its solutions.

You can also use the graph to determine the number of T-shirts Gracie needs to sell to earn $100.

WORKED EXAMPLE

To determine the number of T-shirts sales it takes to earn $100 using your graph, you need to determine the intersection of the two lines represented by the equation $100 = 15t$.

First, graph the function defined by each side of the equation, and then determine the intersection point of the two graphs.



$h(t) = 15t$

$100 = 15t$

$y = 100$ $\quad$ $y = 15x$

Solution: (6.67, 100)

In terms of the graph, Gracie needs to sell 6.67 T-shirts to earn $100. In terms of the context, she needs to sell 7 T-shirts.

7. Consider the equation and graphical representations. What are the limitations of using each to answer questions about the number of T-shirts sold or the amount of money earned?

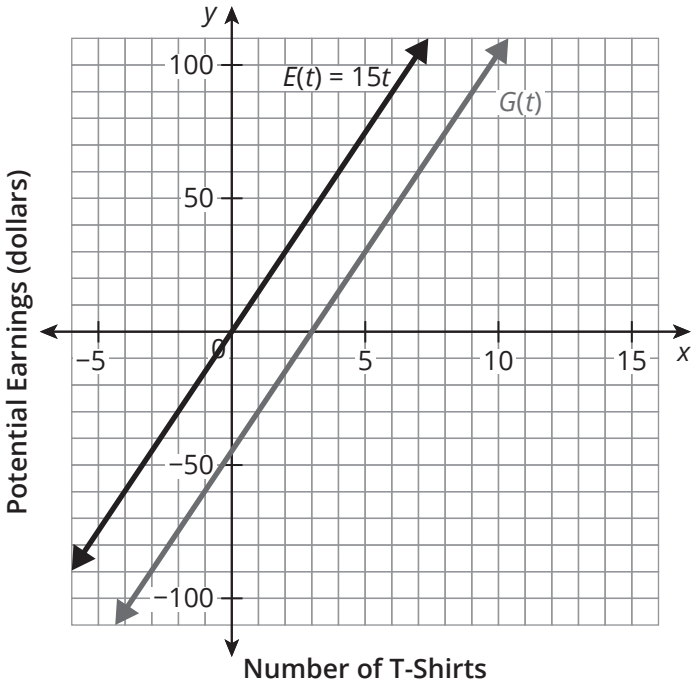
a. Equation

b. Graph

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ACTIVITY
6.3 Interpreting Changes to the Graph of a Linear Function

For the next festival, Gracie's mom suggests that she still sells each T-shirt for $15, but should give away 3 T-shirts in a raffle. This new relationship, $G(t)$, is shown on the graph.



1. Compare the two graphs. What do you notice?

   a. How do the graphs show the selling price per T-shirt remains the same?

   b. Determine and interpret the meaning of $y = G(0)$ in terms of the graph and this scenario. Label the point on the graph.

Ask Yourself . . .
Does $G(t)$ vary directly with t?

   c. Determine and interpret the meaning of $G(t) = 0$ in terms of the graph and this scenario. Label the point on the graph.

Brianna and Hailey each wrote an equation to describe the effect of giving away three T-shirts.

Brianna



Gracie is giving away 3 T-shirts, so she has 3 fewer shirts to sell.

$G(t) = 15(t - 3)$

Hailey



The cost of giving away three shirts is $45.

$G(t) = 15t - 45$

2. Verify the two equation representations are equivalent.

3. How many T-shirts will Gracie need to sell to earn $100? Use the graph and an equation.

4. Consider the expressions in the first two rows that define the quantities of the function and then the parts of each equation written by Brianna and Hailey to complete the table. First, determine the unit of measure for each expression. Then, describe the contextual meaning and the mathematical meaning of each part of the function.

		What It Means	
Expression	Unit	Contextual Meaning	Mathematical Meaning
t			
$G(t)$			
15			
$(t - 3)$			
$15t$			
-45			

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The linear functions that Brianna and Hailey each wrote are equivalent; however, they are written in different forms. The linear function $G(t) = 15(t - 3)$ is written in factored form and $G(t) = 15t - 45$ is written in general form.

A linear function can also be referred to as a *polynomial* function. A **polynomial** is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. The **degree** of a polynomial is the greatest variable exponent in the expression. The **leading coefficient** of a polynomial is the numeric coefficient of the term with the greatest power.

Polynomial comes from *poly-* meaning "many" and *-nomial* meaning "term," so it means "many terms."

WORKED EXAMPLE

A few examples of polynomial functions.

Polynomial Functions		Degree
Constant	$P(x) = 7$	0
Linear	$P(x) = 2x - 5$	1
Quadratic	$P(x) = 3x^2 - 2x + 4$	2
Cubic	$P(x) = 4x^3 - 2$	3

When you graph a polynomial the degree tells you the maximum number of times the graph can cross the *x*-axis.

The structure of each linear function tells you important information about the graph. Let's consider the general form of a linear function, $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. In this form, the a-value is the leading coefficient, which describes the steepness and direction of the line. The b-value describes the y-intercept.

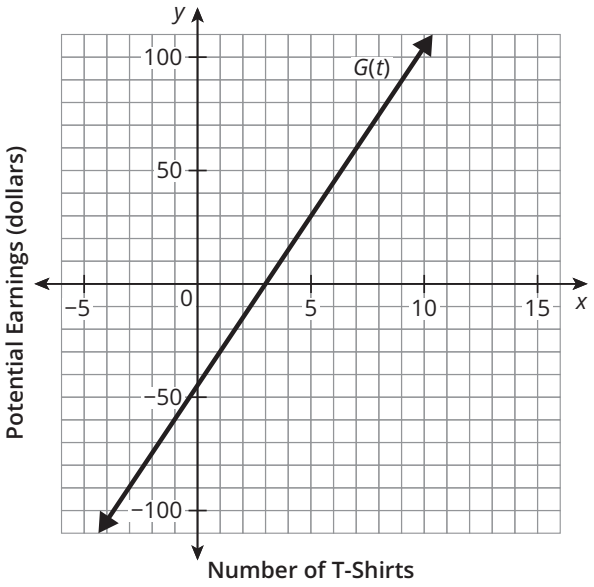
The variables used to represent any real number in the general linear form are irrelevant. Think about the position of the number as either the leading coefficient or a constant and the potential effect on the function.

You know the form $y = mx + b$ as slope-intercept form, where m represents the slope and b represents the y-intercept. Notice that the general form has the same structure. The general form shows that a linear equation is a polynomial of degree 1. You will learn more about polynomials as you progress through future mathematical courses.

5. Consider the general form of the linear function $G(t)$.

   a. Label a on the graph.

   b. Label b on the graph.



Next, consider the factored form of a linear function, $f(x) = a(x - c)$, where a and c are real numbers and $a \neq 0$. When a polynomial is in factored form, the value of x that makes the factor $(x - c)$ equal to zero is the x-intercept. This value is called the *zero of the function*. A **zero of a function** is a real number that makes the value of the function equal to zero, or $f(x) = 0$.

You can set $(x - c)$ equal to zero and determine the point where the graph crosses the x-axis.

6. Consider the factored form of the linear function $G(t)$.

   a. Label a on the graph.

   b. Label c on the graph.

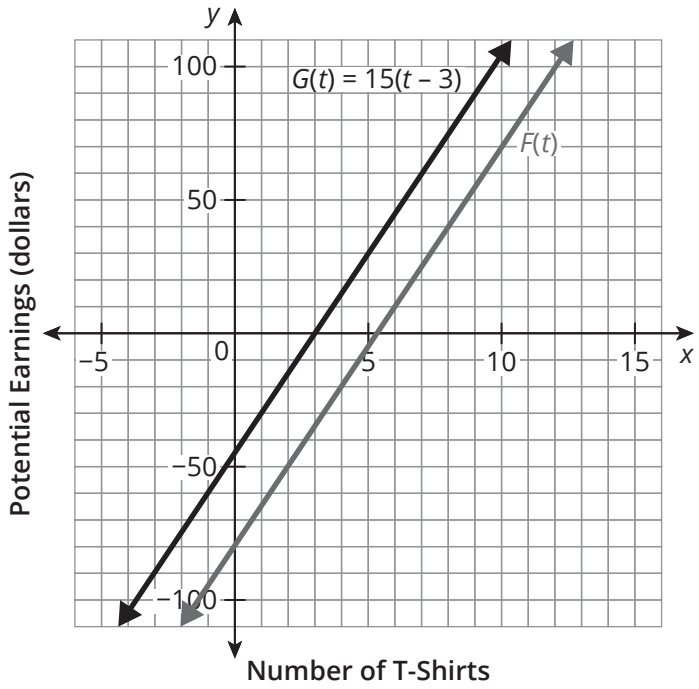
7. What is the zero of $G(t)$? Explain your reasoning.

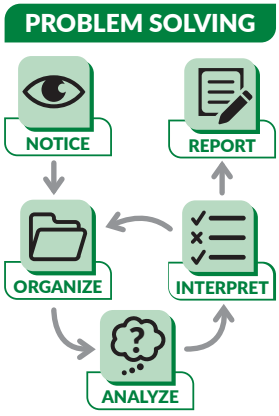
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ACTIVITY
6.4 Interpreting More Changes to the Graph of a Linear Function

The next festival that Gracie is attending charges a $35 fee to rent a booth. She is still selling her mom's T-shirts for $15 each and giving 3 away in a raffle. The graph shows this new relationship, $F(t)$.

Ask Yourself . . .
Is $F(t)$ directly proportional to t?





1. Consider the relationship between graphs of $G(t)$ and $F(t)$.

   a. How do the graphs show that the selling price per T-shirt remains the same?

   b. How did the new booth fee of $35 change the graph?

c. How many T-shirts will Gracie need to sell before she will have any money to return to her mom? Explain your reasoning.

d. How many T-shirts will Gracie need to sell to earn $100?

2. Consider the relationship between the equations of $G(t)$ and $F(t)$.

   a. Write the function $F(t)$ in terms of $G(t)$.

   b. Rewrite $F(t)$ in general form. Then describe how the a- and b-values are represented on the graph.

   c. Rewrite $F(t)$ in factored form. Use a fraction to represent the c-value. Then describe how the a- and c-values are represented on the graph.

The general form of a linear function is $f(x) = ax + b$.

The factored form of a linear function is $f(x) = a(x - c)$.

The values a, b, and c are real numbers and $a \neq 0$.

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Talk the Talk

Reading Between the Lines

Complete each "I can" sentence using *always*, *sometimes*, or *never*.

1. Suppose you are given a dependent value and need to calculate an independent value of a linear function.

   a. I can _____ use a table to determine an *approximate* value.

   b. I can _____ use a table to calculate an *exact* value.

   c. I can _____ use a graph to determine an *approximate* value.

   d. I can _____ use a graph to calculate an *exact* value.

   e. I can _____ use an equation to determine an *approximate* value.

   f. I can _____ use an equation to calculate an *exact* value.

2. Write the function that models each table of values. Then, evaluate the function for each independent and dependent value.

   a.

x	1	2	3	4	5
$f(x)$	−20	5	30	55	80

   $f(x) =$

   $f(12) =$

   $f(x) = -145$

b.

x	1	3	5	7	9
$g(x)$	18	6	−6	−18	−30

$g(x) =$

$g(-9) =$

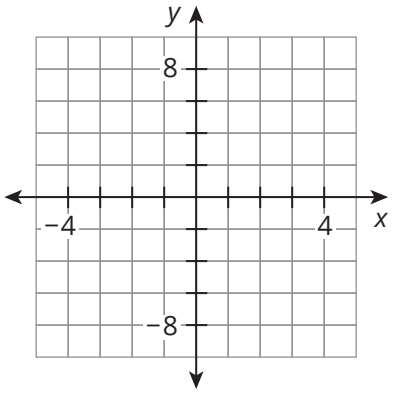
$g(x) = -54$

3. Complete the graphic organizer located at the end of this lesson for the linear function $f(x) = 2x - 8$.

   a. Write $f(x)$ in general form. Then, describe the information given in this form.

   b. Write $f(x)$ in factored form. Then, describe the information given in this form.

   c. Graph $f(x)$. Describe how you know this graph can cross the x-axis only one time.



   d. Create a table of values for $f(x)$.

x	y

Graphic Organizer

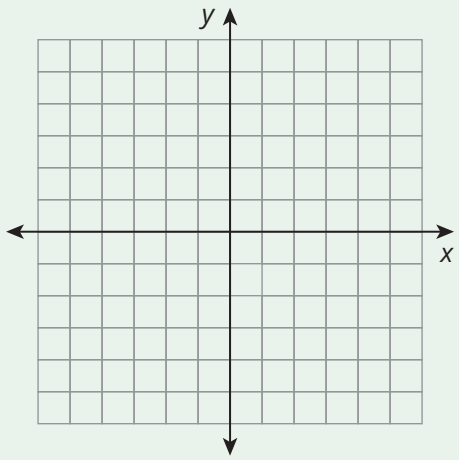
General Form

$f(x) = ax + b$, where a and b are real numbers and $a \neq 0$

Factored Form

$f(x) = a(x - c)$, where a and c are real numbers and $a \neq 0$

$f(x) = 2x - 8$



Graph

x	y

Table

Lesson 6 Assignment

Write

Describe a zero of a function in your own words.

Remember

The general form of a linear function is $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. In this form, the a-value is the leading coefficient which describes the steepness and direction of the line. The b-value describes the y-intercept.

The factored form of a linear function is $f(x) = a(x - c)$, where a and c are real numbers and $a \neq 0$. In this form, the a-value is the slope and the value of x that makes the factor $(x - c)$ equal to zero is the x-intercept.

Practice

Determine whether the table of values represents a linear function. If so, write the function.

1.

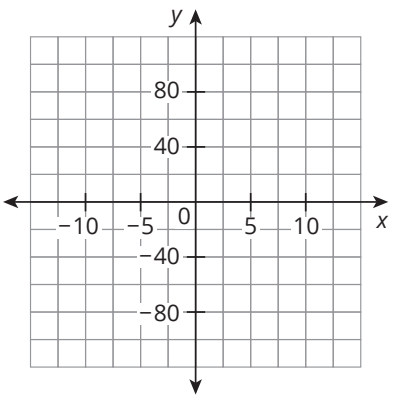
x	y
-2	$5\frac{2}{3}$
0	5
2	$4\frac{1}{3}$
4	$3\frac{2}{3}$

2.

x	y
-5	-27
0	-2
5	20
10	48

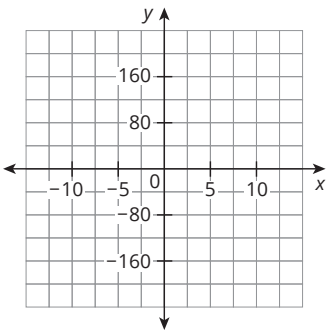
For each scenario, write a linear function in factored form and in general form. Then, sketch a graph and label the x- and y-intercepts. Finally, answer each question.

3. Omar prints and sells T-shirts for $14.99 each. Each month 5 T-shirts are misprinted and cannot be sold. How much money will he earn when he prints 22 T-shirts? How many T-shirts will he need to sell to earn $200?

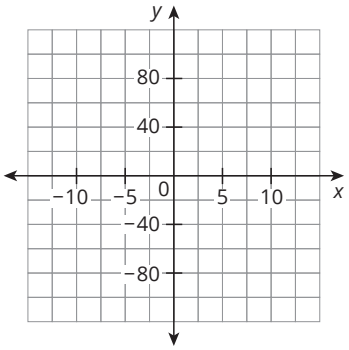


Lesson 6 Assignment

4. Juliana paints and sells ceramic vases for $35 each. Each month, she typically breaks 3 vases in the kiln. How much money will she earn if she sells 17 ceramic vases? How many ceramic vases will she need to sell to earn $600?



5. Antonio builds and sells homemade wooden toys for $12 each. The festival he is attending charges $50 to set up his booth. How much money will he earn if sells 35 wooden toys? How many wooden toys will he need to sell to earn $250?



Prepare

Determine each value, given the function $h(x) = -2x - 5$.

1. $h(3)$

2. $2 \cdot h(-2)$

3 $-1 \cdot h(1) + 5$

4. $5 \cdot h(0) + 5$

TOPIC 1 SELF-REFLECTION

Name: _____

Linear Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Linear Functions* topic by:

TOPIC 1: *Linear Functions*	Beginning of Topic	Middle of Topic	End of Topic
determining when a data set should be modeled by a linear function.	○	○	○
using technology to determine the linear regression function for a data set.	○	○	○
making predictions using the linear regression function.	○	○	○
interpreting the meaning of the slope and *y*-intercept of a linear regression function in terms of the problem context.	○	○	○
explaining the correlation coefficient as a measure of how well a function fits a data set.	○	○	○
understanding that the sign of a correlation coefficient indicates the direction of the association and that the magnitude indicates the strength of the fit.	○	○	○
analyzing a function on a scatterplot and its correlation coefficient to determine whether a function is an appropriate fit for a data set.	○	○	○
recognizing that correlation does not imply causation.	○	○	○

continued on the next page

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TOPIC 1 SELF-REFLECTION *continued*

TOPIC 1: *Linear Functions*	Beginning of Topic	Middle of Topic	End of Topic
recognizing that all arithmetic sequences are linear functions and rewriting an arithmetic sequence as a linear function using function notation.	○	○	○
writing equations of lines in the appropriate form: slope-intercept, point-slope, or standard form.	○	○	○
graphing lines from an equation written in slope-intercept, point-slope, or standard form.	○	○	○
writing and solving equations involving direct variation.	○	○	○
connecting average rate of change of a linear function to the slope of a line.	○	○	○
constructing a linear function from a scenario, table of values, graph, or arithmetic sequence.	○	○	○
interpreting the real-world meaning of the coefficients and constants in a linear function.	○	○	○
recognizing a linear relationship and writing an equation to represent that situation.	○	○	○
identifying the variables and quantities of a linear function.	○	○	○
determining key features of graphs of linear functions including domain, range, x-intercept, y-intercept, zeros, and slope.	○	○	○

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Linear Functions* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

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Linear Functions Summary

LESSON 1 Least Squares Regression

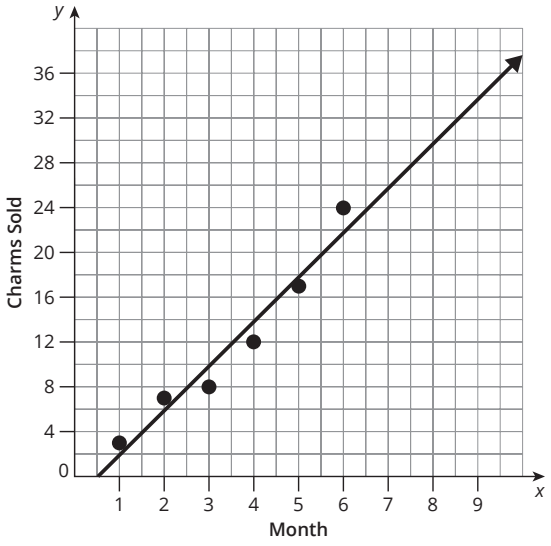
Technology calculates the line of best fit of data on a scatterplot using the **Least Squares Method**. The line includes the **centroid**, or the point whose x value is the mean of all x values and whose y value is the mean of all y values. The **linear regression function** has the smallest possible vertical distance from each given data point to the line. The sum of the squares of these distances is at a minimum with the linear regression function.

When there is a linear association between the independent and dependent variables, you can use a linear regression function to make predictions within the data set. Using a linear regression function to make predictions within the data set is called **interpolation**. To make predictions outside the data set is called **extrapolation**.

For example, consider the situation of Lucia selling charms to her classmates. The table records the sales of her charms over the months since she began selling them.

Month	Charms Sold
1	3
2	7
3	8
4	12
5	17
6	24

The linear regression function modeling the situation is graphed on the scatterplot shown.



The linear regression function is $y = 3.97x - 2.07$.

Using the equation to interpolate, Lucia should sell about 14 charms in the fourth month.

$y = 3.97(4) - 2.07$
$= 13.81$

Using the equation to extrapolate, Lucia should sell about 30 charms in the eighth month.

$y = 3.97(8) - 2.07$
$= 29.69$

NEW KEY TERMS

- Least Squares Method
- centroid [centroide]
- linear regression function [función de regresión lineal]
- interpolation [interpolación]
- extrapolation [extrapolación]
- correlation [correlación]
- correlation coefficient [coeficiente de correlación]
- coefficient of determination [coeficiente de determinación]
- causation [causalidad]
- necessary condition [condición necesaria]
- sufficient condition [condición suficiente]
- common response [respuesta común]
- confounding variable [variable de confusión]
- conjecture [conjetura]
- first differences

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NEW KEY TERMS

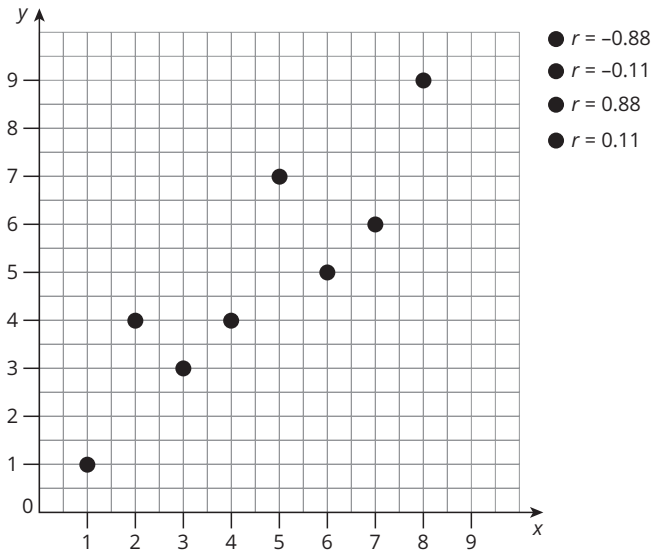
- average rate of change
- point-slope form
- standard form [forma estándar/ general]
- polynomial [polinomio]
- degree
- leading coefficient
- zero of a function [cero de una función]

LESSON 2

Correlation

The measure of how well a regression fits a set of data is called **correlation**. When dealing with regression functions, the variable r is used to represent a value called the **correlation coefficient**. The correlation coefficient indicates how close the data are to the graph of the regression function. The correlation coefficient falls between -1 and 0 when the data show a negative association or between 0 and 1 when the data show a positive association. The closer the r-value is to 1 or -1, the stronger the relationship is between the two. The **coefficient of determination**, r^2, measures how well the regression line fits the data. It represents the percentage of variation of the observed values of the data points from their predicted values.

For example, consider the possible r-values for a linear regression function given for the data graphed in the scatterplot.



The data has a positive correlation. Because of this, r-value must be positive. Also, the data are fairly close to forming a straight line, so of the choices, $r = 0.88$ would be the most accurate. Technology can be used to verify the correlation coefficient. The coefficient of determination for this data set is 0.7744.

When interpreting the correlation between two variables, you are looking at the association between the variables. While an association may exist, that does not mean there is causation between the variables. **Causation** is when one event causes a second event. A correlation is a **necessary condition** for causation, but a correlation is not a **sufficient condition** for causation. Correlation may be due to a **common response**, which is when another reason may cause the same result, or a **confounding variable**, which is when other variables are either unknown or unobserved.

For example, consider an experiment conducted by a group of college students that found that more class absences correlated to rainy days. The group concluded that rain causes students to be sick. However, this correlation does not imply causation. Rain is neither a necessary condition (because students can get sick on days it does not rain) nor a sufficient condition (because not every student who is absent is necessarily sick) for students being sick.

LESSON 3 Making Connections Between Arithmetic Sequences and Linear Functions

You can make a conjecture that the graphs of arithmetic sequences represent linear functions. A **conjecture** is a mathematical statement that appears to be true, but has not been formally proven. An explicit formula for an arithmetic sequence can be rewritten in function notation to prove your conjecture.

For example, to write the explicit formula $a_n = 2 + 4(n - 1)$ as a linear function in general form, first use function notation to represent a_n.

$a_n = 2 + 4(n - 1)$
$f(n) = 2 + 4(n - 1)$

Next, rewrite the expression $2 + 4(n - 1)$.

$f(n) = 2 + 4n - 4$ distributive property
$= 4n + 2 - 4$ commutative property
$= 4n - 2$ combine like terms

So, $a_n = 2 + 4(n - 1)$ written in function notation is $f(n) = 4n - 2$.

One strategy to determine if a table of values represents a linear function is to examine first differences. **First differences** are the values determined by subtracting consecutive output values when the input values have an interval of 1. If the first differences of a table of values are constant, the relationship is linear.

x	y	
1	22	$17 - 22 = -5$
2	17	$12 - 17 = -5$
3	12	$7 - 12 = -5$
4	7	

For example, the first differences of the table shown are constant, so the relationship is linear.

The slope, a, of a linear function is equal to the constant difference of an arithmetic sequence. Another name for the slope of a linear function is **average rate of change**. The formula for the average rate of change is $\frac{f(t) - f(s)}{t - s}$. This represents the change in the output as the input changes from s to t.

You can determine slope of a line when given two points using the formula $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{\Delta y}{\Delta x}$. Delta ($\Delta$) is a greek letter used in mathematics to represent "change in". So $\frac{\Delta y}{\Delta x}$ means "change in y" divided by "change in x".

LESSON 4 Point-Slope Form of a Line

You can determine the equation of a line from a table of values without knowing the *y*-intercept using the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

For example, to write an equation of a line from a table of values, you can use the slope formula.

x	y
2	6
4	5
6	4

- First, calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 6}{4 - 2}$$
$$= \frac{-1}{2} = -\frac{1}{2}$$

- Next, choose any point from the table. Let's use (2, 6).
- Then, substitute what you know into the slope formula: $m = -\frac{1}{2}$, (2, 6), and the unknown point (*x*, *y*).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$-\frac{1}{2} = \frac{y - 6}{x - 2}$$

- Finally, rewrite the equation with no variables in a denominator.

$$-\frac{1}{2} = \frac{y - 6}{x - 2}$$
$$-\frac{1}{2}(x - 2) = y - 6$$

The equation is $y - 6 = -\frac{1}{2}(x - 2)$.

The linear equation you wrote is written in point-slope form. The **point-slope form** of a linear equation is $y - y_1 = m(x - x_1)$, where *m* is the slope of the line and (x_1, y_1) is any point on the line. For example, given a slope of 3 and the point (1, 4), you can write the equation for the line in point-slope form as $y - 4 = 3(x - 1)$.

The slope of a horizontal line is 0. The slope of a vertical line is undefined.

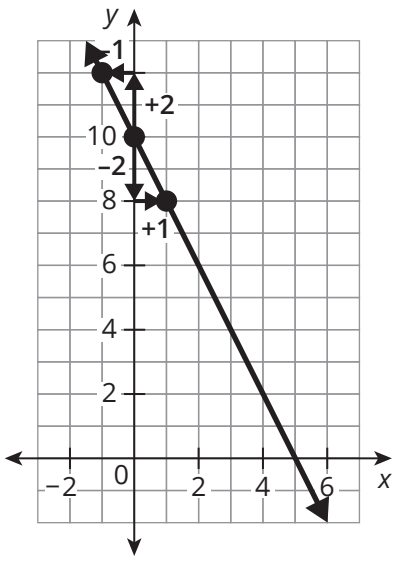
LESSON 5 Using Linear Equations

You can use an equation in slope-intercept form to graph a relationship without first creating a table of values using the *y*-intercept and the slope.

For example, graph the equation $y = -2x + 10$. Begin by plotting the *y*-intercept, (0, 10). Use the slope and count from the *y*-intercept to graph two more points on the line.

- For $m = \frac{-2}{1}$, go down 2 units and to the right 1 unit.
- For $m = \frac{-2}{1}$, go up 2 units and to the left 1 unit.

Connect the points to form a straight line.



Another way to write a linear equation is in standard form. The **standard form** of a linear equation is $Ax + By = C$, where A, B, and C are integers, and A and B are not both zero.

When an equation is in standard form, you can calculate a y-intercept, where a line crosses the y-axis, by substituting 0 for x and solving the equation for y. To calculate an x-intercept, where the line crosses the x-axis, substitute 0 for y and solve the equation for x.

To write a linear equation in standard form when given two points, first write the equation in point-slope form. Then, rewrite the equation in standard form.

Write a linear equation in standard form, $Ax + By = C$, that goes through the points $(-1, 3)$ and $(-4, -3)$.

Begin by determining the slope of the line.

$$m = \frac{(-5 - 3)}{(-4 - (-1))} = \frac{(-6)}{(-3)} = 2$$

Pick one of the points and write an equation in point-slope form for the line.

$$y - 3 = 2(x - (-1))$$
$$y - 3 = 2(x + 1)$$

Rewrite the equation in standard form.

$$y - 3 = 2x + 2$$
$$y = 2x + 5$$
$$-2x + y = 5$$

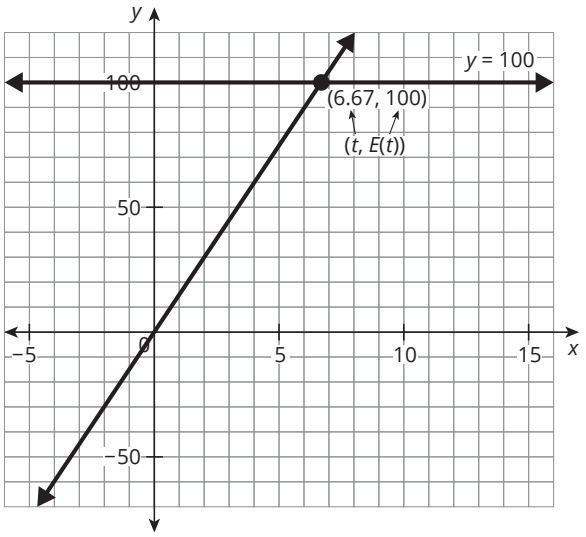
LESSON 6 Making Sense of Different Representations of a Linear Function

To determine whether a table of values represents a linear function, the slope, or average rate of change, needs to be constant between all given points.

You can use substitution or a graph to determine the output for a given input of a function.

For example, consider the function $E(t) = 15t$ which models the amount of money Gracie earns for selling t T-shirts. To determine the number of shirts she needs to sell to earn \$100, substitute \$100 for $E(t)$ and solve.

$$E(t) = 15t$$
$$100 = 15t$$
$$6.67 = t$$



Or you can determine the intersection of the graphs of the two lines represented by the equation $100 = 15x$.

Gracie needs to sell 6.67 T-shirts to earn \$100. In terms of the context, she needs to sell 7 T-shirts.

A linear function can also be referred to as a polynomial function. A **polynomial** is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. The **degree** of a polynomial is the greatest variable exponent in the expression. The **leading coefficient** of a polynomial is the numeric coefficient of the term with the greatest power.

The chart shows a few examples of polynomial functions.

Polynomial Function		Degree
Constant	$P(x) = 7$	0
Linear	$P(x) = 2x - 5$	1
Quadratic	$P(x) = 3x^2 - 2x + 4$	2
Cubic	$P(x) = 4x^3 - 2$	3

The structure of each linear function provides important information about the graph. The general form of a linear function is $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. In this form, the a-value is the leading coefficient, which describes the steepness and direction of the line. The b-value describes the y-intercept.

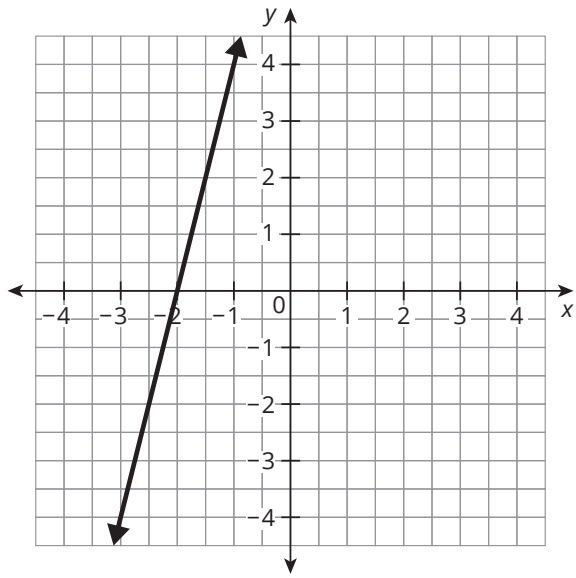
When you graph a polynomial, the degree indicates the maximum number of times the graph can cross the x-axis. A linear function has a degree of 1, so it crosses the x-axis at most one time.

The factored form of a linear function is $f(x) = a(x - c)$, where a and c are real numbers and $a \neq 0$. When a linear function is in factored form, the value of x that makes the factor $(x - c)$ equal to zero is the x-intercept. This value is called the zero of the function. A **zero of a function** is a real number that makes the value of the function equal to zero, $f(x) = 0$.

You can set the factor $(x - c)$ equal to zero to determine the point where the graph crosses the x-axis.

For example, the linear function $f(x) = 4x + 8$ in factored form is $f(x) = 4(x + 2)$. Set the factor $x + 2$ equal to zero and then solve for x to determine the zero of the function, which is the point at which the graph of the function will cross the x-axis.

$$x + 2 = 0$$
$$x = -2$$



TOPIC 2



A region can be described as above, below, to the left, or to the right of a line.

Transforming and Comparing Linear Functions

LESSON 1 Transforming Linear Functions 269

LESSON 2 Vertical and Horizontal Transformations of Linear Functions 287

LESSON 3 Determining Slopes of Perpendicular Lines 305

LESSON 4 Comparing Linear Functions in Different Forms . . . 319

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1 Transforming Linear Functions

OBJECTIVES

- Determine the effects on the graph of a linear function when $f(x)$ is replaced by $f(x) + d$ or $a \cdot f(x)$.
- Graph linear function transformations expressed symbolically and show intercepts.
- Identify key characteristics of the graphs of linear functions, such as slope and y-intercept, in terms of quantities from a verbal description.
- Prove that a translated line and its pre-image have the same slope and are, therefore, parallel.
- Write equations of parallel lines.

NEW KEY TERM

- parent function

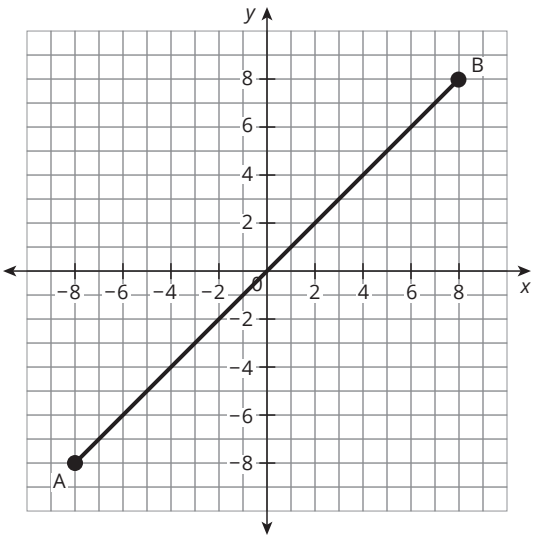
You have learned about linear functions and their characteristics, including slope and y-intercept.

How can you transform a function? What effects do different transformations have on the characteristics of linear functions?

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Getting Started

Returning to Transformation Station



Consider $\overline{AB}$ with coordinates $A(-8, -8)$ and $B(8, 8)$.

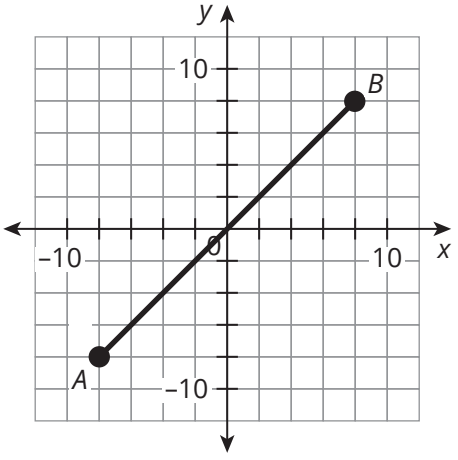
Follow your teacher's instructions to copy this line segment on a coordinate plane on the floor of your classroom, with different students standing at different points of the line segment.

1. Move on the coordinate plane to translate the entire line segment up 4 units and then down 4 units. Describe how the student "points" move.

2. How do the translations affect the coordinates of the figure?

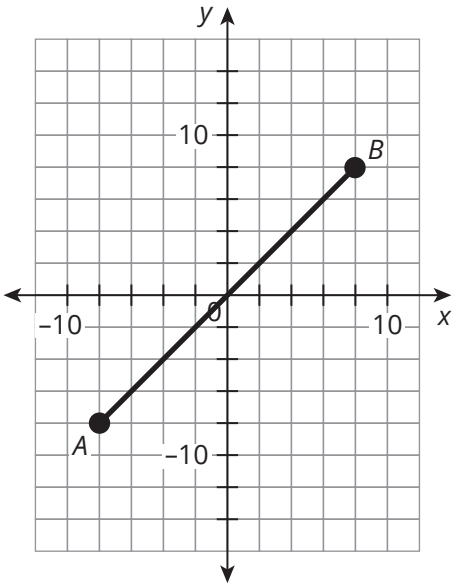
Original Point	4 Units Up	4 Units Down
(x, y)		

3. Draw $\overline{CD}$ on the coordinate plane so that it is a vertical translation of $\overline{AB}$ up 4 units. Compare the two line segments. Describe how they are related.

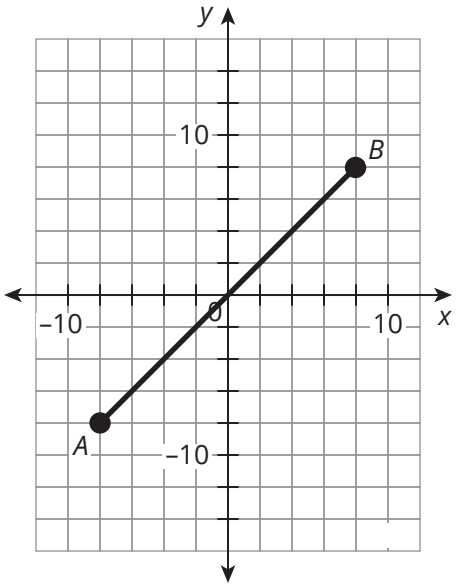


4. Move back to where you started. Then, multiply all the *y*-coordinates by 2 and then by −2. Describe how the student "points" move.

5. Draw $\overline{EF}$ on the coordinate plane by multiplying all the *y*-coordinates of $\overline{AB}$ by 2. Compare the two line segments. Describe how they are related.



6. Draw $\overline{GH}$ on the coordinate plane by multiplying all the *y*-coordinates of $\overline{AB}$ by −2. Compare the two line segments. Describe how they are related.



ACTIVITY
1.1 Vertical Translations of Functions

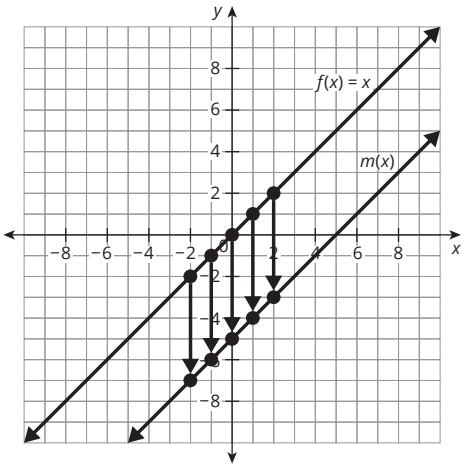
Let's determine how translations impact the graph of the linear function $f(x) = x$. The function $f(x) = x$ is the *parent function* for the linear function family. A **parent function** is the simplest function of its type.

For the family of linear functions having the general form $f(x) = ax + b$, where a and b are any real numbers, the function $f(x) = x$, where $a = 1$ and $b = 0$, is the parent function. It is the simplest linear function.

WORKED EXAMPLE

You can translate the graph of $f(x)$ down 5 units by moving each point 5 units down. The transformed graph is labeled as $m(x)$.

To translate the point $(-2, -2)$ on $f(x)$, subtract 5 units from the output value, or y-value. The input value, or x-value, remains unchanged. The coordinates of the translated point on $m(x)$ are $(-2, -7)$. The coordinates of four additional points on $f(x)$ are translated for you.



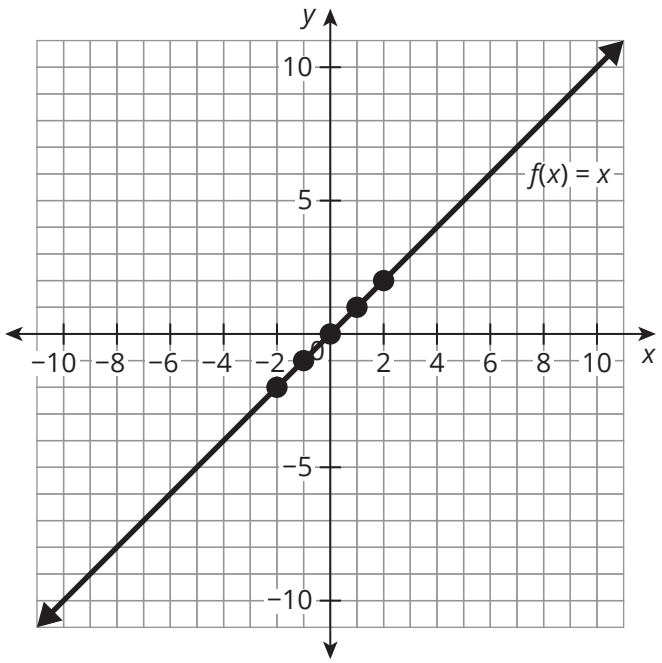
Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$m(x)$
−2	−7
−1	−6
0	−5
1	−4
2	−3

1. Consider the translated function, $m(x)$. Identify the slope and y-intercept of the graph of the function. Then, write the equation for the function in general form.

2. Translate $f(x)$ again to create a new function, $p(x)$.

   a. Translate the graph of $f(x)$ up 5 units. Label your graph as $p(x)$. Complete the table of corresponding points on $p(x)$.



Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$p(x)$

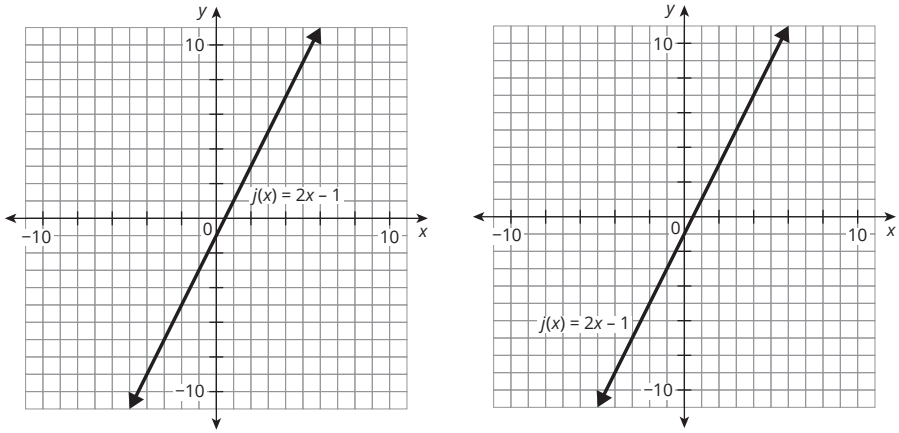
   b. Identify the slope and y-intercept of the graph of the function. Then, write the equation for the function in general form.

For the parent function $f(x) = x$, the transformed function $y = f(x) + d$ shows a vertical translation of the function. This translation affects the output values, or y-values, of the function. For $d > 0$, the resulting graph vertically shifts up. For $d < 0$, the resulting graph vertically shifts down. The distance the graph is shifted is the absolute value of d, or $|d|$.

3. Compare the values of $f(2)$ and $p(2)$. How did the transformation of the function affect the value of the function at $x = 2$?

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4. Consider the graph of $j(x) = 2x - 1$.



a. Translate the graph of $j(x)$ up 4 units. Label the graph as $q(x)$. Then, write an equation for $q(x)$ in terms of $j(x)$.

b. Translate the graph of $j(x)$ down 4 units. Label the graph as $r(x)$. Then, write an equation for $r(x)$ in terms of $j(x)$.

c. Rewrite $q(x)$ and $r(x)$ in general form.

Ask Yourself . . .
Will the graphs of $j(x)$, $q(x)$, and $r(x)$ ever intersect?

d. Compare the equations and graphs of $j(x)$, $q(x)$, and $r(x)$. What do you notice?

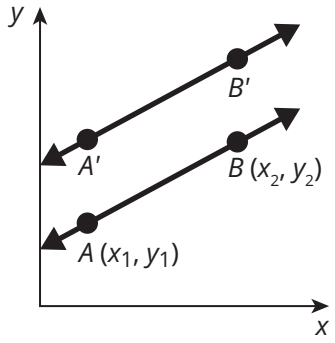
ACTIVITY
1.2 Slopes of Parallel Lines

In the previous activity you translated the function $j(x) = 2x - 1$ up 4 units to create $q(x) = 2x + 3$.

1. Will the graphs of $j(x)$ and $q(x)$ ever intersect? Explain your reasoning.

You can algebraically prove that a line and its translation are parallel to each other.

2. Line AB was translated a units up to create line $A'B'$.



   a. Identify the x- and y-coordinates of each corresponding point on the image.

   b. Use the slope formula to calculate the slope of the pre-image.

   c. Use the slope formula to calculate the slope of the image.

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Ask Yourself . . .
The original figure is the pre-image and the transformation of the figure is the image.

d. How does the slope of the image compare to the slope of the pre-image?

e. How would you describe the relationship between the graph of the image and the graph of the pre-image?

Catalina and James each write the equation of a line that is parallel to $y = \frac{1}{2}x + 3$ and passes through the point (2, 9).

Catalina

I know that since the lines are parallel, the new line is a translation of $y = \frac{1}{2}x + 3$. I can substitute the x-value from (2, 9) into $y = \frac{1}{2}x + 3$ to determine the corresponding point on the pre-image.

$$y = \frac{1}{2}(2) + 3 = 4$$

The point that corresponds to the point (2, 9) on $y = \frac{1}{2}x + 3$ is (2, 4). Going from (2, 4) to (2, 9) is a translation up five units.

Since $y = \left(\frac{1}{2}x + 3\right) + 5 = \frac{1}{2}x + 8$, the equation of the parallel line is $y = \frac{1}{2}x + 8$.

James

To write the equation of the line, all I need to know is the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

I used $m = \frac{1}{2}$ and (2, 9) for (x, y).

$\frac{1}{2} = \frac{y - 9}{x - 2}$.

Then, I rearranged the equation to write it in general form.

$$2(y - 9) = x - 2$$
$$y - 9 = \frac{1}{2}(x - 2)$$
$$y = \frac{1}{2}(x - 2) + 9$$
$$y = \frac{1}{2}x - 1 + 9$$
$$y = \frac{1}{2}x + 8$$

3. Which student's method do you prefer and why?

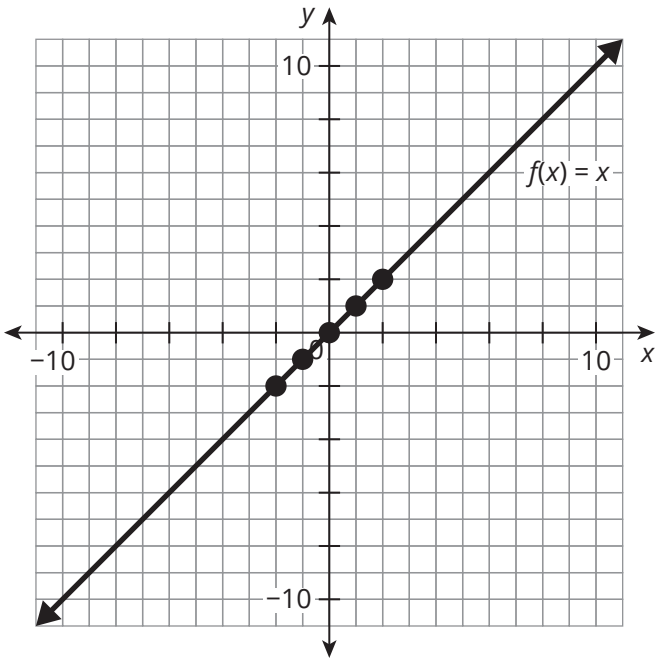
4. Write the equation of a line that is parallel to $y = -3x - 1$ and passes through the point $(-1, 5)$.

ACTIVITY
1.3 Vertical Dilations of Functions

In this activity, let's consider how dilations impact the graph of the linear function $f(x) = x$.

1. Suppose the output values of $f(x)$ are changed by a factor of 4 to create $a(x)$.

   a. Sketch the graph of $a(x)$ and complete the table of values.



Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

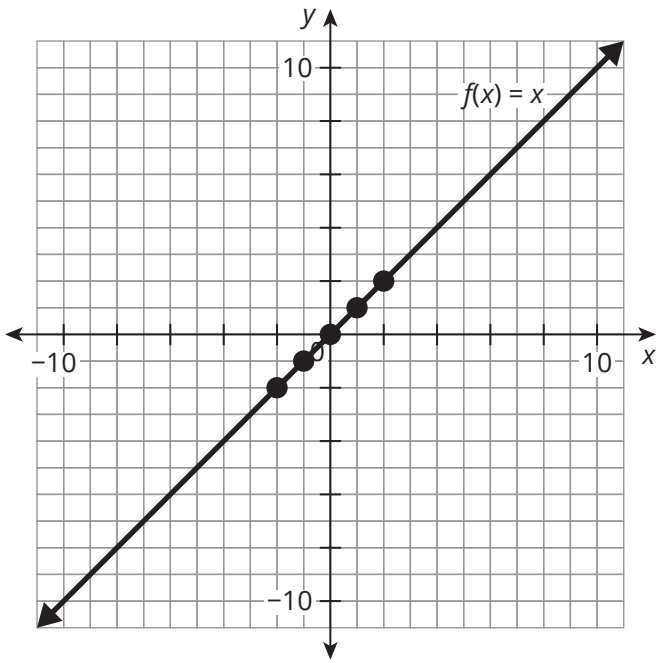
Transformed Graph	
x	$a(x)$
−2	
−1	
0	
1	
2	

   b. Identify the slope and y-intercept of the function $a(x)$. Then, write the equation for the function $a(x)$ in general form.

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2. Suppose the output values of $f(x)$ are changed by a factor of $\frac{2}{3}$ to create $b(x)$.

   a. Sketch the graph of $b(x)$ and complete the table of values.



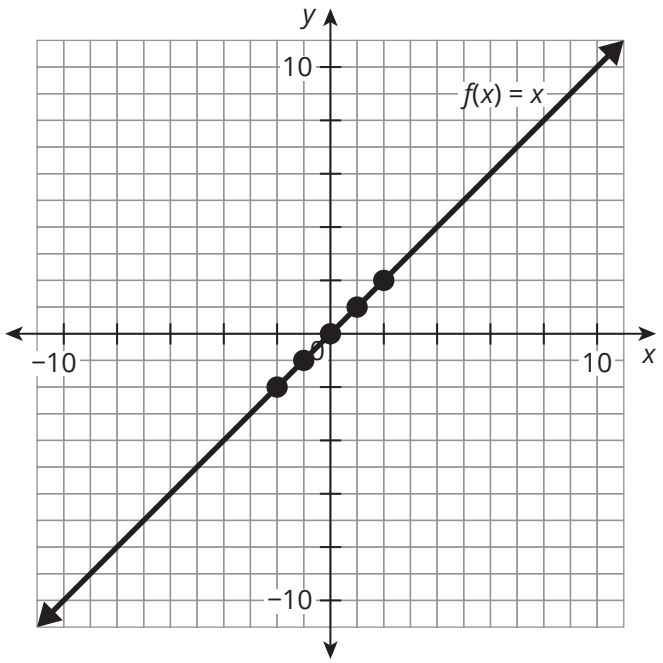
Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$b(x)$
−2	
−1	
0	
1	
2	

   b. Identify the slope and y-intercept of the function $b(x)$. Then, write the equation for the function $b(x)$ in general form.

3. Suppose the output values of $f(x)$ are changed by a factor of -4 to create $c(x)$.

   a. Sketch the graph of $c(x)$ and complete the table of values.



Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$c(x)$
−2	
−1	
0	
1	
2	

   b. Identify the slope and y-intercept of the function $c(x)$. Then, write the equation for the function $c(x)$ in general form.

For the parent function $f(x) = x$, the transformed function $y = a \cdot f(x)$ shows a vertical dilation of the function. This dilation affects the output values, or y-values, of the function. For $|a| > 1$, the resulting graph vertically stretches by a factor of a units. For $0 < |a| < 1$, the resulting graph vertically compresses by a factor of a units. For $a < 0$, the resulting graph is vertically stretched or compressed and is reflected across the x-axis.

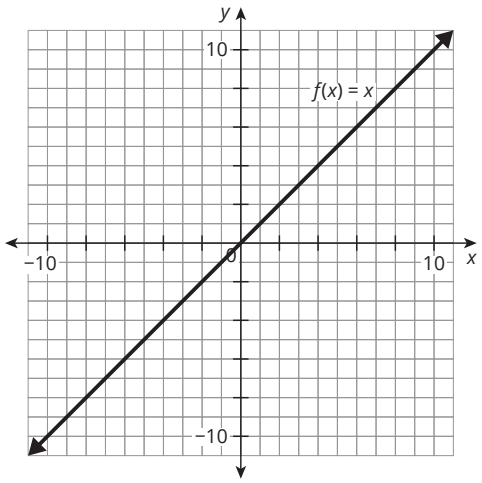
4. Compare the values $f(-1)$ and $c(-1)$. How did the transformation of the function affect the value of the function at $x = -1$?

ACTIVITY
1.4 Vertical Dilations and Vertical Translations of Functions

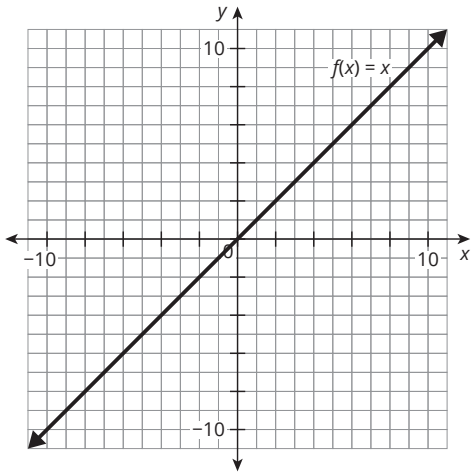
Let's consider more transformations of the parent function $f(x) = x$.

1. Describe the transformations performed on $f(x)$ to produce $g(x)$. Then, graph $g(x)$. Write the function equation in general form.

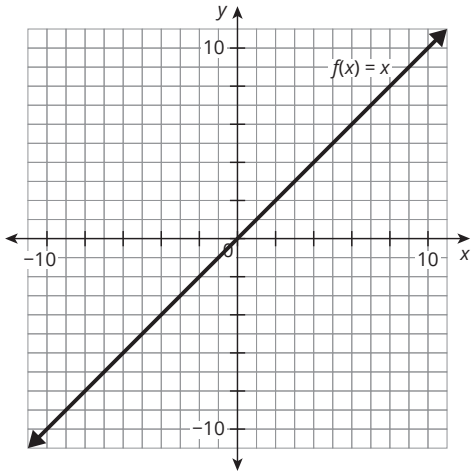
a. $g(x) = 2 \cdot f(x) + 7$



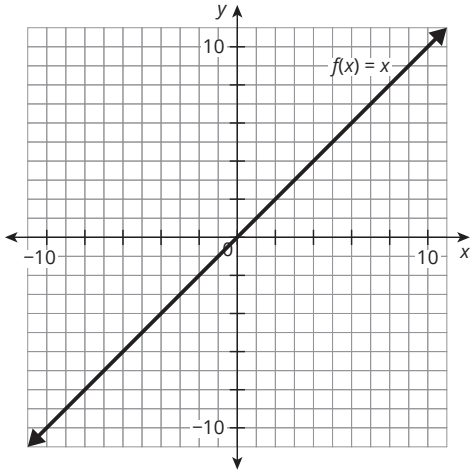
b. $g(x) = 3 \cdot f(x) - 1$



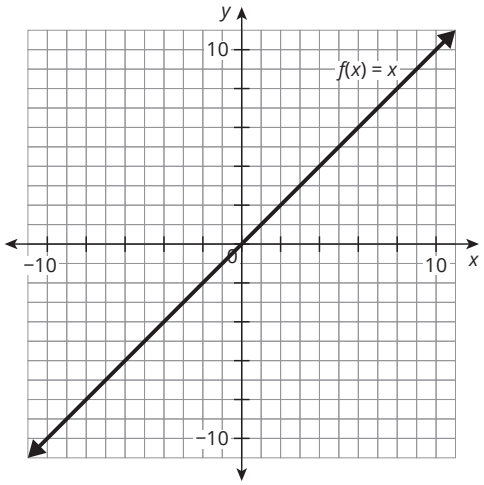
c. $g(x) = \frac{1}{3} \cdot f(x) + 2$



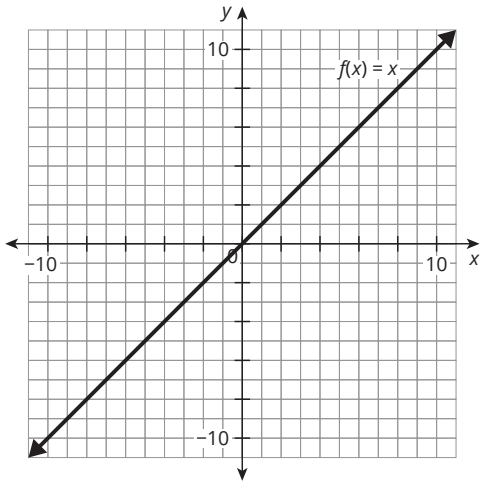
d. $g(x) = \frac{1}{2} \cdot f(x) - 3$



e. $g(x) = -1 \cdot f(x) - 4$



f. $g(x) = -\frac{2}{3} \cdot f(x) + 5$



When a function is both translated and stretched vertically, the resulting function can be written in the form $a \cdot f(x) + d$, where d represents the vertical translation of $f(x)$, and a represents the vertical dilation of $f(x)$.

2. Consider the function $g(x) = a \cdot f(x) + d$.

   a. How does changing the a-value affect the slope of the function? The y-intercept of the function?

   b. How does changing the d-value affect the slope of the function? The y-intercept of the function?

PROBLEM SOLVING

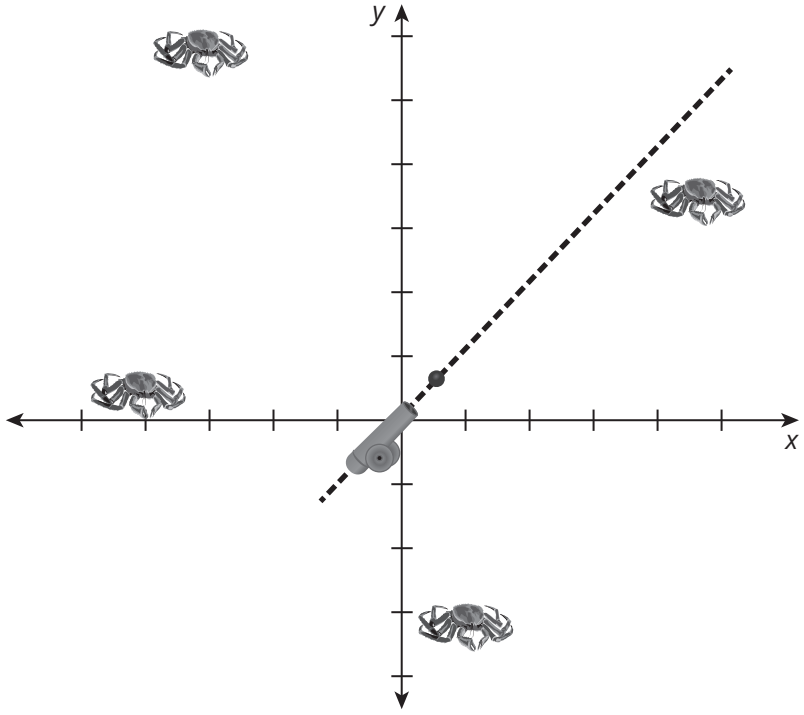


ACTIVITY 1.5

Applying Linear Function Transformations

Your company is developing a new video game for kids, which involves scoring points by shooting targets with a cannon. The cannon starts at the origin of a coordinate system as shown.

By default it shoots along the line $f(x) = x$. Players can move the cannon up and down the y-axis and also change the angle of the cannon. When a player shoots the cannon, the game program determines the values of a and d for the linear function $a \cdot f(x) + d$. If the target is on that line, the program will show an animation of the cannonball hitting the target.



Ask Yourself . . .
The cannon can shoot in only one direction. Does that affect the domain or range of the linear function?

1. Test the program. Determine values for a and d for four linear functions that should hit the targets shown.

2. Identify the domain on which each graph hits the target.

The game developer is also testing two other versions of the game, with different abilities for the cannon.

Version B
The cannon can only shoot along a line parallel to $f(x) = x$ or $f(x) = -x$. It can move up and down the y-axis.

Version C
The cannon can change angles but remains fixed at the origin.

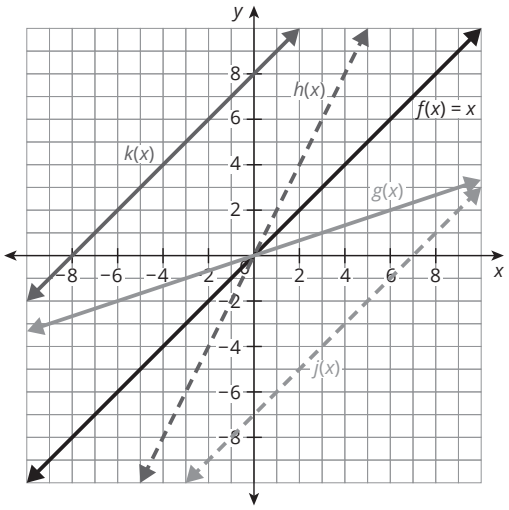
3. Determine values for a and d, along with their corresponding domains, for four linear functions that can hit the targets in each version of the game. Show your work.

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Talk the Talk

Function Matching

The graph shows the linear function $f(x) = x$ and four transformations of $f(x)$.



Transformations
$\frac{1}{3} \cdot f(x)$
$2 \cdot f(x)$
$f(x) - 7$
$f(x) + 8$

Ask Yourself . . .
Did you justify your mathematical reasoning?

Match each transformed graph to one of the transformations in the table. Explain your reasoning.

1. $k(x) =$ _____

2. $h(x) =$ _____

3. $g(x) =$ _____

4. $j(x) =$ _____

Lesson 1 Assignment

Write

Describe the term *parent function* in the context of transformations using your own words.

Remember

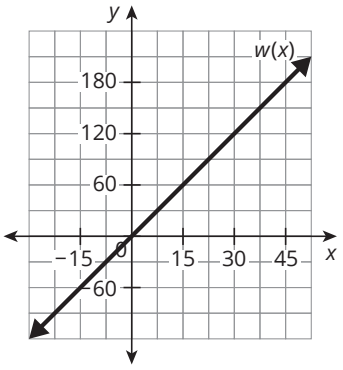
For the parent function $f(x) = x$, the transformed function $y = f(x) + d$ shows a vertical translation of the function. For $d > 0$, the resulting graph vertically shifts up. For $d < 0$, the resulting graph vertically shifts down. The parent function and the resulting graph are parallel because they have the same slope but different y-intercepts.

The transformed function $y = af(x)$ shows a vertical dilation of the function. For $|a| > 1$, the resulting graph vertically stretches by a factor of a units. For $0 < |a| < 1$, the resulting graph vertically compresses by a factor of a units. For $a < 0$, the resulting graph is vertically stretched or compressed and is reflected across the x-axis.

Practice

1. Given $w(x) = 4x$.

   a. Graph $r(x) = \frac{1}{2} \cdot w(x)$. Then, complete the table of corresponding points on $r(x)$.



x	$w(x)$	$r(x)$
0		
15		
30		
45		

   b. Describe the transformation performed on $w(x)$ to produce $r(x)$.

   c. Write the equation for the function $r(x)$ in general form.

2. Write the equation of a line parallel to the line $y = 2x$ that passes through the given points.

   a. (0, 4)

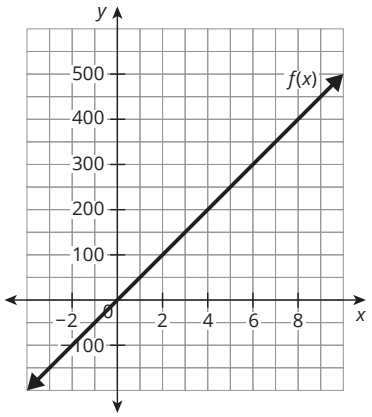
   b. (−2, −1)

   c. (2, 0)

Lesson 1 Assignment

3. Given $f(x) = 50x$.

   a. Graph $b(x) = f(x) - 150$. Then, complete the table of corresponding points on $b(x)$.



x	f(x)	b(x)
2		
4		
6		
8		

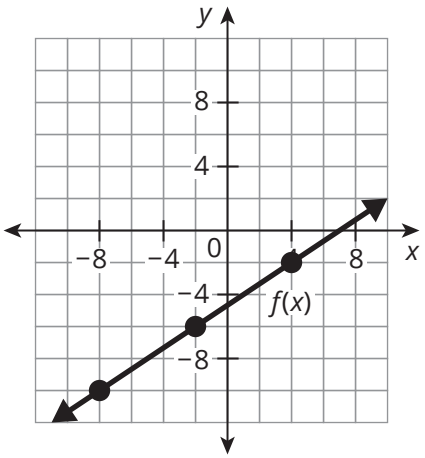
   b. Write the equation for the function $b(x)$ in general form.

   c. Describe the transformation performed on $f(x)$ to produce $b(x)$.

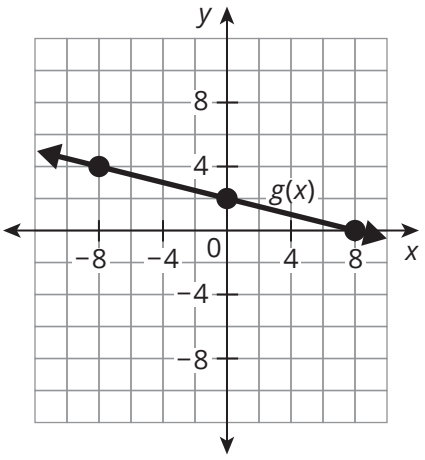
Prepare

Write a linear function for each graph.

1.



2.



2 Vertical and Horizontal Transformations of Linear Functions

OBJECTIVES

- Identify key characteristics of linear functions.
- Determine the effects on the graph of a linear function when $f(x)$ is replaced by $f(x) + d$, $f(x - c)$, $af(x)$, and $f(bx)$.

You have learned about the effects of vertical transformations on linear functions.

What effect do horizontal transformations have on linear functions? How do they compare to vertical transformations?

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Getting Started

Hey, I Remember This!

Graph each linear function and describe the key characteristics of the graph.

1. $g(x) = 3x - 8$

   Domain: _____

   Range: _____

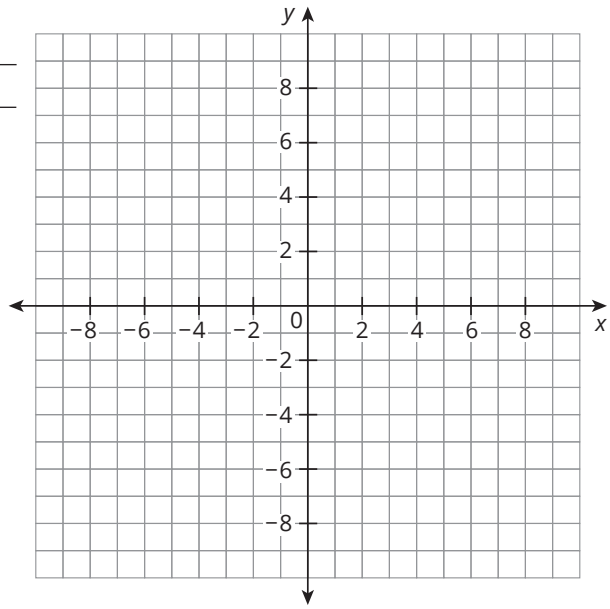
   x-intercept: _____

   y-intercept: _____

   Slope: ____

   Increasing/Decreasing:

   _____



2. $h(x) = -\frac{2}{3}x + 5$

   Domain: _____

   Range: _____

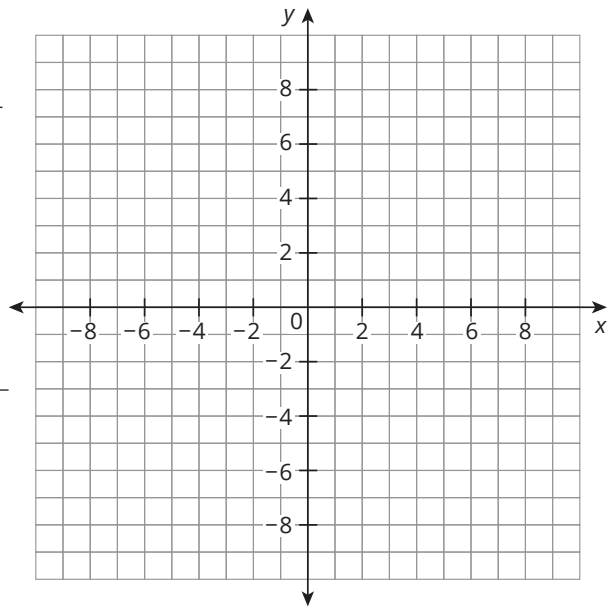
   x-intercept: _____

   y-intercept: _____

   Slope: _____

   Increasing/Decreasing:

   _____



ACTIVITY 2.1

Comparing Vertical and Horizontal Translations

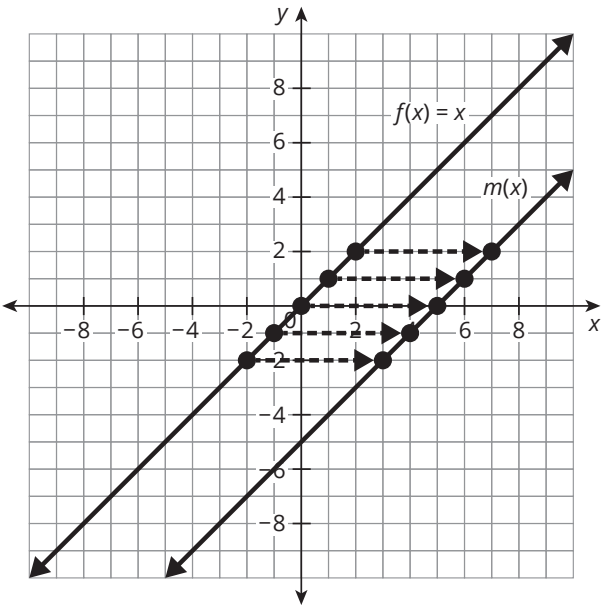
Let's determine how translations impact the graph of the linear function $f(x) = x$.

WORKED EXAMPLE

You can translate the graph of $f(x)$ to the right 5 units by moving each point 5 units to the right. The transformed graph is labeled as $m(x)$.

To translate the point $(-2, -2)$ on $f(x)$, add 5 units to the input value, or x-value. The output value, or y-value, remains unchanged. The coordinates of the translated point on $m(x)$ are $(3, -2)$.

The coordinates of four additional points on $f(x)$ are translated for you.

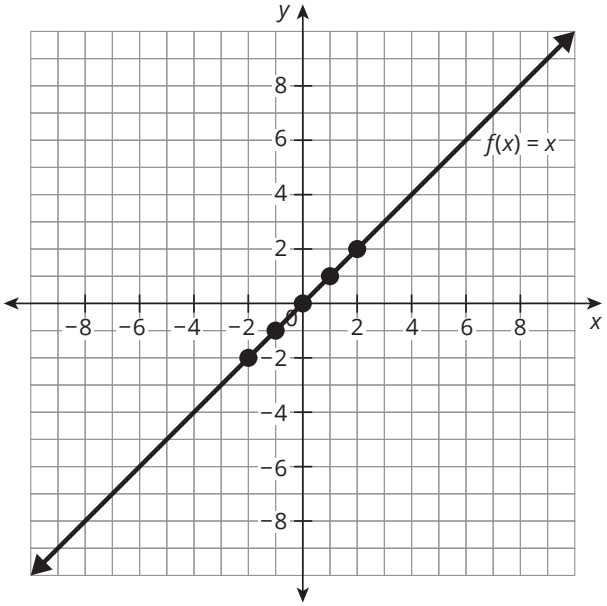


Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$m(x)$
3	−2
4	−1
5	0
6	1
7	2

1. Given the graph and a table of values for $f(x) = x$, complete the following transformations.

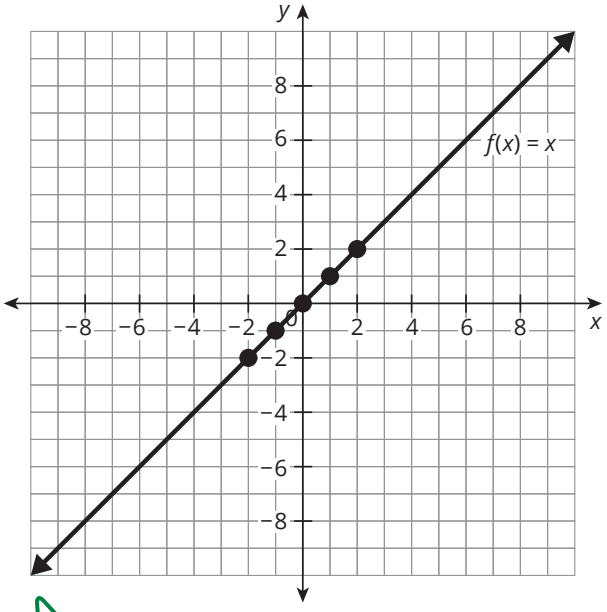
   a. Translate the graph of $f(x)$ left 3 units. Label your graph as $t(x)$. Complete the table of corresponding points on $t(x)$.



Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$t(x)$

   b. Translate the graph of $f(x)$ up 3 units. Label your graph as $w(x)$. Complete the table of corresponding points on $w(x)$.



Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$t(x)$

Ask Yourself . . .
What observations can you make?

c. Compare the table of values for $f(x)$ and $t(x)$. Are the input values or output values affected by this horizontal translation?

d. Compare the table of values for $f(x)$ and $w(x)$. Are the input values or output values affected by this vertical translation?

e. Compare the graphs and tables of values for $t(x)$ and $w(x)$. What do you notice?

2. Which transformation(s) of the parent function $f(x) = x$ affect the input values of the corresponding points on the transformed function?

3. Which transformation(s) of the parent function $f(x) = x$ affect the output values of the corresponding points on the transformed function?

4. Based on your observations, horizontally translating the parent function $f(x) = x$ right n units is equivalent to what other transformation?

5. Based on your observations, vertically translating the parent function $f(x) = x$ up n units is equivalent to what other transformation?

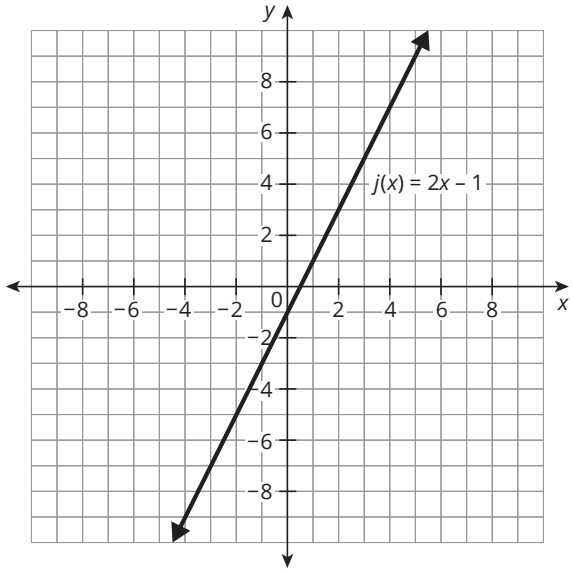
For the parent function $f(x) = x$, the transformed function $y = f(x - c)$ affects the input values, or x-values, of the function. For $c > 0$, the resulting graph horizontally shifts right c units. For $c < 0$, the resulting graph horizontally shifts left c units.

For the parent function $f(x) = x$, the transformed function $y = f(x) + d$ affects the output values, or y-values, of the function. For $d > 0$, the resulting graph vertically shifts up d units. For $d < 0$, the resulting graph vertically shifts down d units.

You have determined that for the parent function $f(x) = x$, a horizontal translation right n units produces the same graph as a vertical translation down n units. In addition, a horizontal translation left n units produces the same graph as a vertical translation up n units.

Let's see whether these translations have similar effects on the graph of any linear function.

6. Consider the graph of $j(x) = 2x - 1$.



   a. Sketch the graph of $j(x)$ shifted up 4 units. Label the graph as $q(x)$.

   b. Write an equation for $q(x)$ in terms of $j(x)$.

   c. Sketch the graph of $j(x)$ shifted left 4 units. Label the graph as $r(x)$.

   d. Write an equation for $r(x)$ in terms of $j(x)$.

e. Compare the graphs of $q(x)$ and $r(x)$. What do you notice?

f. The equation for $q(x)$ is simplified for you. Simplify the equation for $r(x)$. What do you notice?

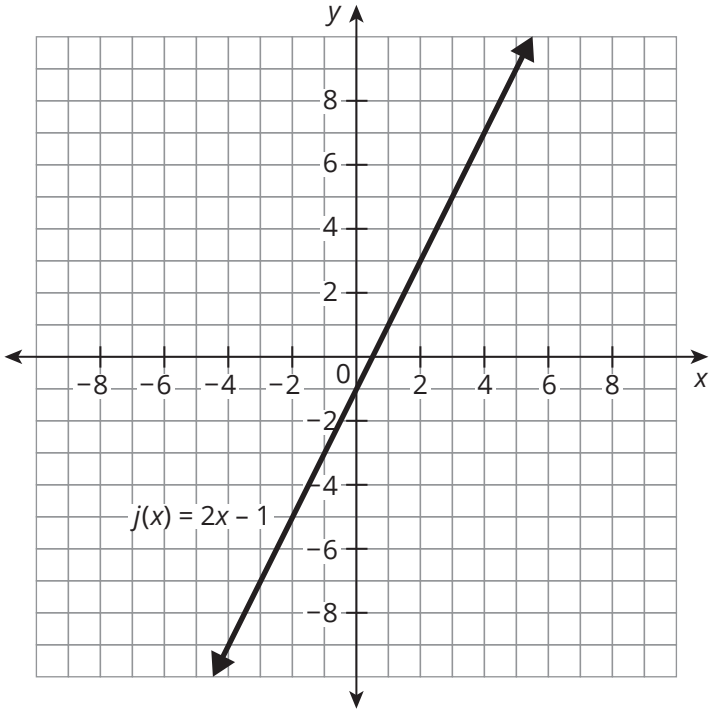
$q(x) = j(x) + 4$ $\qquad\qquad$ $r(x) = j(x) + 4$

$= 2x - 1 + 4$

$q(x) = 2x + 3$

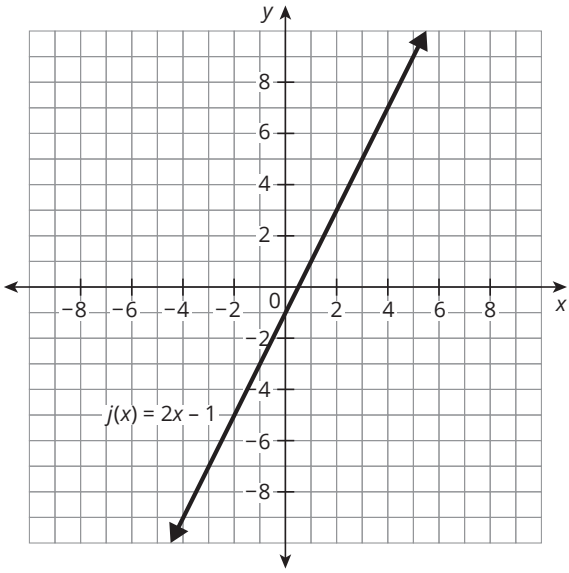
7. Again, consider the graph of $j(x) = 2x - 1$.

a. Sketch the graph of $j(x)$ shifted right 2 units. Label the graph as $b(x)$.



b. Write an equation for $b(x)$ in terms of $j(x)$.

c. Sketch the graph of $j(x)$ shifted down 2 units. Label the graph as $v(x)$.



d. Write an equation for $v(x)$ in terms of $j(x)$.

e. Compare the graphs of $b(x)$ and $v(x)$. What do you notice?

f. Simplify the equations for $b(x)$ and $v(x)$. What do you notice?

$b(x) =$ $v(x) =$

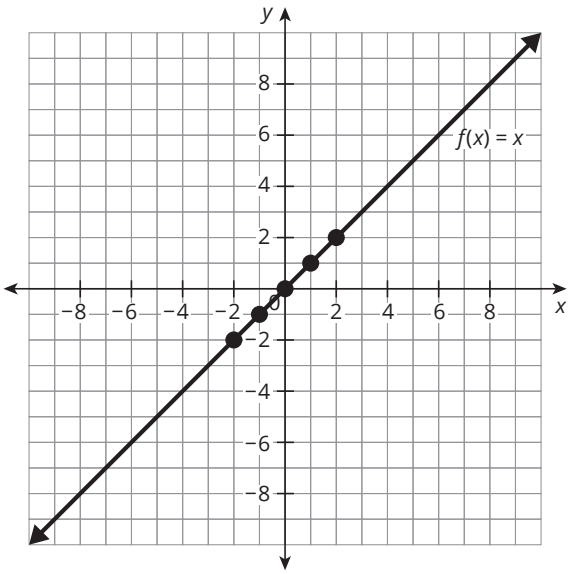
8. For any linear function, does a horizontal translation right n units produce the same graph as a vertical translation down n units?

9. For any linear function, does a horizontal translation left n units produce the same graph as a vertical translation up n units?

ACTIVITY
2.2 Comparing Vertical and Horizontal Dilations

1. The graph and a table of values for $f(x) = x$ are provided for you.

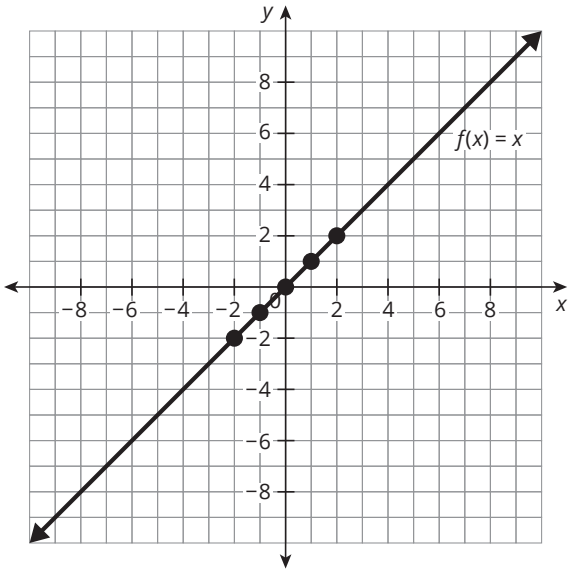
   a. Sketch the graph of $a(x)$, if the output values of $a(x)$ are changed by a factor of 2. Complete the table of corresponding points on $a(x)$.



Original Graph	
x	*f(x)*
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	*a(x)*
−2	
−1	
0	
1	
2	

   b. Sketch the graph of $b(x)$, if the input values of $b(x)$ are changed by a factor of $\frac{1}{2}$. Complete the table of corresponding points on $b(x)$.



Original Graph	
x	*f(x)*
−2	−2
−1	−1
0	0
1	1
2	2

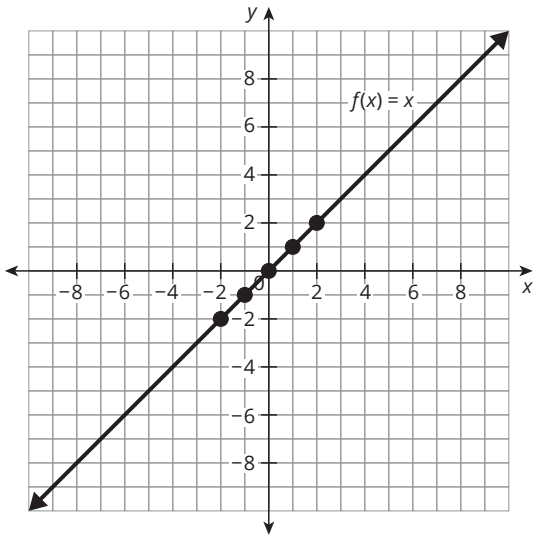
Transformed Graph	
x	*b(x)*
	−2
	−1
	0
	1
	2

   c. Compare the graphs and tables of values for $a(x)$ and $b(x)$. What do you notice?

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2. Again, the graph and a table of values for $f(x) = x$ are provided for you.

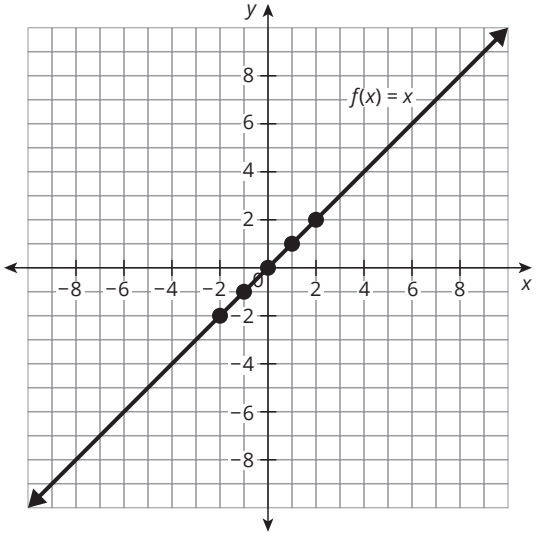
   a. Sketch the graph of $c(x)$, if the output values of $c(x)$ are changed by a factor of $\frac{4}{5}$. Complete the table of corresponding points on $c(x)$.



Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$c(x)$
−2	
−1	
0	
1	
2	

   b. Sketch the graph of $d(x)$, if the input values of $d(x)$ are changed by a factor of $\frac{5}{4}$. Complete the table of corresponding points on $d(x)$.



Original Graph	
x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2

Transformed Graph	
x	$d(x)$
	−2
	−1
	0
	1
	2

   c. Compare the graphs and tables of values for $c(x)$ and $d(x)$. What do you notice?

   d. Based on your observations, changing the output values of the parent function $f(x) = x$ by a factor of n units is equivalent to changing the input values by what factor?

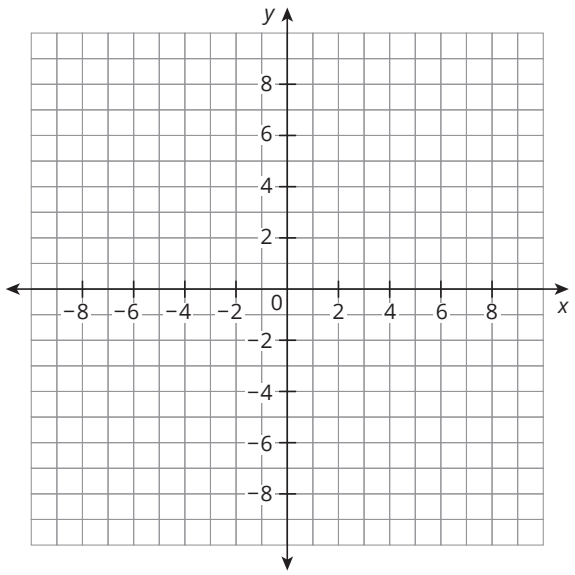
e. Based on your observations, changing the input values of the parent function $f(x) = x$ by a factor of n units is equivalent to changing the output values by what factor?

For the parent function $f(x) = x$, the transformed function $y = af(x)$ affects the output values, or y-values, of the function. For $|a| > 1$, the resulting graph vertically stretches by a factor of a units. For $0 < |a| < 1$, the resulting graph vertically compresses by a factor of a units. For $a < 0$, the resulting graph reflects across the x-axis.

For the parent function $f(x) = x$, the transformed function $y = f(bx)$ affects the input values, or x-values, of the function. For $|b| > 1$, the resulting graph horizontally compresses by a factor of $\frac{1}{|b|}$ units. For $0 < |b| < 1$, the resulting graph horizontally stretches by a factor of $\frac{1}{|b|}$ units. For $b < 0$, the resulting graph reflects across the y-axis.

You have determined that for the parent function $f(x) = x$, a vertical dilation by a factor of n units produces the same graph as a horizontal dilation by a factor of $\frac{1}{n}$ units. In addition, a horizontal dilation by a factor of n units produces the same graph as a vertical dilation by a factor of $\frac{1}{n}$ units.

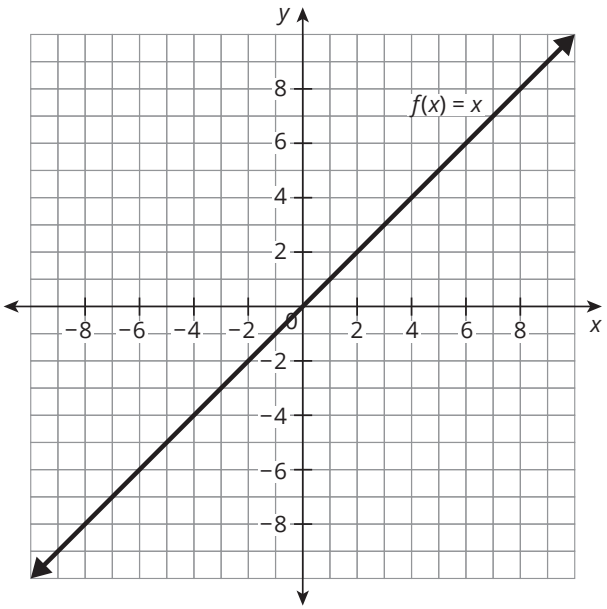
3. Does this relationship between horizontal and vertical dilations hold true for any linear function? Explain your reasoning. Provide a counterexample if necessary.



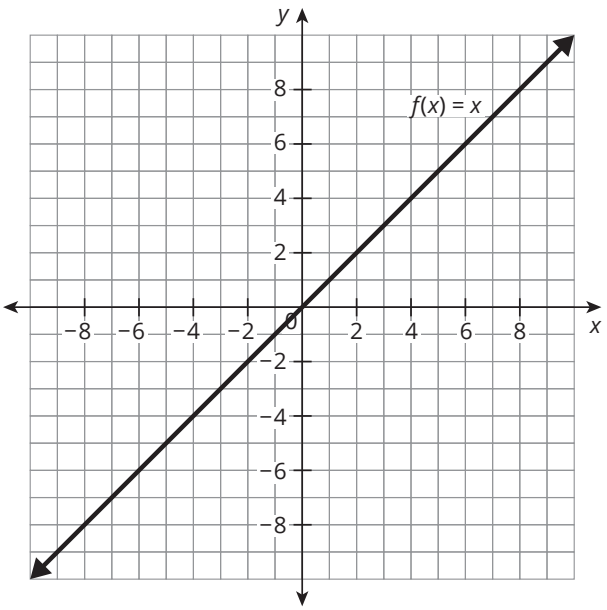
ACTIVITY
2.3 Graphing Linear Transformations

The equation of the parent linear function $f(x)$ is given. The equation for the transformed function $g(x)$ is also given. Describe the transformation(s) performed on $f(x)$ to produce $g(x)$. Then, graph $g(x)$.

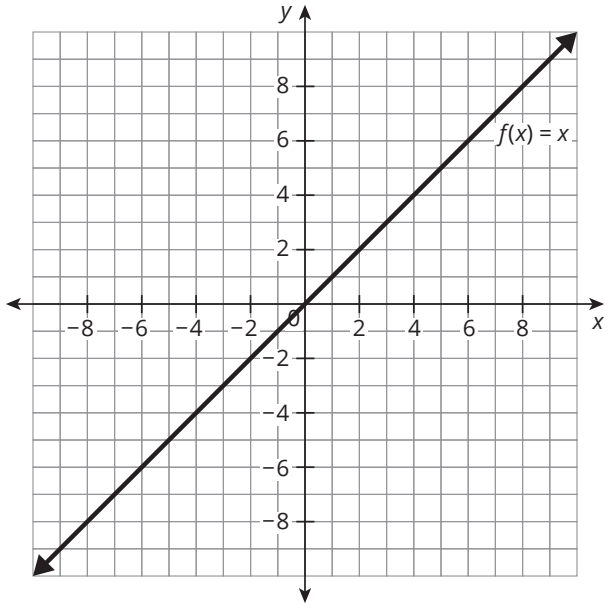
1. $f(x) = x$
   $g(x) = f(x - 7)$



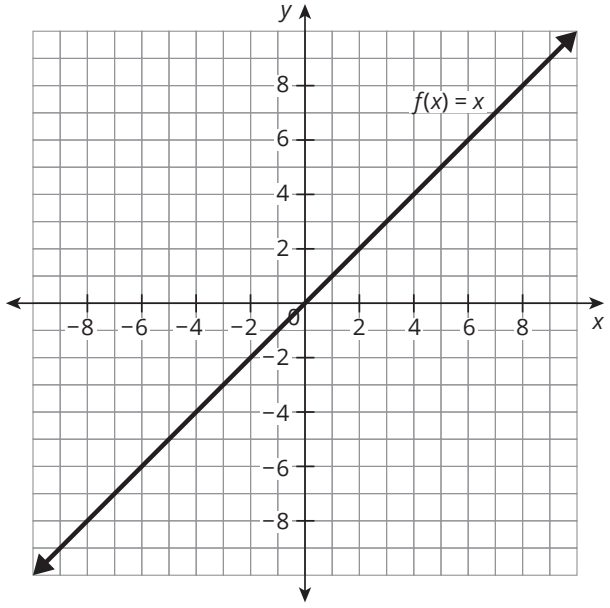
2. $f(x) = x$
   $g(x) = 3f(x)$



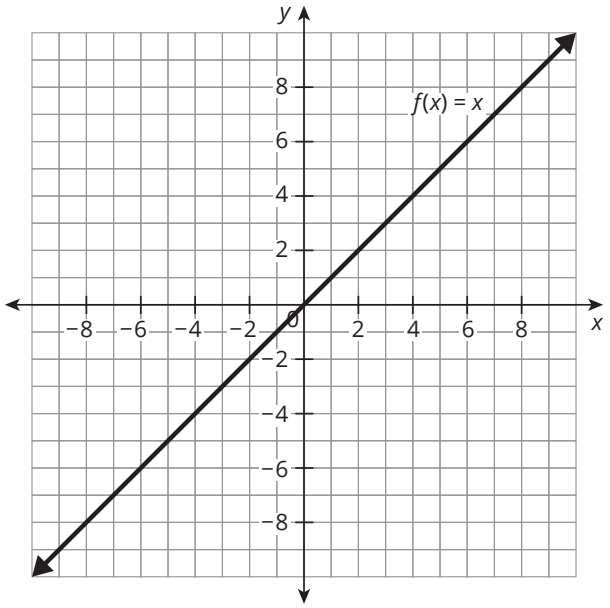
3. $f(x) = x$
$g(x) = f(x) + 2$



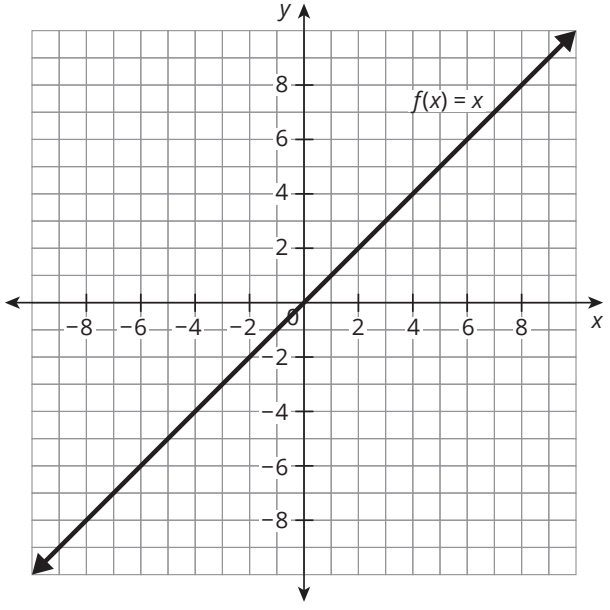
4. $f(x) = x$
$g(x) = f(\frac{1}{2}x)$



5. $f(x) = x$
$g(x) = f(x) - 4$



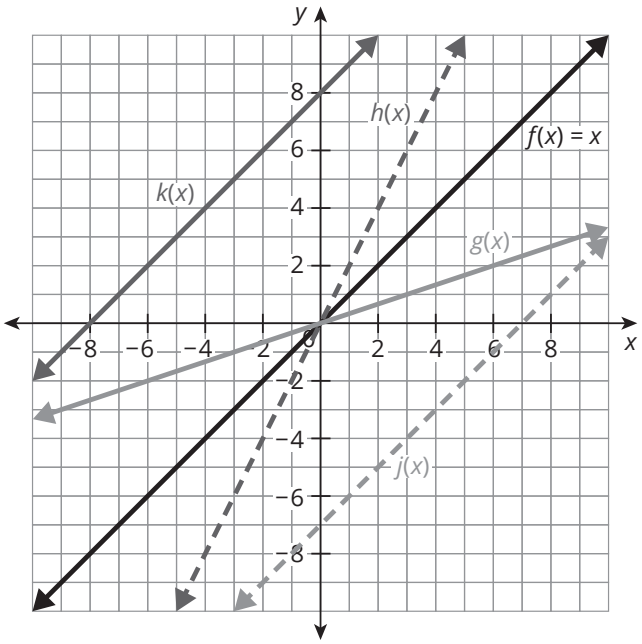
6. $f(x) = x$
$g(x) = f(x + 5)$



Talk the Talk

Show Me What You've Got!

The graph shows the linear function $f(x) = x$ and four transformations of $f(x)$.



Equation
$f(\frac{1}{3}x)$
$2f(x)$
$f(x - 7)$
$f(x) + 8$

Match each transformed graph to the equation in the table, written in terms of $f(x)$.

1. $k(x) =$ _____

2. $h(x) =$ _____

3. $g(x) =$ _____

4. $j(x) =$ _____

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Lesson 2 Assignment

Write

For the parent function, $f(x) = x$, how do horizontal transformations compare to vertical transformations?

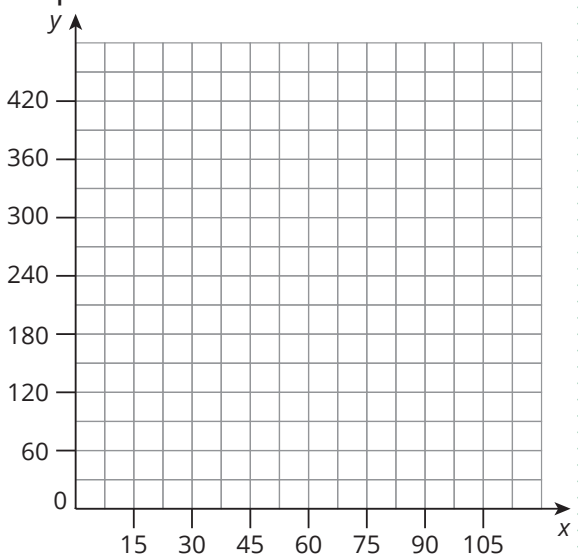
Remember

When $f(x)$ is replaced by $af(b(x - c)) + d$, b and c affect the input values of the function.

Practice

1. Ricardo started an exercise program. He researched the number of calories burned by walking and calisthenics. He learned that walking burns 4 calories per minute. The number of calories Ricardo can burn by walking is modeled by the function $w(x) = 4x$, where x represents the number of minutes he walks.

   a. Graph the linear function.



   b. Describe the key characteristics of the graph.
   Domain:
   Range:
   x-intercept/zero:

   c. Do the domain and range make sense in terms of the problem situation? Explain your reasoning. If the domain and range do not make sense, how could they be changed to better model the situation?

Lesson 2 Assignment

d. Ricardo burned 330 calories doing calisthenics before going for a walk. Let $c(x)$ be the function that represents Ricardo's calories burned from doing calisthenics and walking. Write an equation for $c(x)$ in terms of $w(x)$.

e. Simplify the equation for $c(x)$.

f. Describe the transformation performed on $w(x)$ to produce $c(x)$.

Prepare

Determine the reciprocal of each value.

1. 3

2. -10

3. $\frac{1}{5}$

4. $-\frac{5}{4}$

5. $-c$

6. $\frac{a}{b}$

3 Determining Slopes of Perpendicular Lines

OBJECTIVES

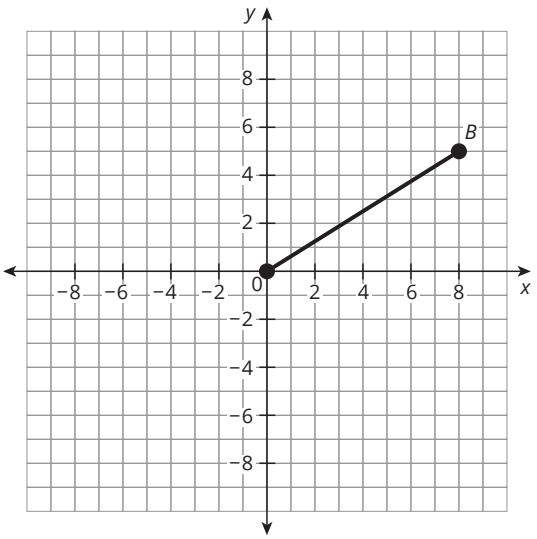
- Identify and write the equations of lines perpendicular to given lines.
- Identify and write the equations of horizontal and vertical lines.

You have translated the graphs of linear equations to determine the slopes of parallel lines. How can you rotate the graphs of linear equations to determine the slopes of perpendicular lines?

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Getting Started

Coordinate Rotation



Consider $\overline{AB}$ with coordinates A (0, 0) and B (8, 5).

1. Suppose the line segment is rotated through an angle with the origin as the center of rotation. Complete the table with the coordinates of the rotated figure.

Original Line Segment	90° Counterclockwise	180°	270° Counterclockwise
$\overline{AB}$	$\overline{A'B'}$	$\overline{A''B''}$	$\overline{A'''B'''}$
A (0, 0)			
B (8, 5)			

2. What do you notice about the coordinates of the endpoints of each rotated figure?

3. Determine the slope of each line segment.

   a. $\overline{AB}$

   b. $\overline{A'B'}$

   c. $\overline{A''B''}$

   d. $\overline{A'''B'''}$

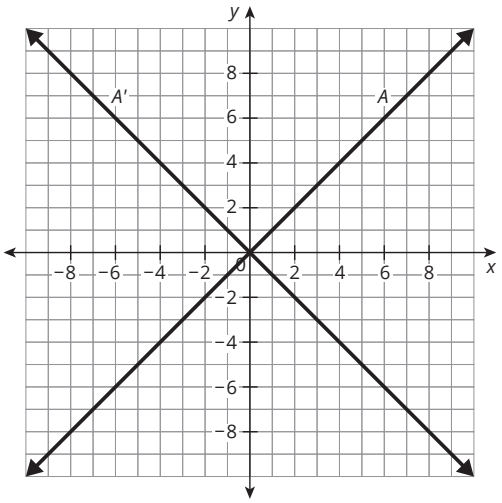
4. **Describe** any patterns you notice in the slopes of the figures.

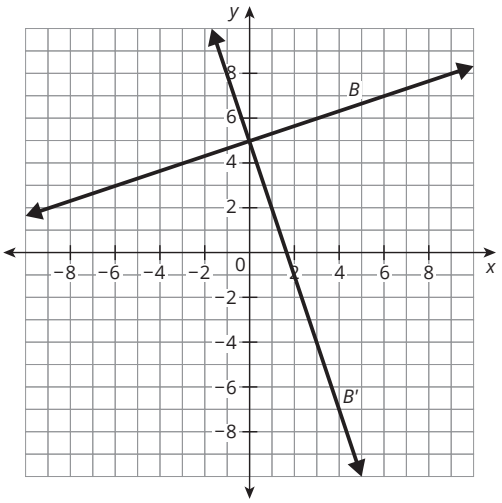
Review the definition of **describe** in the Academic Glossary.

ACTIVITY
3.1 Slopes of Perpendicular Lines

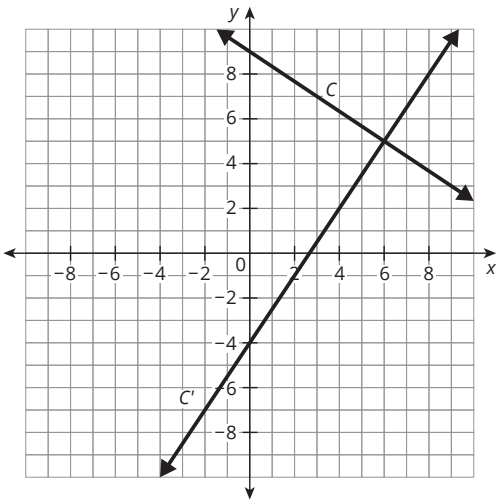
Recall that perpendicular lines or line segments form a right angle at the point of intersection.

Consider the three graphs shown. Each shows a line and its rotation 90° about a point, which is also the point of intersection.





1. Are the lines in each graph perpendicular? Explain your reasoning.



2. Write the equation for each line and its transformation. What do you notice?

Remember . . .

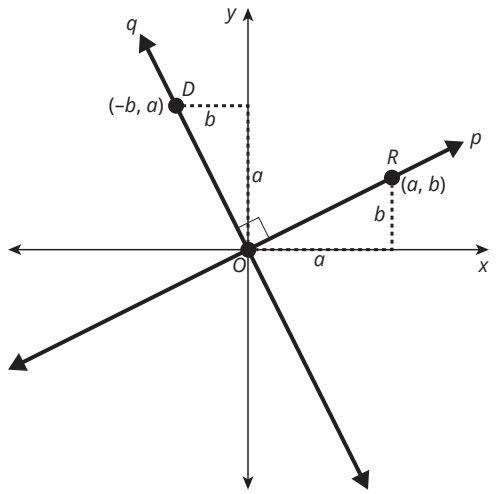
The reciprocal of a number $\frac{a}{b}$ is the number $\frac{b}{a}$, where a and b are nonzero numbers. Because the product of a number and its reciprocal is one, reciprocal numbers are also known as multiplicative inverses.

Consider the theorem.

WORKED EXAMPLE

Theorem: If two lines are perpendicular, their slopes are negative reciprocals.

Use the graph and the proof shown to analyze the validity of the theorem.



Given: $p \perp q$

Let m_1 = slope of line p and let m_2 = slope of line q.

Point R lies on line p.

Prove: $m_1 = -\frac{1}{m_2}$

Rotate point R 90° counterclockwise using point O as the center of rotation. Since p and q are perpendicular, the image (point D) will lie on line q under this 90° rotation.

Since this rotation maps the positive x-axis to the positive y-axis and the positive y-axis to the negative x-axis, then the coordinates of $R(a, b)$ are transformed into the coordinates of $D(-b, a)$. Graphically, you can see the movement of lengths a and b under the rotation.

The graph shows the slope of line p, $m_1 = \frac{b}{a}$, and the slope of line q, $m_2 = \frac{a}{-b}$.

Using these slopes, you can demonstrate that $m_1 = -\frac{1}{m_2}$.

$$\frac{b}{a} = -\frac{1}{\frac{a}{-b}}$$

$$= -1 \cdot \frac{-b}{a}$$

$$= \frac{b}{a}$$

The slope of line q is the negative reciprocal of the slope of line p.

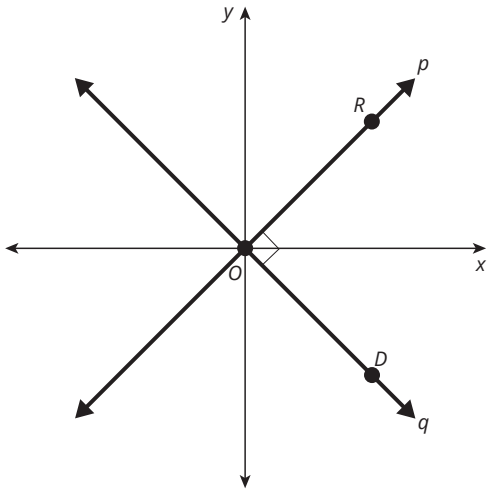
Remember . . .
The symbol $\perp$ means *is perpendicular* to.

The product of the slopes of perpendicular lines is -1.

The proof shown is a paragraph proof. You will learn different formats of proof as you investigate more properties of lines and figures.

There is often more than one way to prove a theorem. Suppose that point R is rotated 90° clockwise using point O as the center of rotation.

3. Rewrite the proof using the clockwise rotation of point R.



4. Line j and line k are perpendicular. Given each slope of line j, determine the slope of line k.

   a. $m = \frac{2}{3}$

   b. $m = -\frac{4}{5}$

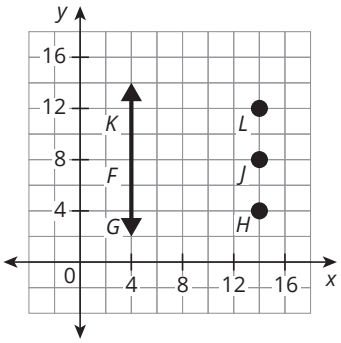
   c. $m = -3$

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ACTIVITY
3.2 Horizontal and Vertical Lines

Consider the graph shown.

1. Use a straightedge to extend $\overline{GK}$ to create line p, extend $\overline{GH}$ to create line q, extend $\overline{FJ}$ to create line r, and extend $\overline{KL}$ to create line s.



2. Consider the three horizontal lines you drew for Question 1. For any horizontal line, if x increases by one unit, by how many units does y change?

3. Describe the slope of any horizontal line. Explain your reasoning.

4. Consider the vertical line you drew in Question 1. Suppose that y increases by one unit. By how many units does x change?

5. Describe the slope of any vertical line. Explain your reasoning.

6. Determine whether each of the given statements is always, sometimes, or never true. Explain your reasoning.

   a. All vertical lines are parallel.

   b. All horizontal lines are parallel.

7. Describe the relationship between any vertical line and any horizontal line.

8. Write an equation for a horizontal line and an equation for a vertical line that pass through the point $(2, -1)$.

9. Write an equation for a line that is perpendicular to the line given by $x = 5$ and passes through the point $(1, 0)$.

10. Write an equation for a line that is perpendicular to the line given by $y = -2$ and passes through the point $(5, 6)$.

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ACTIVITY
3.3 Writing Equations of Perpendicular Lines

Remember . . .
You can use point-slope form to write an equation for any line when you know its slope and one point on that line.

In a previous lesson you wrote the equation of a line parallel to a given line that passes through a given point. You can write the equation of a perpendicular line using what you know about the slope of that line and any point on that line.

1. Write the equation of the line perpendicular to $y = 2x + 1$ that passes through the point (6, 2).

2. Write the equation of the line perpendicular to $y = -\frac{3}{4}x$ that passes through the point (3, −8).

3. Write the equation of the line that passes through the point (6, 2) and is perpendicular to a line that passes through the points (−5, 3) and (−1, −9).

4. Write the equation of a line that passes through the point $(-2, 7)$ and is perpendicular to a line that passes through the points $(-6, 1)$ and $(0, 4)$.

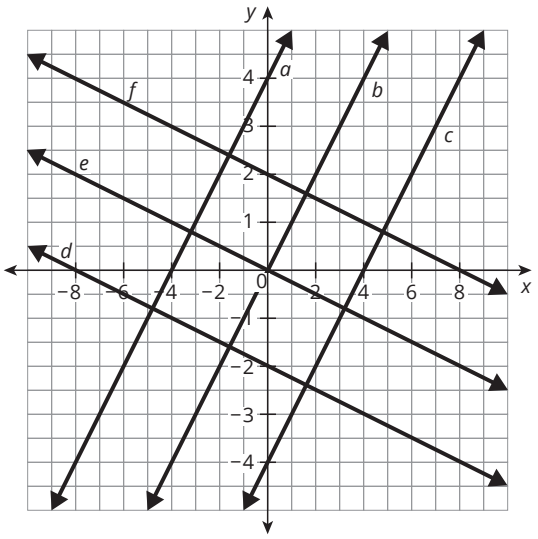
5. A pair of perpendicular lines intersect at the point $(5, 9)$. Write the equation of the line that is perpendicular to the line that also passes through point $(-4, 4)$.

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Talk the Talk

Things Aren't Always What They Seem

Consider the graphs of six linear equations shown.



PROBLEM SOLVING



1. Linh says lines *a*, *b*, and *c* are parallel to each other and perpendicular to lines *d*, *e*, and *f*. Jackson agrees that lines *a*, *b*, and *c* are parallel to each other but says they are not perpendicular to lines *d*, *e*, and *f*. Who is correct? Justify your reasoning.

Lesson 3 Assignment

Write

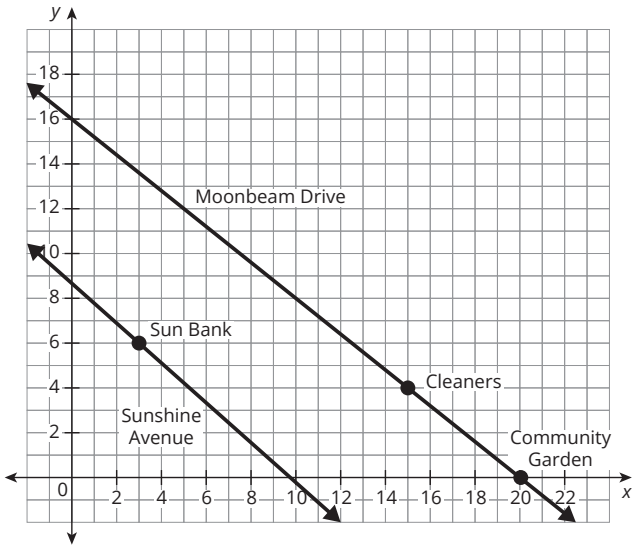
Explain in your own words why the slope of a vertical line is undefined.

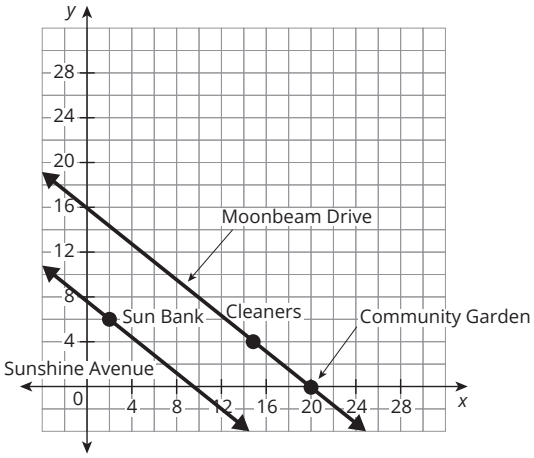
Remember

The slopes of perpendicular lines are negative reciprocals. Any vertical line is perpendicular to any horizontal line.

Practice

Nakota is a developer and plans to build a new development. Use the grid to help Nakota create a map for his development. Each gridline represents one block.





1. There are two main roads that pass through the development, Moonbeam Drive and Sunshine Avenue. Are these two roads parallel to each other? Explain your reasoning.

Lesson 3 Assignment

2. Nakota wants to build a road named Stargazer Boulevard that will be parallel to Moonbeam Drive. On this road, he will build a diner located seven blocks north of the community garden. Determine the equation of the line that represents Stargazer Boulevard. Show your work. Then, draw and label Stargazer Boulevard on the coordinate plane.

3. Nakota wants to build a road named Rocket Drive that connects Sun Bank to Moonbeam Drive. He wants this road to be as short as possible. Determine the equation of the line that represents Rocket Drive. Show your work. Then, draw and label Rocket Drive on the coordinate plane.

4. Two office buildings are to be located at the points (8, 4) and (12, 10). Would the shortest road between the two office buildings be a line that is perpendicular to Moonbeam Drive? Explain your reasoning.

Lesson 3 Assignment

5. A straight road named Planet Drive is planned that will connect the diner and the community garden. What is the equation of the line that represents Planet Drive? Show your work. Draw and label Planet Drive on the coordinate plane.

6. Nakota decides that another road to be named Saturn Avenue is needed that will go past the cleaners and be perpendicular to Planet Drive. Determine the equation of the line that represents Saturn Avenue. Show your work. Draw and label Saturn Avenue on the coordinate plane.

Prepare

Determine whether each set of ordered pairs represents a function. Explain your reasoning.

1. {(−1, −1), (0, 0) (1, 1), (2, 2)}

2. {(−1, −2), (0, 0) (1, 2), (2, 4)}

3. {(−1, −1), (0, −1) (1, −1), (2, −1)}

4. {(−1, −1), (−1, 0) (−1, 1), (−1, 2)}

4 Comparing Linear Functions in Different Forms

OBJECTIVES

- Compare linear functions represented algebraically, graphically, in tables, or with verbal descriptions.
- Choose and interpret appropriate units to represent independent and dependent quantities in situations modeled by linear functions.
- Choose and interpret appropriate scales and origins for graphs of linear functions.

You have represented linear functions in a variety of different ways. How does each linear function representation compare with the others?

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Getting Started

Odd One Out

1. Choose the function that does not belong with the others and justify your choice.

Function A

x	y
−5	10
−1.5	3
0	0
2.5	−5

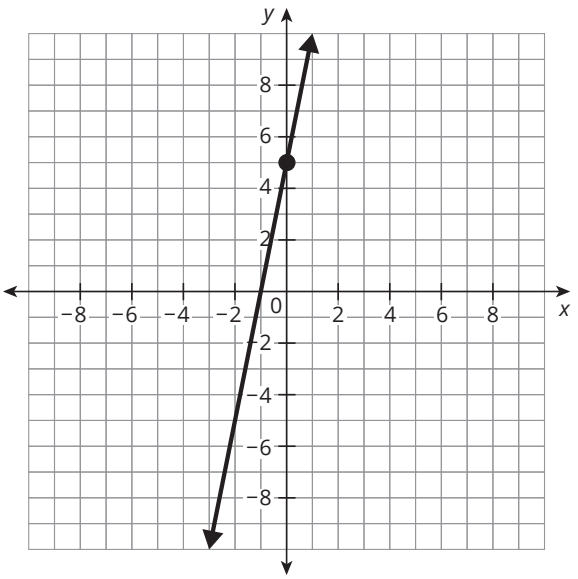
Function B

During one year at a high school, the ratio of 9th-graders to 10th-graders was 1 : 1.

Function C

$c(x) = \frac{1}{5}x$

Function D



ACTIVITY
4.1 Slopes of Linear Relationships

Consider the line that represents the equation $y = 4x - 3$.

1. Determine whether the graph of each linear relationship is parallel, perpendicular, or neither parallel nor perpendicular to the graph of $y = 4x - 3$.

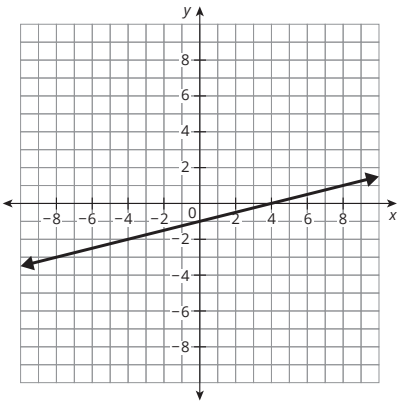
a.

x	y
−2.5	−6
0	4
0.5	6
3	16

b. An elevator descends 4 feet every second.

c. $y = \frac{1}{4}x + 5$

d.



e.

x	y
−4	3
0	2
2	1.5
8	0

f. For every hour Chiletso bakes, she makes 4 dozen cookies.

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PROBLEM SOLVING



ACTIVITY 4.2

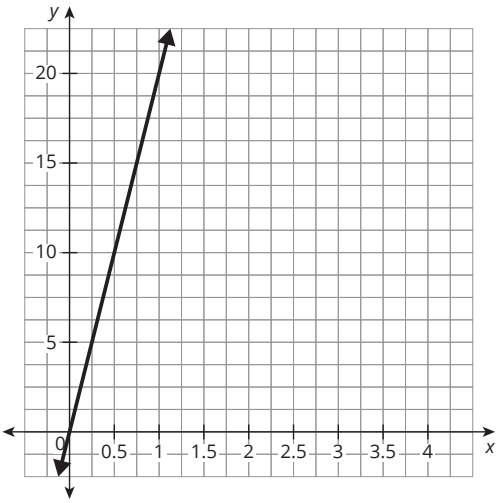
Comparing Tables and Graphs

Large recycling centers pay out different amounts per ton of cardboard recycled. Travis County Recycling Center lists its payout amounts in a table, and Hays County uses a graph to advertise its payout amounts. The two centers' payout amounts are close but not equal.

Travis County Recycling

0.5	$7.50
1	$15
1.5	$22.50
2	$30

Hays County Recycling



Ask Yourself . . .
How else can you represent this information?

1. Each representation shows a functional relationship between quantities. Label the quantities and their units in the table and on the graph.

2. Let $w(x)$ represent the function for Travis County, and let $j(x)$ represent the function for Hays County. Use $>$, $<$, or $=$ to complete each statement.

   a. $w(0.5)$ _____ $j(0.5)$ b. $w(1)$ _____ $j(1)$ c. $w(3)$ _____ $j(3)$

3. Compare the y-intercepts of each relationship. What does each y-intercept mean in terms of the relationship between the quantities?

Ask Yourself . . .
Did you complete all the steps in the problem-solving model?

4. Which recycling center would you choose? Provide evidence to justify your choice.

ACTIVITY
4.3 Comparing Equations and Tables

Xavier and Joey are friends who live in different countries. When they chat online and talk about the weather, one of them always uses temperature in degrees Celsius while the other talks about temperature using degrees Fahrenheit.

Xavier uses an equation to convert Joey's temperatures, and Joey looks at a table to convert Xavier's temperatures. The equation and part of the table are shown.

Remember . . .
$0°C = 32°F$

Joey's Table

10	−12.2
30	−1.1
50	10
75	23.9

Xavier's Equation

$J(x) = \frac{9}{5}x + 32$

1. Identify quantities in the table and equation. Who is converting from °C to °F, and who is converting from °F to °C? Explain your reasoning.

2. Compare the given characteristics for each function in terms of the quantities.

   a. slope

   b. y-intercept

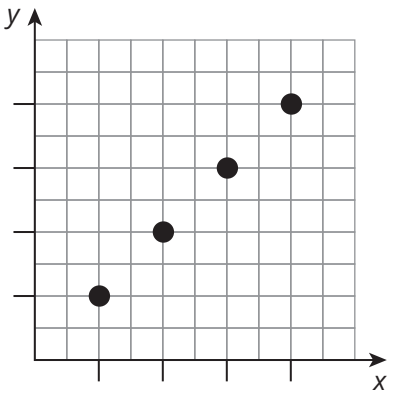
Think About . . .
The units of the quantities are very important.

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ACTIVITY
4.4 Comparing Descriptions and Graphs

Lauren and Miguel both opened bank accounts at the same time. Miguel's account balance is shown in the graph. He started with $0 and deposited the same amount each month. In the 4th month, he had $80 saved. Michelle opened her bank account on September 1st with $25 and continues to deposit $25 each month.

Miguel



Lauren

Lauren opened her bank account on September 1st with $25 and continues to deposit $25 each month.

1. Use what you know about Miguel's account to determine the scale of each axis and interpret the origin on the graph. Explain your reasoning.

2. Compare the given characteristics for each function in terms of the quantities.

   a. slope

   b. y-intercept

Ask Yourself . . .
How can you use comparing graphs, tables, and descriptions in everyday life?

Talk the Talk

Function Maker Space

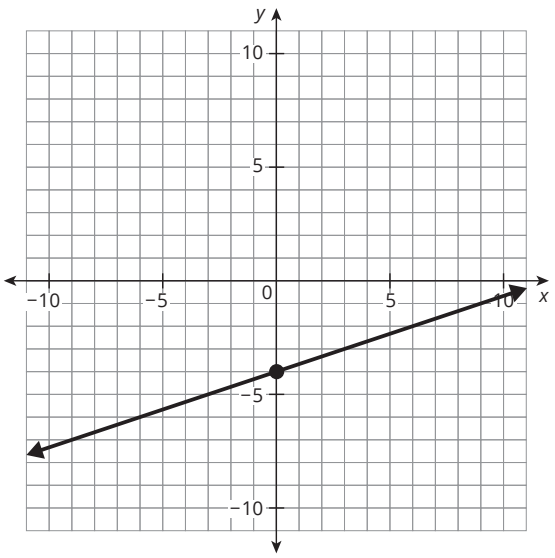
Consider the table of values.

x	−10	10	20	25
y	0	5	7.5	8.75

1. Create a situation to represent the table of values shown.

2. Write an equation that has a slope that is less steep than the relationship in the table.

3. How does the slope of the graph shown compare to the slope from the table of values and your equation?



4. What strategies did you use to create your linear functions and to compare the slopes?

Lesson 4 Assignment

Write

Describe how to compare the slopes and *y*-intercepts of two linear functions if one is represented as a graph and one is represented as a table.

Remember

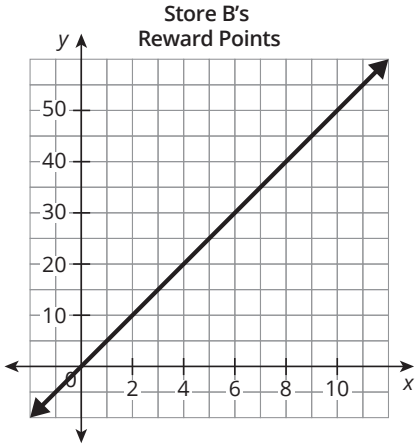
A linear function can be represented using an equation, a table, a graph, or with a verbal description. Characteristics of linear functions, such as slope, *y*-intercepts, and independent and dependent quantities, can be understood from different representations of functions.

Practice

1. Bookstores specializing in selling used books award different amounts of points to customers who supply them with used books. The points are used toward the purchase of other books in the store. Store A lists its point values in a table, and Store B uses a graph to post its point values.

Store A's Reward Points

2	12
4	24
6	36
8	48



a. Label each column of values in the table and label the *x*-axis and *y*-axis on the graph with the appropriate variable quantities.

b. Compare the slope of each function and explain what each represents in context.

Lesson 4 Assignment

c. Compare the y-intercepts of each function and explain what each represents in context.

2. Hailey and Ben live in different cities. They are planning to meet in Nashville, but they will both need to drive several days to get there. They have each calculated the distance to Nashville from their homes, but one calculated the distance in miles and the other calculated the distance in kilometers.

   Hailey uses an equation to convert the distances Ben plans to drive each day, and Ben uses a table to convert the distances Sherry plans to drive each day. The equation and part of the table are shown.

Ben's Table

300	482.80
382	614.77
426	685.58
475	764.44

Hailey's Equation

$y = 0.6214x$

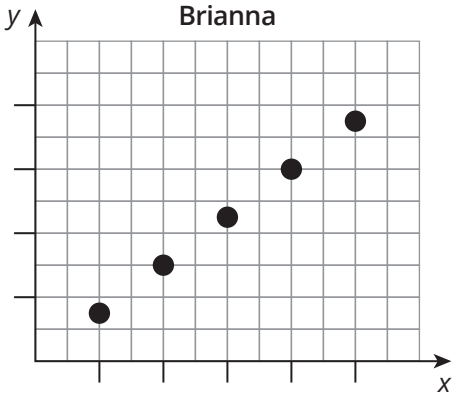
a. Label each column of quantities in Ben's table and identify the meaning of x and y in Hailey's equation. Who is converting from miles to kilometers and who is converting from kilometers to miles? Explain your reasoning.

Lesson 4 Assignment

b. Compare the slope for each function. Explain what each slope represents in context.

c. Compare the *y*-intercepts of each function and explain what each *y*-intercept represents in context.

3. Antonio and Brianna collect movies. Brianna's movie collection is shown in the graph. She started with 0 movies and added the same number of movies to her collection each month. In the 5th month, she had 15 movies. Antonio's movie collection is given by a description.



Antonio

Antonio started his collection with 27 movies he inherited from his uncle and continues to buy 2 movies each month.

a. Use what you know about Brianna's movie collection to determine the scale of her graph. Label the *x*- and *y*-axis, the origin, and the intervals on both axes. Explain your reasoning.

Lesson 4 Assignment

b. Compare the slope for each function. Explain what each represents in context.

c. Compare the *y*-intercepts of each function and explain what each represents in context.

Prepare

Solve each equation for x.

1. $\frac{1}{3}x = 8$

2. $5 + x = 12.7$

3. $2x - 9 = 6$

4. $12 + 2x = 3x - 1$

TOPIC 2 SELF-REFLECTION

Name: _____

Transforming and Comparing Linear Functions

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Transforming and Comparing Linear Functions* topic by:

TOPIC 2: *Transforming and Comparing Linear Functions*	Beginning of Topic	Middle of Topic	End of Topic
recognizing that the parent function for linear functions is $f(x) = x$.	○	○	○
determining the effects on the graph of the parent function $f(x) = x$ when $f(x)$ is replaced by $a \cdot f(x)$, $f(x) + d$, $f(x - c)$ and $f(bx)$.	○	○	○
describing the transformations performed on a parent function to create a new function in transformation form $a \cdot f(b(x - c)) + d$.	○	○	○
writing the equation of a line that contains a given point and is parallel or perpendicular to a given line.	○	○	○

continued on the next page

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TOPIC 2 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Transforming and Comparing Linear Functions* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

Transforming and Comparing Linear Functions Summary

NEW KEY TERM
- parent function

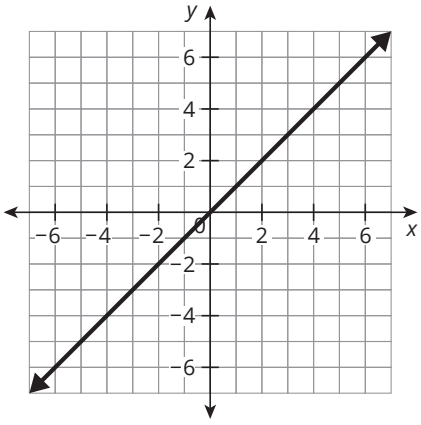
LESSON 1 Transforming Linear Functions

A **parent function** is the simplest function of its type. For example, $f(x) = x$ is the simplest linear function. It is in the form $f(x) = ax + b$, where $a = 1$ and $b = 0$.

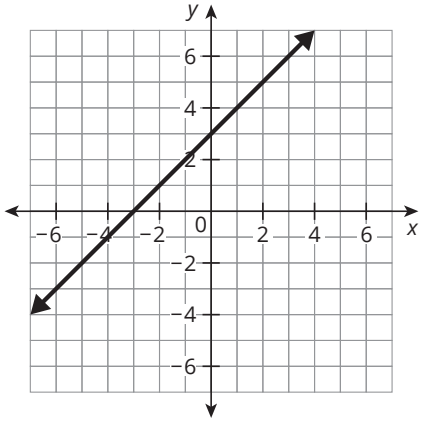
For the parent function $f(x) = x$, the transformed function $y = f(x) + d$ affects the output values of the function. For $d > 0$, the graph vertically shifts up. For $d < 0$, the graph vertically shifts down. The amount of shift is given by $|d|$.

For example, the function $y = x + 3$ translates the graph of $y = x$ vertically up 3 units.

$y = x$



$y = x + 3$



You can algebraically prove that a line and its translation are parallel to each other.

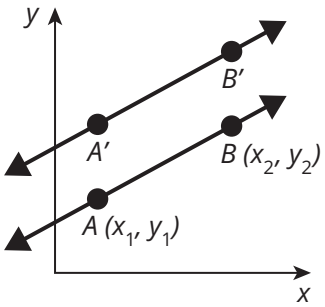
Line AB was translated a units up to create line $A'B'$.

The coordinates of point A' are $(x_1, y_1 + a)$ and the coordinates of point B' are $(x_2, y_2 + a)$.

$$\text{slope of line } AB = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$\text{slope of line } A'B' = \frac{[(y_2 + a) - (y_1 + a)]}{(x_2 - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

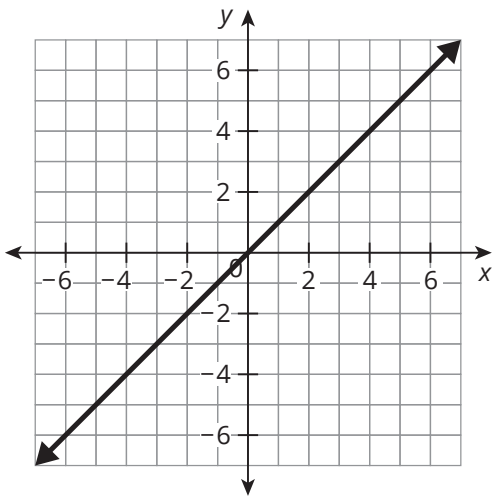
Since the slope of line AB is equal to the slope of line $A'B'$, the lines are parallel.



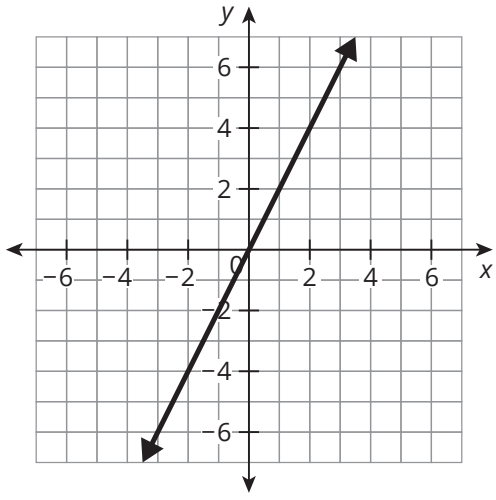
For the parent function $f(x) = x$, the transformed function $y = a \cdot f(x)$ affects the output values of the function. For $|a| > 1$, the graph vertically stretches by a factor of a units. For $0 < |a| < 1$, the graph vertically compresses by a factor of a units. For $a < 0$, the graph reflects across the x-axis.

For example, the function $y = 2x$ dilates the graph of $y = x$ by a factor of 2.

$y = x$



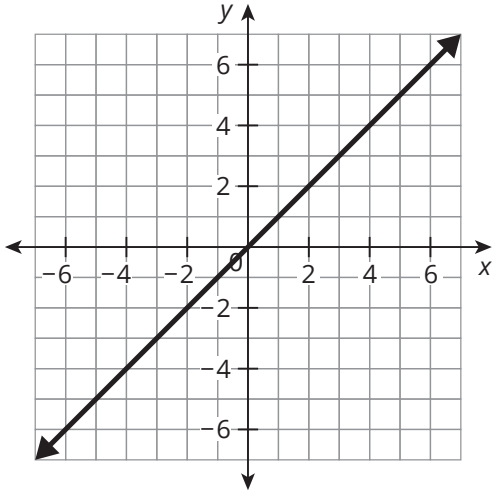
$y = 2x$



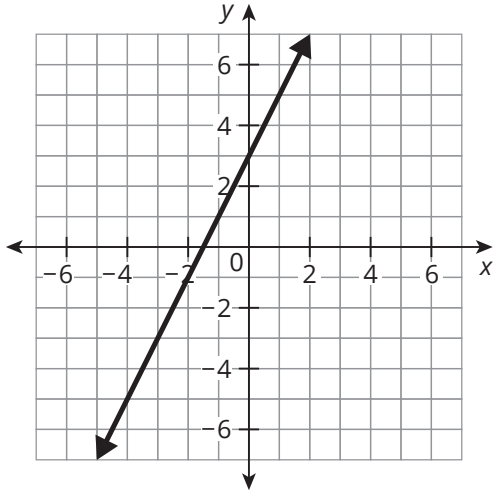
When a function is both translated and vertically dilated, the resulting function can be written in the form $a \cdot f(x) + d$, where d represents the vertical translation of $f(x)$ and a represents the vertical dilation of $f(x)$.

For example, the graph of $y = 2x + 3$ represents both a vertical translation of 3 units and vertical dilation by a factor of 2.

$y = x$



$y = 2x + 3$



LESSON 2

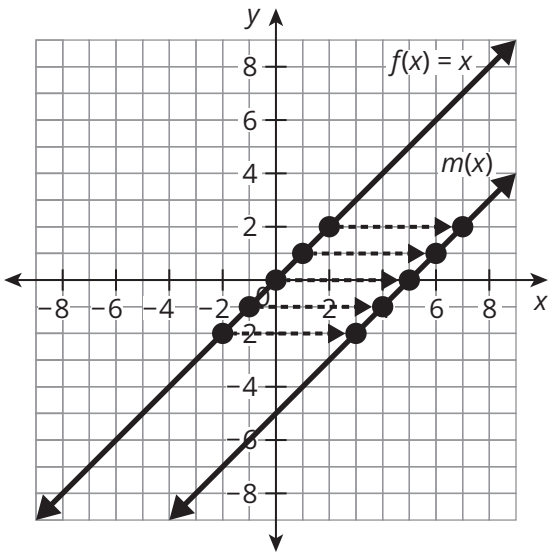
Vertical and Horizontal Transformations of Linear Functions

For the parent function $f(x) = x$, the transformed function $y = f(x - c)$ affects the input values, or x-values, of the function. For $c > 0$, the resulting graph horizontally shifts right c units. For $c < 0$, the resulting graph horizontally shifts left c units.

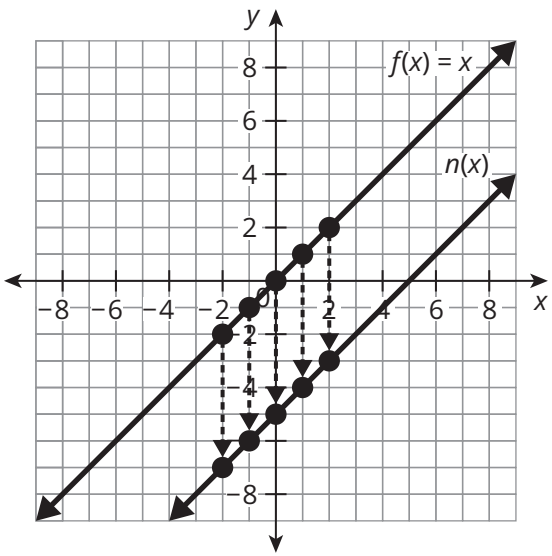
For the parent function $f(x) = x$, the transformed function $y = f(x) + d$ affects the output values, or y-values, of the function. For $d > 0$, the resulting graph vertically shifts up d units. For $d < 0$, the resulting graph vertically shifts down d units.

You have determined that for the parent function $f(x) = x$, a horizontal translation right n units produces the same graph as a vertical translation down n units. In addition, a horizontal translation left n units produces the same graph as a vertical translation up n units.

For example, you can translate the graph of $f(x) = x$ to the right 5 units by moving each point 5 units to the right. The transformed graph is labeled as $m(x)$. $m(x) = f(x - 5)$ which is $m(x) = x - 5$. You can translate the graph of $f(x) = x$ down 5 units by moving each point 5 units down. The transformed graph is labeled as $n(x)$. $n(x) = f(x) + (-5)$ which is $n(x) = x - 5$, thus producing the same equation and graph as $m(x)$.



Original Graph	
x	***f(x)***
−2	−2
−1	−1
0	0
1	1
2	2



Transformed Graph	
x	***m(x)***
3	−2
4	−1
5	0
6	1
7	2

Transformed Graph	
x	***n(x)***
−2	−7
−1	−6
0	−5
1	−4
2	−3

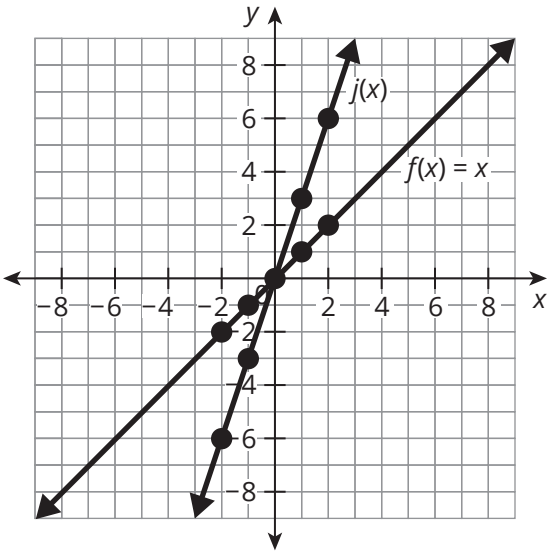
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For the parent function $f(x) = x$, the transformed function $y = a \cdot f(x)$ affects the output values, or y-values, of the function. For $|a| > 1$, the resulting graph vertically stretches by a factor of a units. For $0 < |a| < 1$, the resulting graph vertically compresses by a factor of a units. For $a < 0$, the resulting graph reflects across the x-axis.

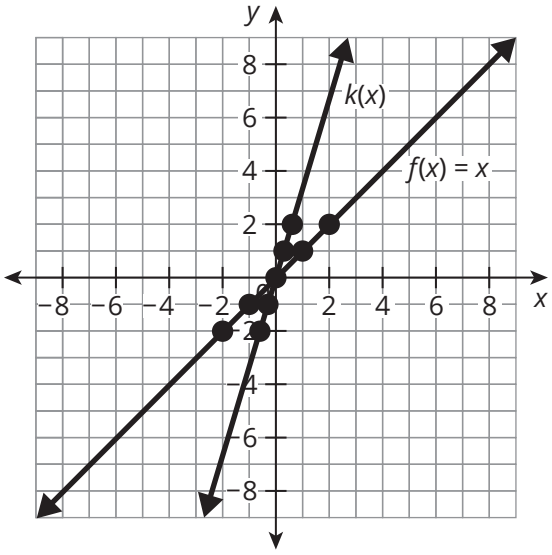
For the parent function $f(x) = x$, the transformed function $y = f(bx)$ affects the input values, or x-values, of the function. For $|b| > 1$, the resulting graph horizontally compresses by a factor of $\frac{1}{|b|}$ units. For $0 < |b| < 1$, the resulting graph horizontally stretches by a factor of $\frac{1}{|b|}$ units. For $b < 0$, the resulting graph reflects across the y-axis.

You have determined that for the parent function $f(x) = x$, a vertical dilation by a factor of n units produces the same graph as a horizontal dilation by a factor of $\frac{1}{n}$ units. In addition, a horizontal dilation by a factor of n units produces the same graph as a vertical dilation by a factor of $\frac{1}{n}$ units.

For example, you can vertically dilate the graph of $f(x) = x$ by a scale factor of 3. The transformed graph, $j(x)$, is a result of vertically stretching the graph of $f(x)$ by a factor of 3 units; $j(x) = 3f(x)$ which is $j(x) = 3x$. You can horizontally dilate the graph of $f(x) = x$ by a scale factor of $\frac{1}{3}$. The transformed graph, $k(x)$, is a result of horizontally compressing the graph $f(x)$ by a factor of $\frac{1}{|b|}$ units or 3 units in this case. $k(x) = f\left(\frac{1}{3}x\right)$,which is $k(x) = 3x$.



Original Graph	
x	*f(x)*
−2	−2
−1	−1
0	0
1	1
2	2



Transformed Graph	
x	*j(x)*
−2	−6
−1	−3
0	0
1	3
2	6

Transformed Graph	
x	*k(x)*
$-\frac{2}{3}$	−2
$-\frac{1}{3}$	−1
0	0
$\frac{1}{3}$	1
$\frac{2}{3}$	2

LESSON 3

Determining Slopes of Perpendicular Lines

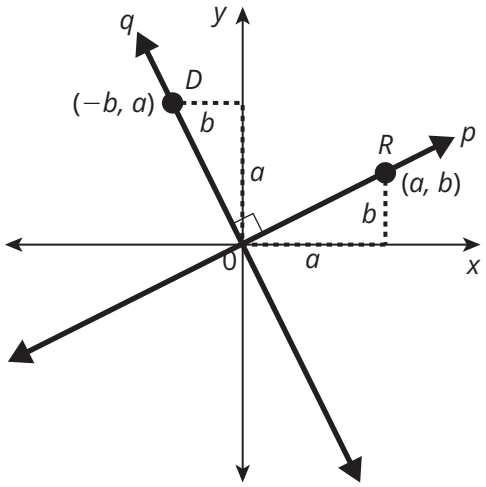
Perpendicular lines or line segments form a right angle at the point of intersection. You can think of perpendicular lines as a line and its rotation 90° about a point, which is also the point of intersection.

The product of the slopes of perpendicular lines is -1.

For example, consider the theorem.

Theorem: If two lines are perpendicular, their slopes are negative reciprocals.

Use the graph and the proof shown to analyze the validity of the theorem.



Given: $p \perp q$

Let m_1 = slope of line p and let m_2 = slope of line q.

Point R lies on line p.

Prove: $m_1 = -\frac{1}{m_2}$

Rotate point R 90° counterclockwise using point O as the center of rotation. Since p and q are perpendicular, the image (point D) will lie on line q under this 90° rotation.

Since this rotation maps the positive x-axis to the positive y-axis and the positive y-axis to the negative x-axis, then the coordinates of $R(a, b)$ are transformed into the coordinates of $D(-b, a)$. Graphically, you can see the movement of lengths a and b under the rotation.

The graph shows the slope of line p, $m_1 = \frac{b}{a}$, and the slope of line q, $m_2 = \frac{a}{-b}$.

Using these slopes, we can demonstrate that $m_1 = \frac{-1}{m_2}$.

$$\begin{aligned} \frac{b}{a} &= \frac{-1}{\frac{a}{-b}} \\ &= -1 \cdot \frac{-b}{a} \\ &= \frac{b}{a} \end{aligned}$$

The slope of line q is the negative reciprocal of the slope of line p.

The slope of any horizontal line is 0 since no matter the change for x, there is 0 change for y.

$$\frac{0}{x_2 - x_1} = 0$$

The slope of any vertical line is undefined since no matter the change for y, there is 0 change for x.

$\frac{y_2 - y_1}{0}$ is undefined since a value cannot be divided by zero.

All horizontal lines are parallel to each other since their slopes are equal and all vertical lines are parallel since their slopes are equal. A horizontal and a vertical line are always perpendicular to each other.

For example, to write an equation for a line that passes through the point $(-4, -2)$ and is perpendicular to the line $y = 3$, first determine that the line given by $y = 3$ is a horizontal line. Therefore, a line that is perpendicular to $y = 3$ is a vertical line. A vertical line that passes through the point $(-4, -2)$ has the equation $x = -4$.

You can use what you know about the slopes of perpendicular lines and slope-intercept form to write the equation of a perpendicular line.

For example, consider the line $y = 4x - 1$. Write the equation of the line that passes through the point $(-4, 2)$ and is perpendicular to $y = 4x - 1$.

The slope of $y = 4x - 1$ is 4. The slope of the line perpendicular to $y = 4x - 1$ must have a slope of $-\frac{1}{4}$, since $4 \cdot -\frac{1}{4} = -1$.

Using the given point and slope-intercept form, you can set up an equation to solve for b, the y-intercept.

$$2 = -\frac{1}{4}(-4) + b$$

$$2 = 1 + b$$

$$b = 1$$

Therefore, the equation of the line that passes through the point $(-4, 2)$ and is perpendicular to $y = 4x - 1$ is $y = -\frac{1}{4}x + 1$.

LESSON 4

Comparing Linear Functions in Different Forms

Functions can be represented using tables, equations, graphs, and with verbal descriptions. Features of linear functions, such as *y*-intercepts, slopes, and independent and dependent quantities, can be determined from different representations of functions.

A table can help you calculate solutions given a few specific input values. A graph can help you determine exact solutions if the graph of the function crosses the grid lines exactly. A function can be solved for any value, so any and all solutions can be determined. Technology can allow for more accuracy when using a graph to determine a solution.

For example, suppose Omar had $100 in his car fund. He earns $7.50 per hour at his after-school job. He works 3 hours each day, including weekends. Omar puts all of his earned money in his car fund. How many days will it take him to have enough money to buy a car that costs $3790?

A table can be used to estimate that it will take between 100 and 175 days to buy the car. A graph can be used to estimate that it will take about 160 days to buy the car. A function will give an exact solution. It will take exactly 164 days to buy a car that costs $3790.

d	$100 + 22.50d$
0	100
10	325
20	550
50	1225
100	2350
175	4037.5

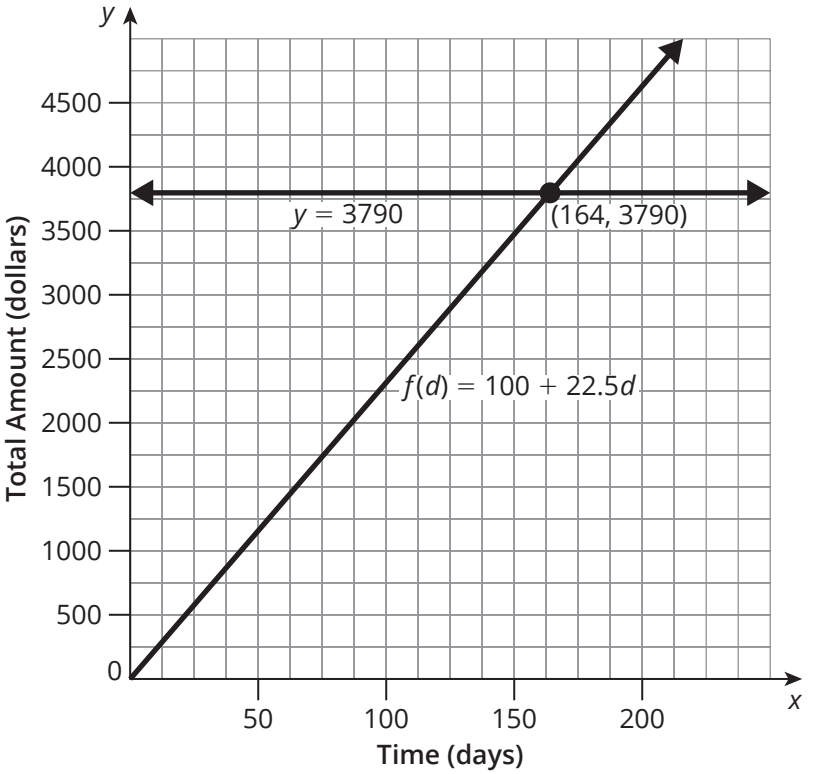
$f(d) = 100 + 22.50d$

$3790 = 100 + 22.50d$

$3690 = 22.50d$

$\frac{3690}{22.50} = \frac{22.50d}{22.50}$

$164 = d$



MODULE
3

Modeling Linear Equations and Inequalities

TOPIC 1 Linear Equations and Inequalities 343

TOPIC 2 Systems of Linear Equations and Inequalities 391

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TOPIC 1



A region can be described as above, below, to the left, or to the right of a line.

Linear Equations and Inequalities

LESSON 1 Solving Linear Equations **345**

LESSON 2 Literal Equations **357**

LESSON 3 Modeling Linear Inequalities **371**

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1 Solving Linear Equations

OBJECTIVES

- Write equivalent equations using properties of equality.
- Use properties of equality to solve linear equations and justify a solution method.
- Determine whether an equation has one solution, no solution, or infinite solutions.
- Solve linear equations with variables on both sides.

NEW KEY TERMS

- solution
- no solution
- infinite solutions

You know that equations are one way to represent a linear function, and you have used equations to evaluate linear functions for a given input value.

How can you use equations to solve for unknown input values of a function?

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Getting Started

Equation Creation

An *equation* is a mathematical sentence that uses an equals sign to show that two expressions are equivalent. When one of those expressions contains a variable, you can solve the equation.

A **solution** to an equation is a value for the variable that makes the equation a true statement.

Consider the equation $x = 2$. You can substitute the value 2 for x to create the true statement $2 = 2$. Because this is the only value that makes the statement true, 2 is the only solution to the equation $x = 2$.

By performing the same operation on each side of an equation, you can create more complex equations that have the same solution.

1. Consider the equation $x = 2$.
   a. Choose any constant. Add that constant to each side of the equation and simplify.

   b. Choose any constant other than 0. Multiply each side of the equation you created in part (a) by that constant and simplify.

   c. Choose any number other than 0 to represent a in the expression ax. Subtract the term ax from each side of the equation you created in part (b) and simplify.

   d. Choose any constant other than 0. Divide each side of the equation you created in part (c) by that constant and simplify.

Think about . . .
What strategies can you use to verify a solution?

2. Have a partner solve the equation you created in Question 1 to verify that $x = 2$ is the solution.

ACTIVITY
1.1 Using Properties to Justify Solutions

Recall that the properties of equality are rules that allow you to maintain balance and rewrite equations to isolate the variable.

Properties of Equality	For all numbers a, b, and c
addition property of equality	If $a = b$, then $a + c = b + c$.
subtraction property of equality	If $a = b$, then $a - c = b - c$.
multiplication property of equality	If $a = b$, then $ac = bc$.
division property of equality	If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

Harper and Sofia both created new equations starting from the solution statement $x = 2$.

Harper

$-3x + 14 = 18 - 5x$

Sofia

$2x = 5 - \frac{1}{2}x$

1. Verify that both equations are equivalent to $x = 2$ using the given strategy.

   a. substitution

   b. properties of equality

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There are also basic number properties that you are already familiar with that can be used to justify your steps when solving equations.

Number Properties	For all numbers a, b, and c
commutative property	$a + b = b + a$ $ab = ba$
associative property	$a + (b + c) = (a + b) + c$ $a(bc) = (ab)c$
distributive property	$a(b + c) = ab + ac$

Justifying your steps with the properties of equality will better prepare you for writing geometrical proofs in future courses.

2. Solve each equation and check your solution. Write the properties that justify each step of your solving strategy.

   a. $24x + 7 = -4x$

   b. $0.6(2x + 1) = 4x + 6.2$

   c. $\frac{1}{2}x - 6 = 2 + (2x + 1)$

3. Compare the solution strategies used by Daniel and Chloe. What do you notice about the properties each student used?

Daniel

$$4(3x + 2) = 8x + 4$$
$$12x + 8 = 8x + 4$$
$$4x = -4$$
$$x = -1$$

Chloe

$$4(3x + 2) = 8x + 4$$
$$\frac{4(3x + 2)}{4} = \frac{8x + 4}{4}$$
$$3x + 2 = 2x + 1$$
$$x = -1$$

ACTIVITY
1.2 Solutions to Linear Equations

In the Getting Started, you built equations from $x = 2$. In this activity, you will build equations from two mathematical sentences and compare your results.

1. Consider the mathematical sentence $2 = 2$.
   Is there any variable or constant that you could add, subtract, multiply, or divide to both sides using the properties of equality to make this true sentence false? Choose variables and constants to create new mathematical sentences to justify your conclusion.

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2. Consider the mathematical sentence $2 = 3$.
   Is there any variable or constant that you could add, subtract, multiply, or divide to both sides using the properties of equality to make this false sentence true? Choose variables and constants to create new mathematical sentences to justify your conclusion.

A linear equation can have *one solution*, *no solution*, or *infinite solutions*. The equations you solved in the previous activity are examples of linear equations with one solution. A linear equation with **no solution** means that there is no value for the variable that makes the equation true. A linear equation with **infinite solutions** means that any value for the variable makes the equation true.

3. Consider the equations you created in this activity.

   a. Explain whether the equation(s) you created in Question 1 have one solution, no solution, or infinite solutions.

   b. Explain whether the equation(s) you created in Question 2 have one solution, no solution, or infinite solutions.

4. Consider the equation $2x = 3x$. Does this equation have one solution, no solution, or infinite solutions? Explain your reasoning.

ACTIVITY
1.3 Tic-Tac-Bingo

In this activity, you are going to play a game called Tic-Tac-Bingo. The object of the game is to match two expressions to create an equation with specific solution types. Use the Tic-Tac-Bingo sheet located at the end of the lesson.

Prepare the board.

The board has 9 spaces. Three spaces are already designated. Fill each remaining space with one of the solution types listed. Each option must be used at least once.

Solution Types
- positive rational solution
- negative rational solution
- non-zero integer solution

Play the game.

Your teacher will assign you an expression. When you and a classmate have created an equation with one of the solution types, write your equation in the corresponding box.

Try to be the first person to fill three spaces in a row. Then, try to be the first person to completely fill your board with equations.

1. Reflect on the equations you created.

   a. How can you look at an equation and determine that it has no solution?

   b. How can you look at an equation and determine that it has infinite solutions?

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Talk the Talk

One Step at a Time

1. Rectangle A has a width of $(5 + 2x)$ inches and a length of 6 inches. Rectangle B has a width of 8 inches and a length of $(x + 5)$ inches. Write and solve an equation to determine the length of x when both rectangles have the same area. Write the property that justifies each step to solve the equation.

Tic-Tac-Bingo Board

Equation: Solution:	**Solution is neither positive nor negative** Equation: Solution:	Equation: Solution:
No solution Equation: Solution:	**FREE SPACE**	Equation: Solution:
Equation: Solution:	Equation: Solution:	**Infinite solutions** Equation: Solution:

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Why is this page blank?

So you can cut out the Bingo Board on the other side

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Lesson 1 Assignment

Write

Explain how you know when an equation has no solution and when it has infinite solutions.

Remember

To solve an equation, use the properties of equality to isolate the variable. A linear equation can have one solution, no solution, or infinite solutions.

Practice

1. Solve each equation. Write the properties that justify each step in the solution method.

   a. $3x - 8 = -7x + 18$

   b. $-2(4 - x) = 12x - 3$

   c. $\frac{1}{2}(-10x + 4) = -4(-3 + 2x) + 8$

   d. $\frac{(-2x - 4)}{5} + \frac{8}{5} = 3(x - 1)$

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Lesson 1 Assignment

e. $\frac{4}{3}x + 2\left(9 - \frac{1}{3}x\right) = -\frac{7}{3}x + 9$

2. Determine whether each equation has one solution, no solution, or infinite solutions. Explain your reasoning.

a. $-2(x + 5) = -6x + 4(x - 2)$

b. $4(0.2x - 1.2) = -0.5x + 4.3$

c. $\frac{\left(\frac{1}{2}x - 5\right)}{2} = -3x + 4$

d. $2(x - 4) + x = 3(x - 2) - 2$

e. $3 - \frac{2}{5}x - \frac{12}{5} = \frac{10 - 2x}{5}$

f. $6(x - 1) + 21 = 6x + 15$

Prepare

The formula for the circumference of a circle is $C = 2\pi r$. Determine the radius for each circle with the given circumference. Use 3.14 for π and round to the nearest tenth of a unit, if necessary.

1. C = 62.8 in.

2. C = 10 cm

3. C = 15.7 ft

4. C = 48 mm

2 Literal Equations

OBJECTIVES

- Rewrite linear equations in different forms.
- Analyze the structure of different forms of linear equations.
- Recognize and use literal equations.
- Rearrange literal equations to highlight quantities of interest.

NEW KEY TERM

- literal equation

You have used different properties to solve linear equations to determine which value makes the equation true.

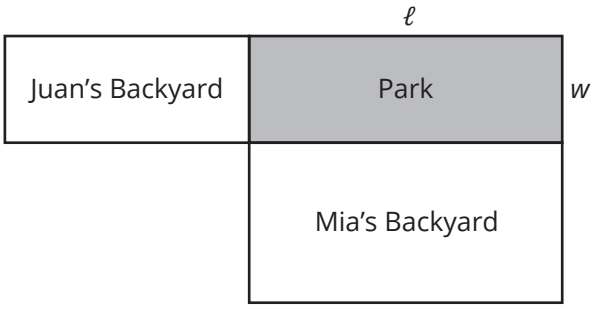
How can you use those same properties to solve for one specific variable in an equation that has multiple variables?

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Getting Started

Perimeter Perspectives

Juan and Mia have rectangular backyards that each share a side with a neighborhood park. The sides shared with the park are adjacent to each other as shown.



Remember . . .

The formula for the perimeter of a rectangle is $P = 2\ell + 2w$.

Juan and Mia are each responsible for constructing their own fence to separate their yard from the park. The city gives them the measure of the perimeter of the park in feet. So that each knows how much fencing to buy, Juan needs to determine the width of the park, and Mia needs to determine the length of the park.

1. How can you write an equation for the perimeter of the park from each person's perspective to help them determine the information they need?

2. Show that the two equations are equivalent.

ACTIVITY
2.1 Equations in Different Forms

The slope, *x*-intercept, and *y*-intercept are each important characteristics of linear functions. The structure of equations can reveal these characteristics of a function.

1. Consider the equation $y = -\frac{2}{3}x + 4$. Determine each characteristic and then explain your work.

   a. Slope

   b. *y*-intercept

   c. *x*-intercept

   d. How did you determine each characteristic?

2. Consider the equation $y = 4(x - 2)$. Determine each characteristic and then explain your work.

   a. Slope

   b. *y*-intercept

   c. *x*-intercept

   d. How did you determine each characteristic?

Remember . . .

- You can calculate slope using $\frac{y_2 - y_1}{x_2 - x_1}$.
- You can determine the *y*-intercept by substituting 0 for *x* and solving for *y*.
- You can determine the *x*-intercept by substituting 0 for *y* and solving for *x*.

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3. Consider the equation $3x - 2y = 8$. Determine each characteristic and then explain your work.

   a. Slope

   b. y-intercept

   c. x-intercept

   d. How did you determine each characteristic?

In slope-intercept and point-slope form, the values of m, x_1, and y_1 can be any real numbers. In standard form, however, there are constraints on the variables: A, B, and C must be integers, and A and B cannot both be 0.

Recall that are three useful forms of linear equations.

Slope-Intercept Form	**Point-Slope Form**	**Standard Form**
$y = mx + b$	$y - y_1 = m(x - x_1)$	$Ax + By = C$

4. Identify the form for each equation in Questions 1 through 3.

Consider how the structure of the standard form of a linear function reveals its key characteristics.

5. For the equation $Ax + By = C$, determine the slope, x-intercept, and y-intercept.

6. Which form of a linear equation is more efficient for determining each characteristic?

   a. Slope

   b. a point on the line

   c. y-intercept

7. If you want to graph an equation using technology, which form is more efficient? Explain your reasoning.

8. Rewrite each equation in slope-intercept form.

   a. $y - 2 = 5(x - 7)$

   b. $8x - 2y = 12$

9. Rewrite each equation in standard form.

   a. $y + 7 = 0.5(x - 1)$

   b. $y = -4x + 3$

10. Write the equation for a line in standard form that passes through the given point with the identified slope.

    a. Slope $= \frac{1}{2}$
    Passes through $(-3, 5)$

    b. Slope $= -4$
    Passes through $(2, 1)$

ACTIVITY
2.2 Rewriting Literal Equations

Literal equations are equations in which the variables represent specific measures. You most often see literal equations when you study formulas. These literal equations can be manipulated in order to allow you to solve for one specific variable.

You have used a common literal equation, the formula for converting degrees Fahrenheit to degrees Celsius.

$$C = \frac{5}{9}(F - 32)$$

Jayden is talking on the phone to his friend Mateo who lives in Europe. Mateo is used to describing temperatures in °C, while Jayden is used to describing temperatures in °F. Mateo can use the formula above to quickly convert the temperatures Jayden describes to °C.

1. How can you rewrite the formula so that Jayden can quickly convert the temperatures that Mateo describes to °F? Justify your solution method.



2. Jayden tells Mateo that the temperature where he lives is currently 77°F. Mateo tells Jayden that the temperature where he lives is currently 30°C. Jayden says it is warmer where he lives and Mateo says it is warmer where he lives. Who is correct? Explain your reasoning.

3. Isabella, Ethan, and Minh each convert the given formula to degrees Fahrenheit.

Isabella



$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}F - \frac{160}{9}$$
$$9(C) = 9\left(\frac{5}{9}F - \frac{160}{9}\right)$$
$$9C = 5F - 160$$
$$9C + 160 = 5F$$
$$\frac{9C}{5} + \frac{160}{5} = \frac{5F}{5}$$
$$\frac{9}{5}C + 32 = F$$

a. Explain Isabella's reasoning.

Ethan



$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}F - 32$$
$$9(C) = 9\left(\frac{5}{9}F - 32\right)$$
$$9C = 5F - 288$$
$$9C + 288 = 5F$$
$$\frac{9C}{5} + \frac{288}{5} = \frac{5F}{5}$$
$$\frac{9}{5}C + 57.6 = F$$

b. Explain the error in Ethan's work.

Minh



$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}F - \frac{160}{9}$$
$$9(C) = 9\left(\frac{5}{9}F - \frac{160}{9}\right)$$
$$9C = 45F - 1440$$
$$9C + 1440 = 45F$$
$$\frac{9C}{45} + \frac{1440}{45} = \frac{45F}{45}$$
$$\frac{1}{5}C + 32 = F$$

c. Explain the error in Minh's work.

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WORKED EXAMPLE

You can use multiplication to rewrite an equation without fractions. Consider the final equation from Isabella's work.

$\frac{9}{5}C + 32 = F$

$5\left(\frac{9}{5}C + 32\right) = (5)F$ Multiply both sides of the equation by 5 to eliminate the fraction.

$9C + 160 = 5F$ Subtract $9C$ from both sides of the equation.

$160 = 5F - 9C$

4. Use the strategy from the worked Example to rewrite Isabella's original equation $C = \frac{5}{9}(F - 32)$ without fractions.

ACTIVITY 2.3

Solving Literal Equations for Specific Variables

Convert each literal equation to solve for the given variable.

1. Think Inside the Box is manufacturing new boxes for Pack-n-Ship. Pack-n-Ship told Think Inside the Box that the boxes must have a specific volume and area of the base. However, Pack-n-Ship did not specify a height for the boxes.

   a. Write a literal equation to calculate the volume of a box. Then, convert the volume formula to solve for height.

   b. Pack-n-Ship specified the volume of the box must be 450 cubic inches and the area of the base must be 75 square inches. Use your formula to determine the height of the new boxes.

Remember . . .

Volume is measured in cubic units since it's calculated using three dimensions. Height measures only one dimension.

2. The volume of an ice cream cone is the measure of how much ice cream fits inside the cone. An ice cream cone company wants to make an ice cream cone with a greater height that still holds the same amount of ice cream.

   a. Write an equation to calculate the volume of a cone. Then, convert the equation to solve for the height.

   b. Explain how your equation determines a linear measurement when the original equation determined a cubic measurement.

The formula for the volume of a cone is $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$, where B is the area of the base of the cone, r is the radius and h is the height.

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3. The formula for the area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height and b_1 and b_2 are the lengths of each base.

   a. Convert the area formula to solve for the height.

   b. Use your formula to determine the height of a trapezoid with an area of 24 square centimeters and base lengths of 9 cm and 7 cm.

4. For the given literal equation $Z = \frac{A}{B} + \frac{C}{D}$, solve for each variable given. Justify your solution method.

   a. A

   b. D

Talk the Talk

Rearrange It

1. The formula for the lateral surface area of a cylinder is $S = 2\pi rh$, where S is the lateral surface area, r is the length of the radius of the base, and h is the height. Convert the formula to solve for the height.

2. The lateral surface area of a can of soup is 37.68 square inches, and the length of the radius of the base of the can is 1.5 inches. Use your formula to determine the height of the can of soup. Use 3.14 for π.

3. The formula for the surface area of a cylinder is $S = 2\pi r^2 + 2\pi rh$, where S is the surface area, r is the length of the radius of the base, and h is the height. Convert the formula to solve for the height.

4. The surface area of a can of soup is 51.81 square inches, and the length of the radius of the base is 1.5 inches. Use your formula to determine the height of the can of soup. Use 3.14 for π.

Lesson 2 Assignment

Write

Define the term *literal equation* in your own words.

Remember

Literal equations can be rewritten using properties of equality to allow you to solve for one specific variable.

Practice

1. Rewrite each equation in the given form.

   a. $y = \frac{3}{4}x - 8$ in standard form

   b. $y - 12 = -3(x - 1)$ in slope-intercept form

   c. $5x - 2y = -18$ in slope-intercept form

   d. $y + 4 = 0.5(x - 14)$ in standard form

Lesson 2 Assignment

2. Shoe boxes are often used in other capacities once the shopper has bought the shoes. Sometimes the boxes are used to hold other items, so it is helpful to know the volume of the box.

   a. Write the equation to solve for the volume of the shoe box.

   b. If the area of the base of the box is 112 square inches and the height is 3.5 inches, what is the volume of the box?

   c. Rewrite the equation to solve for height.

   d. A box has a volume of 456 cubic inches, with a length of 1 foot and a height of 4 inches. Determine the width of the box.

3. Solve each equation for the specified variable.

   a. $V = \frac{1}{3}Bh$ for B

   b. $I = prt$ for r

   c. $\frac{x+y}{3} = 6$ for y

   d. $A + B + C = 180$ for C

Prepare

Solve each statement for x.

1. $x + 1 = 6$

2. $x - 2 > 5$

3. $\frac{x}{3} \leq -1$

4. $-x > 4$

3 Modeling Linear Inequalities

OBJECTIVES

- Write and solve inequalities.
- Analyze a graph on a coordinate plane to solve problems involving inequalities.
- Interpret how a negative rate affects how to solve an inequality.

NEW KEY TERM

- linear inequality

You have used horizontal lines on a graph and properties of equality with equations to solve problems that can be modeled with linear equations.

Can you use these same methods to solve problems involving linear inequalities?

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Getting Started

Fundraising Function

Ask yourself . . .
How will you represent the $25 credit in your function?

Elijah's camping troop is selling popcorn to earn money for an upcoming camping trip. Each camper starts with a credit of $25 toward his sales, and each box of popcorn sells for $3.75.

1. Write a function, $f(b)$, to show Elijah's total sales as a function of the number of boxes of popcorn he sells, b.

2. Analyze the function you wrote.

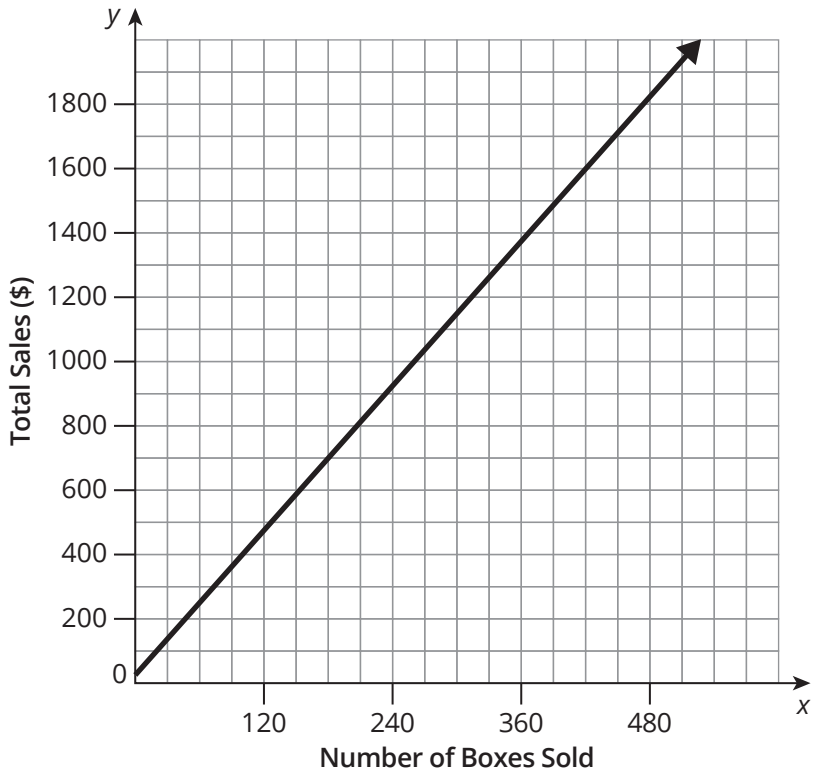
   a. Identify the independent and dependent quantities and their units.

   b. What is the slope of the function? What does it represent in this problem situation?

   c. What is the y-intercept? What does it represent in this problem situation?

ACTIVITY
3.1 Modeling Linear Inequalities

The graph shown represents the function you wrote in the previous activity, $f(b) = 3.75b + 25$.



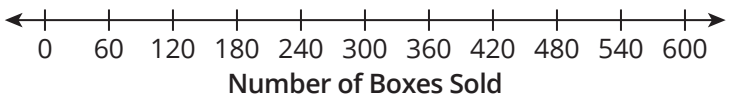
1. Suppose Elijah has a sales goal of $1600.

   a. Draw a horizontal line from the *y*-axis until it intersects with the graphed function to determine the point on the graph that represents $1600 in total sales.

   b. How many boxes would Elijah have to sell to make $1600 in total sales? Explain your reasoning.

Ask yourself . . .
Will the points you graph on the number line be open or closed?

   c. Use the number line to represent the number of boxes sold if the total sales is equal to $1600.



A **linear inequality** is a non-equal comparison of two linear expressions.

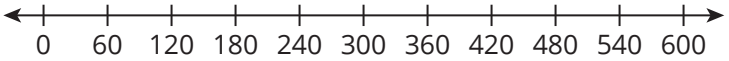
2. Analyze the region of the graph that lies below the horizontal line you drew, up to and including the point intersected by the line.

   a. What does this region of the graph represent?

   b. Write an inequality to represent this region.

Remember . . .
There are five inequality signs, less than ($<$), greater than ($>$), less than or equal ($\leq$), greater than or equal ($\geq$) and not equal ($\neq$). The solution set of an inequality is all the values that make the inequality statement true.

   c. Use the number line to represent the number of boxes that are solutions to the inequality you wrote.



Number of Boxes Sold

   d. Do all the solutions make sense in context of the problem? Explain your reasoning.

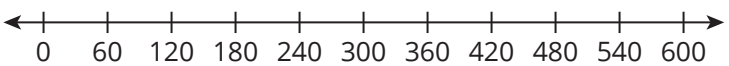
3. Analyze the region of the graph that lies above the horizontal line you drew, not including the point intersected by the line.

   a. What does this region of the graph represent?

   b. Write an inequality to represent this region.

Graphing the solutions to inequalities on number lines will prepare you for graphing linear inequalities in two variables.

   c. Use the number line to represent the number of boxes that are solutions to the inequality you wrote.



Number of Boxes Sold

   d. Do all the solutions make sense in context of the problem? Explain your reasoning.

4. Explain the difference between the open and closed circles on your number lines.

Ask Yourself . . .
How does determining the intersection point help you determine your answers?

5. Use the graph to answer each question. Write an equation or inequality statement for each.

   a. How many boxes would Elijah have to sell to earn at least $925?

   b. How many boxes would Elijah have to sell to earn less than $2050?

   c. How many boxes would Elijah have to sell to earn exactly $700?

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ACTIVITY
3.2 Solving Two-Step Linear Inequalities

To solve an inequality means to determine the values of the variable that make the inequality true.

Another way to determine the solution set of any inequality is to solve it algebraically. The objective when solving an inequality is similar to the objective when solving an equation: You want to isolate the variable on one side of the inequality symbol.

To make the first deposit on the trip, Elijah's total sales, $f(b)$, need to be at least \$1100.

WORKED EXAMPLE

You can set up an inequality and solve it to determine the number of boxes Elijah needs to sell.

$$f(b) \geq 1100$$
$$3.75b + 25 \geq 1100$$

Solve the inequality in the same way you would solve an equation.

$$3.75b + 25 \geq 1100$$
$$3.75b + 25 - 25 \geq 1100 - 25$$
$$3.75b \geq 1075$$
$$\frac{3.75b}{3.75} \geq \frac{1075}{3.75}$$
$$b \geq 286.\overline{6}$$

Think about . . .
How accurate does your answer need to be?

1. How many boxes of popcorn does Elijah need to sell to make the first deposit? Explain your reasoning.

2. Write and solve an inequality for each. Show your work.

   a. What is the greatest number of boxes Elijah could sell and still not have made $600 in sales?

   b. At least how many boxes would Elijah have to sell to make $1500 in sales?

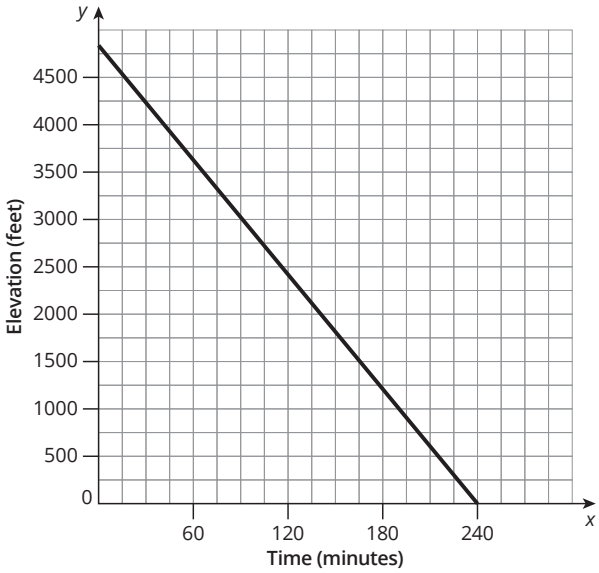
3. The Worked Example showed how to solve an inequality of the form, $y = mx + b$. How does your method change if the inequality is in a different form?

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ACTIVITY
3.3 Reversing Inequality Signs

Sea level has an elevation of 0 feet.

Elijah's camping troop hikes down from their campsite at an elevation of 4800 feet to the base of the mountain, which is at sea level. They hike down at a rate of 20 feet per minute.



1. Write a function, $h(m)$, to show the troop's elevation as a function of time in minutes. Label the function on the coordinate plane.

Ask yourself . . .
What are the independent and dependent quantities and their units?

2. Analyze the function. Identify each characteristic and explain what it means in terms of this problem situation.

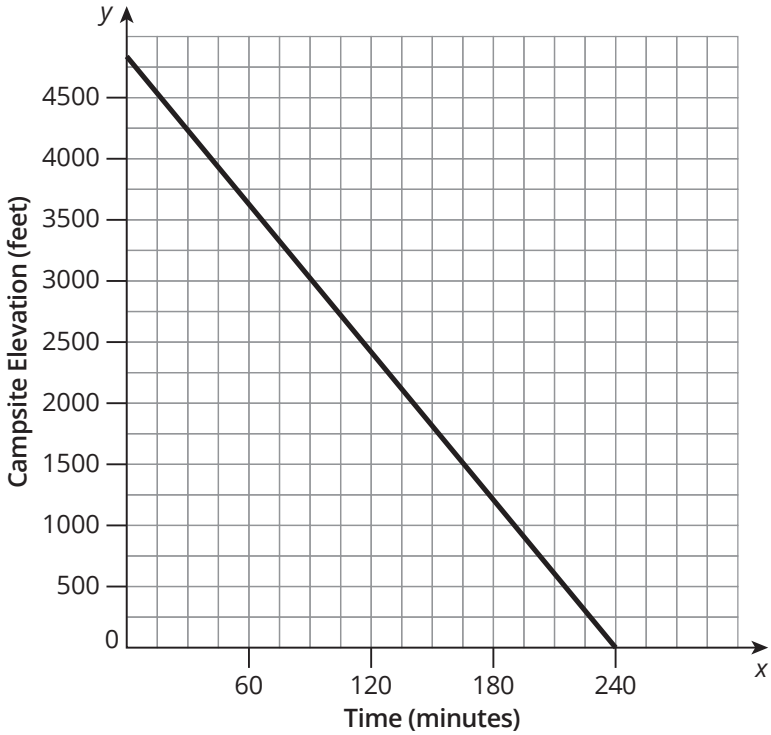
   a. Slope

   b. y-intercept

   c. x-intercept

   d. Domain of the function

3. Use the graph to determine how many minutes passed if the troop is below 3200 feet. Draw an oval on the graph to represent this part of the function and write the corresponding inequality statement.



4. Write and solve an inequality to verify the solution set you interpreted from the graph.

5. Compare and contrast the solution sets you wrote using the graph and the function. What do you notice?

6. Analyze the relationship between the inequality statements representing $h(m)$ and m.

   a. Complete the table by writing the corresponding inequality statement that represents the number of minutes for each height.

   b. Compare each row in the table shown. What do you notice about the inequality signs?

$h(m)$	m
$h(m) > 3200$	
$h(m) \geq 3200$	
$h(m) = 3200$	
$h(m) < 3200$	
$h(m) \leq 3200$	

   c. Explain your answer from part (b). Use what you know about solving inequalities when you have to multiply or divide by a negative number.

ACTIVITY
3.4 Solving Other Linear Inequalities

The inequalities that you have solved in this lesson so far have all been two-step inequalities. Let's consider inequalities in different forms.

Victoria and Kai each solved the inequality $5x + 2 \geq 3x - 10$.

You can rewrite Kai's solution to show it is equivalent to Victoria's. It is sometimes preferred to have the variable on the left side of the inequality. So, $-6 \leq x$ is equivalent to $x \geq -6$.

Victoria

$$5x + 2 \geq 3x - 10$$
$$5x - 3x + 2 \geq 3x - 3x - 10$$
$$2x + 2 \geq -10$$
$$2x + 2 - 2 \geq -10 - 2$$
$$2x \geq -12$$
$$\frac{2x}{2} \geq \frac{-12}{2}$$
$$x \geq -6$$

Kai

$$5x + 2 \geq 3x - 10$$
$$5x - 5x + 2 \geq 3x - 5x - 10$$
$$2 \geq -2x - 10$$
$$2 + 10 \geq -2x - 10 + 10$$
$$12 \geq -2x$$
$$\frac{12}{-2} \leq \frac{-2x}{-2}$$
$$-6 \leq x$$

Remember. . .
When multiplying or dividing by a negative number while solving inequalities, the inequality sign reverses.

1. Describe the process each student used to solve the inequality.
   a. Victoria
   b. Kai

2. How does the process of solving an inequality with variables on both sides compare to the process of solving an equation with variables on both sides?

Aaliyah and Ava each solved the inequality $-4(x - 6) < 22$.

Aaliyah

$$-4(x - 6) < 22$$

$$-4x + 24 < 22$$

$$-4x + 24 - 24 < 22 - 24$$

$$-4x < -2$$

$$\frac{-4x}{-4} > \frac{-2}{-4}$$

$$x > \frac{1}{2}$$

Ava

$$-4(x - 6) < 22$$

$$\frac{-4(x - 6)}{-4} > \frac{22}{-4}$$

$$x - 6 > -5\frac{1}{2}$$

$$x > \frac{1}{2}$$

3. Describe the process each student used to solve the inequality.

   a. Aaliyah

   b. Ava

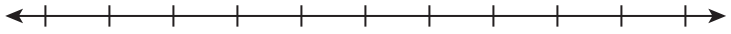
4. How does the process of solving an inequality using the distributive property compare to the process of solving an equation using the distributive property?

Talk the Talk

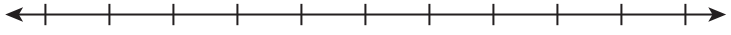
It's About Solutions, More or Less

1. Solve each inequality and graph the solution on the number line.

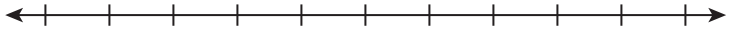
   a. $-\frac{2}{3}x \geq 7$

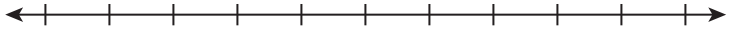
   b. $32 > 23 - x$

   c. $15 - 4x \leq 3x - 6$

   d. $2(x + 6) < 14 + 4x$

2. If $A < B$, identify the constraints to make each statement true.

   a. $A + C < B + C$

   b. $AC > BC$

   c. $-A < -B$

Lesson 3 Assignment

Write

Describe how to solve an inequality in your own words.

Remember

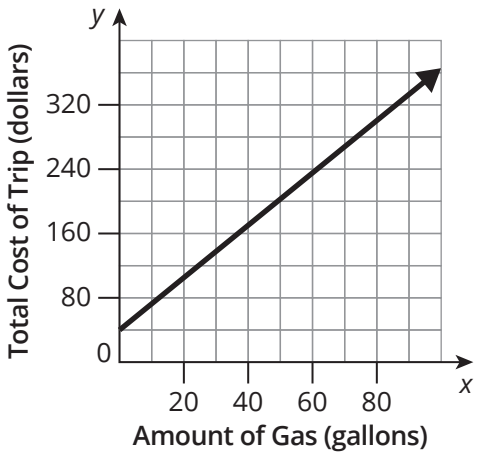
The methods for solving linear inequalities are similar to the methods for solving linear equations. Be sure to reverse the direction of the inequality symbol when multiplying or dividing both sides by a negative number.

Practice

1. Noah is going on a trip to visit some friends from summer camp. He will use $40 for food and entertainment. He will also need money to cover the cost of gas. The price of gas at the time of his trip is $3.25 per gallon.
   a. Write a function to represent the total cost of the trip as a function of the number of gallons used.

   b. Identify the rate of change and the *y*-intercept. Explain their meanings in terms of the problem situation.

   c. Graph the function representing this situation on a coordinate plane.



   d. Use the graph to determine how many gallons of gas Noah can buy if he has $170 saved for the trip. Draw an oval on the graph to represent the solution. Then, write your answer in words and as an inequality.

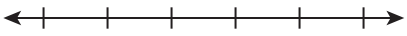
Lesson 3 Assignment

e. Verify the solution set you interpreted from the graph.

f. Noah's mom gives him some money for his trip. He now has a total of $220 saved for the trip. What is the greatest number of gallons of gas he can buy before he runs out of money? Show your work and graph your solution on a number line.



g. If Noah spent more than $92 on his trip, how much gas could he have bought? Show your work and graph your solution on a number line.

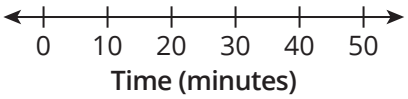


2. Noah is on his way to visit his friends at camp. Halfway to his destination, he realizes there is a slow leak in one of the tires. He checks the pressure and it is at 26 psi. It appears to be losing 0.1 psi per minute.

   a. Write a function to represent the tire's pressure as a function of time in minutes.

Lesson 3 Assignment

b. Noah knows that if the pressure in a tire goes below 22 psi, it may cause a tire blowout. What is the greatest amount of time that he can drive before the tire pressure hits 22 psi? Show your work and graph the solution.



3. Solve each inequality for the unknown value.

a. $13 + 4x > 9$

b. $3(4 - 5x) > 8x - 149$

c. $99 - 5d \geq 4d$

d. $3k - 9 \leq -6k - 225$

Prepare

Determine an ordered pair that represents a solution to each equation.

1. $4x + 7y = 24$

2. $5x - 2y = -6$

3. $\frac{1}{2}x + \frac{3}{4}y = 10$

TOPIC 1 SELF-REFLECTION

Name: _____

Linear Equations and Inequalities

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represent **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Linear Equations and Inequalities* topic by:

TOPIC 1: *Linear Equations and Inequalities*	Beginning of Topic	Middle of Topic	End of Topic
solving linear equations in one variable, including those for which the application of the distributive property is necessary and for which variables are included on both sides.	○	○	○
constructing an argument to justify a solution process for a one-variable equation.	○	○	○
using and interpreting units when solving literal equations for a given variable.	○	○	○
solving literal equations for a specified variable.	○	○	○
recognizing linear relationships from problem situations and writing equations or inequalities to model linear relationships.	○	○	○
solving linear inequalities in one variable, including those for which the application of the distributive property is necessary and for which variables are included on both sides.	○	○	○
interpreting solutions in the context of a problem situations and determining whether they are reasonable.	○	○	○
representing constraints to a problem situation as an inequality.	○	○	○

continued on the next page

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TOPIC 1 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Linear Equations and Inequalities* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?



TOPIC 1 SUMMARY

Linear Equations and Inequalities Summary

NEW KEY TERMS

- solution [solución]
- infinite solutions [soluciones infinitas]
- no solution [sin solución]
- literal equation [ecuación literal]
- linear inequality [desigualdad lineal]

LESSON 1 Solving Linear Equations

A **solution** to an equation is any variable value that makes the equation true. Solving equations requires the use of number properties and the properties of equality. The properties of equality state that if an operation is performed on both sides of the equation, to all terms of the equation, the equation maintains its equality. When the properties of equality are applied to an equation, the transformed equation will have the same solution as the original equation.

A linear equation in the form $y = mx + b$ can have *one solution*, *infinite solutions*, or *no solution*.

Linear equations with **infinite solutions** are created by equating two equivalent expressions.

$$\begin{aligned} 2x + 6 &= 2(x + 3) \\ &= (2 \cdot x) + (2 \cdot 3) \\ &= 2x + 6 \end{aligned}$$

Linear equations with **no solution** are created by equating expressions of the form $mx + b$, with the same value for m and different values for b.

$$\begin{aligned} 4x + 2 &= 4x + 5 \\ 4x - 4x + 2 &= 4x - 4x + 5 \\ 2 &\neq 5 \end{aligned}$$

Linear equations with the solution $x = 0$ are created by equating expressions of the form $mx + b$, with different values for m and the same value for b.

$$\begin{aligned} 2x + 3 &= 5x + 3 \\ 2x - 2x + 3 &= 5x - 2x + 3 \\ 3 &= 3x + 3 \\ 3 - 3 &= 3x + 3 - 3 \\ 0 &= 3x \\ 0 &= x \end{aligned}$$

LESSON 2 Literal Equations

There are three useful forms of linear equations.

Slope-Intercept	Point-Slope Form	Standard Form
$y = mx + b$	$y - y_1 = m(x - x_1)$	$Ax + By = C$

You can use any real numbers for the values of m, b, x_1, *and* y_1 in slope-intercept and point-slope from. In standard form, however, there are constraints on the variables: A, B, and C must be integers, and A and B cannot both be 0.

It is often necessary to change between forms, as the structure of each form reveals different key characteristics of the equation, such as the slope and x- and y-intercepts. Remember, to convert to slope-intercept form from standard form, solve for y. To convert to standard form from slope-intercept form, isolate both variables on the same side of the equation and the constant on the other side of the equation.

A **literal equation** is an equation in which the variables represent specific measures. Literal equations are most often seen when studying formulas. These literal equations can be manipulated in order to solve for one specific variable.

For example, a common literal equation is the formula for converting degrees Fahrenheit to degrees Celsius, $C = \frac{5}{9}(F - 32)$. You can use the properties of equality to rewrite the equation to convert degrees Celsius to degrees Fahrenheit, $F = \frac{9}{5}(C + 32)$.

LESSON 3 Modeling Linear Inequalities

To solve a **linear inequality**, first write a function to represent the problem situation. Then, write the function as an inequality based on the independent quantity. To determine the solution, identify the values of the variable that make the inequality true. The objective when solving an inequality is similar to the objective when solving an equation: isolate the variable on one side of the inequality symbol. Finally, interpret the meaning of the solution.

For example, consider the situation in which Diego has $25 in his gift fund that he is going to use to buy graduation gifts. Graduation is 9 weeks away. He would like to have at least $70 to buy gifts, how much should he save each week?

The function is $f(x) = 25 + 9x$, so the inequality is $25 + 9x \geq 70$.

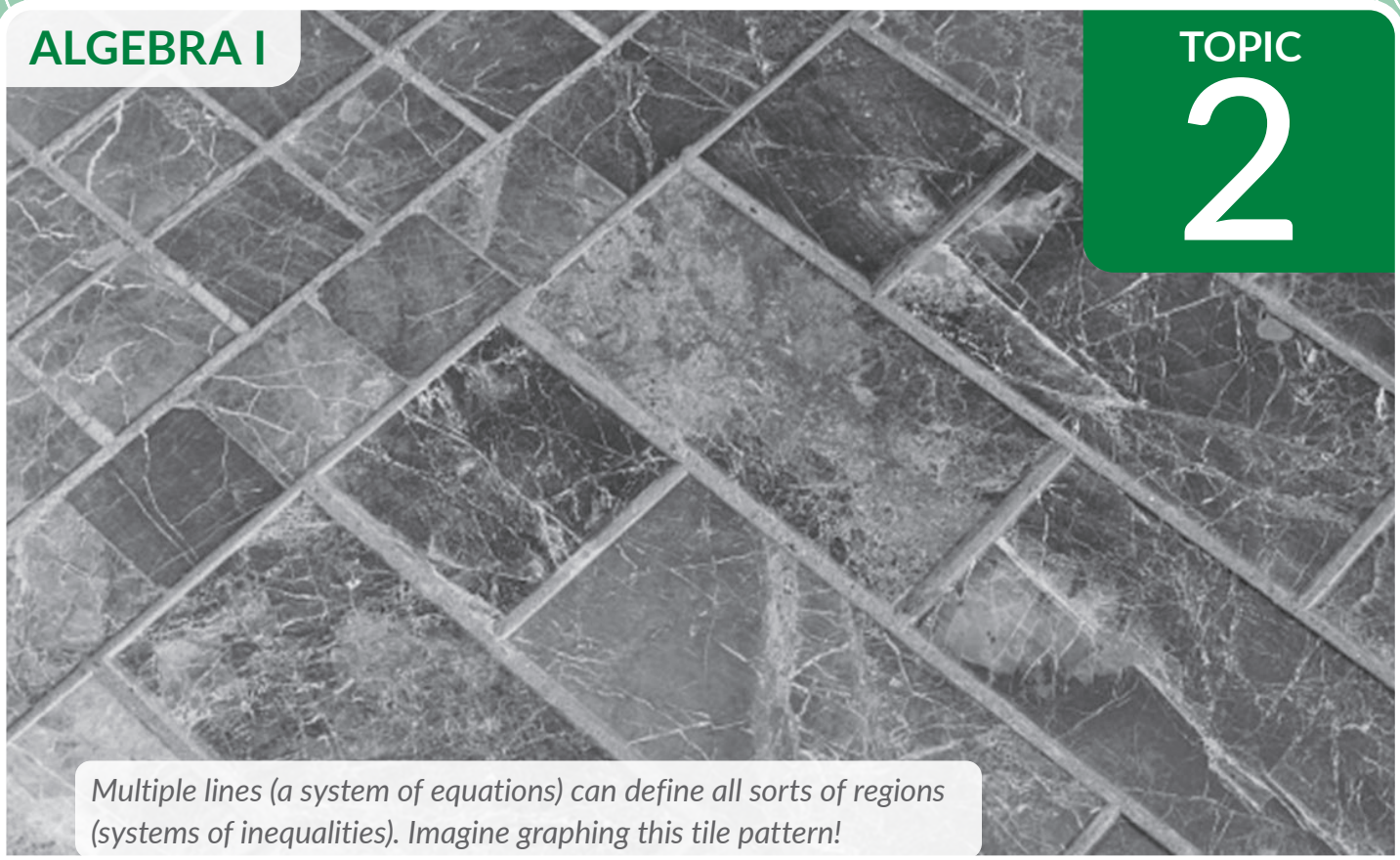
$$\begin{aligned} 25 + 9x &\geq 70 \\ 25 + 9x - 25 &\geq 70 - 25 \\ 9x &\geq 45 \\ \frac{9x}{9} &\geq \frac{45}{9} \\ x &\geq 5 \end{aligned}$$

Diego needs to save at least $5 each week to meet his goal.

When you divide each side of an inequality by a negative number, the inequality sign reverses. For example, consider the inequality $250 - 9.25x < 398$.

$$\begin{aligned} 250 - 9.25x &< 398 \\ 250 - 9.25x - 250 &< 398 - 250 \\ -9.25x &< 148 \\ \frac{-9.25x}{-9.25} &< \frac{148}{-9.25} \\ x &> -16 \end{aligned}$$

TOPIC 2



Multiple lines (a system of equations) can define all sorts of regions (systems of inequalities). Imagine graphing this tile pattern!

Systems of Linear Equations and Inequalities

LESSON 1 Using Graphing to Solve Systems of Equations **393**

LESSON 2 Using Substitution to Solve Linear Systems **411**

LESSON 3 Using Linear Combinations to Solve a System of Linear Equations **433**

LESSON 4 Graphing Inequalities in Two Variables **449**

LESSON 5 Systems of Linear Inequalities. **467**

LESSON 6 Solving Systems of Equations and Inequalities. **485**

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1 Using Graphing to Solve Systems of Equations

OBJECTIVES

- Write equations in standard form.
- Determine the intercepts of an equation in standard form.
- Use intercepts to graph an equation.
- Write a system of equations to represent a problem context.
- Use a graph to solve a system of linear equations.
- Interpret the solution to a system of equations in terms of a problem situation.
- Use the slope and the *y*-intercept to determine whether the system of two linear equations has one solution, no solution, or infinite solutions.

NEW KEY TERMS

- system of linear equations
- consistent systems
- inconsistent systems

You have examined different linear functions and solved for unknown values.

How can you solve problems that require two linear functions? How many solutions exist when you consider two functions at the same time?

Getting Started

Ticket Tabulation

A high school sells tickets for its basketball games. Students pay $5, and adults pay $10 for a ticket. The high school needs to raise $3000 selling tickets to send the team to an invitational tournament.

1. Write an equation to represent this situation.

2. What combination of student and adult ticket sales would achieve the high school's goal?

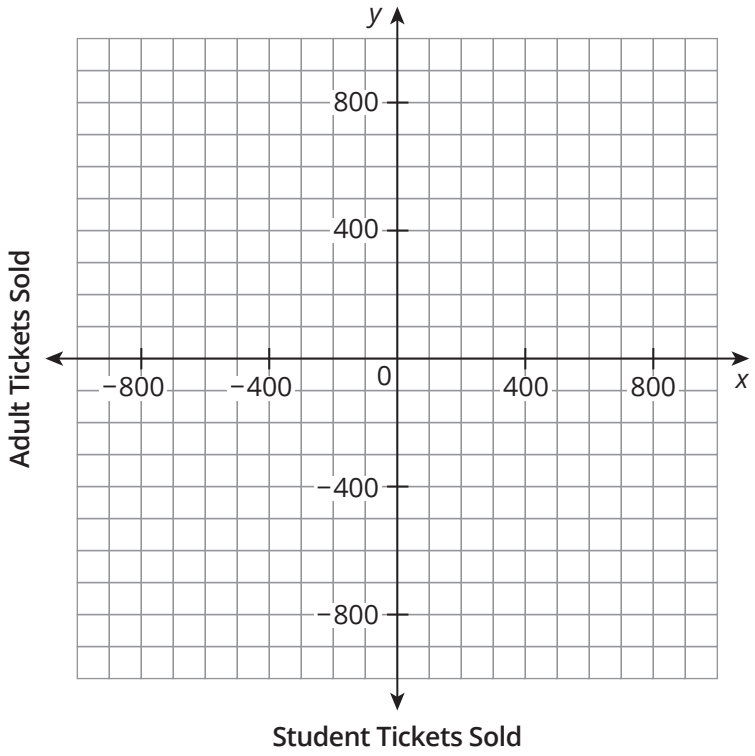
3. Compare your combination of ticket sales with your classmates'. Did you all get the same answer? Explain why or why not.

ACTIVITY
1.1 Analyzing the Graph of an Equation in Standard Form

Consider the goal of the athletic association described in the previous activity. Let s represent the number of student tickets sold, and let a represent the number of adult tickets sold. Written in standard form, the equation that represents the situation is $5s + 10a = 3000$.

One efficient way to graph a linear function in standard form is to use x- and y-intercepts. You can calculate the x-intercept by substituting $y = 0$ and solving for x. You can calculate the y-intercept by substituting $x = 0$ and solving for y.

1. Use the x- and y-intercepts to graph the equation.



Ask Yourself . . .
Which quantity is represented on each axis?

2. Determine the domain and range of each.

   a. The mathematical function

   b. The function modeling the real-world situation

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3. Explain what each intercept means in terms of the problem situation.

4. Identify the slope of the function. Interpret its meaning in terms of the problem situation.

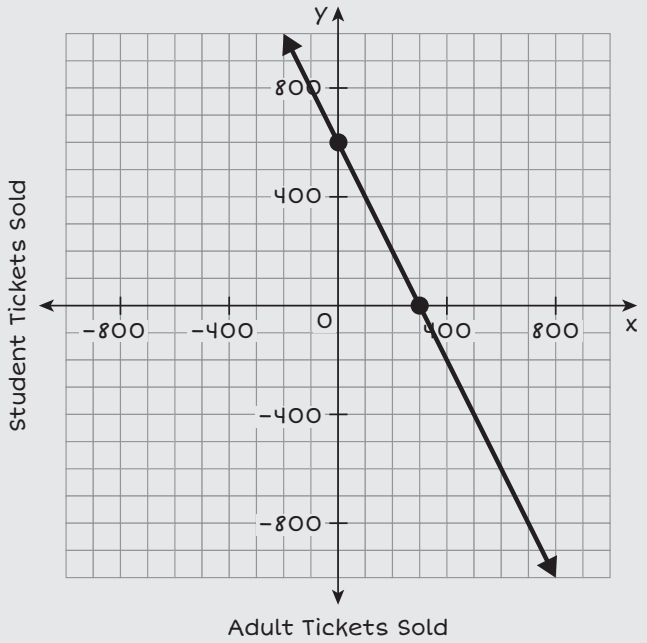
Ask Yourself . . .
What does each point on the graph of an equation represent?

5. How can you use the graph to determine a combination of ticket sales to meet the goal of $3000?

6. Ethan graphed the equation $5s + 10a = 3000$ in a different way. Explain why Ethan's graph is correct.

Ask Yourself . . .
How does representing mathematics in multiple ways help to communicate reasoning?

Ethan



7. Use Ethan's graph to describe the domain and range of each.

   a. The mathematical function

   b. The function modeling the real-world situation

8. Explain what each intercept means in terms of the problem situation.

9. Identify the slope of the function. Interpret its meaning in terms of the problem situation.

10. Compare the domain and range of the two functions that model the real-world situation. What do you notice?

11. Compare the *x*-intercepts and the *y*-intercepts of the two graphs. What do you notice?

12. Is there a way to determine the total amount of money collected from either graph? Explain why or why not.

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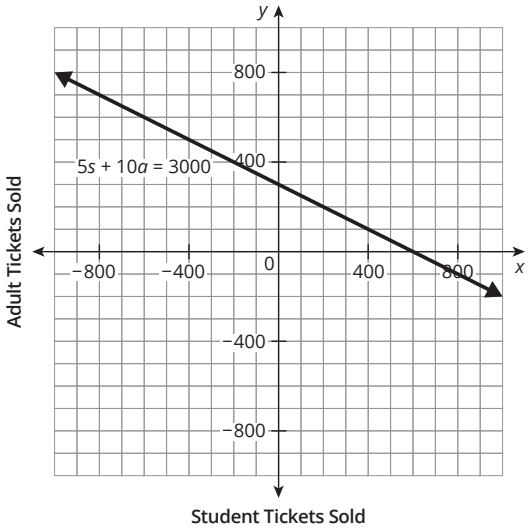
ACTIVITY
1.2 Determining the Solution to a System of Linear Equations

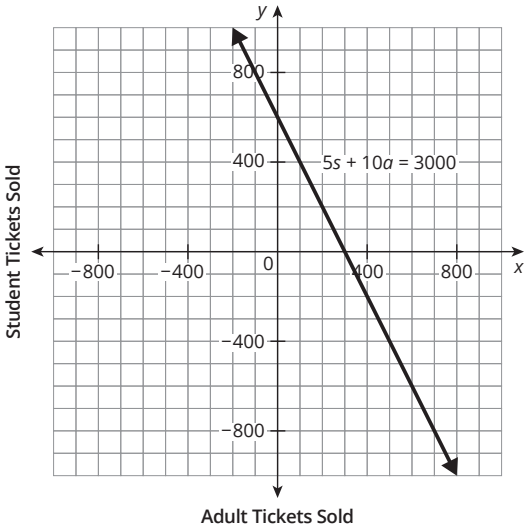
When two or more linear equations define a relationship between quantities, they form a **system of linear equations**.

The athletic director of the high school says that 450 total tickets were sold to the home game.

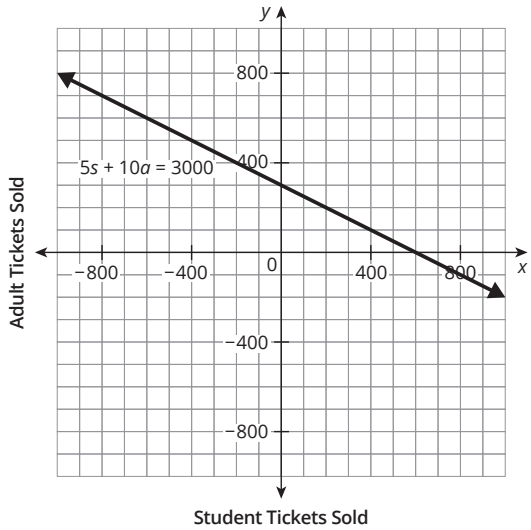
1. Write an equation that represents this situation. Let s represent the number of student tickets sold, and let a represent the number of adult tickets sold.

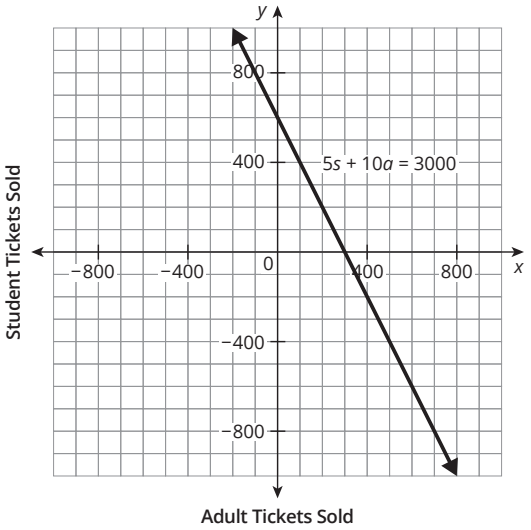
The coordinate planes shown already contain the function that models the earnings from ticket sales.





2. Use x- and y-intercepts to graph the function modeling the total number of tickets sold on each coordinate plane.





3. When the high school reached its goal of $3000 in ticket sales, how many of each type of ticket was sold? Is there more than one solution?

Remember . . .
According to the situation, 450 tickets were sold to the game.

4. Use technology to locate the exact point of intersection. Explain the process you used.

5. Algebraically justify that your solution is correct.

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ACTIVITY
1.3

Systems with No Solution, One Solution, or an Infinite Number of Solutions

Samuel and Diego are in the Robotics Club. They are both saving money to buy materials to build a new robot.

Samuel opens a new bank account. He deposits $25 that he won at a robotics competition. He also plans on depositing $10 a week that he earns from tutoring. Diego decides he wants to save money as well. He already has $40 saved from mowing lawns over the summer. He plans to also save $10 a week from his allowance.

1. Write equations to represent the amount of money Samuel saves and the amount of money Diego saves.

2. Use your equations to predict when Samuel and Diego will have the same amount of money saved.

You can prove your prediction by solving and graphing a system of linear equations.

3. Analyze the equations in your system.

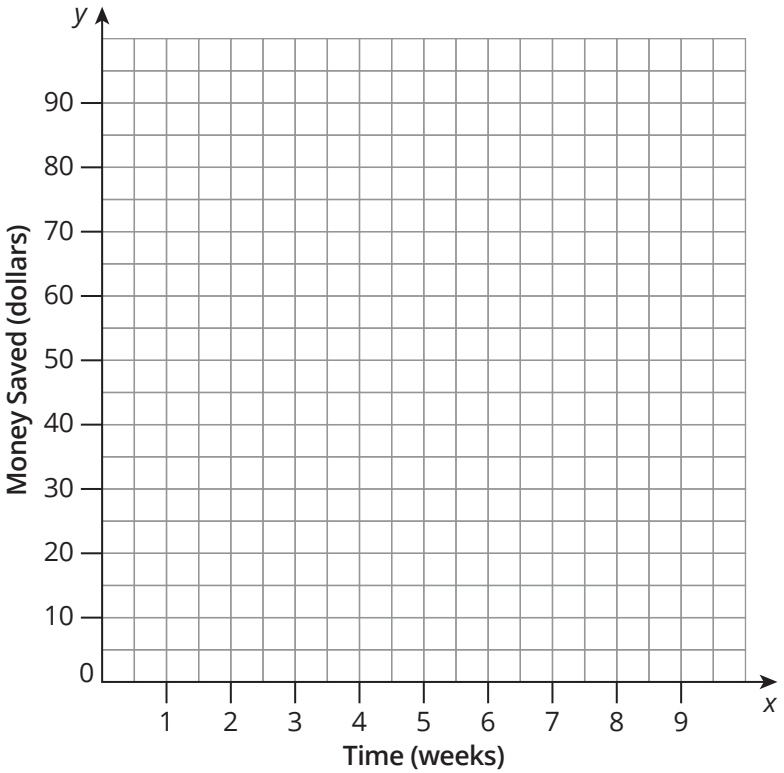
   a. How do the slopes compare? Describe what this means in terms of this problem situation.

   b. How do the *y*-intercepts compare? Describe what this means in terms of this problem situation.

4. Determine the solution of the system of linear equations graphically.

   a. Predict what the graph of this system will look like. Explain your reasoning.

   b. Graph both equations on the coordinate plane.



5. Analyze the graph you created.

   a. Describe the relationship between the graphs.

   b. Does this linear system have a solution? Explain your reasoning.

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6. Was your prediction in Question 2 correct? Explain how you algebraically and graphically proved your prediction.

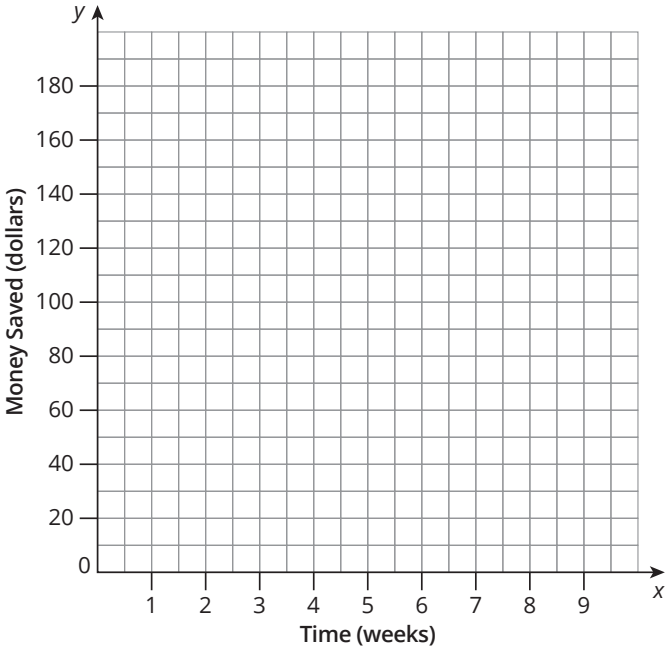
Isabella is also in the Robotics Club and has heard about Samuel's and Diego's savings plans. She wants to be able to buy her new materials before Diego, so she opens her own bank account. She is able to deposit $40 in her account that she has saved from her job as a waitress. Each week, she also deposits $4 from her tips.

Remember . . .
Don't forget to define your variables!

7. Use equations and graphs to determine when Isabella and Diego have saved the same amount of money.

   a. Write a linear system to represent the total amount of money Isabella and Phillip have saved after a certain amount of time.

   b. Graph the linear system on the coordinate plane.



   c. Do the graphs intersect? If so, describe the meaning in terms of this problem situation.

Diego and Isabella went on a shopping spree this weekend and spent all their savings except for $40 each. Diego is still saving $10 a week from his allowance. Isabella now deposits her tips twice a week. On Tuesdays, she deposits $4, and on Saturdays, she deposits $6.



8. Diego claims he is still saving more each week than Isabella.

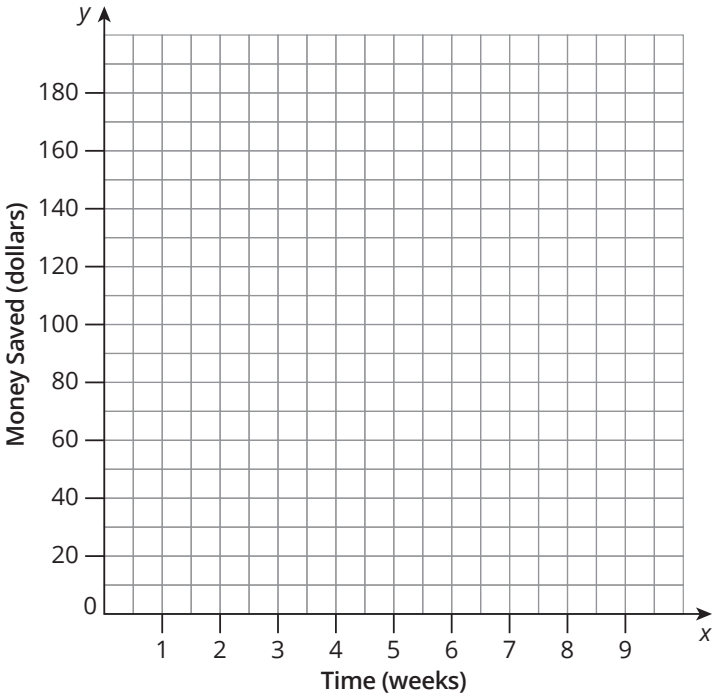
   a. Do you think DIego's claim is true? Explain your reasoning.

   b. How can you prove your prediction?

9. Prove your prediction graphically.

   a. Write a new linear system to represent the total amount of money each friend has after a certain amount of time.

b. Graph the linear system on the coordinate plane.



10. Describe the relationship between the graphs. What does this mean in terms of this problem situation?

11. Was Diego's claim that he is still saving more than Isabella a true statement? Explain why or why not.

Talk the Talk

Beating the System

A system of equations may have one unique solution, no solution, or infinite solutions. Systems that have one solution or many solutions are called **consistent systems**. Systems with no solution are called **inconsistent systems**.

1. Complete the table.

System of Two Linear Equations	Consistent Systems		Inconsistent Systems
Description of *y*-Intercepts	Same or different *y*-intercepts	Same *y*-intercepts	Different *y*-intercepts
Number of Solutions			
Description of Graph			

2. Explain why the *x*- and *y*-coordinates of the points where the graphs of a system intersect are solutions.

Ask Yourself . . .
What patterns do you notice?

Lesson 1 Assignment

Write

Define each term in your own words.

1. Consistent systems
2. Inconsistent systems

Remember

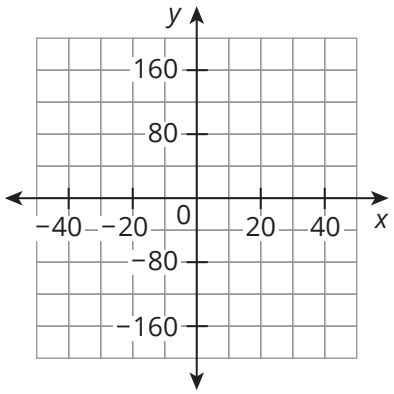
When two or more equations define a relationship between quantities, they form a system of linear equations. The point of intersection of a graphed system of linear equations is the solution to both equations. A system of linear equations can have one solution, no solution, or infinite solutions.

Practice

1. Mr. Nguyen gives his class 50-question multiple choice tests. Each correct answer is worth 2 points, while a half point is deducted for each incorrect answer. If the student does not answer a question, that question does not get any points.

   a. A student needs to earn 80 points on the test in order to keep an A grade for the semester. Write an equation in standard form that represents the situation. Identify three combinations of correct and incorrect answers that satisfy the equation.

   b. Determine the x- and y-intercepts of the equation and use them to graph the equation. Explain what each intercept means in terms of the problem situation.



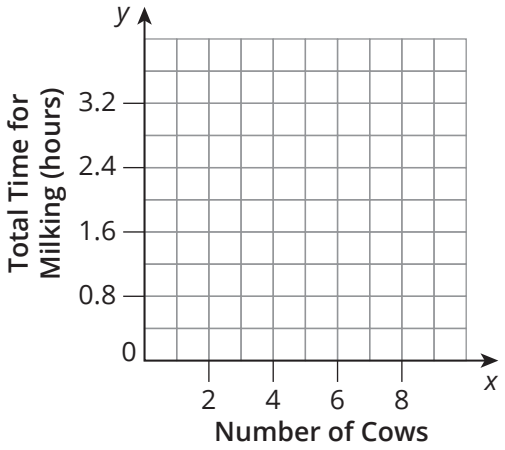
Lesson 1 Assignment

2. Harper owns a dairy farm. In the morning, it takes her 0.3 hour to set up for milking the cows. Once she has set up, it takes Harper 0.2 hour to milk each cow by hand. She is contemplating purchasing a milking machine in hopes that it will speed up the milking process. The milking machine she is considering will take 0.4 hour to set up each morning and takes 0.05 hour to milk each cow.

   a. Write a system of linear equations that represents the total amount of time Harper will spend milking the cows using the two different methods.

   b. Graph both equations on a coordinate plane.

   c. Estimate the point of intersection. Explain how you determined your answer.



   d. What does the point of intersection represent in this problem situation?

   e. Does the solution make sense in terms of this problem situation? Explain your reasoning.

   f. Is this system of equations consistent or inconsistent? Explain your reasoning.

Lesson 1 Assignment

3. Identify whether each system is consistent or inconsistent. Explain your reasoning.

a. $\begin{cases} -3x = 4y = 3 \\ -12x + 16y = 8 \end{cases}$

b. $\begin{cases} 7x + 3y = 0 \\ -14x + 6y = 0 \end{cases}$

c. $\begin{cases} 6x + y = 1 \\ -6x - 4y = -4 \end{cases}$

Prepare

Analyze each system of equations. What can you conclude about the value of y in each?

1. $\begin{cases} x = 12 \\ y = x + 22 \end{cases}$

2. $\begin{cases} x = 0 \\ y = x - 45 \end{cases}$

3. $\begin{cases} x = y \\ y = 2x - 10 \end{cases}$

4. $\begin{cases} x = y + 3 \\ y = 2x - 10 \end{cases}$

2 Using Substitution to Solve Linear Systems

OBJECTIVES

- Write a system of equations to represent a problem context.
- Algebraically solve a system of equations using substitution.
- Interpret a solution to a system of linear equations in terms of the problem situation.
- Solve real-world and mathematical problems with two linear equations in two variables.

NEW KEY TERM

- substitution method

You have graphed systems of equations.

Suppose you graph a system of equations, but the point of intersection is not clear from the graph. How can you determine the solution to the system?

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PROBLEM SOLVING



Getting Started

Goats, Chickens, and Pigs

At the county fair, farmers bring some of their animals to trade with other farmers. To make all trades fair, a master of trade oversees all trades. Assume all chickens are of equal value, all goats are of equal value, and all pigs are of equal value.

- In the first trade of the day, 4 goats were traded for 5 chickens.
- In the second trade, 1 pig was traded for 2 chickens and 1 goat.
- In the third trade, Farmer Lyndi put up 3 chickens and 1 pig against Farmer Simpson's 4 goats.

1. Is this a fair trade? If not, whose animals are worth more? How could this be made into a fair trade?

ACTIVITY
2.1 Introduction to Substitution

Sofia is helping her mother make potato salad for the county fair and is asked to go to the market to buy fresh potatoes and onions. Sweet onions cost $1.25 per pound, and potatoes cost $1.05 per pound. Her mother told her to use the $30 she gave her to buy these two items.

1. Write an equation that relates the number of pounds of potatoes and the number of pounds of onions that Sofia can buy for $30. Use x to represent the number of pounds of onions and y to represent the number of pounds of potatoes that Sofia can buy. Then, rewrite your equation in standard form.

2. Sofia's mother told her that the number of pounds of potatoes should be 8 times greater than the number of pounds of onions in the salad. Write an equation in x and y that represents this situation.

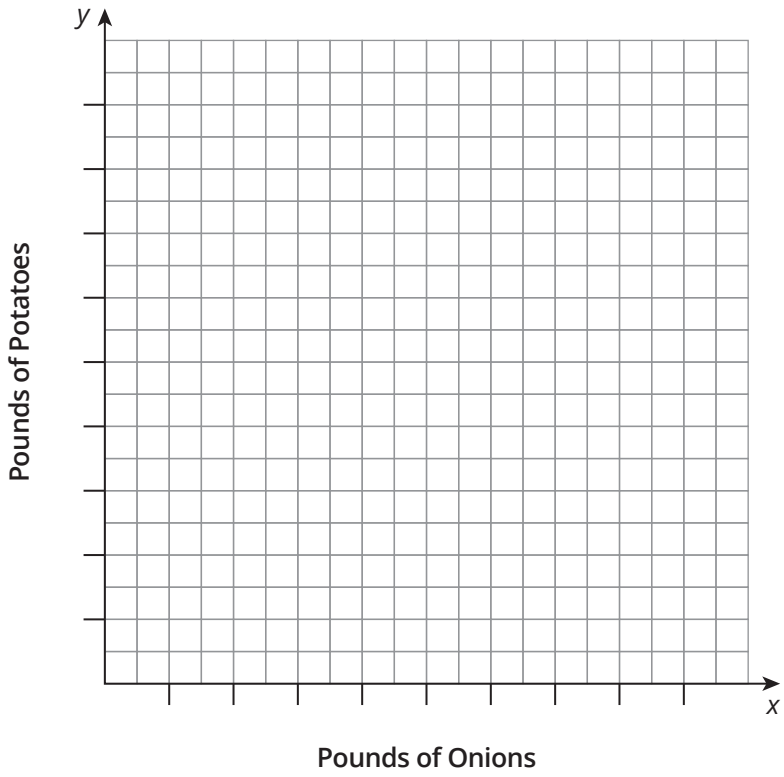
3. Will 1 pound of onions and 8 pounds of potatoes satisfy both equations? Explain your reasoning.

The standard form of a linear equation is $Ax + By = C$, where A, B, and C are integers and A and B are not both zero.

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4. Create graphs of both equations. Choose your bounds and intervals for each quantity.

Variable Quantity	Lower Bound	Upper Bound	Interval



5. Can you determine the exact solution of this linear system from your graph? Explain your reasoning.

6. Estimate the point of intersection from your graph.

In many systems, it is difficult to determine the solution from the graph. There is an algebraic method that can be used called the *substitution method*. The **substitution method** is a process of solving a system of equations by substituting a variable in one equation with an equivalent expression.

WORKED EXAMPLE

Let's consider the system you wrote.

$$\begin{cases} 1.25x + 1.05y = 30 \\ y = 8x \end{cases}$$

Step 1: To use the substitution method, begin by choosing one equation and isolating one variable. Because $y = 8x$ is in slope-intercept form, use this as the first equation.

Step 2: Now, substitute the expression equal to the isolated variable into the second equation.

Substitute $8x$ for y in the equation $1.25x + 1.05y = 30$.

Write the new equation.

$$\begin{aligned} 1.25x + 1.05y &= 30 \\ 1.25x + 1.05(8x) &= 30 \end{aligned}$$

You have just created a new equation with only one unknown.

Step 3: Solve the new equation.

$$\begin{aligned} 1.25x + 8.40x &= 30 \\ 9.65x &= 30 \\ x &\approx 3.1 \end{aligned}$$

Therefore, Sofia should buy approximately 3.1 pounds of onions.

Now, substitute the value for x into $y = 8x$ to determine the value of y.

$$y = 8(3.1) = 24.8$$

Therefore, Sofia should buy approximately 24.8 pounds of potatoes.

Step 4: Check your solution by substituting the values for both variables into the original system to show that they make both equations true.

The slope-intercept form of a linear equation is $y = mx + b$, where m is the slope of the line and b is the y-intercept of the line.

Ask Yourself . . .
Keep in mind what the value represents.

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7. Check that the solution is correct. Show your work.

8. What is the solution to the system? What does it represent in terms of the problem situation?

9. Compare your solution using the substitution method to the solution on your graph. What do you notice?

ACTIVITY
2.2

Substitution with Special Systems

Daniel and Chloe are helping to set up the booths at the fair. They are each paid $7 per hour to carry the wood that is needed to build the various booths. Daniel arrives at 7:00 A.M. and begins working immediately. Chloe arrives 90 minutes later and starts working.

1. Write an equation that gives the amount of money that Daniel will earn, *y*, in terms of the number of hours he works, *x*.

2. How much money will Daniel earn after 90 minutes of work?

3. Write an equation that gives the amount of money Chloe will earn, *y*, in terms of the number of hours since Daniel started working, *x*.

4. How much money will each student earn by noon?

5. Will Chloe ever earn as much money as Daniel? Explain your reasoning.

6. Write a system of linear equations for this problem situation.

Think About . . .
How is this similar to solving linear equations with no solution or with infinite solutions?

7. Analyze the system of linear equations. What do you know about the solution of the system by observing the equations? Explain your reasoning.

Let's see what happens when we solve the system algebraically.

8. Since both equations are written in slope-intercept form as expressions for y in terms of x, substitute the expression from the first equation into the second equation.

   a. Write the new equation.

   b. Solve the equation for x.

   c. Does your result for x make sense? Explain your reasoning.

9. What is the result when you algebraically solve a linear system that contains parallel lines?

On Monday night, the fair is running a special for the the local schools: if tickets are purchased from the school, you can buy student tickets for $4 and adult tickets for $4. You buy 5 tickets and spend $20.

10. Write an equation that relates the number of student tickets, x, and the number of adult tickets, y, to the total amount spent.

11. Write an equation that relates the number of student tickets, x, and the number of adult tickets, y, to the total number of tickets purchased.

12. Write both equations in slope-intercept form.

13. Analyze the system of linear equations. What do you know about the solution of the system by looking at the equations?

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Let's see what happens when you solve the system algebraically.

14. Since both equations are now written in slope-intercept form as expressions for y in terms of x, substitute the expression from the first equation into the second equation.

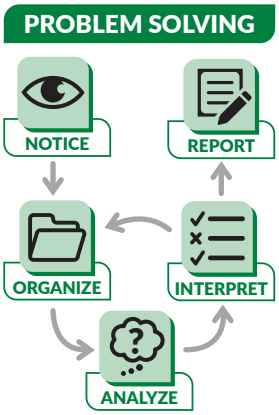
    a. Write the new equation and solve the equation for x.

    b. Does your result for x make sense? Explain your reasoning.

15. How many student tickets and adult tickets did you purchase?

16. What is the result when you algebraically solve a linear system that contains two lines that are actually the same line?

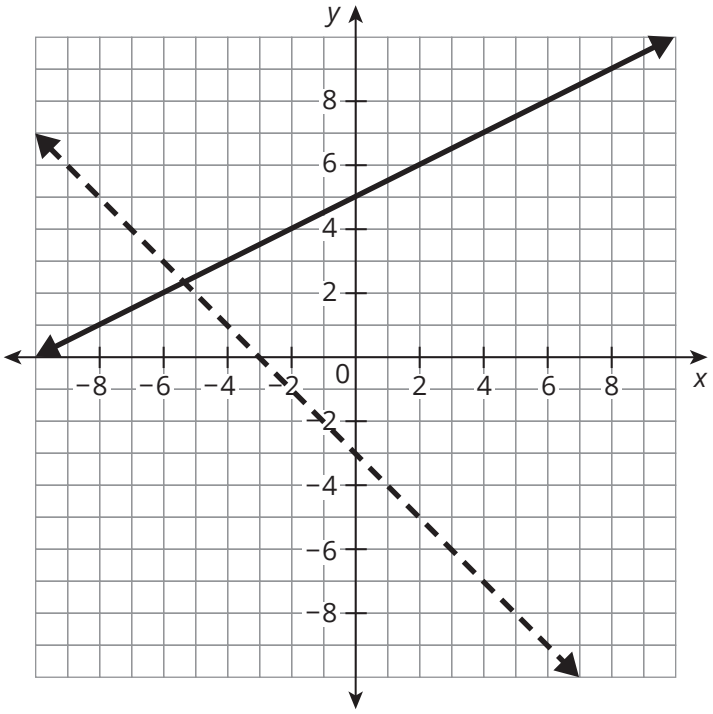
ACTIVITY
2.3 Writing and Solving Systems



Write a system of equations for each graph, table, or description.

1. The admission fee for a fair is $7 for an adult and $5 for children under the age of 5. There are 449 people at the fair. They collect $2768 in admission fees. How many adults, *x*, and children, *y*, are at the fair?

2. Write a system of equations from the graph.



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3. Write a system of linear equations from the tables provided.

Line *A*	
x	*y*
−2	−20
0	−15
2	−10
4	−5

Line *B*	
x	*y*
−4	16
−1	10
2	4
5	−2

Write and solve a system of equations to solve each problem.

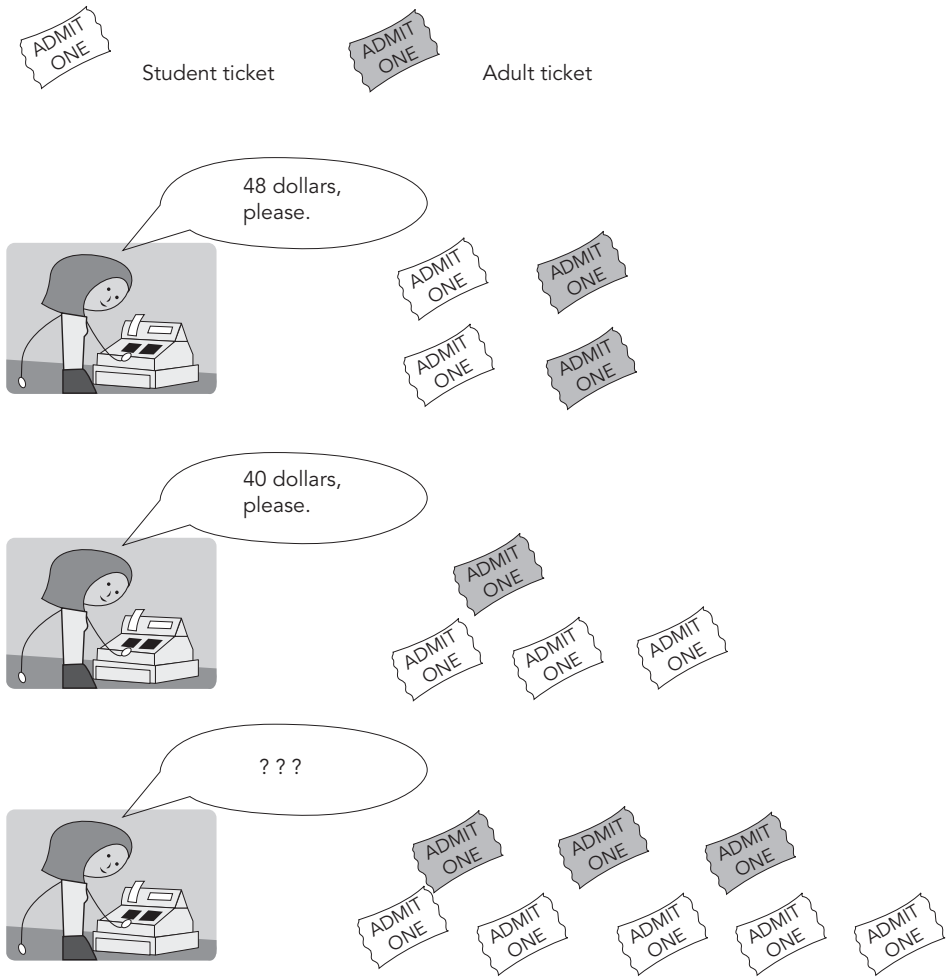
4. The business manager for a band must make $236,000 from ticket sales to cover costs and make a reasonable profit. The auditorium where the band will play has 4000 seats, with 2800 seats on the main level and 1200 on the upper level. Attendees will pay $20 more for main-level seats.

   a. Write a system of equations with *x* representing the cost of the main-level seating and *y* representing the cost of the upper-level seating.

   b. Without solving the system of linear equations, interpret the solution.

   c. Solve the system of equations using the substitution method. Then, interpret the solution of the system in terms of the problem situation.

5. Mia is working as a cashier at the sports arena. What should she tell the next person in line?



Write and solve a system of equations that represents the problem situation. Define the variables. Then, determine the cost of each type of ticket. Finally, state the amount Mia charges the third person.

Talk the Talk

The Substitution Train

1. Determine the solution to each linear system by using the substitution method. Check your answers algebraically.

   a. $\begin{cases} 2x + 3y = 34 \\ y = 5x \end{cases}$

   b. $\begin{cases} y = 4x + 2 \\ y = 3x - 2 \end{cases}$

c. $\begin{cases} 3x + 2y = 4 \\ 2x - y = 5 \end{cases}$

d. $\begin{cases} 3x + y = 8 \\ 6x + 2y = 10 \end{cases}$

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Lesson 2 Assignment

Write

Explain how to use the substitution method to solve systems of linear equations.

Remember

When a system has no solution, the equation resulting from the substitution step has no solution.

When a system has infinite solutions, the equation resulting from the substitution step has infinite solutions.

Practice

1. Victoria is trying to become more environmentally conscious by making her own cleaning products. She researches different cleaners and decides to make furniture polish using olive oil and lemon juice. She wants to make enough to fill two 24-fluid-ounce bottles.

   a. Write an equation in standard form that relates the amount of olive oil and lemon juice to the total amount of mixture Victoria wants to make. Use x to represent the amount of lemon juice and y to represent the amount of olive oil.

   b. The recommendation for the mixture is that the amount of olive oil be twice the amount of lemon juice. Write an equation in terms of x and y as defined in part (a) that represents this situation.

   c. Use substitution to solve the system of equations. Check your answer.

   d. What does the solution of the system represent in terms of the mixture?

Lesson 2 Assignment

e. The best price Victoria can find for lemon juice is $0.25 per fluid ounce. The best price she can find for olive oil is $0.39 per fluid ounce. She buys a total of 84 fluid ounces of lemon juice and olive oil and spends $29.40. Write equations in standard form for this situation. Use x to represent the amount of lemon juice she buys, and use y to represent the amount of olive oil she buys.

f. Solve the system of equations you wrote using the substitution method. Check your answer. Describe the solution in terms of the problem situation.

2. In an effort to eat healthier, Juan is tracking his food intake by using an application on his phone. He records what he eats, then the application indicates how many calories he has consumed.

   One day, Juan eats 10 medium strawberries and 8 vanilla wafer cookies as an after-school snack. The caloric intake from these items is 192 calories. The next day, she eats 20 medium strawberries and 1 vanilla wafer cookie as an after-school snack. The caloric intake from these items is 99 calories.

   a. Write a system of equations for this problem situation. Define your variables.

   b. Without solving the system of linear equations, interpret the solution.

Lesson 2 Assignment

c. Solve the system of equations using the substitution method. Check your work.

d. Interpret the solution of the system in terms of the problem situation.

e. Juan's friend Jayden also has a calorie counting application on his phone. The two friends decide to compare the two programs. Juan eats 1 banana and 5 pretzel rods, and his application tells him he consumed 657 calories. Jayden eats 1 banana and 5 pretzel rods, and his application tells him he consumed 656 calories. The boys want to know how many calories are in each food. Write a system of equations for this problem. Define your variables.

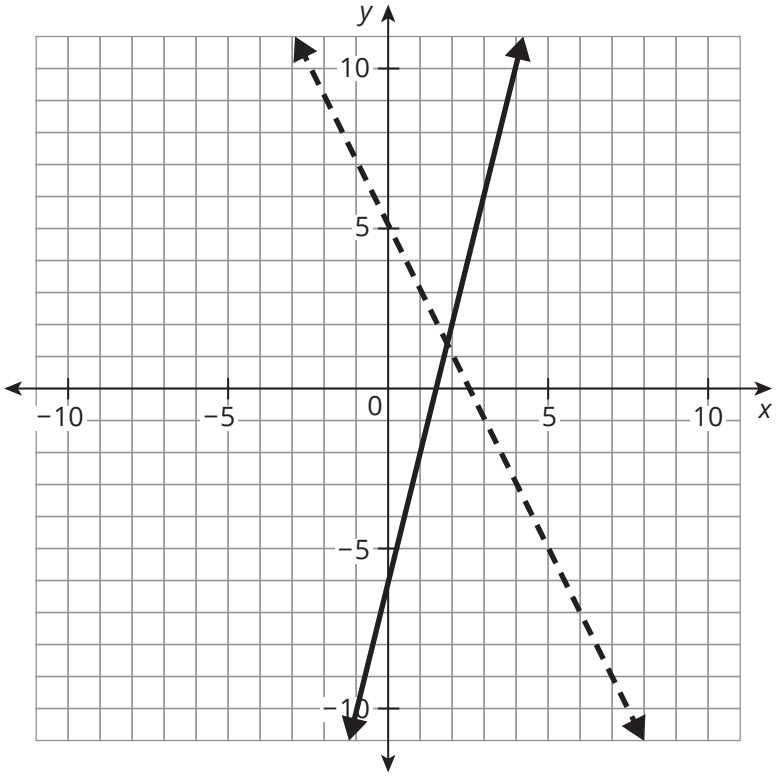
f. Solve the system of equations using the substitution method. Interpret your answer in terms of the problem.

3. Write a system of linear equations for each graph, table, or description.

   a. Mateo has 13 coins. The coins are nickels and quarters. The coins have a total value of $2.05. Let n represent the number of nickels, and let q represent the number of quarters.

Lesson 2 Assignment

b.



c.

Line *A*	
x	*y*
−3	−15
0	−3
3	9
6	15

Line *B*	
x	*y*
−4	12
−2	5
2	−9
6	−23

Lesson 2 Assignment

4. Use the substitution method to determine the solution of each system of linear equations. Check your solutions.

a. $\begin{cases} 9x + y = 16 \\ y = 7x \end{cases}$

b. $\begin{cases} 3x + \frac{1}{2}y = -3.5 \\ y = -6x + 11 \end{cases}$

c. $\begin{cases} y = -5x \\ 21x - 7y = 28 \end{cases}$

d. $\begin{cases} 2x + 4y = -32 \\ y = -\frac{1}{2}x - 8 \end{cases}$

Prepare

Determine the additive inverse for each expression.

1. 4
2. x
3. $20x$
4. $-9x$
5. $78.5x$

3 Using Linear Combinations to Solve a System of Linear Equations

OBJECTIVES

- Write a system of equations to represent a problem context.
- Solve a system of equations algebraically using linear combinations.
- Interpret the solution to a system of equations in terms of a problem situation.

NEW KEY TERM

- linear combinations method

You have solved a system of linear equations graphically and algebraically, using the substitution method.

What are other algebraic strategies for solving systems of equations?

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PROBLEM SOLVING



Getting Started

Coding Class

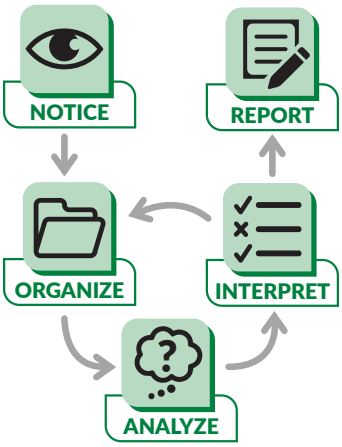
There are a total of 324 students taking a coding class. The number of 9th-grade students outnumber the number of 10th-grade students by 34.

1. Use reasoning to determine the number of 9th- and 10th-grade students who joined the coding class.

2. How can you check that your solution is correct?

ACTIVITY
3.1 Combining Linear Systems

PROBLEM SOLVING



Consider the scenario from the Getting Started. Let's explore an algebraic strategy to determine a solution.

1. Write the system that represents the problem situation. Use x to represent the 9th-grade students in the class, and use y to represent the 10th-grade students in the class. Write the equations in standard form.

Mateo and Minh used different strategies to solve the system of equations in a similar way. Analyze each student's reasoning.

Remember . . .
As long as you maintain equality you can rewrite equations any way you want.

Mateo

I can use the substitution method to solve this system.

$x + y = 324$
$x - y = 34 \rightarrow x = y + 34$
$(y + 34) + y = 324$
$2y + 34 = 324$
$2y = 290$
$y = 145$

$x = 145 + 34$
$x = 179$

Minh

You can eliminate one of the quantities by adding the two equations together.

$$\begin{array}{r} x + y = 324 \\ +\ x - y = 34 \\ \hline 2x = 358 \\ x = 179 \end{array}$$

$179 + y = 324$
$y = 145$

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2. Explain why each student is correct in their reasoning.

3. Examine the structure of the system. What characteristic of the system made Minh's strategy efficient?

4. Identify the solution to the linear system as an ordered pair. Then, interpret the solution in terms of this problem situation.

The algebraic method used by Minh to solve the linear system is called the *linear combinations method*. The **linear combinations method** is a process used to solve a system of equations by adding two equations together, resulting in an equation with one variable. You can then determine the value of that variable and use it to determine the value of the other variable.

5. Solve each system of equations using the linear combinations method.

   a. $\begin{cases} -4x + y = 10 \\ -2x - y = -1 \end{cases}$

   b. $\begin{cases} -\frac{1}{2}x + 5y = -6 \\ \frac{1}{2}x + y = -6 \end{cases}$

ACTIVITY
3.2 Using Additive Inverses to Combine Linear Systems

In the system of equations from the previous activity, one of the variables in both equations has coefficients that are additive inverses. What if a system doesn't have variables that are additive inverses? Let's use the strategy of linear combinations to solve other systems.

WORKED EXAMPLE

Consider this system of equations: $\begin{cases} 7x + 2y = 24 \\ 4x + y = 15 \end{cases}$.

$$\begin{aligned} 7x + 2y &= 24 \\ -2(4x + y) &= -2(15) \end{aligned}$$

Multiply the second equation by a constant that results in coefficients that are additive inverses for one of the variables.

$$\begin{aligned} 7x + 2y &= 24 \\ +\ -8x - 2y &= -30 \\ \hline -x &= -6 \\ x &= 6 \end{aligned}$$

Now that the y-values are additive inverses, you can solve this linear system for x.

$$\begin{aligned} 7(6) + 2y &= 24 \\ 42 + 2y &= 24 \\ 2y &= -18 \\ y &= -9 \end{aligned}$$

Substitute the value for x into one of the equations to determine the value for y.

The solution to the system of linear equations is $(6, -9)$.

1. In the Worked Example, only one equation needs to be rewritten to solve using the linear combinations method. Why?

Remember . . .
Two numbers with a sum of zero are called *additive inverses*.

Ask Yourself . . .
How can you solve the system of equations by transforming the first equation instead of the second?

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Now, let's consider a system where both equations need to be rewritten.

Ask Yourself . . .
If you multiply both sides of an equation by the same number, is the equation still true?

WORKED EXAMPLE

$$\begin{cases} 4x + 2y = 3 \\ 5x + 3y = 4 \end{cases}$$

$3(4x + 2y) = 3(3)$
$-2(5x + 3y) = -2(4)$

Multiply each equation by a constant that results in coefficients that are additive inverses for one of the variables.

$12x + 6y = 9$
$-10x - 6y = -8$

2. Determine the solution for the linear system shown in the second Worked Example.

3. How could you have solved this system by creating x-values that are additive inverses?

4. Describe the first step needed to solve each system using the linear combinations method. Identify the variable that will be solved for when you add the equations.

   a. $\begin{cases} 5x + 2y = 10 \\ 3x + 2y = 6 \end{cases}$

   b. $\begin{cases} x + 3y = 15 \\ 5x + 2y = 7 \end{cases}$

c. $\begin{cases} 4x + 3y = 12 \\ 3x + 2y = 4 \end{cases}$

5. Analyze each system. How would you rewrite the system to solve for one variable? Explain your reasoning.

a. $\begin{cases} \frac{1}{2}x - 5y = -45 \\ -\frac{1}{2}x + 10y = -20 \end{cases}$

b. $\begin{cases} 4x + 3y = 24 \\ 3x + y = -2 \end{cases}$

c. $\begin{cases} 3x + 5y = 17 \\ 2x + 3y = 11 \end{cases}$

d. $\begin{cases} 6x + 3y = 5 \\ 2x + y = 1 \end{cases}$

e. $\begin{cases} x + 2y = -6 \\ 2x + 4y = -12 \end{cases}$

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ACTIVITY
3.3

Applying the Linear Combinations Method

A snow resort offers two winter specials: the Get-Away Special and the Extended Stay Special. The Get-Away Special offers two nights of lodging and four meals for $270. The Extended Stay Special offers three nights of lodging and eight meals for $435. Determine what Let It Snow charges per night of lodging and per meal.

1. Write the system of linear questions that represents the problem situation. Let *n* represent the cost for one night of lodging at the resort and *m* represent the cost for each meal. Write the equations in standard form.

2. How are these equations the same? How are these equations different?

3. Solve the system comparing the two winter specials.

4. Interpret the solution of the linear system in the problem situation.

5. Check your solution algebraically.

6. Is the Extended Stay Special the better deal? Explain why or why not.

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ACTIVITY
3.4 Fractions and Linear Combinations

The School Spirit Club is making beaded friendship bracelets with the school colors to sell in the school store. The bracelets are black and orange and come in two lengths: 5 inches and 7 inches. The club has enough beads to make a total of 84 bracelets. So far, they have made 49 bracelets, which represents $\frac{1}{2}$ the number of 5-inch bracelets plus $\frac{3}{4}$ the number of 7-inch bracelets they plan to make and sell. Determine how many 5-inch and 7-inch bracelets the club plans to make.

1. Let x represent the number of 5-inch bracelets, and let y represent the number of 7-inch bracelets. Write a system of equations in standard form to represent this problem situation.



2. Aaliyah says that the first step to solve this system is to multiply the second equation by the least common denominator (LCD) of the fractions. Liam says that the first step is to multiply the first equation by $-\frac{1}{2}$. Who is correct? Explain your reasoning.

3. Rewrite the equation containing fractions as an equivalent equation without fractions.

4. Determine the solution to the system of equations by using linear combinations and check your answer.

5. Interpret the solution of the linear system in terms of this problem situation.

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Talk the Talk

There's a Method in My Madness

You have used three different methods for solving systems of equations: graphing, substitution, and linear combinations.

1. Describe how to use each method and the characteristics of the system that makes this method most appropriate.

   a. Graphing method:

   b. Substitution method:

   c. Linear combinations method:

Lesson 3 Assignment

Write

Explain how you would combine the two equations to solve for x and y. Use the following terms in your explanation: *linear combination* and *additive inverses*.

$$3x + 2y = -25$$
$$x - 4y = 5$$

Remember

You can use the linear combinations method to solve a system of equations by adding two equations together, resulting in an equation with one variable. You can then determine the value of that variable and use it to determine the value of the other variable.

Practice

1. Two high schools are taking field trips to the state capital. A total of 408 students from East High will be going in 3 vans and 6 buses. A total of 516 students from West High will be going in 6 vans and 7 buses. Each van has the same number of passengers, and each bus has the same number of passengers.

   a. Write a system of equations that represents this problem situation. Let x represent the number of students in each van, and let y represent the number of students in each bus.

   b. How are the equations in the system the same? How are they different?

   c. Describe the first step needed to solve the system using the linear combinations method. Identify the variable that will be eliminated as well as the variable that will be solved for when you add the equations.

Lesson 3 Assignment

d. Solve the system of equations using the linear combinations method. Show your work.

e. Interpret the solution of the linear system in terms of the problem situation.

f. Check your solution algebraically.

2. Solve each system of linear equations.

a. $\begin{cases} 3x + y = 9 \\ 7x + y = 32 \end{cases}$

b. $\begin{cases} \frac{2}{3}x + \frac{1}{4}y = 18 \\ \frac{1}{6}x - \frac{3}{8}y = -6 \end{cases}$

Lesson 3 Assignment

c. $\begin{cases} 5x - 8y = 25 \\ -x + 4y = -8 \end{cases}$

d. $\begin{cases} 5x + 4y = -14 \\ 3x + 6y = 6 \end{cases}$

Prepare

Determine if each point is a solution to $y > x$, $y < x$, or $y = x$.

1. $(8, -2)$

2. $(0, 7)$

3. $(-1, -1)$

4. $(-4, -3)$

5. $(9, 9)$

6. $(-3, -10)$

4 Graphing Inequalities in Two Variables

OBJECTIVES

- Write an inequality in two variables.
- Graph an inequality in two variables on a coordinate plane.
- Determine whether a solid or dashed boundary line is used to graph an inequality on a coordinate plane.
- Interpret the solutions of inequalities mathematically and in the context of real-world problems.

NEW KEY TERMS

- half-plane
- boundary line

You have graphed linear inequalities in one variable.

What does the graph of a linear inequality in two variables look like? How does it compare to the graph of a linear equation?

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Getting Started

Making a Statement

Consider each solution statement.

$$x = 2$$

$$x < 2 \qquad x \leq 2$$

$$x > 2 \qquad x \geq 2$$

1. Compare the solution statements. What does each one mean?

2. Choose a solution statement and write a scenario to represent it. Then, modify the scenario so the resulting interpretation is one of the other four solution statements.

ACTIVITY
4.1 Linear Inequalities in Two Variables

1. Coach Garcia is analyzing the scoring patterns of players on his basketball team. Kai is averaging 20 points per game from scoring on two-point and three-point shots.

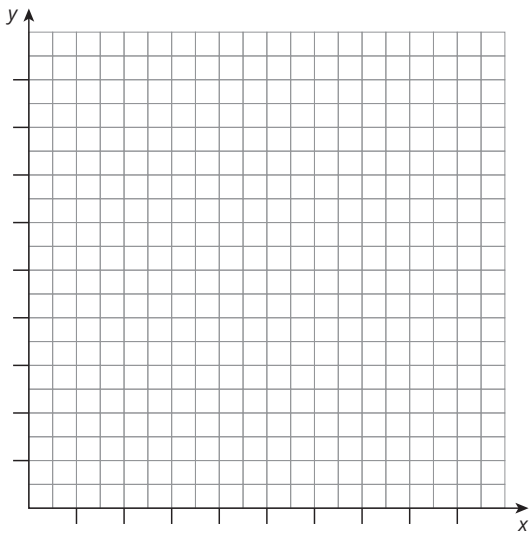
   a. If she scores 6 two-point shots and 2 three-point shots, will Kai meet her points-per-game average?

   b. If she scores 7 two-point shots and 2 three-point shots, will Kai meet her points-per-game average?

   c. If she scores 7 two-point shots and 4 three-point shots, will Kai meet her points-per-game average?

2. Write an equation to represent the number of two-point shots and the number of three-point shots that total 20 points.

3. Graph the equation you wrote on the coordinate plane.



Ask Yourself . . .
How should you label the graph?

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4. Coach Garcia believes that the team can win the district playoffs if Kai scores at least 20 points per game.

   a. How can you rewrite the equation you wrote in Question 2 to represent that Kai must score at least 20 points per game?

Remember. . .
An inequality is a statement formed by placing an inequality symbol ($>$, $\geq$, $<$, $\leq$,) between two expressions.

   b. Write an inequality in two variables that represents this problem situation.

5. Complete the table of values. Then, add the ordered pairs in the table to the graph in Question 3. If the number of total points scored does not meet or exceed Kai's points-per-game average, use an "x" to plot the point. If the number of total points scored meets or exceeds Kai's points-per-game average, use a dot to plot the point.

Number of Two-Point Shots Scored	Number of Three-Point Shots Scored	Number of Total-Points Scored
4	1	
6	1	
7	1	
8	2	
6	4	
9	5	

6. What do you notice about your graph?

7. What can you interpret about the solutions of the inequality from the graph?

8. Choose a different ordered pair located above the line and a different ordered pair that is located below the line. How do these points confirm your interpretation of the situation? Explain your reasoning.

9. Shade the side of the graph that contains the combinations of shots that are greater than or equal to Kai's points-per-game average.

10. How do the solutions of the linear equation $2x + 3y = 20$ differ from the solutions of the linear inequality $2x + 3y \geq 20$?

11. Does the ordered pair (6.5, 5.5) make sense as a solution in the context of this problem situation? Explain why or why not.

Like linear equations, linear inequalities take different forms. Each of the linear inequalities in two variables shown represents a different relationship between the variables.

$$ax + by < c \qquad ax + by > c$$
$$ax + by \leq c \qquad ax + by \geq c$$

ACTIVITY
4.2 Determining the Graphs of Linear Inequalities

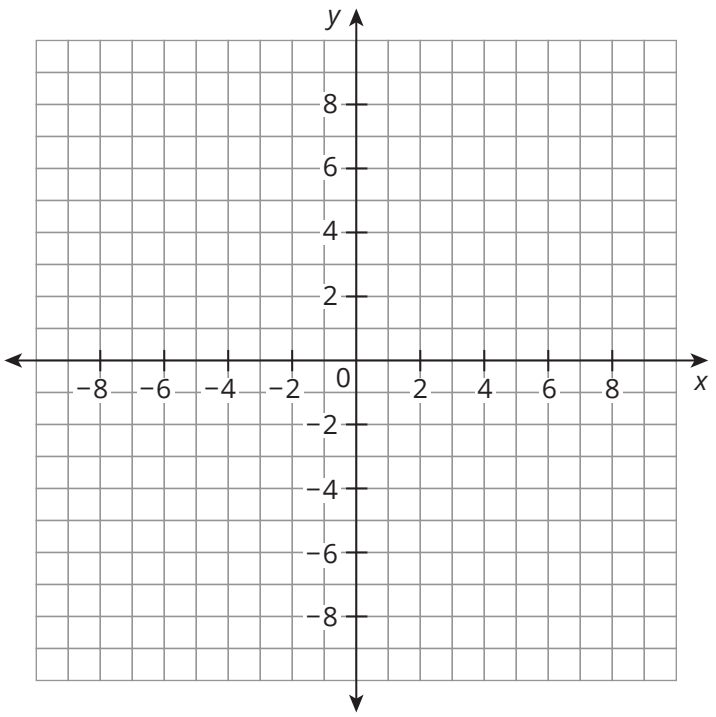
If the inequality symbol is $\geq$ or $\leq$, the boundary line is a solid line because all points on the line are part of the solution set. If the symbol is $>$ or $<$, the boundary line is a dashed line because no point on that line is a solution.

The graph of a linear inequality in two variables is a **half-plane**, or half of a coordinate plane. A **boundary line**, determined by the inequality, divides the plane into two half-planes, and the inequality symbol indicates which half-plane contains all the solutions. These solutions are represented by shading the appropriate half-plane.

Consider the linear inequality $y > 4x - 6$. The boundary line that divides the plane is determined by the equation $y = 4x - 6$.

1. Should the boundary line in this graph be a solid line or a dashed line? Explain your reasoning.

2. Graph the boundary line on the coordinate plane shown.



After you graph the inequality with either a solid or a dashed boundary line, you need to decide which half-plane to shade. To make your decision, consider the point (0, 0). If (0, 0) is a solution, then the half-plane that contains (0, 0) contains the solutions and should be shaded. If (0, 0) is *not* a solution, then the half-plane that does not contain (0, 0) contains the solutions and should be shaded.

3. Decide which half-plane to shade.

   a. Is (0, 0) a solution? Explain your reasoning.

   b. Shade the correct half-plane on the coordinate plane.

Think About . . .

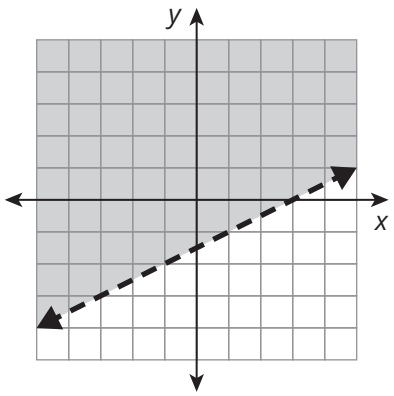
It's a good idea to check points in both half-planes to verify your solution.

4. Match each graph to one of the inequalities given. In part (d), graph the inequality that was not graphed in parts (a) through (c).

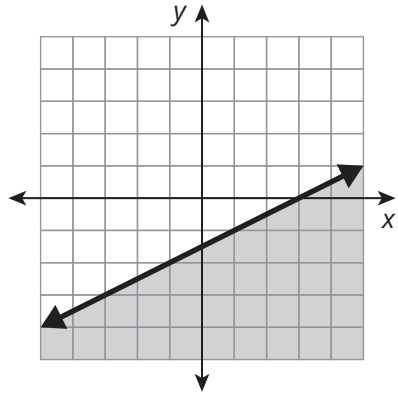
$y \geq \frac{1}{2}x - 3$ $\quad$ $y \leq \frac{1}{2}x - 3$

$y > \frac{1}{2}x - 3$ $\quad$ $y < \frac{1}{2}x - 3$

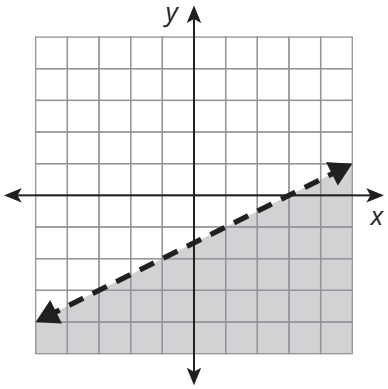
a.



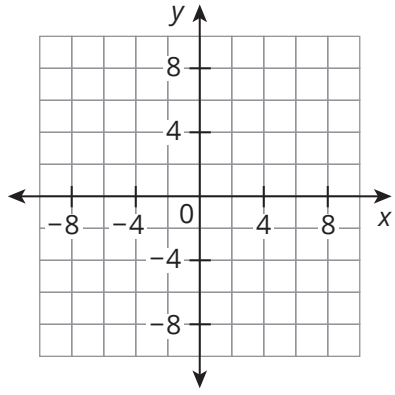
b.



c.



d.

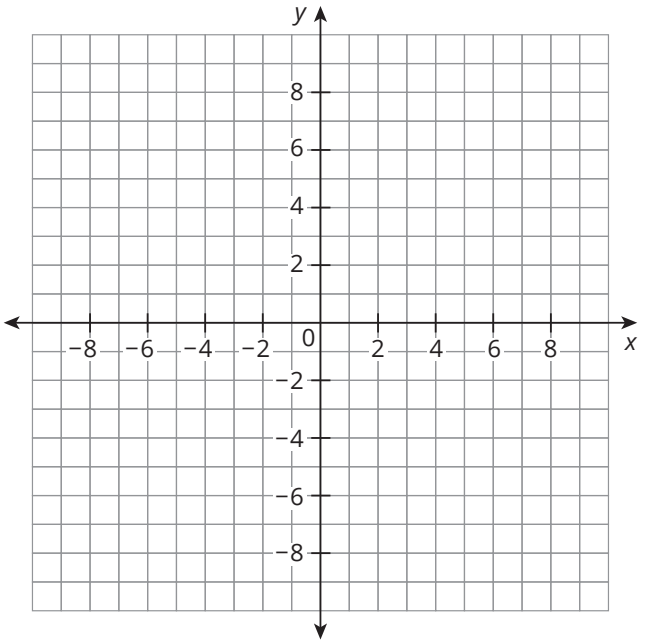


Think About . . .

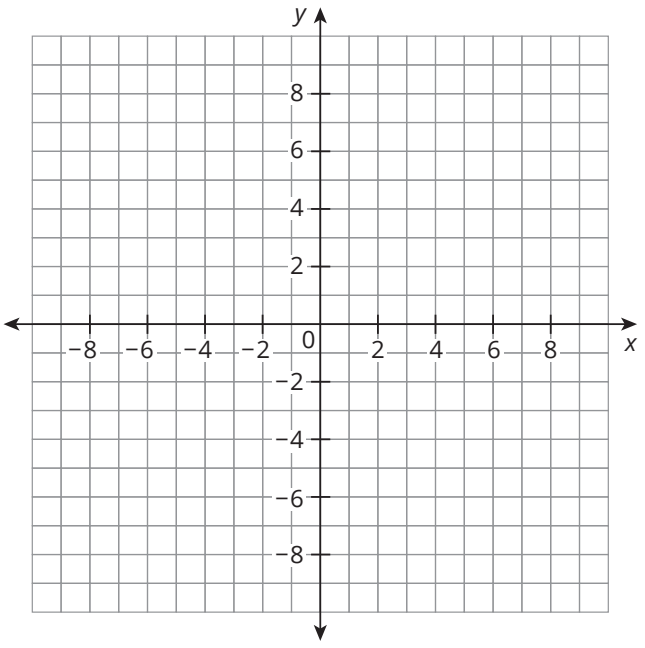
Consider the inequality symbol and which half-plane will be shaded before you test any points.

5. Graph each linear inequality.

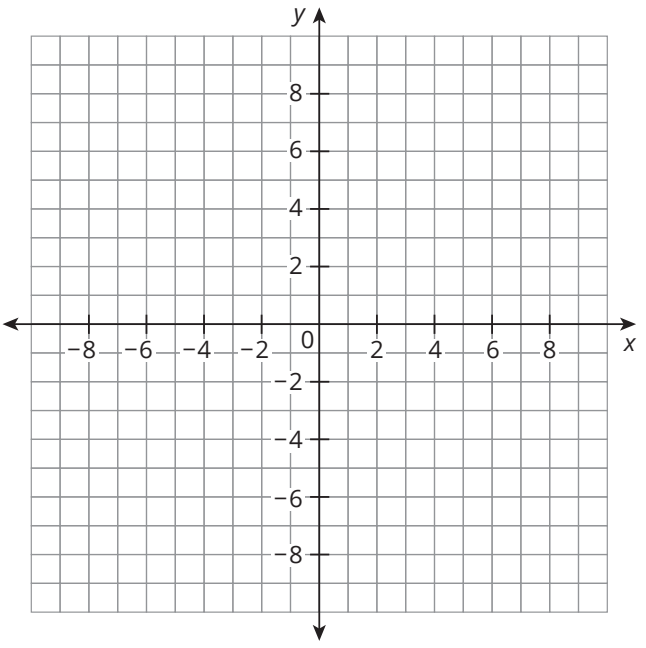
a. $y > x + 3$



b. $y \leq -\frac{1}{3}x + 4$



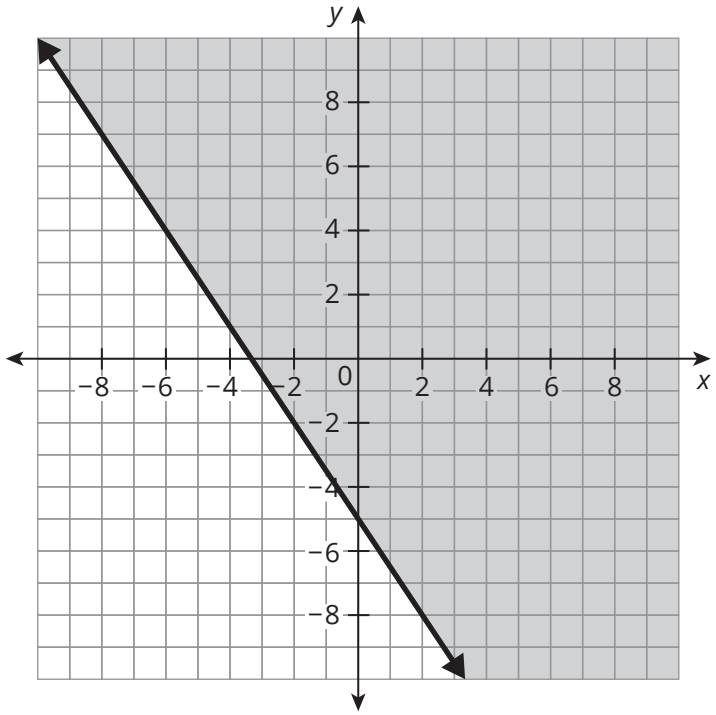
c. $2x - y < 4$



Previously, you have written a linear equation given various representations, including two points, one point and the slope, a table of values, or a graph. You can use a similar approach when writing a linear inequality.

WORKED EXAMPLE

Write a linear equality for the graph.



You can use what you have previously learned about the graphs of linear equations to determine that the boundary line is represented by the equation $y = -\frac{3}{2}x - 5$. Now, you must decide which inequality symbol should replace the equals sign in the equation.

Since the graph shows a solid boundary line and the half-plane above the line is shaded, use the symbol $\geq$.

$$y \geq -\frac{3}{2}x - 5$$

Test a point in the solution set to check the linear inequality.

Test the point (0, 0):

$$0 \stackrel{?}{\geq} -\frac{3}{2}(0) - 5$$

$$0 \geq -5 \checkmark$$

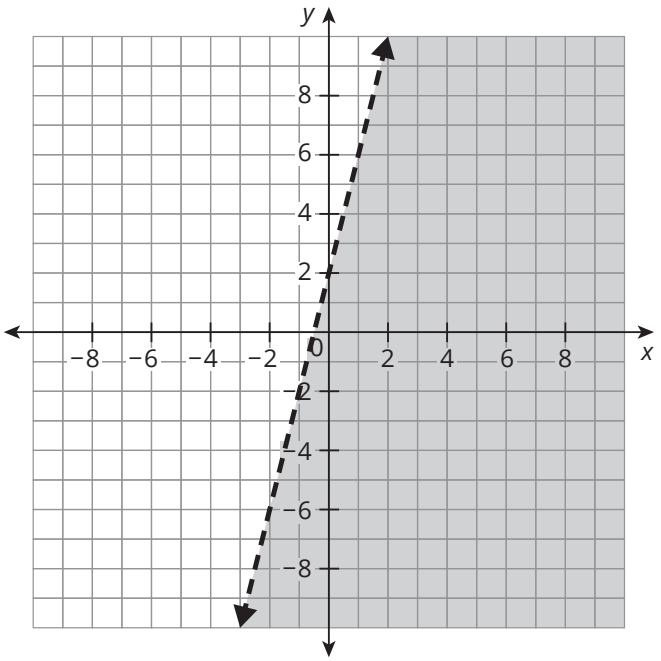
Remember. . .

The point (0, 0) can be used as a test point, unless the boundary line passes through (0, 0).

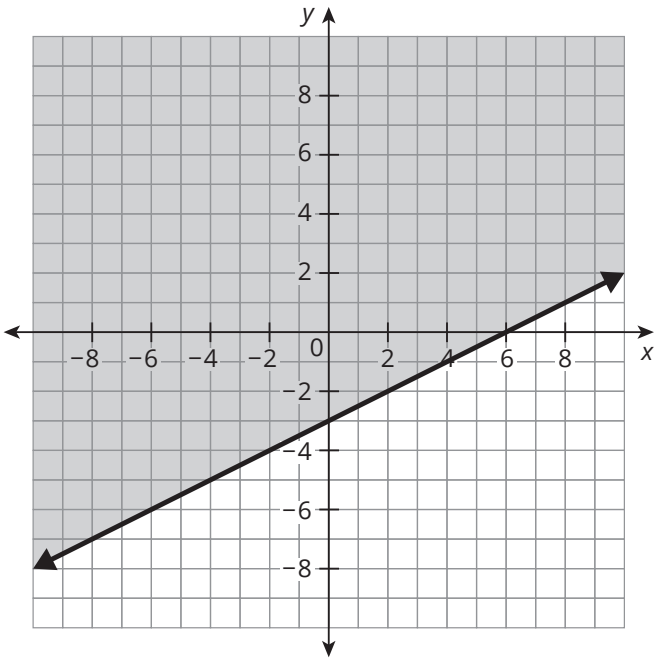
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6. Write a linear inequality for each graph.

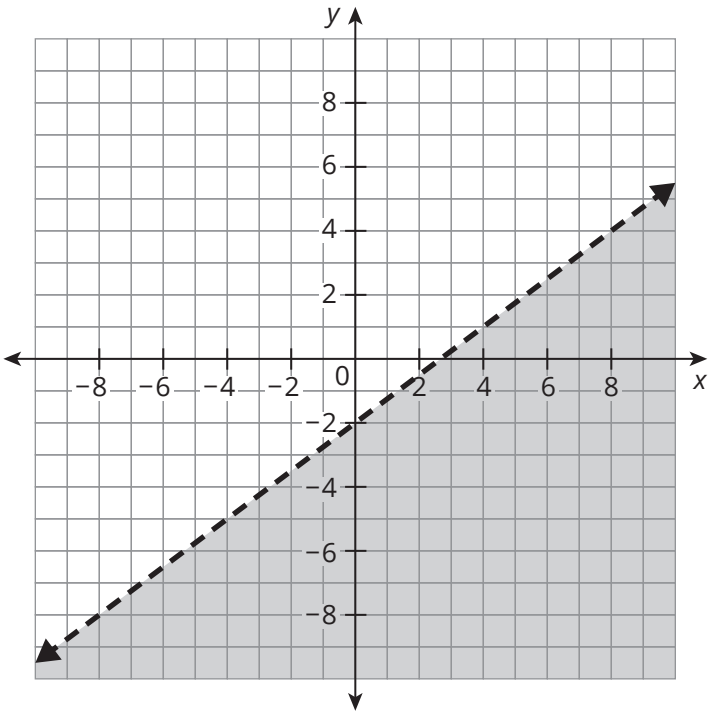
a.



b.



c.



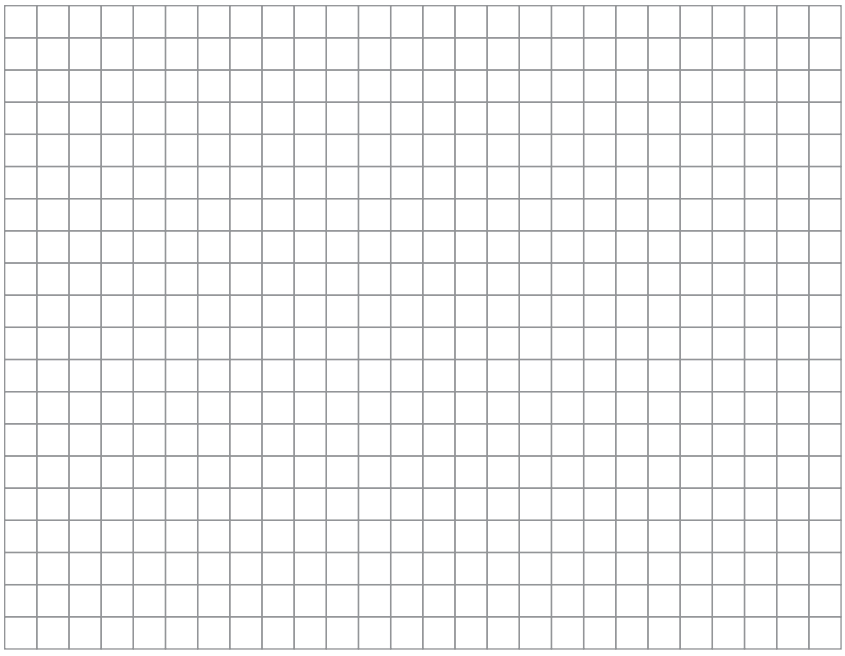
ACTIVITY
4.3 Interpreting the Graph of a Linear Inequality

César has relatives living in both the United States and Mexico. He is given a prepaid phone card worth $50 for his birthday. The table of values shows combinations of minutes for calls within the United States, x, and calls to Mexico, y, that expend his $50 prepaid phone card.

Length of Calls Within United States (minutes)	Length of Calls to Mexico (minutes)
0	100
50	80
140	44
200	20
240	4

1. Write an inequality modeling the number of minutes César can use for calls within the United States and for calls to Mexico.

2. Graph your inequality on the given coordinate grid. Be sure to label your axes.



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3. If César speaks with his aunt in Guadalajara, Mexico, for 70 minutes using his phone card, how long can he speak with his cousin in New York using the same card?

4. Can César call his uncle in San Antonio for 100 minutes and also call his grandmother in Juárez, Mexico, for 80 minutes using his phone card? Explain your reasoning.

Ask Yourself . . .
How can you use linear inequalities in everyday life?

5. Can César call his brother in Mexico City, Mexico, for 55 minutes and also call his sister in Denver for 90 minutes using his phone card? Explain your reasoning.

6. Interpret the meaning of each.

   a. Points on the line

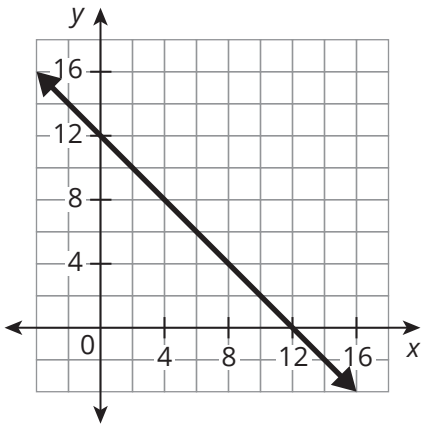
   b. Points above the line

   c. Points below the line

Talk the Talk

There's a Fine Line

Consider the graph of the linear equation $x + y = 12$.



Use the graph to answer each question.

1. Describe how to graph $x + y < 12$ and choose a point to test this region.

2. Describe how to graph $x + y \leq 12$ and choose a point to test this region.

3. Describe how to graph $x + y > 12$ and choose a point to test this region.

4. Complete the table.

Equation or Inequality	Description of the Solution Set
$x + y = 0$	
$x + y \geq 0$	
$x + y \leq 0$	
$x + y > 0$	
$x + y < 0$	

Lesson 4 Assignment

Write

Describe a half-plane in your own words.

Remember

The graph of a linear inequality in two variables is the half-plane that contains all the solutions. If the inequality symbol is $\leq$ or $\geq$, the graph shows a solid boundary line because the line is part of the solution set. If the symbol is $<$ or $>$, the boundary line is a dashed line because no point on the line is a solution.

Practice

1. Elijah is working two jobs to save money for his college education. He makes $8 per hour working for his uncle at a pizzeria bussing tables and $10 per hour tutoring peers after school in math. His goal is to make $160 per week.

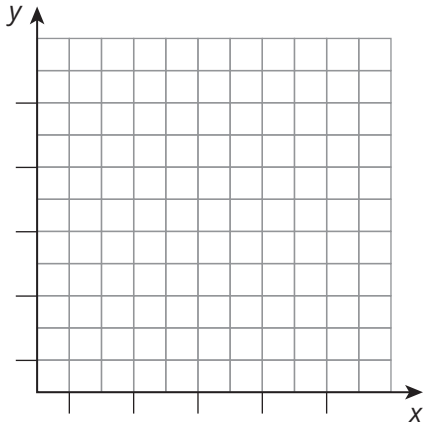
   a. If Elijah works 8 hours at the pizzeria and tutors 11 hours during the week, does he reach his goal?

   b. Write an expression to represent the total amount of money Elijah makes in a week from working both jobs. Let x represent the number of hours he works at the pizzeria and y represent the number of hours he tutors.

   c. After researching the costs of colleges, Elijah decides he needs to make more than $160 each week. Write an inequality with two variables to represent the amount of money Elijah needs to make.

Lesson 4 Assignment

d. Graph the inequality from part (c).



e. Is the point (0, 0) in the shaded region of the graph? Explain why or why not.

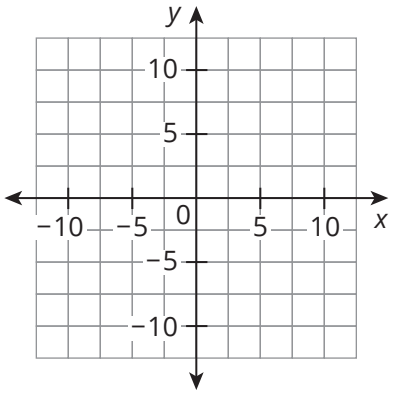
f. According to the graph, if Elijah works 5 hours at the pizzeria and tutors for 10 hours, will he make more than $160? Explain why or why not.

g. Due to days off from school, Elijah will only be tutoring for 6 hours this week. Use the graph to determine the least amount of full hours he must work at the pizzeria to still reach his goal. Then, show that your result satisfies the inequality.

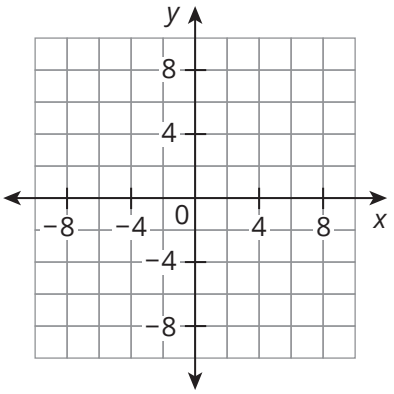
Lesson 4 Assignment

2. Graph each inequality on a coordinate plane.

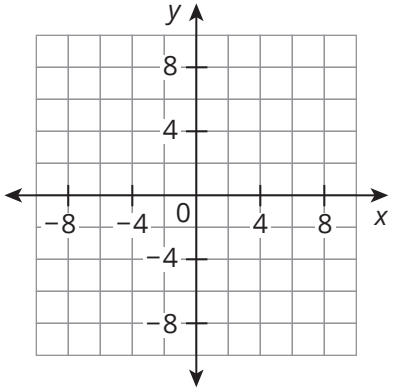
a. $x + 3y > 9$



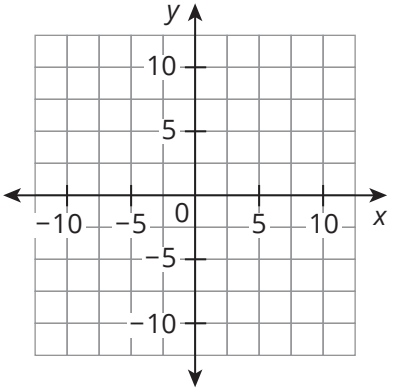
b. $2x - 6y \leq 15$



c. $2x + y < 6$



d. $3x - y \geq 1$



Prepare

Determine an ordered pair (x, y) that satisfies each inequality.

1. $x + y < 18$

2. $x - y > -7$

3. $2x + 3y \leq -5$

4. $-5x - 2y \geq 10$

5 Systems of Linear Inequalities

OBJECTIVES

- Represent constraints in a problem situation with systems of inequalities.
- Write and graph systems of linear inequalities.
- Graph the solutions to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.
- Verify solutions to systems of linear inequalities algebraically.

NEW KEY TERMS

- constraints
- solution of a system of linear inequalities

You have graphed a linear inequality in two variables and interpreted the solutions.

What does the graph of a system of linear inequalities look like, and how can you describe the solution set?

Getting Started

A River Runs Through It

Noah is an experienced whitewater rafter who guides groups of adults and children out on the water for amazing adventures. The raft he uses can hold 800 pounds of weight. Any weight greater than 800 pounds can cause the raft to sink, hit more rocks, and/or maneuver more slowly.

Think About . . .
Does Chase count himself when determining the weight and the cost?

Noah estimates the weight of each adult as approximately 200 pounds and the weight of each child as approximately 100 pounds. Noah charges adults $75 and children $50 to ride down the river with him. His goal is to earn at least $150 each rafting trip.

1. Write an inequality to represent the most weight Noah can carry in terms of rafters. Define your variables.

2. Write an inequality to represent the minimum amount of money Noah wants to collect for each rafting trip.

3. Write a system of linear inequalities to represent the maximum weight of the raft and the minimum amount of money Noah wants to earn per trip.

ACTIVITY
5.1 Determining Solutions to Systems of Linear Inequalities

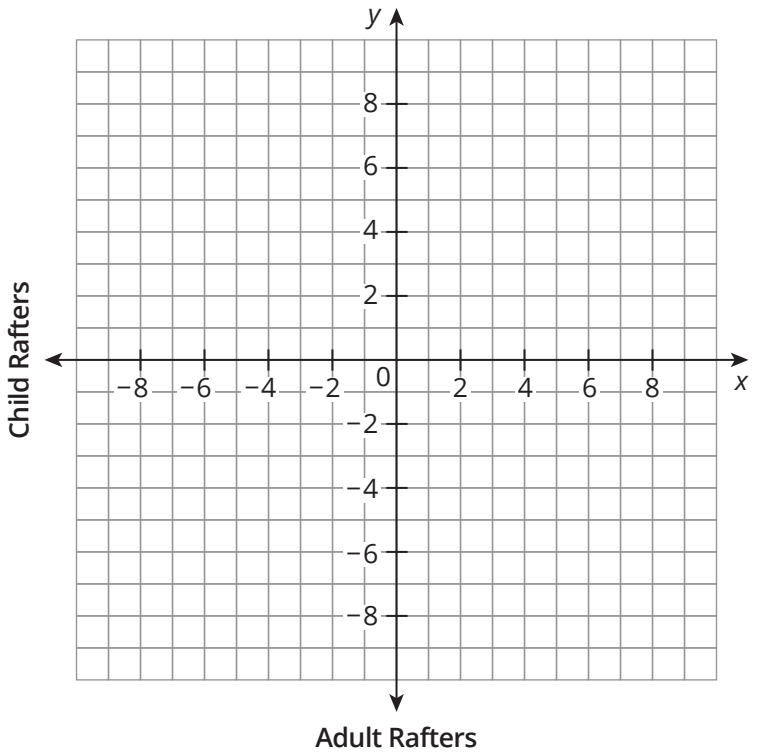
In a system of linear inequalities, the inequalities are known as **constraints** because the values of the expressions are "constrained" to lie within a certain region on the graph.

1. Let's consider two trips that Noah guides. Determine whether each combination of rafters is a solution of the system of linear inequalities. Then, describe the meaning of the solution in terms of this problem situation.

   a. First Trip: Noah guides 2 adults and 2 children.

   b. Second Trip: Noah guides 5 adults.

2. Graph the system of linear inequalities.



Shade the half-plane of each inequality differently. You can use colored pencils or simply vertical and horizontal lines.

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The **solution of a system of linear inequalities** is the intersection of the solutions to each inequality. Every point in the intersection region satisfies all inequalities in the system.

3. Analyze your graph.

   a. Describe the possible number of solutions for a system of linear inequalities.

   b. Is the intersection point a solution to this system of inequalities? Why or why not?

   c. Identify three different solutions of the system of linear inequalities you graphed. What do the solutions represent in terms of the problem situation?

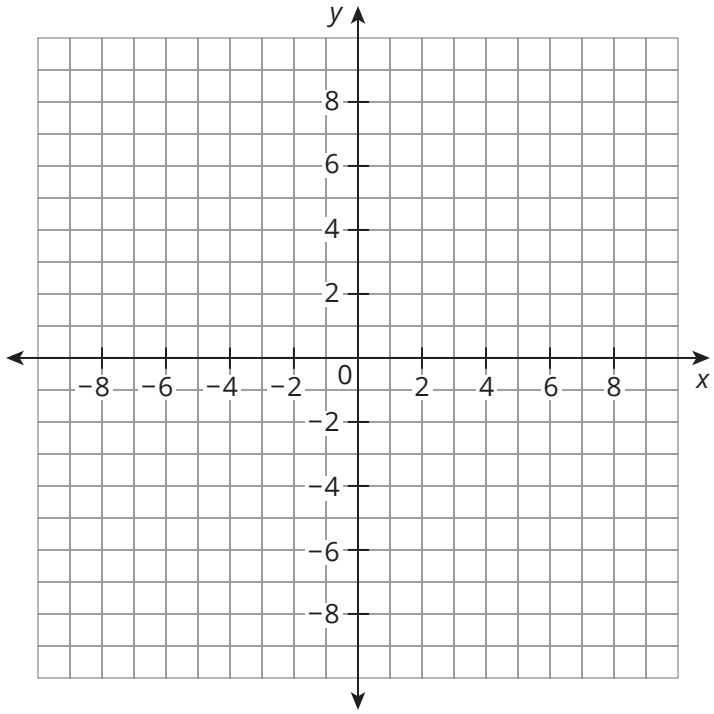
   d. Determine one combination of adults and children that is not a solution for this system of linear inequalities. Explain your reasoning.

ACTIVITY
5.2 Analyzing Graphs of Systems of Linear Inequalities

Determine the solution set of the given system of linear inequalities.

$$\begin{cases} x + y > 1 \\ -x + y \leq 3 \end{cases}$$

1. Graph the system of linear inequalities.



Think About . . .

Notice the inequality symbols. How does this affect your graph?

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2. Choose a point in each shaded region of the graph. Determine whether each point is a solution of the system. Then, describe how the shaded region represents the solution.

Point	$x + y > 1$	$-x + y \leq 3$	Description of location
$(-8, 2)$	$-8 + 2 > 1$ $-6 > 1$ ✗	$-(-8) + 2 \leq 3$ $10 \leq 3$ ✗	The point is not a solution to either inequality and it is located in the region that is not shaded by either inequality.

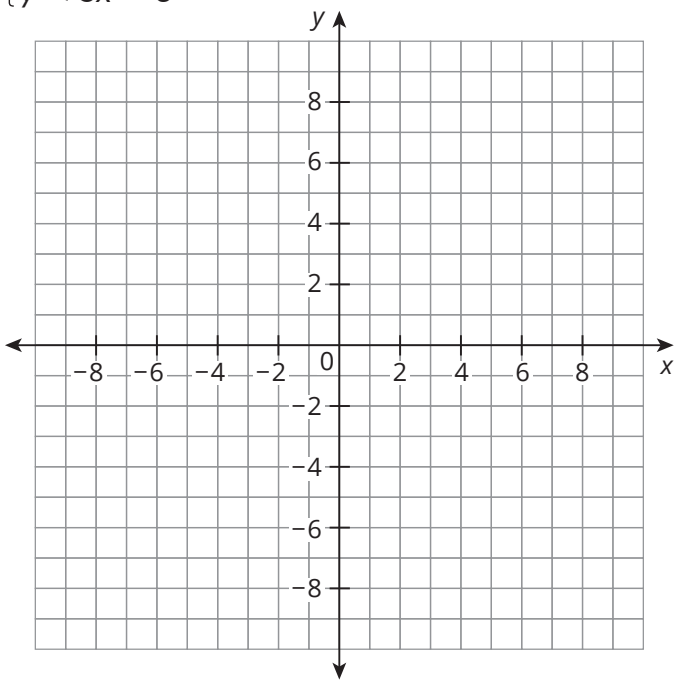
3. Ava makes the statement about the intersection point of a system of inequalities. Explain why Ava's statement is incorrect.

Ava

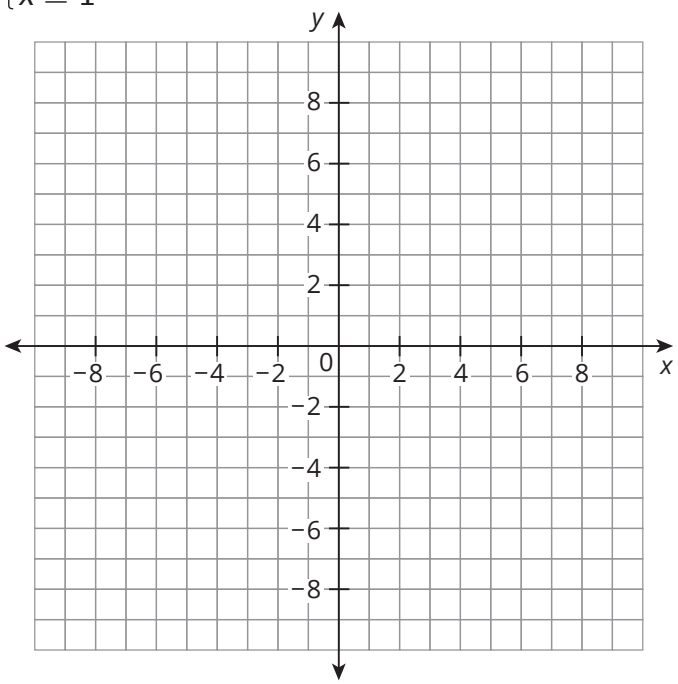
The intersection point is always a solution to a system of inequalities because that is where the two lines meet.

4. Solve each system of linear inequalities by graphing the solution set. Then, identify two points that are solutions of the system.

a. $\begin{cases} y > 5x + 3 \\ y < 5x - 3 \end{cases}$



b. $\begin{cases} x \geq -5 \\ x \geq 1 \end{cases}$



Ask Yourself . . .
Did you share your solution(s) with others?

ACTIVITY
5.3 Applying Systems of Linear Inequalities

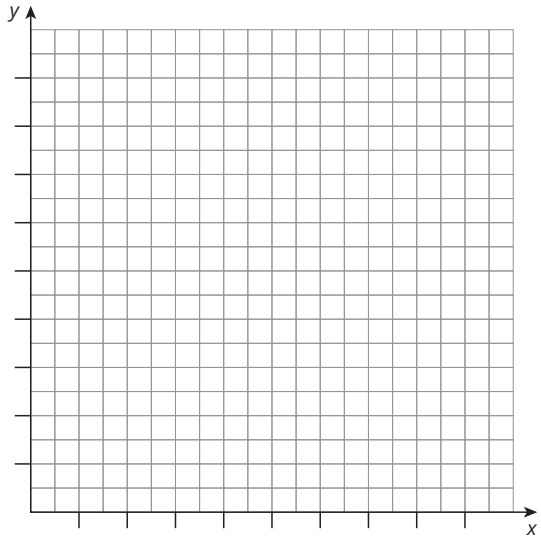
Emma and a group of friends decide to use the fitness room after school. A sign on the wall provides the information shown.

Exercise	Calories Burned per Minute
Treadmill—light effort	7.6
Treadmill—vigorous effort	12.4
Stair Stepper—light effort	6.9
Stair Stepper—vigorous effort	10.4
Stationary Bike—light effort	5.5
Stationary Bike—vigorous effort	11.1

Emma decides to use the stair stepper. She has, at most, 45 minutes to exercise and he wants to burn at least 400 calories.

1. Write a system of linear inequalities to represent Emma's workout. Define your variables.

2. Graph the system of inequalities from Question 1 on the coordinate plane. Be sure to label your axes.



Use technology to graph your inequalities and check your answer.

3. Analyze your graph.

   a. Identify two different solutions of the system of inequalities.

   b. Interpret your solutions in terms of Emma's workout.

   c. Algebraically prove that your solutions satisfy the system of linear inequalities.

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ACTIVITY
5.4 Identifying Systems of Linear Inequalities

Consider the four systems shown.

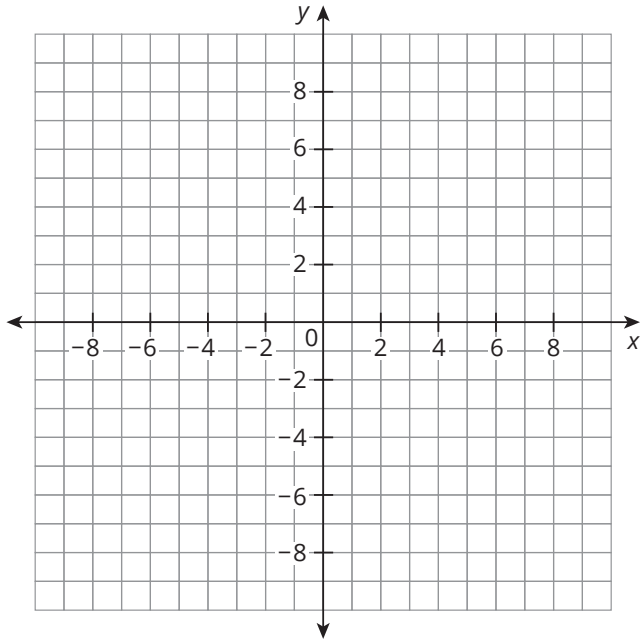
System A	System B	System C	System D
$\begin{cases} y < \frac{3}{5}x + 3 \\ y > -\frac{3}{5}x + 3 \end{cases}$	$\begin{cases} y > \frac{3}{5}x + 3 \\ y > -\frac{3}{5}x + 3 \end{cases}$	$\begin{cases} y > \frac{3}{5}x + 3 \\ y < -\frac{3}{5}x + 3 \end{cases}$	$\begin{cases} y < \frac{3}{5}x + 3 \\ y < -\frac{3}{5}x + 3 \end{cases}$

1. Analyze Isabella's statement and explain why it is incorrect.

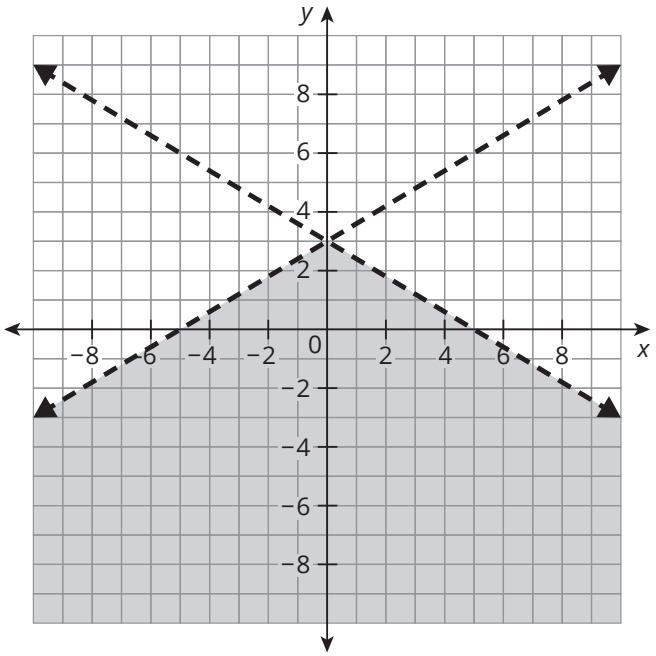
Isabella

Since the equations in each system are the same, the graphs and solutions should all be identical.

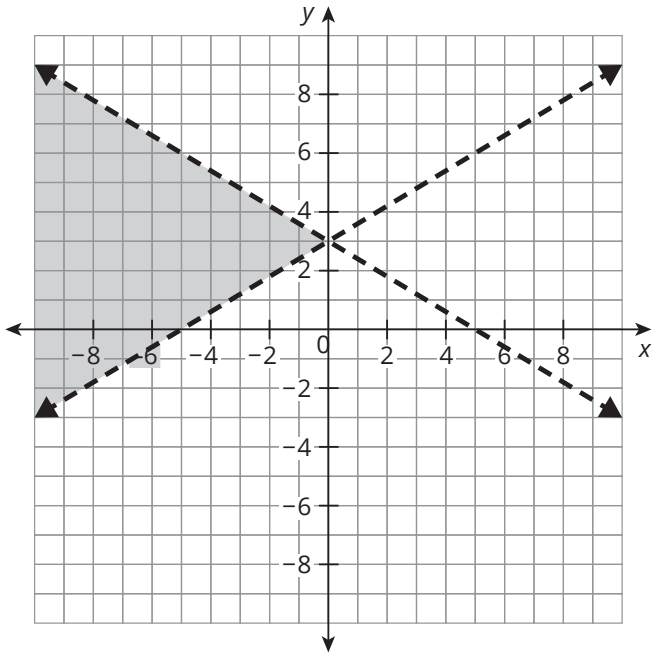
2. Match a graph and possible solution to each given system of linear inequalities. Complete the blank graph and partial solution set to make four complete sets.



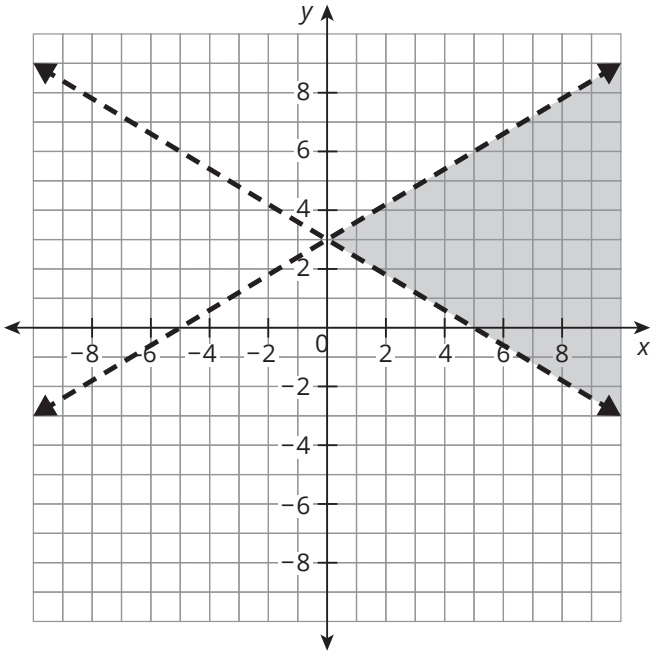
Graph A



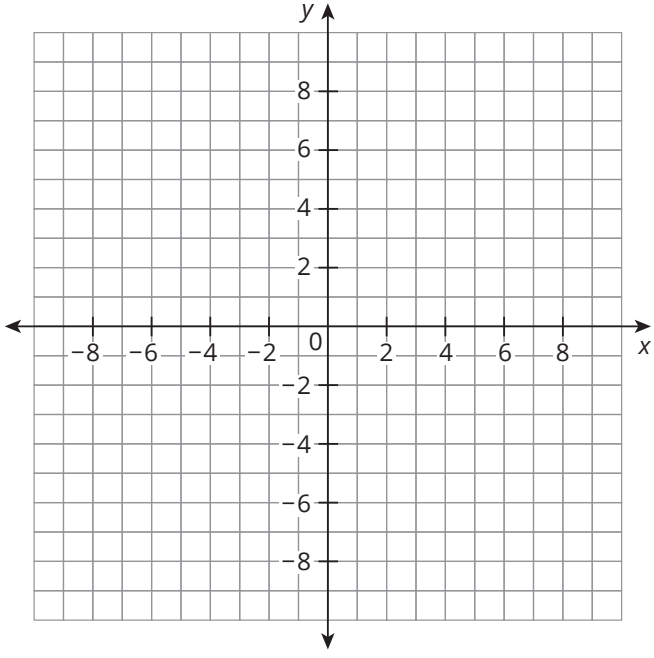
Graph B



Graph C



Graph D



Possible Solutions A	Possible Solutions B	Possible Solutions C	Possible Solutions D
(0, −6) and (4, −4)	(0, 7.5) and (−4, 10)	(−6, 4) and (−10, 8)	_____and_____

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Talk the Talk

Get to Know the Region

The solution set to a system of inequalities can be any of four regions on the coordinate plane.

1. Consider each system of linear inequalities and decide which region represents the solution set. Explain your reasoning.
   - A region above both lines.
   - A region below both lines.
   - A region between the lines.
   - No solution.

   a. $y < 8 + 2x$
      $3 + 2x > y$

   b. $5 + x < y$
      $y > 7 + x$

   c. $y > 12 - 3x$
      $10 - 3x > y$

   d. $6 - x < y$
      $y < 9 - x$

2. How is the solution to a system of linear inequalities the same as or different from the solution to a system of linear equations?

Lesson 5 Assignment

Write

Describe how you know which region, if any, represents the solution to a system of linear inequalities.

Remember

The solution of a system of linear inequalities is the intersection of the solutions to each inequality. Every point in the intersection region satisfies all inequalities in the system.

Practice

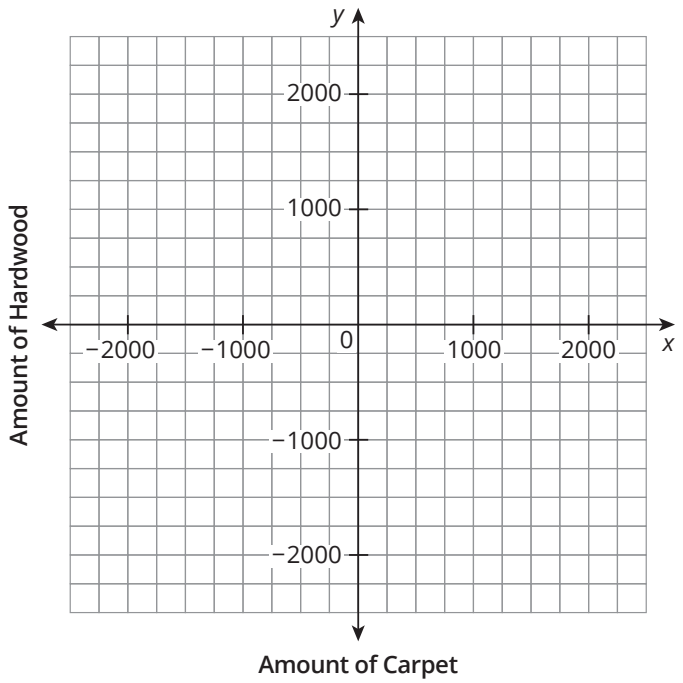
1. Samuel is remodeling his basement. One part of the planning involves the flooring. He knows that he would like both carpet and hardwood, but he isn't sure how much of each he will use. The most amount of flooring area he can cover is 2000 square feet. The carpet is $4.50 per square foot, and the hardwood is $8.25 per square foot. Both prices include labor costs. Samuel has budgeted $10,000 for the flooring.

   a. Write a system of inequalities to represent the maximum amount of flooring needed and the maximum amount of money Samuel wants to spend.

   b. One idea Samuel has is to make two rooms—one having 400 square feet of carpeting and the other having 1200 square feet of hardwood. Determine whether this amount of carpeting and hardwood are solutions to the system of inequalities. Explain your reasoning in terms of the problem situation.

Lesson 5 Assignment

c. Graph this system of inequalities.



d. Determine the intersection point of the two lines. Is this a solution to this system of inequalities in terms of the problem situation?

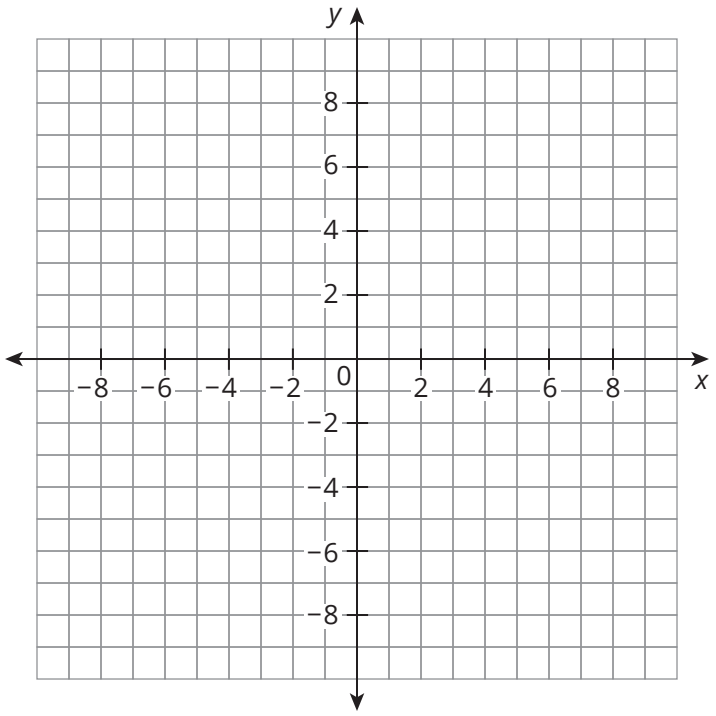
e. Identify two different solutions to the system of inequalities. Explain what the solutions represent in terms of the problem situation.

Lesson 5 Assignment

f. Determine one combination of amounts of carpet and hardwood that is not a solution for the system of inequalities. Explain your reasoning.

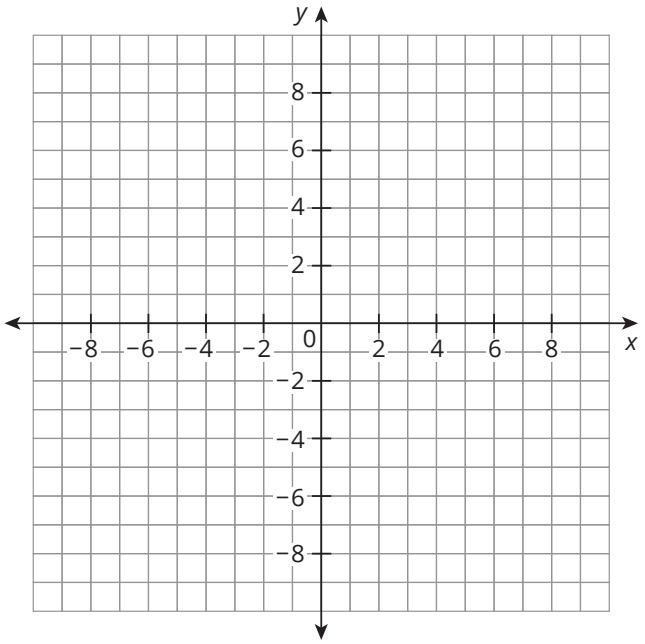
2. Solve each system of linear inequalities.

a. $\begin{cases} -x + 3y \leq -6 \\ -5x + 3y \geq 6 \end{cases}$

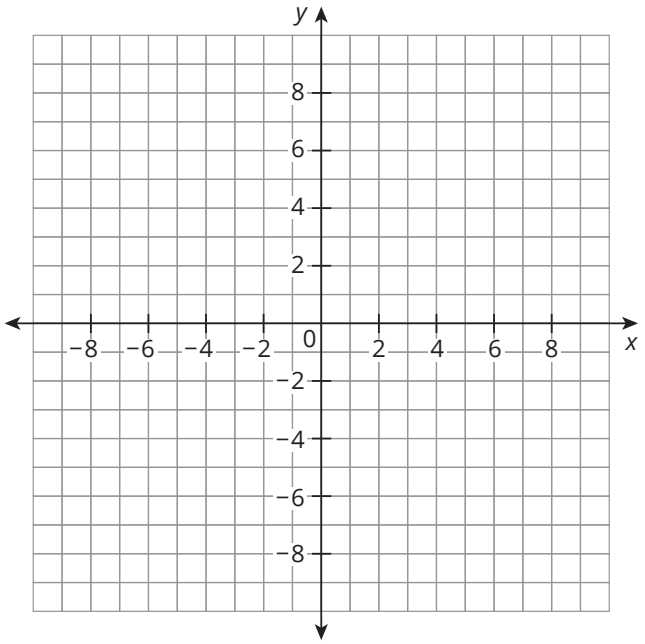


Lesson 5 Assignment

b. $\begin{cases} -x + 2y < 6 \\ 3x + 2y \leq 2 \end{cases}$



c. $\begin{cases} -x + 3y \leq 18 \\ x \leq 3 \end{cases}$



Lesson 5 Assignment

Prepare

Determine whether the point (1, 7) is a solution to each system.

1. $\begin{cases} 4x - y = -3 \\ -2x + y = 5 \end{cases}$

2. $\begin{cases} x + y > 4 \\ 4x - y < -4 \end{cases}$

3. $\begin{cases} y = -3.5x - 2 \\ y = 4.5x - 10 \end{cases}$

4. $\begin{cases} -2x + y < 8 \\ x - y > -8 \end{cases}$

6 Solving Systems of Equations and Inequalities

OBJECTIVES

- Use various methods of solving systems of linear equations to determine the better buy or the better job offer.
- Solve systems of linear inequalities with more than two inequalities.
- Graph the solutions to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

You have solved systems of linear equations by graphing, by using substitution, and by using linear combinations. You have also graphed systems of linear inequalities to determine possible solutions.

How can you use these various methods to reason about real-world problems?

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Getting Started

Systems for Summer Savings

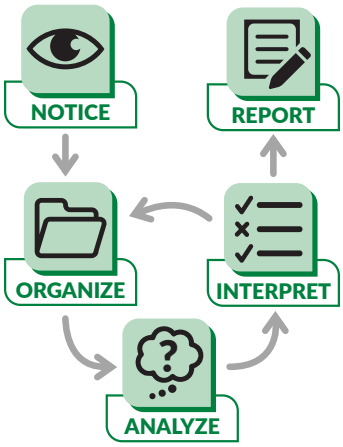
A neighborhood pool club offers two membership plans. Plan A includes a seasonal sign-up fee of $100 and charges $2 each time you use the pool. Plan B has no sign-up fee but charges $6 each time you use the pool. Harper chooses Plan A. She has a budget of $200 to spend on pool fees during the months of July and August.

Harper wants to be sure she uses the pool enough times so that the plan she chooses works out to be the better deal between the two plans, but she does not want to go over her budget.

1. Use a system of linear equations and a system of linear inequalities to make a recommendation to Harper as to how often she should use the pool in July and August.

ACTIVITY
6.1 Determining the Better Deal

PROBLEM SOLVING



A bicycle company is planning to make a low price ultra-light bicycle. There are two different plans being considered for building this bicycle. The first plan includes a cost of $125,000 to design and build a prototype bicycle. The combined materials and labor costs for each bike made under the first plan will be $225. The second plan includes a cost of $100,000 to design and build the prototype. The combined materials and labor costs for each bike made under the second plan will be $275.

1. You recently got a job at the bicycle company as a financial analyst. Analyze the costs for each proposed bicycle prototype and determine which plan they should follow. Provide evidence for your proposal.

Ask Yourself . . .
Can others follow and understand your process and reasoning?

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PROBLEM SOLVING



ACTIVITY 6.2

Determining the Better Buy

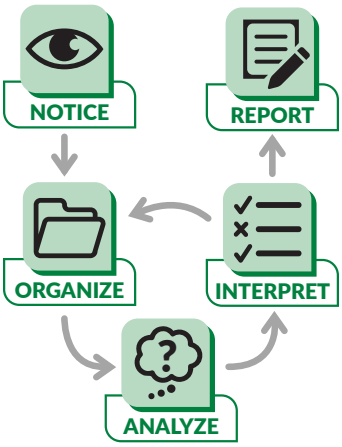
Diego is in search of a new cell phone plan. He is considering two different cell phone services from two different providers.

One plan is for a phone with GPS capabilities that has a monthly fee of $99.99 and 2000 MB of data per month. Once a user exceeds the free monthly data allowance, each additional MB of data used is $0.05. The other plan is for a phone without GPS capabilities for a monthly fee of $79.99 and 1500 MB of data per month. Once a user exceeds the free monthly data allowance, each additional MB of data used is $0.08. Diego is unsure which plan to choose. He wasn't very careful with his last contract and paid a lot of extra money in charges for data.

1. Write an email to advise Diego which plan to choose. Provide evidence in your response.

ACTIVITY
6.3 Solving a System of Linear Inequalities with Four Constraints

PROBLEM SOLVING



Daniel's eye doctor informed him that he needs glasses. Luckily, the local vision store is having a sale on all eyeglass frames. The advertisement in the window is as shown.

Previously, you solved a system containing two linear inequalities. However, systems can consist of more than two linear inequalities.

Save 60% to 75% on All Frames Regularly Priced at $120—$360

1. Use the advertisement to write two inequalities that represent the regular price of eyeglass frames. Let r represent the regular price of the frames.

2. Use the same advertisement to write two inequalities that represent the reduced price of the eyeglass frames. Let s represent the sales price of the frames in terms of r.

Remember . . .
When an item is 20% off the regular price, you can think of that item costing 80% of the regular price.

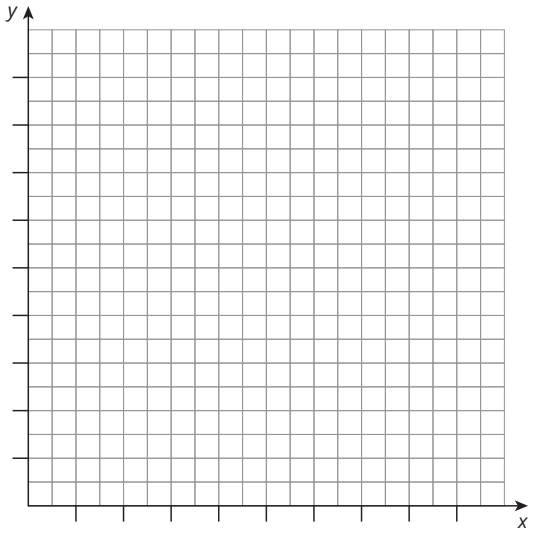
3. Sofia wrote this system of linear inequalities for the problem situation. Explain why it is incorrect.

Sofia

$$\begin{cases} r \geq 120 \\ r \leq 360 \\ s \leq 0.6r \\ s \geq 0.75r \end{cases}$$

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4. Graph each inequality on the grid.



5. Shade the portion of the graph that satisfies the system of linear inequalities. What shape does the solution region resemble?

6. About how much would Daniel expect to spend if he purchases eyeglasses that are regularly priced at $320?

Remember . . .
When graphing a system of linear inequalities, you must determine the portion of the graph that satisfies all the inequalities in the system.

7. Daniel is definitely going to purchase a pair of eyeglasses that are on sale. What is the least amount of money Daniel can expect to spend? What is the greatest amount he can expect to spend?

8. Daniel decides on a pair of eyeglasses that are regularly priced at $240.

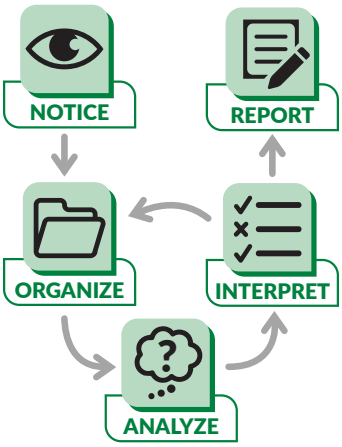
   a. Can Daniel expect to save more or less than $140 off the purchase price of this pair of eyeglasses? Use your graph to determine an approximate answer.

The graph shows you the sale price of the eyeglasses, but how can you determine how much he will save?

   b. Use algebra to determine the greatest amount of money Daniel can save by purchasing eyeglasses that are regularly priced at $240.

ACTIVITY
6.4 Determining the Better Job Offer

PROBLEM SOLVING



Juan interviewed for two different sales positions at competing companies. Company A has offered Juan a salary of $31,200 per year, plus a 9% commission on his total sales. Company B will offer him $26,000 per year, plus a 15% commission on his total sales.

Juan isn't sure which offer to accept. He's great at making a sale, but he's just not sure which job will be better in terms of his pay. He is confident that he can make at least $2000 worth of sales each week.

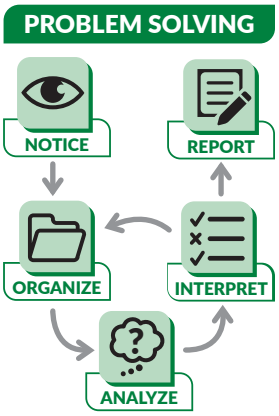
1. Write an email to Juan with your recommendation of which job offers better compensation. Provide evidence in your response.

Ask Yourself . . .
What tools or strategies can you use to solve this problem?

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Talk the Talk

Which Cab Is More Fab?



Cab Company A charges $3.50 upon entry and an additional $1.25 per mile driven. Cab Company B charges $5 upon entry and an additional $1.15 per mile driven.

1. Emma needs to take a cab to the airport. Which company should she use if she wants to minimize the cost? Use any method to solve.

Lesson 6 Assignment

Write

You have used three methods to solve systems: graphing, substitution, and linear combinations. Describe the characteristics you would look for when determining which method to use.

Remember

The solution set to a system of inequalities with more than two constraints can be described as the region where all the graphs overlap.

Practice

1. Jayden wants to subscribe to a service that will allow him to rent DVDs and stream movies online. Company A offers a subscription for $14.25 a month. With this subscription, Antonio can check out as many DVDs as he wants each month and must pay $1.40 for each movie he streams online. Company B offers a subscription for $8.50 a month. With this subscription, Jayden can checkout as many DVDs as he wants each month and must pay $3.25 for each movie he streams online.

   a. Write a system of linear equations to represent this problem situation.

   b. Analyze the two subscription plans and determine which one is the better deal. Use any or all of the methods you have learned to determine your answer.

   c. Write a short paragraph recommending which subscription Antonio should choose.

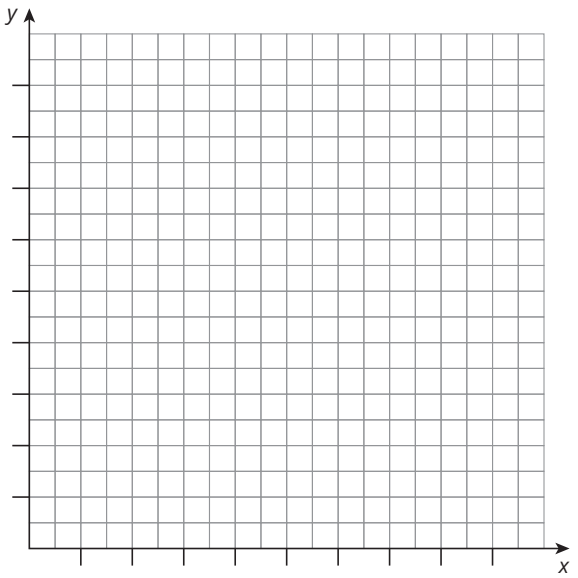
Lesson 6 Assignment

d. Which method do you think provides the quickest way to analyze a system of equations to determine which one is the better deal? Explain your reasoning.

2. A ballet company needs to rent a venue for their production. There are a number of arenas they are considering. The arenas have seating capacities that range from 800 to 1776 seats. The management of the company knows the ticket sales may not be good this year, but their goal is to sell between 65% and 90% of the available seats. Whichever arena they choose, 100 seats must be set aside for the company's donors.

   a. Write a system of inequalities that represents the problem situation. Define your variables.

   b. Graph each inequality on a coordinate plane.



Lesson 6 Assignment

c. One of the arenas they are considering has 1200 available seats. Determine the minimum and maximum number of seats they would need to sell for management to reach their goal.

d. If the company sells 900 seats, what is the range of seating capacities for the arenas they may rent?

e. If they rent an arena that has a 1300-seat capacity and sell 800 tickets, would management reach their goal? Explain your reasoning.

Prepare

Simplify each expression.

1. $(-10)(-10)(-10)$
2. $(-10)(-10)(-10)(-10)$
3. $(-1)(2)(-3)(4)(-5)$
4. $(-2)(-3)(-4)(-5)$

TOPIC 2 SELF-REFLECTION

Name: _____

Systems of Linear Equations and Inequalities

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me**, 2 represents **I am building proficiency of the skill**, and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Systems of Linear Equations and Inequalities* topic by:

TOPIC 2: *Systems of Linear Equations and Inequalities*	Beginning of Topic	Middle of Topic	End of Topic
understanding that the graph of an equation in two variables is the set of all its solutions plotted on the coordinate plane.	○	○	○
writing an equation, an inequality, or a system of linear equations or inequalities that best models a problem.	○	○	○
graphing systems of equations on a coordinate grid with appropriate labels and scales.	○	○	○
solving a system of two linear equations in two variables approximately (graphically) and exactly (algebraically).	○	○	○
interpreting the meaning of the solution to a system of linear equations in terms of a problem situation.	○	○	○
identifying linear systems that have no solution and explain why they have no solution.	○	○	○
identifying linear systems that have infinite solutions and explain why they have infinite solutions.	○	○	○

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TOPIC 2 SELF-REFLECTION *continued*

TOPIC 2: *Systems of Linear Equations and Inequalities*	Beginning of Topic	Middle of Topic	End of Topic
writing and graphing an inequality and a system of inequalities in two variables on a coordinate plane.	○	○	○
understanding that the solution of a system of linear inequalities is the intersection of the shaded regions of both inequalities.	○	○	○
interpreting the solution to an inequality in two variables graphed on a coordinate plane.	○	○	○

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Systems of Linear Equations and Inequalities* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

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TOPIC 2 SELF-REFLECTION *continued*

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

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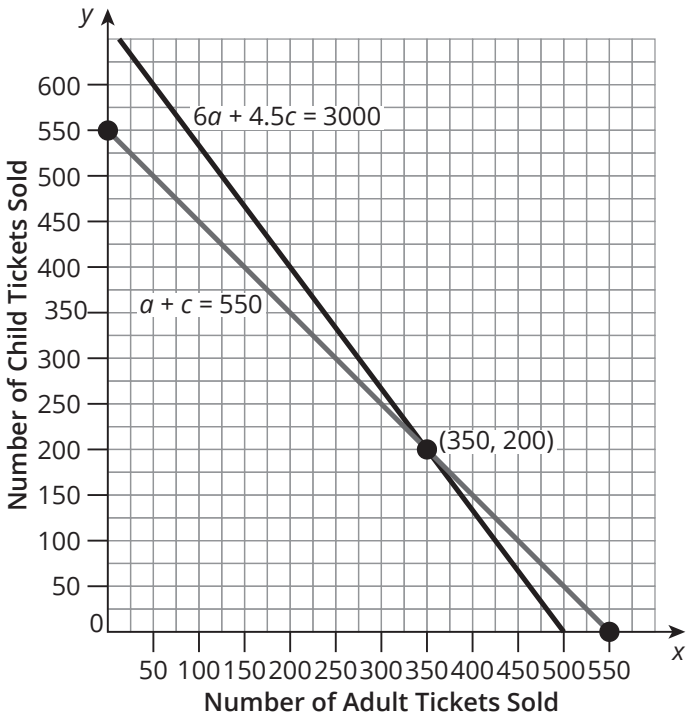
Systems of Equations and Inequalities Summary

LESSON 1 Using Graphing to Solve Systems of Equations

When two or more linear equations define a relationship between quantities, they form a **system of linear equations**. The solution of a linear system is an ordered pair (x, y) that is a solution to both equations in the system. One way to predict the solution to a system of equations is to graph both equations and identify the point at which the two graphs intersect.

For example, suppose a fundraiser sells 550 tickets for a spaghetti dinner to raise $3000 for charity. An adult ticket costs $6, and a child ticket costs $4.50. To determine how many adult tickets and child tickets were sold, let a represent the number of adult tickets purchased, and let c represent the number of child tickets purchased.

$$\begin{cases} a + c = 550 \\ 6a + 4.5c = 3000 \end{cases}$$



NEW KEY TERMS

- system of linear equations [sistema de ecuaciones lineales]
- consistent systems [sistemas consistentes]
- inconsistent systems [sistemas inconsistentes]
- standard form of a linear equation [forma estándar/genera de una ecuación lineal]
- substitution method [método de sustitución]
- linear combinations method [método de combinaciones lineales]
- half-plane
- boundary line
- constraints
- solution of a system of linear inequalities [solución de un sistema de desigualdades lineales]

You can use x- and y-intercepts to graph each of the two equations to determine how many of each type of tickets were sold.

The intersection point appears to be (350, 200). There were 350 adult tickets and 200 child tickets sold.

A system of equations may have one unique solution, no solution, or infinite solutions. Systems that have one or many solutions are called **consistent systems**. Systems with no solution are called **inconsistent systems**.

LESSON 2 Using Substitution to Solve Linear Systems

The **standard form of a linear equation** can be written as $Ax + By = C$, where A, B, and C are integers, and A and B are not both zero.

Sometimes, a system of equations may not be accurately solved using graphs. There is an algebraic method that can be used called the *substitution method*. The **substitution method** is a process of solving a system of equations by substituting a variable in one equation with an equivalent expression.

For example, consider this system of equations.

$$\begin{cases} 1.25x + 1.05y = 30 \\ y = 8x \end{cases}$$

Step 1: To use the substitution method, begin by choosing one equation and isolating the variable. This will be considered the first equation. Because $y = 8x$ is in slope-intercept form, use this as the first equation.

Step 2: Now, substitute the expression equal to the isolated variable into the second equation. Substitute $8x$ for y in the equation $1.25x + 1.05y = 30$. Write the new equation: $1.25x + 1.05(8x) = 30$.

Step 3: Solve the new equation.

$$\begin{aligned} 1.25x + 8.40x &= 30 \\ 9.65x &= 30 \\ x &\approx 3.1 \end{aligned}$$

Now, substitute the value for x into $y = 8x$ to determine the value for y.

$y \approx 8(3.1) \approx 24.8$

Step 4: Check your solution by substituting the values for both variables into the original system to show that they make both equations true.

When a system has no solution, the equation resulting from the substitution step has no solution. When a system has infinite solutions, the equation resulting from the substitution step has infinite solutions.

LESSON 3 Using Linear Combinations to Solve a System of Linear Equations

The **linear combinations method** is a process used to solve a system of equations by adding two equations so they result in an equation with one variable. Once the value of this variable is determined, it can be used to calculate the value of the other variable. In many cases, one or both of the equations may need to be multiplied by a constant so that the coefficients of the term containing either x or y are opposites. Then, when the equations are added, the result is an equation in one variable.

For example, consider this system of linear equations.

$$\begin{cases} 7x + 2y = 24 \\ x + y = 15 \end{cases}$$

$$\begin{aligned} 7x + 2y &= 24 \\ -2(4x + y) &= -2(15) \end{aligned}$$

Multiply the bottom equation by a constant that results in opposite coefficients for one of the variables.

$$\begin{aligned} 7x + 2y &= 24 \\ +\ -8x - 2y &= -30 \\ \hline -x &= -6 \\ x &= 6 \end{aligned}$$

Now that the y-values are opposite, you can add the equations and solve this linear system.

$$\begin{aligned} 7(6) + 2y &= 24 \\ 42 + 2y &= 24 \\ -2y &= -18 \\ y &= -9 \end{aligned}$$

Substitute the value for x into one of the equations to calculate the value for y.

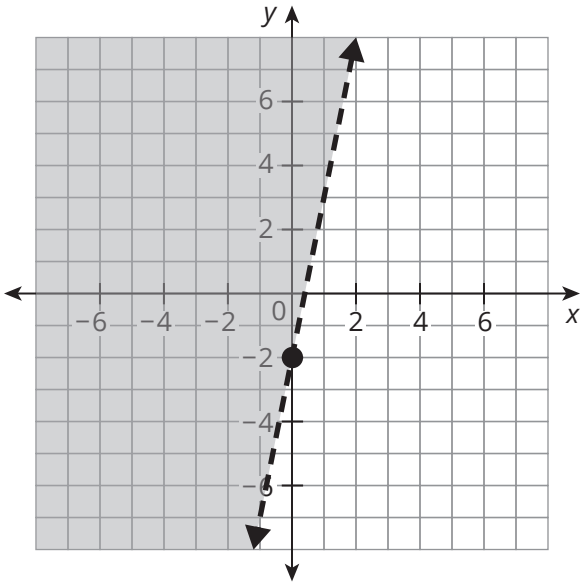
The solution to the system of linear equation is $(6, -9)$.

LESSON 4 Graphing Inequalities in Two Variables

The graph of a linear inequality in two variables is a **half-plane**, or half of a coordinate plane. A **boundary line**, determined by the inequality, divides the plane into two half-planes. The inequality symbol identifies which half-plane contains the solutions. When the symbol is $\leq$ or $\geq$, the boundary line is part of the solution and is solid. When the symbol is $<$ or $>$, the boundary line is not part of the solution and is represented with a dashed line. Use (0, 0) as a test point to determine which half of the plane is the solution of the inequality and should, therefore, be shaded.

For example, the graph shows the solution to the inequality $y < 5x - 2$. Since the inequality symbol in the solution is $<$, the shaded half-plane does not include points on the line. Since (0, 0) is not a solution, then the region to the right of the dashed line is the solution set.

$$y < 5x - 2$$
$$0 \stackrel{?}{<} 5(0) - 2$$
$$0 \stackrel{?}{<} -2$$



LESSON 5 Systems of Linear Inequalities

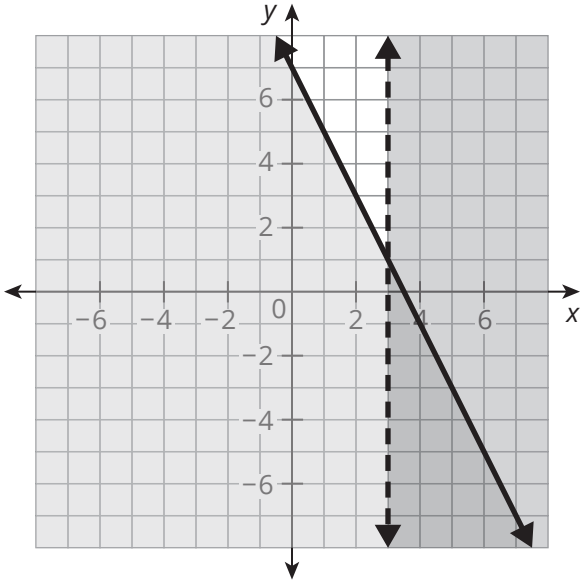
In a system of linear inequalities, the inequalities are known as **constraints** because the values of the expressions are "constrained" to lie within a certain region.

The **solution of a system of linear inequalities** is the intersection of the solutions of each inequality. Every point in the intersection region satisfies the system. To determine the solution set of the system, graph each inequality on the same coordinate plane. The region that overlaps is the solution to the system.

For example, the overlapping region of the graphs of the inequalities $2x + y \leq 7$ and $x > 3$ is the solution to the system.

$$\begin{cases} 2x + y \leq 7 \\ x > 3 \end{cases}$$

Two points that are solutions of the system are (4, −5) and (5, −8).



LESSON 6

Solving Systems of Equations and Inequalities

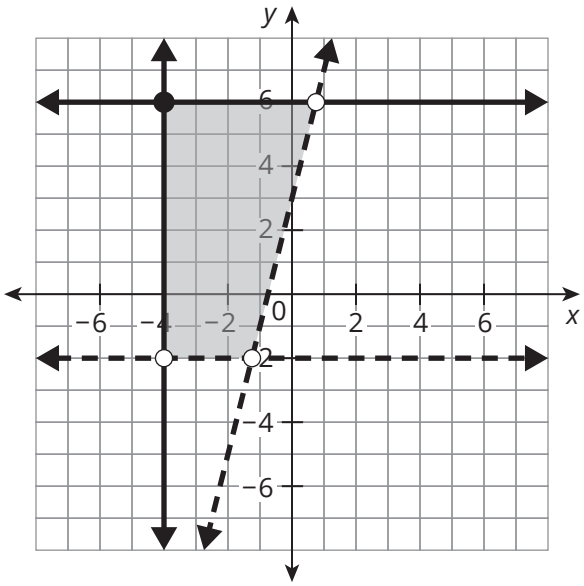
Substitution, linear combinations, and graphing are three methods for determining the solution of a linear system of equations. Use substitution when a variable in one equation can easily be isolated or when the equations are in slope-intercept form. Use the linear combinations method when the coefficients of like terms are opposites or can be easily made into opposites by multiplication. Use the graphing method when the numbers are convenient to graph or when an exact solution is not needed.

To solve a system of linear inequalities with more than two inequalities, graph each linear inequality and shade the correct region that contains the solution sets that satisfy all the inequalities in the system. Also, determine all the points of intersection for the boundary lines that make the vertices of the solution region. Use a closed point when the point is part of the solution region, and use an open dot when the point is not part of the solution region but is a point of intersection.

The graphs show the solution to a system of four inequalities.

$$\begin{cases} x \geq -4 \\ y > -2 \\ y \leq 6 \\ y > 4x + 3 \end{cases}$$

One solution of the system of inequalities is $(-2, 2)$.



Math Glossary

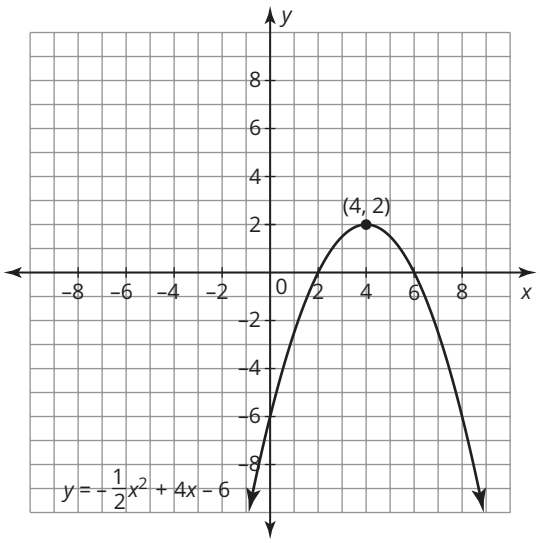
A

absolute maximum

A function has an absolute maximum if there is a point that has a y-coordinate that is greater than the y-coordinates of every other point on the graph.

Example

The ordered pair (4, 2) is the absolute maximum of the graph of the function $f(x) = -\frac{1}{2}x^2 + 4x - 6$.

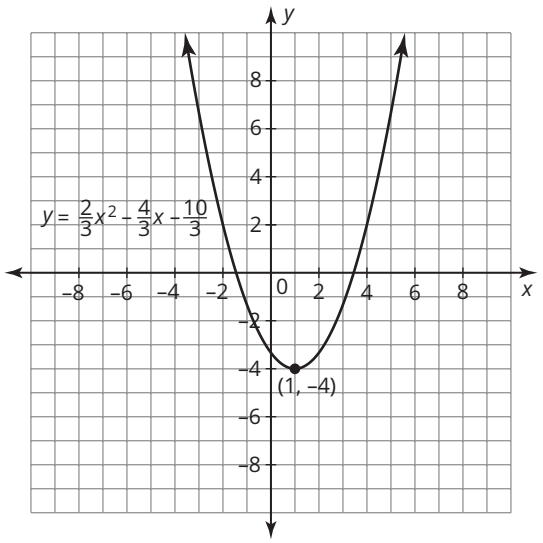


absolute minimum

A function has an absolute minimum if there is a point that has a y-coordinate that is less than the y-coordinates of every other point on the graph.

Example

The ordered pair (1, −4) is the absolute minimum of the graph of the function $y = \frac{2}{3}x^2 - \frac{4}{3}x - \frac{10}{3}$.



argument of a function

The argument of a function is the variable on which the function operates.

Example

In the function $f(x + 5) = 32$, the argument is $x + 5$.

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arithmetic sequence

An arithmetic sequence is a sequence of numbers in which the difference between any two consecutive terms is a constant.

Example

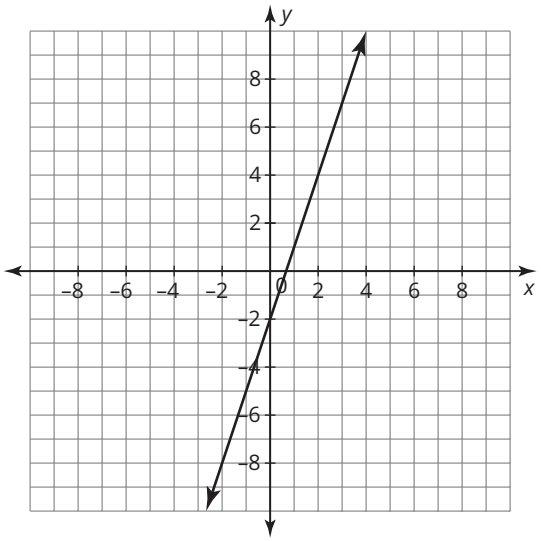
The sequence 1, 3, 5, 7 is an arithmetic sequence with a common difference of 2.

average rate of change

Another name for the slope of a linear function is average rate of change. The formula for the average rate of change is $\frac{f(t) - f(s)}{t - s}$.

Example

The average rate of change of the function shown is 3.

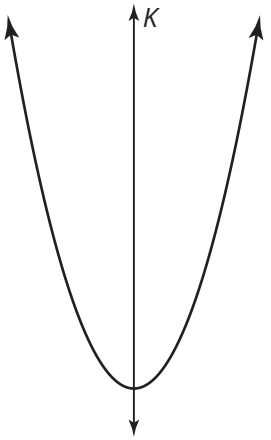


axis of symmetry

The axis of symmetry of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images.

Example

Line *K* is the axis of symmetry of this parabola.



B

base

The base of a power is the expression that is used as a factor in the repeated multiplication.

binomial

Polynomials with exactly two terms are binomials.

Example

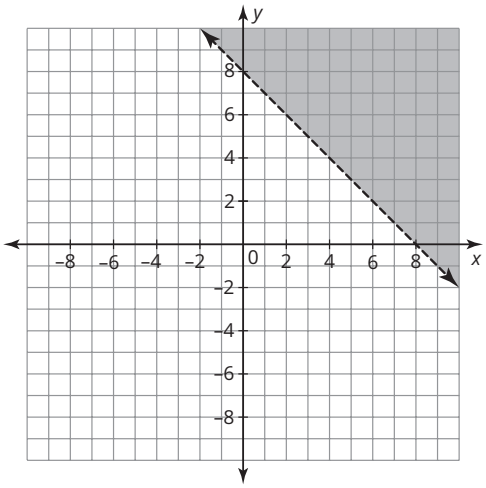
The polynomial $3x + 5$ is a binomial.

boundary line

A boundary line, determined by the inequality in a linear inequality, divides the plane into two half-planes and the inequality symbol indicates which half-plane contains all the solutions.

Example

For the linear inequality $y > -x + 8$, the boundary line is a dashed line because no point on that line is a solution.



C

causation

Causation is when one event affects the outcome of a second event.

centroid

The centroid is a point in which *x*-value is the mean of all the *x*-values of the points on the scatterplot and its *y*-value is the mean of all the *y*-values of the points on the scatterplot.

Example

For the data points (1, 3), (1, 7), (2, 6), (3, 5), and (3, 4), the centroid is (2, 5).

closed (closure)

When an operation is performed on any of the numbers in a set and the result is a number that is also in the same set, the set is said to be closed (or to have closure) under that operation.

Example

The set of whole numbers is closed under addition. The sum of any two whole numbers is always another whole number.

coefficient of determination

The coefficient of determination measures how well the graph of a regression fits the data. It is calculated by squaring the correlation coefficient and represents the percentage of variation of the observed values of the data points from their predicted values.

Example

The correlation coefficient for a data set is −0.9935. The coefficient of determination for the same data set is approximately 0.987, which means 98.7% of the data values should fall on the graph.

common difference

The difference between any two consecutive terms in an arithmetic sequence is called the common difference. It is typically represented by the variable *d*.

Example

The sequence 1, 3, 5, 7 is an arithmetic sequence with a common difference of 2.

common ratio

The ratio between any two consecutive terms in a geometric sequence is called the common ratio. It is typically represented by the variable *r*.

Example

The sequence 2, 4, 8, 16 is a geometric sequence with a common ratio of 2.

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common response

A common response is when a variable other than the ones measured cause the same result as the one observed in the experiment.

completing the square

Completing the square is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

compound interest

In a compound interest account, the balance is multiplied by the same amount at each interval.

Example

Sonya opens a savings account with $100. She earns $4 in compound interest the first year. The compound interest y is found by using the equation $y = 100(1 + 0.04)^t$, where t is the time in years.

concave down

A graph that opens downward is identified as being concave down.

concave up

A graph that opens upward is identified as being concave up.

confounding variable

A confounding variable is when there are other variables in an experiment that are unknown or unobserved.

conjecture

A conjecture is a mathematical statement that appears to be true but has not been formally proven.

consistent systems

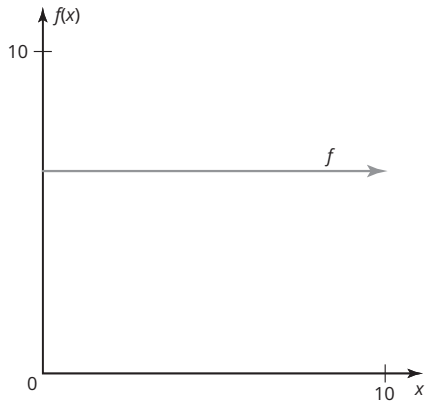
Systems that have one or many solutions are called consistent systems.

constant function

If the dependent variable of a function does not change or remains constant over the entire domain, then the function is called a constant function.

Example

The function shown is a constant function.



constraints

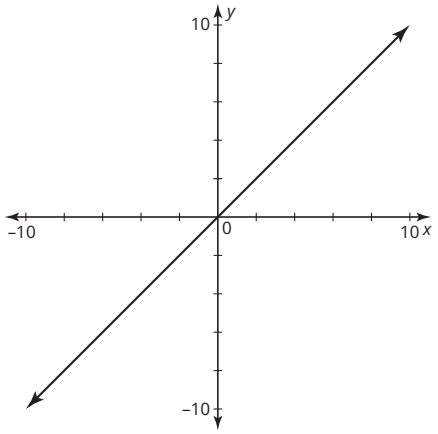
In a system of linear inequalities, the inequalities are known as constraints because the values of the expressions are "constrained" to lie within a certain region on the graph.

continuous graph

A continuous graph is a graph of points that are connected by a line or smooth curve on the graph. Continuous graphs have no breaks.

Example

The graph shown is a continuous graph.



correlation

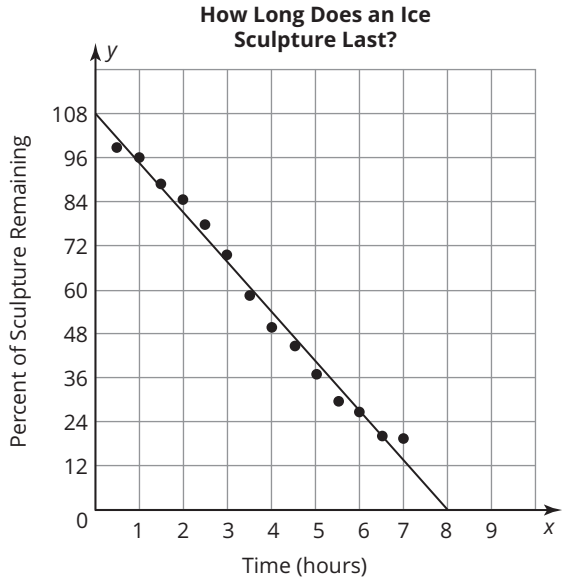
A measure of how well a regression fits a set of data is called a correlation.

correlation coefficient

The correlation coefficient is a value between −1 and 1, which indicates how close the data are to the graph of the regression function. The closer the correlation coefficient is to −1 or 1, the stronger the relationship is between the two variables. The variable r is used to represent the correlation coefficient.

Example

The correlation coefficient for these data is −0.9935. The value is negative because the equation has a negative slope. The value is close to −1 because the data are very close to the graph of the equation of the line.



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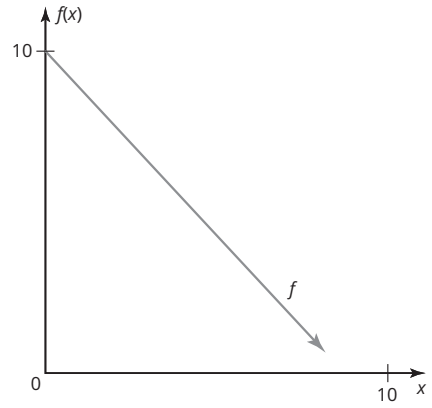
D

decreasing function

If a function decreases across the entire domain, then the function is called a decreasing function.

Example

The function shown is a decreasing function.



degree

The degree of a polynomial is the greatest variable exponent in the expression.

degree of a polynomial

The greatest exponent for any variable term in a polynomial determines the degree of the polynomial.

Example

The polynomial $2x^3 + 5x^2 - 6x + 1$ has a degree of 3.

dependent quantity

When one quantity is determined by another in a problem situation, it is said to be the dependent quantity.

Example

In the relationship between driving time and distance traveled, distance is the dependent quantity, because distance depends on the driving time.

difference of two squares

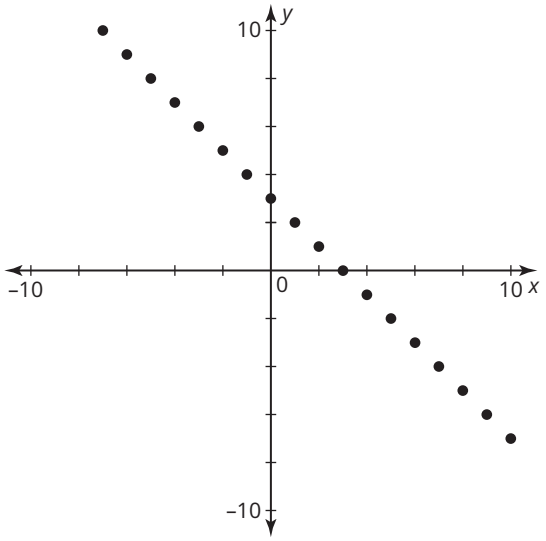
The difference of two squares is an expression in the form $a^2 - b^2$ that can be factored as $(a + b)(a - b)$.

discrete graph

A discrete graph is a graph of isolated points.

Example

The graph shown is a discrete graph.



discriminant

The discriminant is the radicand expression in the quadratic formula which "discriminates" the number of real roots of a quadratic equation.

Example

The discriminant in the quadratic formula is the expression $b^2 - 4ac$.

domain

The domain is the set of input values in a relation.

Example

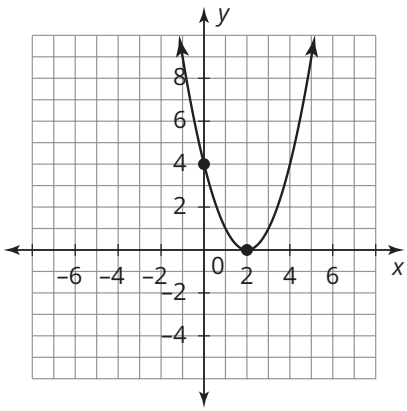
The domain of the function $y = 2x$ is the set of all real numbers.

double root

The root of an equation indicates where the graph of the equation crosses the x-axis.
A double root occurs when the graph just touches the x-axis but does not cross it.

Example

The quadratic equation $y = (x - 2)^2$ has a double root at $x = 2$.



E

explicit formula

An explicit formula of a sequence is a formula for calculating the value of each term of a sequence using the term's position in the sequence. The explicit formula for an arithmetic sequence is $a_n = a_1 + d(n - 1)$. The explicit formula for a geometric sequence is $g_n = g_1 \cdot r^{n-1}$.

Example

The sequence 1, 3, 5, 7, 9, . . . can be described by the rule $a_n = 2n - 1$ where n is the position of the term. The fourth term of the sequence a_4 is $2(4) - 1$, or 7.

exponent

The exponent of a power is the number of times that the base is used as a factor in the repeated multiplication.

exponential decay function

An exponential decay function is an exponential function with a b-value greater than 0 and less than 1 and is of the form $y = a(1 - r)^x$, where r is the rate of decay.

Example

Greenville has a population of 7000. Its population is decreasing at a rate of 1.75%. The exponential decay function that models this situation is $f(x) = 7000 \cdot 0.9825^x$.

exponential functions

The family of exponential functions includes functions of the form $f(x) = ab^x$, where a and b are real numbers, and b is greater than 0 but is not equal to 1.

Example

The function $f(x) = 2^x$ is an exponential function.

exponential growth function

An exponential growth function is an exponential function with a b-value greater than 1 and is of the form $y = a(1 + r)^x$, where r is the rate of growth.

Example

Blueville has a population of 7000. Its population is increasing at a rate of 1.4%. The exponential growth function that models this situation is $f(x) = 7000 \cdot 1.014^x$.

extract the square root

To extract a square root, solve an equation of the form $a^2 = b$ for a.

extrapolation

To make predictions for values of x that are outside of the data set is called extrapolation.

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F

Factor theorem

The Factor theorem states that a polynomial function $p(x)$ has $x - r$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$.

factored form

A quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.

Example

The function $h(x) = x^2 - 8x + 12$ written in factored form is $(x - 6)(x - 2)$.

finite sequence

If a sequence terminates, it is called a finite sequence.

Example

The sequence 22, 26, 30 is a finite sequence.

first differences

First differences are the values determined by subtracting consecutive output values in a table when the input values have an interval of 1.

Example

	Time (minutes)	Height (feet)	First Differences
	0	0	
$1 - 0 = 1$	1	1800	$1800 - 0 = 1800$
$2 - 1 = 1$	2	3600	$3600 - 1800 = 1800$
$3 - 2 = 1$	3	5400	$5400 - 3600 = 1800$

function

A function is a relation that assigns to each element of the domain exactly one element of the range.

Example

The equation $y = 2x$ is a function. Every value of x has exactly one corresponding y-value.

function notation

Function notation is a way of representing functions algebraically.

Example

In the function $f(x) = 0.75x$, f is the name of the function, x represents the domain, and $f(x)$ represents the range.

function family

A function family is a group of functions that share certain characteristics.

Example

Linear functions and exponential functions are examples of function families.

G

geometric sequence

A geometric sequence is a sequence of numbers in which the ratio between any two consecutive terms is a constant.

Example

The sequence 2, 4, 8, 16 is a geometric sequence with a common ratio of 2.

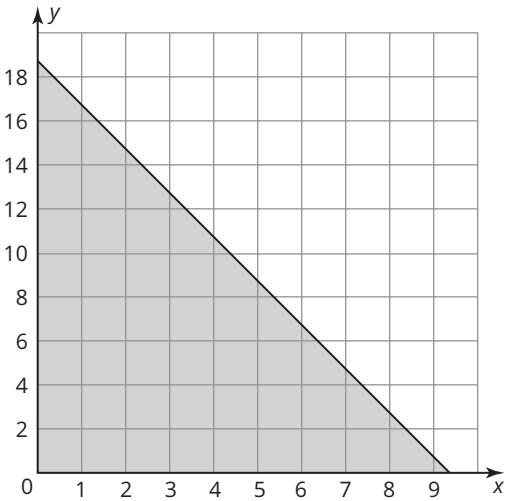
H

half-plane

The graph of a linear inequality is a half-plane, or half of a coordinate plane.

Example

The shaded portion of the graph is a half-plane.

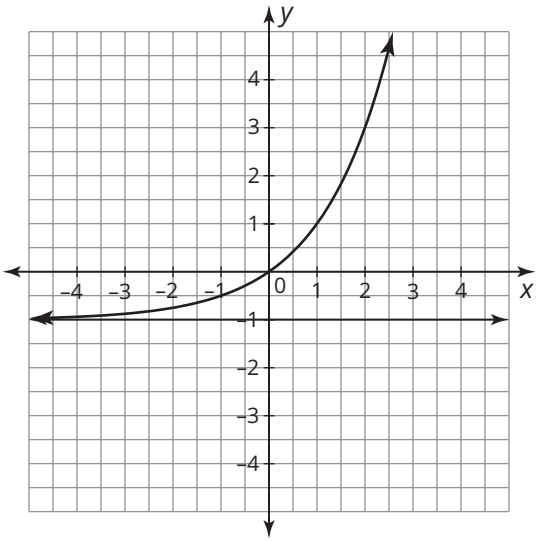


horizontal asymptote

A horizontal asymptote is a horizontal line that a function gets closer and closer to but never intersects.

Example

The graph shows a horizontal asymptote at $y = -1$.



I

inconsistent systems

Systems with no solution are called inconsistent systems.

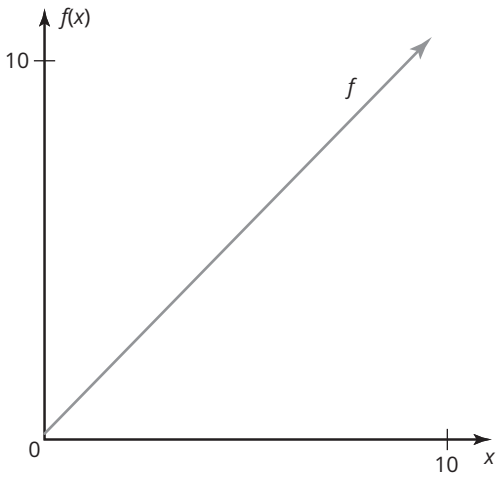
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increasing function

If a function increases across the entire domain, then the function is called an increasing function.

Example

The function shown is an increasing function.



independent quantity

The quantity that the dependent quantity depends upon is called the independent quantity.

Example

In the relationship between driving time and distance traveled, driving time is the independent quantity, because it does not depend on any other quantity.

infinite sequence

If a sequence continues on forever, it is called an infinite sequence.

Example

The sequence 22, 26, 30, 34 . . . is an infinite sequence.

infinite solutions

An equation with infinite solutions means that any value for the variable makes the equation true.

Example

The equation $2x + 1 = 2x + 1$ has infinite solutions.

interpolation

Using a linear regression to make predictions within the data set is called interpolation.

L

leading coefficient

The leading coefficient of a polynomial is the numeric coefficient of the term with the greatest power.

Example

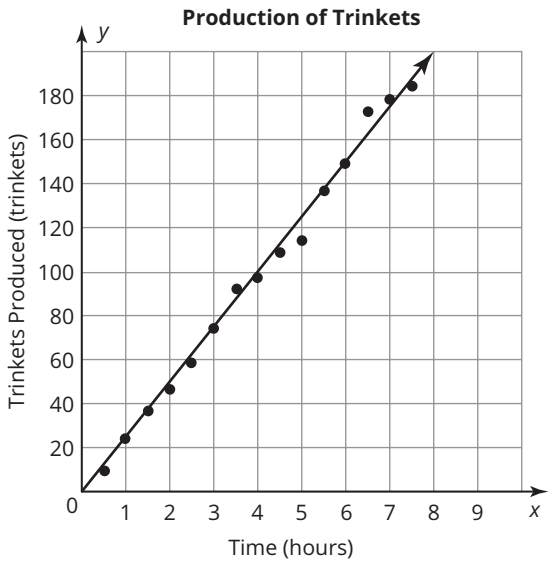
In the polynomial $-7x^2 + x + 25$, the value -7 is the leading coefficient.

Least Squares Method

The Least Squares Method is a method that creates a regression line for a scatterplot that has two basic requirements: 1) the line must contain the centroid of the data set, and 2) the sum of the squares of the vertical distances from each given data point is at a minimum with the line.

Example

The regression line shown was created using the Least Squares Method.



linear combinations method

The linear combinations method is a process used to solve a system of equations by adding two equations together, resulting in an equation with one variable.

Example

Solve the following system of equations by using the linear combinations method:

$$\begin{cases} 6x - 5y = 3 \\ 2x + 2y = 12 \end{cases}$$

First, multiply the second equation by -3. Then, add the equations and solve for the remaining variable. Finally, substitute $y = 3$ into the first equation and solve for x. The solution of the system is (3, 3).

linear functions

The family of linear functions includes functions of the form $f(x) = ax + b$, where a and b are real numbers.

Example

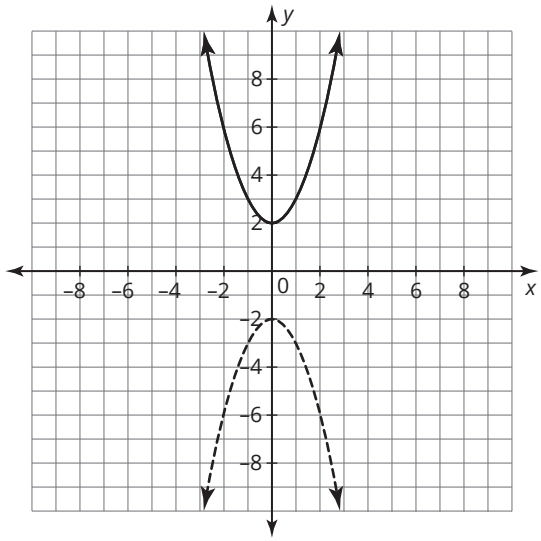
The function $f(x) = 3x + 2$ is a linear function.

line of reflection

A line of reflection is the line that the graph is reflected across.

Example

The graph of $y = x^2 + 2$ was reflected across the line of reflection, $y = 0$.



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literal equation

Literal equations are equations in which the variables represent specific measures.

Example

The equations $I = Prt$ and $A = lw$ are literal equations.

M

mathematical modeling

Mathematical modeling is explaining patterns in the real world based on mathematical ideas.

monomial

Polynomials with only one term are monomials.

Example

The expressions $5x$, 7, $-2xy$, and $13x^3$ are monomials.

N

necessary condition

A correlation is a necessary condition for causation, meaning that for one variable to cause another, they must be correlated.

no solution

An equation with no solution means that there is no value for the variable that makes the equation true.

Example

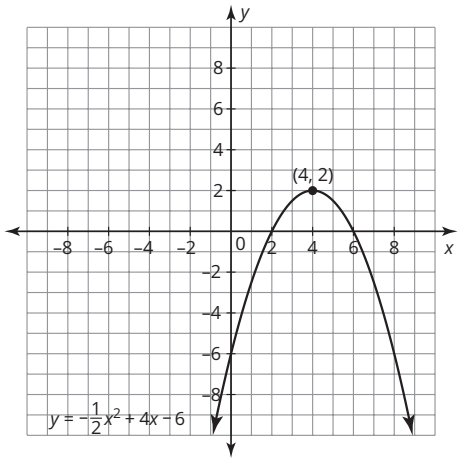
The equation $2x + 1 = 2x + 3$ has no solution.

P

parabola

The shape that a quadratic function forms when graphed is called a parabola. A parabola is a smooth curve with reflectional symmetry.

Example



parent function

A parent function is the simplest function of its type.

Examples

The parent linear function is $f(x) = x$.
The parent exponential function is $g(x) = 2^x$.
The parent quadratic function is $h(x) = x^2$.

perfect square trinomial

A perfect square trinomial is an expression in the form $a^2 + 2ab + b^2$ or in the form $a^2 - 2ab + b^2$.

point-slope form

The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope of the line and (x_1, y_1) is a point on the line.

polynomial

A polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients.

Example

The expression $3x^3 + 5x - 6x + 1$ is a polynomial.

polynomial long division

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division.

Example

Polynomial Long Division

$(2x^2 + 5x - 12) \div (x + 4)$

or

$\frac{2x^2 + 5x - 12}{x + 4}$

$$
\begin{array}{r}
\text{Ⓐ}\ 2x \ \text{Ⓓ}\ -3 \\
x + 4 \overline{)\ 2x^2 + 5x - 12} \\
\text{Ⓑ}\ (2x^2 + 8x) \downarrow \quad \text{Ⓒ} \\
\hline
-3x - 12 \\
-(-3x - 12) \\
\hline
\text{Remainder } 0
\end{array}
$$

A. Divide $\frac{2x^2}{x} = 2x$.
B. Multiply $2x(x + 4)$, and then subtract.
C. Bring down -12.
D. Divide $\frac{-3x}{x} = -3$.
E. Multiply $-3(x + 4)$, and then subtract.

power

A power has a *base* and an *exponent*.

principal square root

The principal square root is a positive square root of a number.

Q

quadratic formula

The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c$, where a, b, and c represent real numbers and $a \neq 0$.

quadratic functions

The family of quadratic functions includes functions of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, and a is not equal to 0.

Examples

The equations $y = x^2 + 2x + 5$ and $y = -4x^2 - 7x + 1$ are quadratic functions.

R

range

The range is the set of output values in a relation.

Example

The range of the function $y = x^2$ is the set of all numbers greater than or equal to zero.

recursive formula

A recursive formula expresses each new term of a sequence based on the preceding term in the sequence. The recursive formula for an arithmetic sequence is $a_n = a_{n-1} + d$. The recursive formula for a geometric sequence is $g_n = g_{n-1} \cdot r$.

Example

The formula $a_n = a_{n-1} + 2$ is a recursive formula. Each successive term is calculated by adding 2 to the previous term. If $a_1 = 1$, then $a_2 = 1 + 2 = 3$.

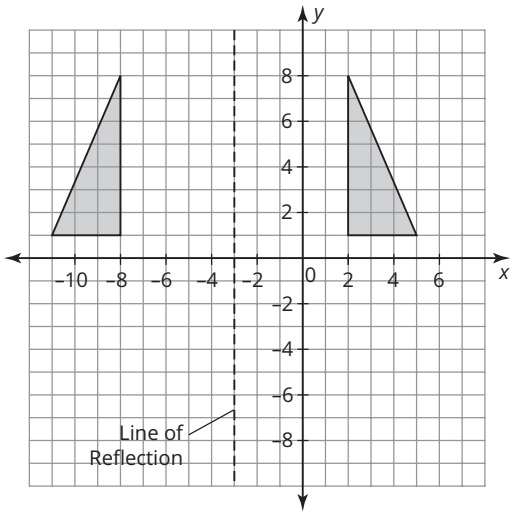
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reflection

A reflection of a graph is a mirror image of the graph about a line of reflection.

Example

The triangle on the right is a reflection of the triangle on the left.



regression function

On a scatterplot, a regression function is a mathematical model that can be used to predict the values of a dependent variable based upon the values of an independent variable.

relation

A relation is the mapping between a set of input values called the domain and a set of output values called the range.

Example

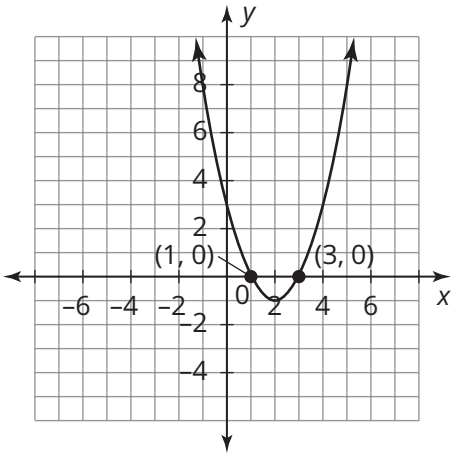
The set of points {(0, 1), (1, 8), (2, 5), (3, 7)} is a relation.

root (roots)

The root or roots of an equation indicate where the graph of the equation crosses the *x*-axis.

Example

The roots of the quadratic equation $x^2 - 4x + 3 = 0$ are $x = 3$ and $x = 1$.



S

second differences

Second differences are the differences between consecutive values of the first differences.

Example

x	*y*	First Differences	Second Differences
−3	−5		
		5	
−2	0		−2
		3	
−1	3		−2
		1	
0	4		−2
		−1	
1	3		−2
		−3	
2	0		−2
		−5	
3	−5		

sequence

A sequence is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects.

Example

The numbers 1, 1, 2, 3, 5, 8, 13 form a sequence.

simple interest

In a simple interest account, the interest earned at the end of each interval is a percent of the starting balance (also known as the principal).

Example

Tonya deposits $200 in a 3-year certificate of deposit that earns 4% simple interest. The amount of interest that Tonya earns can be found using the simple interest formula.

$$I = (200)(0.04)(3)$$
$$I = 24$$

Tonya earns $24 in interest.

solution

The solution to an equation is any value for the variable that makes the equation a true statement.

Example

The solution of the equation $3x + 4 = 25$ is 7 because 7 makes the equation true: $3(7) + 4 = 25$, or $25 = 25$.

solution set of a system of linear inequalities

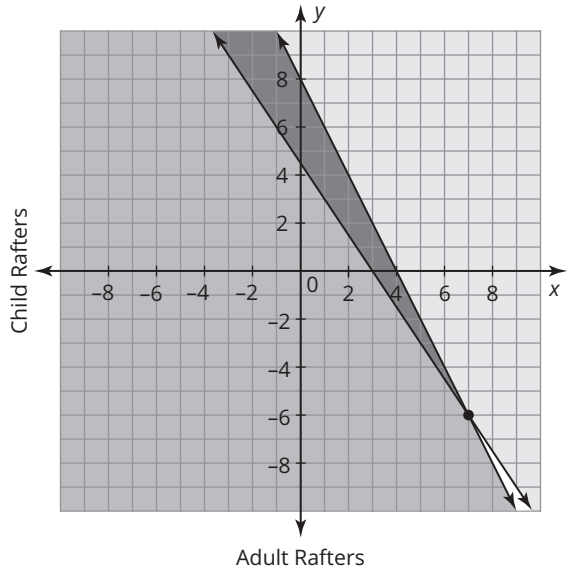
The solution set of a system of linear inequalities is the intersection of the solutions to each inequality. Every point in the intersection region satisfies all inequalities in the system.

Example

The solution set of this system of linear inequalities ...

$$\begin{cases} 200a + 100c \leq 800 \\ 75(a - 1) + 50c \geq 150 \end{cases}$$

... is shown by the shaded region, which represents the intersection of the solutions to each inequality.



solve an inequality

To solve an inequality means to determine the values of the variable that make the inequality true.

Example

The inequality $x + 5 > 6$ can be solved by subtracting 5 from each side of the inequality. The solution is $x > 1$. Any number greater than 1 will make the inequality $x + 5 > 6$ true.

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standard form of a linear equation

The standard form of a linear equation is $Ax + By = C$, where A, B, and C are integers and A and B are not both zero.

standard form of a quadratic function

A quadratic function written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is in standard form.

Example

The function $f(x) = -5x^2 - 10x + 1$ is written in standard form.

sufficient condition

A correlation is not a sufficient condition for causation, meaning that a correlation between two variables is not enough to establish that one variables causes another.

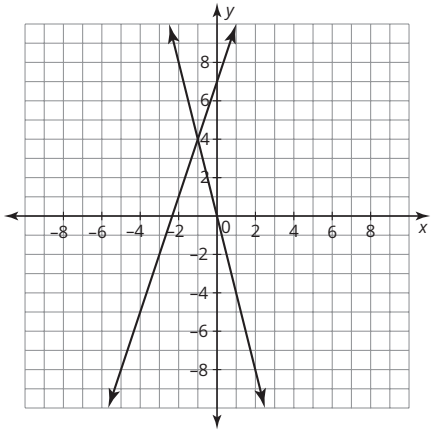
system of linear equations

When two or more linear equations define a relationship between quantities, they form a system of linear equations.

Example

The equations $y = 3x + 7$ and $y = -4x$ are a system of linear equations.

$$\begin{cases} y = 3x + 7 \\ y = -4x \end{cases}$$



T

term of a sequence

A term of a sequence is an individual number, figure, or letter in the sequence.

Example

In the sequence 2, 4, 6, 8, 10, the first term is 2, the second term is 4, and the third term is 6.

trinomial

Polynomials with exactly three terms are trinomials.

Example

The polynomial $5x^2 - 6x + 9$ is a trinomial.

V

vertex form

A quadratic function written in vertex form is in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$.

Example

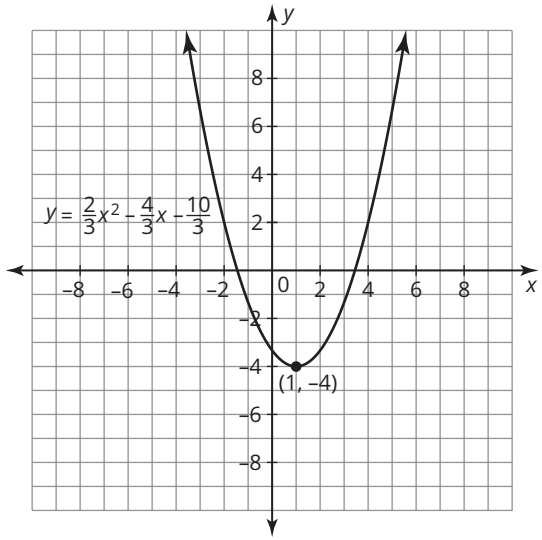
The quadratic equation $y = 2(x - 5)^2 + 10$ is written in vertex form. The vertex of the graph is the point (5, 10).

vertex of a parabola

The vertex of a parabola is the lowest or highest point on the graph of the quadratic function.

Example

The vertex of the graph of $y = \frac{2}{3}x^2 - \frac{4}{5}x - \frac{10}{3}$ is the point $(1, -4)$, the absolute minimum of the parabola.

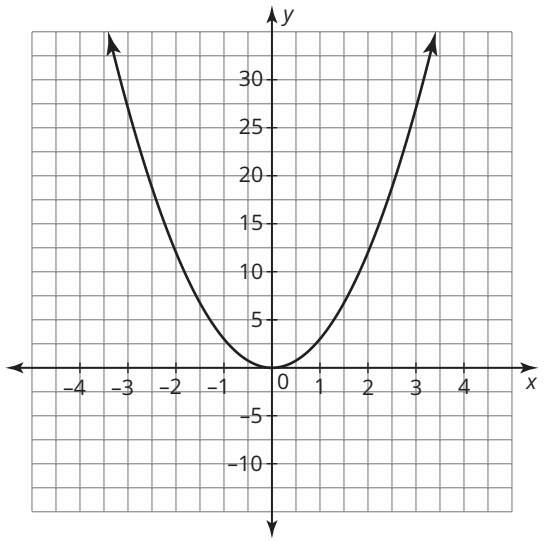


Vertical Line Test

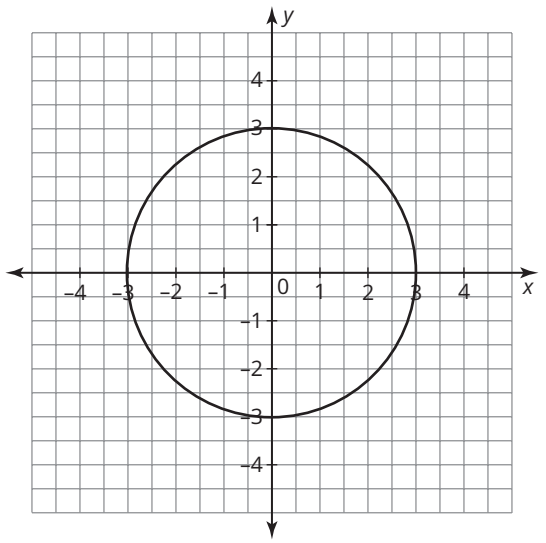
The Vertical Line Test is a visual method used to determine whether a relation represented as a graph is a function.

Example

The equation $y = 3x^2$ is a function. The graph passes the Vertical Line Test because there are no vertical lines that can be drawn that would intersect the graph at more than one point.



The equation $x^2 + y^2 = 9$ is not a function. The graph fails the Vertical Line Test because a vertical line can be drawn that intersects the graph at more than one point.



vertical motion model

A vertical motion model is a quadratic equation that models the height of an object at a given time. The equation is of the form $g(t) = -16t^2 + v_0t + h_0$, where $g(t)$ represents the height of the object in feet, t represents the time in seconds that the object has been moving, v_0 represents the initial velocity (speed) of the object in feet per second, and h_0 represents the initial height of the object in feet.

Example

A rock is thrown in the air at a velocity of 10 feet per second from a cliff that is 100 feet high. The height of the rock is modeled by the equation $y = -16t^2 + 10t + 100$.

X

x-intercept

The point where a graph crosses the *x*-axis is the *x*-intercept.

Y

y-intercept

The point where a graph crosses the *y*-axis is the *y*-intercept.

Z

zero of a function

A zero of a function is a real number that makes the value of the function equal to zero, or $f(x) = 0$.

Example

The zero of the linear function $f(x) = 2(x - 4)$ is (4, 0).

The zeros of the quadratic function $f(x) = -2x^2 + 4x$ are (0, 0) and (2, 0).

zero product property

The zero product property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

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