Rumson-Fair Haven Regional High School Curriculum

Course: *Multivariable Calculus*
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Section I: Course Description

Multivariable Calculus is an advanced course that equates to a college-level Calculus 3 class. The course covers vectors and the geometry of space, vector-valued functions, functions of several variables, partial derivatives, and multiple integrals. Students will make extensive use of a graphing calculator.

Section II: NJSLS: New Jersey Student Learning Standards/Learning Objectives

1. ***2023 New Jersey Student Learning Standards – Mathematics:***
   - "A New Jersey education in Mathematics builds quantitatively and analytically literate citizens prepared to meet the demands of college and career, and to engage productively in an information-driven society; …A high-quality mathematics education fosters a population that…leverages data in decision-making and as a lens for discussing, analyzing, and responding to practical questions, persists to make sense of and model problems arising in everyday life, society, and the workplace, thinks critically and strategically to assess quantitative relationships and to solutions to complex problems, employs precise reasoning and constructs viable arguments to deduce conclusions, recognize false statements and assess peers' reasoning, interprets, evaluates and critiques the mathematics embedded in social, scientific and commercial systems, as well as the claims made in the private and public sectors, communicates precisely when conveying, representing, and justifying both qualitative and quantitative perspectives."
2. ***2023 New Jersey Student Learning Standards English Language Arts:***
   - A New Jersey education in English Language Arts builds readers, writers, and communicators prepared to meet the demands of college and career and to engage as productive American citizens with global responsibilities. …Students will develop the necessary skills in reading, writing, speaking, and listening that are the foundations for creative and purposeful expression in language read rich, challenging texts that build their knowledge of the world, grow their confidence and identities as readers, and develop critical thinking skills and vocabulary necessary for long-term success[; e]ngage in regular, meaningful, writing authentic tasks, exploring valued topics, writing for impact and expression, and sharing their work with others (including authentic audiences) leverage complex texts and digital media to develop comprehension, active listening, and discussion skills ground daily writing and discussion in evidence, fostering an ability to read critically, build arguments, cite evidence, and communicate ideas to contribute meaningfully as productive citizens evaluate the reliability, credibility, and perspective of authors and speakers across all forms of media express ideas and knowledge through a variety of modalities and media, and serve as effective communicators who purposefully read, write, and speak across multiple disciplines [and l]earn to persist in reading complex texts, establishing lifelong habits to read voluntarily for pleasure, for further education, for information on public policy, and for advancement in the workplace.
3. ***Standard 8.1 (Computer Science) and 8.2 (Design Thinking)* of the *2020 NJSLS*:**
   - "The 'Intent and Spirit of the Computer Science and Design Thinking Standards' is to focus on deep understanding of concepts that enable students to think critically and systematically about leveraging technology to solve local and global issues. Authentic learning experiences that enable students to apply content knowledge, integrate concepts across disciplines, develop computational thinking skills, acquire and incorporate varied perspectives, and communicate with diverse audiences about the use and effects of computing prepares New Jersey students for college and careers."
4. ***Standard 9.4 (Life Literacies and Key Skills)* of the *2020 NJSLS*:**
   - "This standard outlines key literacies and technical skills such as critical thinking, global and cultural awareness, and technology literacy* that are critical for students to develop to live and work in an interconnected global economy."

     Climate Change: The state of New Jersey has mandated instruction in, "Climate Change across all content areas, leveraging the passion students have shown for this critical issue and providing them opportunities to develop a deep understanding of the science behind the changes and to explore the solutions our world desperately needs."
5. ****Amistad Law: N.J.S.A. 18A 52:16A-88:***

- The inclusion of lessons and resources/texts dealing with the African slave trade, slavery in America, the vestiges of slavery in this country and the contributions of African-Americans to our society will be implemented in English and Social Studies courses in accordance with state law: “Every board of education shall incorporate the information regarding the contributions of African-Americans to our country in an appropriate place in the curriculum of elementary and secondary school students.”

6. ***Holocaust Law: N.J.S.A. 18A 35-28:***
   - The inclusion of lessons and resources/texts that enable pupils to identify and analyze applicable theories concerning human nature and behavior; to understand that genocide is a consequence of prejudice and discrimination; and to understand that issues of moral dilemma and conscience have a profound impact on life will be implemented in English and Social Studies courses in accordance with state law: “Every board of education shall include instruction on the Holocaust and genocides in an appropriate place in the curriculum of all elementary and secondary school pupils. The instruction shall further emphasize the personal responsibility that each citizen bears to fight racism and hatred whenever and wherever it happens.”
7. ***LGBT and Disabilities Law: N.J.S.A. 18A:35-4.35:***
   - A transformative approach to the inclusion of lessons and resources/texts on the contributions and issues concerning the LGBTQ+ population and people with disabilities will be implemented across all core subjects in accordance with state law: “A board of education shall include instruction on the political, economic, and social contributions of persons with disabilities and lesbian, gay, bisexual, and transgender people, in an appropriate place in the curriculum of middle school and high school students as part of the district’s implementation of the New Jersey Student Learning Standards (N.J.S.A.18A:35-4.36). A board of education shall have policies and procedures in place pertaining to the selection of instructional materials to implement the requirements of N.J.S.A. 18A:35-4.35.”
8. ****Asian American and Pacific Islanders Legislation: N.J.S.A 4021/A6100:***
   - The inclusion of lessons and resources/texts on the history and contributions of Asian Americans and Pacific Islanders, will enable New Jersey’s schools to provide a curriculum that reflects the diversity of our state. In accordance with state law: “A board of education shall include instruction on the history and contributions of Asian Americans and Pacific Islanders in an appropriate place in the curriculum of students in grades kindergarten through as part of the school district’s implementation of the New Jersey Student Learning Standards in Social Studies.”
9. Acquisition/development/refinement of the higher-order critical thinking skills aligned with the *Revised Bloom’s Taxonomy of Cognitive Objectives*

Section III: Curriculum Modifications

The *Honors College Calculus* curriculum is subject to case-by-case modifications to support/advance the needs of all students, including special education students, English language learners, gifted students and those at risk of school failure. These modifications are based on Individualized Learning Programs (IEPs), recommendations made by the district’s English Language Learners (ELL) coordinator, feedback from members of the Intervention & Referral Services Team (*I&RS*) for at-risk students, and 504 Plans.

Coursework and assessments will be modified on an individual basis for students when necessary. Modifications may include but are not limited to those outlined on the Modifications/Accommodations for Mathematics Courses chart.

Section IV: Preparation for Standardized Testing

Instruction in *Multivariable Calculus* is aligned with the requirements of state and national standardized assessments, including the *NJGPA, NJSLA*, the *ACT*, the *PSAT,* and the *SAT*.

Section V: Curriculum Pacing Guide

Curriculum Pacing Guide	
Course Title: *Multivariable Calculus*	**Grade Level:** 12

Unit I: Parametric Equations and Polar Coordinates	Weeks 1-5
Unit II: Vectors and the Geometry of Space	Weeks 6-10
Unit III: Vector-Valued Functions & Motion in Space	Weeks 11-16
Unit IV: Partial Derivatives	Weeks 17-27
Unit V: Multiple Integrals	Weeks 28-33
Unit VI: Integrals and Vector Fields	Weeks 34-40

Section VI: Technology Skills

Students in *Multivariable Calculus* are required to complete the technology skills components of the curriculum:

- TI-89 Calculator
- Desmos
- Geogebra

Section VII: Primary Texts and Year-Long Instructional Resources

The following texts and instructional resources are employed in *Multivariable Calculus*:

- *Common Sense Education*
- Calculus Early Transcendentals 14th Edition: By Thomas, Hass, Heil, Weir

Section VIII: Grading Formula and Assessment Modes

Marking period grades in *Multivariable Calculus* are determined via a percentage weighting model. The specific grading categories and weightings of each will be determined prior to the start of each academic year and will be published in the posted/distributed course syllabi.

Assessments in *Multivariable Calculus* vary greatly in format, scope/content/skills assessed, and alternative assessments, differentiation in assessments and choice will be incorporated as appropriate. Preliminary assessments of each format will be used as benchmarks and summative assessments will be created/revised collaboratively each year and planned by members of the *Dance* instructional team to inform future learning and to measure student growth.

Section IX: Unit Templates

The following unit templates have been established for the *Multivariable Calculus* Curriculum by the *Multivariable Calculus* instructional team:

Unit I: Parametric Equations and Polar Coordinates
Unit Summary
In this unit, students will study new ways to define curves in the plane. Instead of thinking of a curve as the graph of a function or equation, they will consider a more general way of thinking of a curve as the path of a moving particle whose position is changing over time. Then each of the x- and y-coordinates of the particle's position becomes a function of a third variable t. We can also change the way in which points in the plane themselves are described by

using polar coordinates rather than the rectangular or Cartesian system. Both of these new tools are useful for describing motion, like that of planets and satellites, or projectiles moving in the plane or space. In addition, they will review the geometric definitions and standard equations of parabolas, ellipses, and hyperbolas. These curves are called conic sections, or conics, and model the paths traveled by projectiles, planets, or any other object moving under the sole influence of a gravitational or electromagnetic force.
Standards/Core Ideas/Performance Expectations
The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:* • *2023 New Jersey Student Learning Standards: Mathematics* 　○ MP.1-8 　○ N.CN.B.5 • *2023 New Jersey Student Learning Standards English Language Arts* 　○ L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6 • *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking* 　○ 8.1.12.DA.5-6 • *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills* 　○ 9.4.12.CI.1, 9.4.12.CT.1-2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings
• What is a parametrization of a curve in the *xy*-plane? Does a function $y = f(x)$ always have a parametrization? Are parametrizations of a curve unique? • What is a cycloid? What are typical parametric equations for cycloids? What physical properties account for the importance of cycloids? • What is the formula for the slope dy/dx of a parametrized curve x = f(t), y = g(t)? When does the formula apply? When can you expect to be able to find d^2y/dx^2 as well? • How do students find the area bounded by a parametrized curve and one of the coordinate axes? • How do students find the length of a smooth parametrized curve x = f(t), y = g(t)? • How do students find the area of the surface generated by revolving a curve x = f(t), y = g(t), about the *x*-axis? • What are polar coordinates? What equations relate polar coordinates to Cartesian coordinates? Why might you want to change from one coordinate system to the other? • How is the area of a region bounded by two polar curves in the polar coordinate plane found? • How is the length of a polar curve found and under what conditions?	• For any curve, the set of points (x,y) = (f(t), g(t)), where x=f(t), and y=g(t) is a parametric curve. A curve can have several parametrizations. • The curve traced out by a point on the circumference of a wheel as it rolls along a straight line is a Cycloid. Derive the parametric equations of a cycloid. • Use derivatives to obtain the formula for the slope of a parametrized curve. The formula applies if the derivatives of x(t) and y(t) exist and dx/dt is not equal to zero. If x(t) and y(t) are twice differentiable then d^2y/dx^2 exists. • Use derivatives to derive the equation for the area bounded by a parametrized curve. • Use derivatives to derive the length of a smooth parametrized curve. • Use derivatives to derive the equation for the area of a surface of revolution for parametrized curves. • Define polar coordinates in terms of the distance of any point from the origin and the angle made by the line joining the point P to the origin and the x-axis. Discuss examples where polar coordinates will be preferred over cartesian coordinates. • Derive the equation for the area of a region bounded by polar curves. • Use derivatives to derive the equation of the length of a polar curve. The equation works as long as the curve r=f(θ) has a continuous first derivative.

Evidence of Learning		
Formative & Alternative Assessments: • Classwork • Homework • Weekly topical quizzes • Desmos activity • Conics activity • Geogebra activity • Individual student check ins with	**Benchmark & Summative Assessments:** • Parametric Equations Quiz (Benchmark) • Polar Coordinate Quiz • Unit test (Benchmark)	**Resources Needed:** • *Calculus Early Transcendentals* 14th Edition: By Thomas, Hass, Heil, Weir • Desmos • Geogebra

teacher		

Unit II: Vectors and the Geometry of Space
Unit Summary
This unit is foundational to the study of *Multivariable Calculus*. To apply Calculus in many real-world situations and in higher mathematics, students will need analytic geometry to describe three-dimensional space. To accomplish this objective, three-dimensional coordinate systems and vectors are introduced. Building on what is already known about coordinates in the xy-plane, coordinates in space will be established by adding a third axis that measures the distance above and below the xy-plane. Then students will define vectors and use them to study the analytic geometry of space. Vectors provide simple ways to define equations for lines, planes, curves, and surfaces in space. These geometric concepts are used throughout the remainder of the text to study motion in space and the Calculus of functions of several variables and vector fields, with their many important applications in science, engineering, operations research, economics, and higher mathematics.
Standards/Core Ideas/Performance Expectations
The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:* ● *2023 New Jersey Student Learning Standards: Mathematics* ○ MP.1-8 ○ N.VM.A.3, B.4b, B.5a & C.11, F.IF.C.7 ● *2023 New Jersey Student Learning Standards English Language Arts* ○ L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6 ● *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking* ○ 8.1.12.DA.5-6 ● *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills* ○ 9.4.12.CI.1, 9.4.12.CT.1 & 2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings
● When do directed line segments in the plane represent the same vector? ● How are vectors added and subtracted geometrically? Algebraically? ● How are a vector's magnitude and direction found? ● If a vector is multiplied by a positive scalar, how is the result related to the original vector? What if the scalar is zero? Negative? ● Define the dot product (scalar product) of two vectors. Which algebraic laws are satisfied by dot products? When is the dot product of two vectors equal to zero? ● What geometric interpretation does the dot product have? ● What is the vector projection of a vector u onto a vector v? Give an example of a useful application of a vector projection. ● Define the cross product (vector product) of two vectors. Which algebraic laws are satisfied by cross products, and which are not? Give examples. When is the cross product of two vectors equal to zero? ● What geometric or physical interpretations do cross products have? ● What is the determinant formula for calculating the cross product of two vectors relative to the Cartesian i, j, k-coordinate system? ● How does one find equations for lines, line segments, and planes in space? Give examples. Can one express a line in space by a single equation? A	● Two vectors are equal if they have the same magnitude and direction. ● Geometrically, vectors are added by the parallelogram law of addition. Algebraically vectors are added or subtracted by adding or subtracting their components. ● The magnitude of a vector can be found by finding the root of the sum of the squares of the components of the vector. Dividing a vector by its magnitude gives its direction. ● Multiplying a vector by a scalar changes the length of a vector if the scalar is positive. In case the scalar is negative then the scalar multiplication also reverses the direction of the vector. ● Dot product of any two vectors is commutative. If two vectors are orthogonal then their dot product vanishes. ● Dot product gives the projection of one vector in the direction of a second vector. ● Use the dot product to obtain an expression for the vector projection of one vector onto another vector. ● The cross product can be used to calculate the torque exerted by a force. The cross-product is not commutative. If two vectors are parallel, then their cross-product vanishes. ● Obtain the determinant formula for the cross product of two vectors. ● Derive the equation for a line in space given a point and the vector that this line is parallel to. Derive

plane? • How is distance from a point to a line in space found? From a point to a plane? • What are box products? What significance do they have? How are they evaluated? • How is the intersection of two lines in space found? A line and a plane? Two planes? • How are equations for spheres in space found? • What is a cylinder? Give examples of equations that define cylinders in Cartesian coordinates. • What are quadric surfaces? Give examples of different kinds of ellipsoids, paraboloids, cones, and hyperboloids (equations and sketches).	equation of a plane passing through a given point and perpendicular to a given vector. • Using dot and cross products, find the distance between a point and a line, and the distance between a point and a plane. • Define a box product and use it to find the volume of a box spanned by three vectors. • Using cross products, obtain the equation of the line of intersection of two planes. • Derive the equation of a sphere. • Derive the equation of a cylinder. • Define a quadric surface and classify it into ellipsoids, paraboloids, elliptic cones, and hyperboloids.

Evidence of Learning		
Formative & Alternative Assessments: • Classwork • Homework • Weekly topical quizzes • Individual student check ins with teacher	**Benchmark & Summative Assessments:** • Vector Review Quiz • Dot and Cross Product Quiz • Spheres in Space Quiz • Unit Test	**Resources Needed:** • *Calculus Early Transcendentals* 14th Edition: By Thomas, Hass, Heil, Weir • Desmos • Geogebra

Unit III: Vector-Valued Functions & Motion in Space

Unit Summary

In this unit, students will combine what they have learned about vectors and the geometry of space with the earlier study of functions. Here the calculus of vector-valued functions will be introduced. The domains of these functions are sets of real numbers, as before, but their ranges consist of vectors, not scalars. When a vector-valued function changes, the change can occur in both magnitude and direction, so the derivative is itself a vector. The integral of a vector-valued function is also a vector. The calculus of these functions is used to describe the paths and motions of objects moving in a plane or in space, so their velocities and accelerations are given by vectors. New scalars that quantify the turning and twisting in the path of an object moving in space will also be introduced.

Standards/Core Ideas/Performance Expectations

The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:*

- *2023 New Jersey Student Learning Standards: Mathematics*
  - MP.1-8
  - N.VM.A.3 & B.4b
- *2023 New Jersey Student Learning Standards English Language Arts*
  - L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6
- *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking*
  - 8.1.12.DA.5-6
- *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills*
  - 9.4.12.CI.1, 9.4.12.CT.1-2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings
• State the rules for differentiating and integrating vector functions. • How do one define and calculate the velocity, speed, direction of motion, and acceleration of a body moving along a sufficiently differentiable space curve? • What is special about the derivatives of vector functions of constant length? • What are the vector and parametric equations for ideal projectile motion? How	• Derive the rules for differentiating and integrating vector functions. • Use derivatives to calculate velocity and acceleration. Use vectors to determine the speed and direction of motion. • The derivative of a vector function of constant length is orthogonal to the vector function at every point. • Derive the equations of motion for a projectile. Derive the expressions for the maximum, time of flight, and range for a projectile. • Derive the formula to calculate the length of a smooth curve,

are a projectile's maximum height, flight time, and range found? ● How does one define and calculate the length of a segment of a smooth space curve? What mathematical assumptions are involved in the definition? ● What is a differentiable curve's unit tangent vector? ● Define curvature, circle of curvature (osculating circle), center of curvature, and radius of curvature for twice-differentiable curves in the plane. What curves have zero curvature? Constant curvature? ● What is a plane curve's principal normal vector? When is it defined? Which way does it point? ● What is a curve's binormal vector? How is this vector related to the curve's torsion? ● What formulas are available for writing a moving object's acceleration as a sum of its tangential and normal components? ● State Kepler's laws.	provided the first derivative is defined at each point of the path. ● The unit vector in the direction of the velocity vector of a curve is the Unit Tangent Vector. ● The derivative of the unit tangent vector with respect to the arc length is the curvature of the curve. A straight line has zero curvature. The curvature of a circle is constant. ● The principal normal vector is the ratio of the derivative of the unit tangent vector with the curvature. It points perpendicular to the tangent vector and lies in the plane of the curve. ● The binormal vector is the cross product of the tangent and normal vectors. The torsion is defined in terms of the dot product of the normal vector and the derivative of the binormal vector with respect to the arc length. ● Derive formulas for the acceleration in terms of its tangential and normal components. ● Kepler's Laws are the following: ○ A planet's path is an ellipse with the sun at one focus. ○ The radius vector from the sun to a planet sweeps out equal areas in equal times. ○ The orbital period and the orbit's semimajor axis are related by a derived equation.

Evidence of Learning		
Formative & Alternative Assessments: ● Classwork ● Homework ● Weekly topical quizzes ● Kepler's laws activity ● Desmos vector derivative activity ● Individual student check ins with teacher	**Benchmark & Summative Assessments:** ● Vector-valued Function Quiz ● Motion in Space Quiz ● Unit Test	**Resources Needed:** ● *Calculus Early Transcendentals* 14th Edition: By Thomas, Hass, Heil, Weir ● Desmos ● Geogebra

Unit IV: Partial Derivatives

Unit Summary

In this unit, the basic ideas of single-variable differential calculus to functions of several variables will be extended. Their derivatives are more varied and interesting because of the different ways the variables can interact. The applications of these derivatives are also more varied than for single-variable calculus, and in the next unit, students will see that the same is true for integrals involving several variables. Students will be able to define, graph, and identify the basic properties of the limits of a real-valued function of two or three independent variables. Students will use their knowledge of derivatives to work through the Chain rule for up to three independent variables.

Standards/Core Ideas/Performance Expectations

The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:*

- *2023 New Jersey Student Learning Standards: Mathematics*
  - MP.1-8
  - F.IF.B.6
- *2023 New Jersey Student Learning Standards English Language Arts*
  - L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6
- *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking*
  - 8.1.12.DA.5&6
- *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills*
  - 9.4.12.CI.1, 9.4.12.CT.1&2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings

• What is a real-valued function of two independent variables? Three independent variables? • What does it mean for sets in the plane or in space to be open? Closed? How can the values of a function f(x, y) of two independent variables graphically be displayed? How does one do the same for a function f(x, y, z) of three independent variables? • What does it mean for a function f(x, y) to have limit L as (x, y) approaches (x_0 , y_0)? What are the basic properties of limits of functions of two independent variables? • When is a function of two (three) independent variables continuous at a point in its domain? • What can be said about algebraic combinations and composites of continuous functions? • Explain the two-path test for nonexistence of limits. • How are the partial derivatives of a function f(x, y) defined? How are they interpreted and calculated? • How does the relation between first partial derivatives and continuity of functions of two independent variables differ from the relation between first derivatives and continuity for real-valued functions of a single independent variable? • What is the Mixed Derivative Theorem for mixed second-order partial derivatives? • What does it mean for a function f(x, y) to be differentiable? What does the Increment Theorem say about differentiability? • How can one sometimes decide from examining f_x and f_y that a function f(x, y) is differentiable? What is the relation between the differentiability of f and the continuity of f at a point? • What is the general Chain Rule? What form does it take for functions of two independent variables? Three independent variables? Functions defined on surfaces? How can these different forms be diagrammed? What pattern enables one to remember all the different forms? • What is the derivative of a function f(x, y) at a point P_0 in the direction of a unit vector **u**? What rate does it describe? What geometric interpretation does it have? • What is the gradient vector of a differentiable function f(x, y)? How is it related to the function's directional derivatives? State the analogous results for functions of three independent variables. • How is the tangent line at a point on a level curve of a differentiable function f(x, y) found? How is the tangent plane and normal line at a point on a level surface of a differentiable function f(x, y, z) found? Give examples. • How directional derivatives be used to estimate change?	• Define function of two or more variables. • Define open and closed regions in space. • Give the formal definition of the limit of a function of one, two, and three variables. • A function is continuous at a point if the function is defined at the point, the limit of the function exists at the point, and the two are equal to each other. • The sums, differences, constant multiples, products, quotients, and powers of continuous functions are continuous where defined. • If a function f(x, y) has different limits along two different paths in the domain of f as (x, y) approaches (x_0 , y_0), then the limit as (x,y) approaches (x_0 ,y_0) of f(x, y) does not exist. • The partial derivative of f(x, y) with respect to x at the point (x_0, y_0) is $\frac{\delta f}{\delta x}\|(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$, provided the limit exists. • A function f(x, y) can have partial derivatives with respect to both x and y at a point without the function being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. • If f(x, y) and its partial derivatives f_x , f_y , f_{xy} , and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then f_{xy}(a, b) = f_{yx}(a, b). • We call f differentiable if its partial derivatives exist at every point in its domain. • If the partial derivatives of a function are continuous throughout an open region, then the function is differentiable at every point of the open region. If a function is differentiable at a point, then the function is continuous at that point. • Derive the chain rule for functions of one independent variable and two or three intermediate variables. Derive the chain rule for functions with two independent variables and three intermediate variables. Use branch diagrams to evaluate the derivatives using the chain rule. • Obtain an expression for the directional derivative of function of two or more variables. • Define the gradient vector in terms of the partial derivatives of the function. The directional derivative is the dot product of the gradient vector in the direction of the vector. • Use partial derivatives to find the tangent line and the tangent plane at a point on a level curve. • To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction **u**, use the formula $df = (\nabla f\|_P . \mathbf{u})\, ds$ • Use partial derivatives to linearize a differentiable function.

• How does one linearize a function $f(x, y)$ of two independent variables at a point (x_0, y_0)? Why might a student want to do this? How does one linearize a function of three independent variables? • What can be said about the accuracy of linear approximations of functions of two (three) independent variables? • If (x, y) moves from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, how can the estimate of the resulting change in the value of a differentiable function $f(x, y)$ be found? • How does one define local maxima, local minima, and saddle points for a differentiable function $f(x, y)$? • What derivative tests are available for determining the local extreme values of a function $f(x, y)$? How do they enable a narrowing of the search for these values? • How is the extrema of a continuous function $f(x, y)$ on a closed bounded region of the xy-plane found? • Describe the method of Lagrange multipliers and give examples. • How does Taylor's formula for a function $f(x, y)$ generate polynomial approximations and error estimates?	• Obtain an expression for the error incurred in replacing a function by its linearization. • If ones moves from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change $df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$ in the linearization of f is called the total differential of f. • Let $f(x, y)$ be defined on a region R containing the point (a, b). Then.. ○ $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b). ○ $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b). ○ $f(x, y)$ has a saddle point at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. • Use the second derivative test to find local extreme values. • Develop the method of Lagrange Multipliers to calculate the maxima and minima of a function that is constrained to lie within some particular subset of the plane. • Derive the Taylor's formula for a function of two variables at any point and use it to calculate the error in the approximations of functions.

Evidence of Learning

Formative & Alternative Assessments:	Benchmark & Summative Assessments:	Resources Needed:
• Classwork • Homework • Weekly topical quizzes • Desmos activity-two variables • Individual student check-ins with teacher	• Two Function Quiz • Three Function Quiz • Taylor's Formula Quiz • Unit Test	• *Calculus Early Transcendentals* 14th Edition: By Thomas, Hass, Heil, Weir • Desmos • Geogebra

Unit V: Multiple Integration

Unit Summary

In this unit, the double integral of a function of two variables $f(x, y)$ over a region in the plane as the limit of approximating Riemann sums is defined. Just as a single integral represents signed area, so does a double integral represent signed volume. Double integrals can be evaluated using the Fundamental Theorem of Calculus studied earlier, but now the evaluations are done twice by integrating with respect to each of the variables x and y in turn. Double integrals can be used to find areas of more general regions in the plane than those encountered earlier. Moreover, just as the Substitution Rule could simplify finding single integrals, we can sometimes use polar coordinates to simplify computing a double integral. More general substitutions for evaluating double integrals as well will be studied. Triple integrals for a function of three variables $f(x, y, z)$ over a region in space will be defined. Triple integrals can be used to find volumes of still more general regions in space, and their evaluation is like that of double integrals with yet a third evaluation. Cylindrical or spherical coordinates can sometimes be used to simplify the calculation of a triple integral, and we investigate those techniques. Double and triple integrals have a number of additional applications, such as calculating the average value of a multivariable function and finding moments and centers of mass for more general regions than those encountered before.

Standards/Core Ideas/Performance Expectations

The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:*

- *2023 New Jersey Student Learning Standards: Mathematics*

- MP.1-8
- N.CN.B.4, A.CED.A.2
- *2023 New Jersey Student Learning Standards English Language Arts*
  - L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6
- *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking*
  - 8.1.12.DA.5&6
- *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills*
  - 9.4.12.CI.1, 9.4.12.CT.1&2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings
• Define the double integral of a function of two variables over a bounded region in the coordinate plane. • How are double integrals evaluated as iterated integrals? Does the order of integration matter? How are the limits of integration determined? • How are double integrals used to calculate areas and average values? • How can a double integral in rectangular coordinates be changed into a double integral in polar coordinates? Why might it be worthwhile to do so? • Define the triple integral of a function f(x, y, z) over a bounded region in space. • How are triple integrals in rectangular coordinates evaluated? How are the limits of integration determined? • How are double and triple integrals in rectangular coordinates used to calculate volumes, average values, masses, moments, and centers of mass? • How are triple integrals defined in cylindrical and spherical coordinates? Why might one prefer working in one of these coordinate systems to working in rectangular coordinates? • How are triple integrals in cylindrical and spherical coordinates evaluated? How are the limits of integration found? • How are substitutions in double integrals pictured as transformations of two-dimensional regions? • How are substitutions in triple integrals pictured as transformations of three-dimensional regions?	• Using the concept of the limit of Riemann Sums, define the double integral of a function over a bounded region. • Use Fubini's theorem to evaluate double integrals as iterated integrals. • If the function f(x,y) is positive and continuous in a given region R, then the volume of the solid region between R and the surface w=f(x,y) is obtained using the double integral of the function f evaluated over the region R. If f=1, then the double integral gives the area of the region R. The average value of the function f over the region R is defined as the ratio of the double integral of f over R to the area of R. • Derive an expression for changing a double integral in rectangular coordinates to a double integral in polar coordinates. Change of variables typically simplifies the calculation of evaluating the integral. • Using Riemann sums, define the triple integral of a function in rectangular coordinates. To calculate the limits of the integral over a region D, sketch the region D, find the z-limits of integration, and then find the y-limits of integration, and finally the x-limits of integration. • If the function f represents the density of the object, then the triple integral of f over the region D would give the mass of the object. Find the first moments using the formulae: $M_x = \int\int x\delta dA$ and $M_y = \int\int y\delta dA$. • Derive expressions for the triple integral in Cylindrical and spherical coordinates. To find the limits of integration sketch the region of integration, find the limits for each of the coordinates. • Use Jacobians to transform integrals from one region into another region given by the substitution.

Evidence of Learning		
Formative & Alternative Assessments: • Classwork • Homework • Weekly topical quizzes • Desmos activity • Individual student check ins with teacher	**Benchmark & Summative Assessments:** • Double Integral Quiz • Triple Integral Quiz • Cylindrical and Spherical Space Quiz • Unit Test	**Resources Needed:** • *Calculus Early Transcendentals* 14th Edition: By Thomas, Hass, Heil, Weir • Desmos • Geogebra

Unit VI: Integrals and Vector Fields
Unit Summary

In this unit, the theory of integration over coordinate lines and planes to general curves and surfaces in space will be extended. The resulting theory of line and surface integrals gives powerful mathematical tools for science and engineering. Line integrals are used to find the work done by a force in moving an object along a path, and to find the mass of a curved wire with variable density. Surface integrals are used to find the rate of flow of a fluid across a surface. The fundamental theorems of vector integral Calculus will be presented, and their mathematical consequences and physical applications will be discussed. In the final analysis, the key theorems are shown as generalized interpretations of the Fundamental Theorem of Calculus.

Standards/Core Ideas/Performance Expectations

The state standards outlined below, and established by the New Jersey Department of Education, will guide instruction throughout this unit in *Multivariable Calculus:*

- *2023 New Jersey Student Learning Standards: Mathematics*
  - MP.1-8
  - A.CED.A.2
- *2023 New Jersey Student Learning Standards English Language Arts*
  - L.VL.11-12.3.A, W.AW.11-12.1.A & E, W.IW.11-12.2.A, W.NW.11-12.3.A-E, SL.II.11-12.2, SL.PI.11-12.4, RI.MF.11-12.6
- *2020 New Jersey Student Learning Standards: Computer Science and Design Thinking*
  - 8.1.12.DA.5&6
- *2020 New Jersey Student Learning Standards: Career Readiness, Life Literacies and Key Skills*
  - 9.4.12.CI.1, 9.4.12.CT.1&2, 9.4.12.TL.3

Unit Essential Questions	Unit Enduring Understandings
• What are line integrals of scalar functions? How are they evaluated? • How can line integrals be used to find the centers of mass of springs or wires? • What is a vector field? What is the line integral of a vector field? What is a gradient field? • What is the flow of a vector field along a curve? What is the work done by a vector field moving an object along a curve? How can the work done be calculated? • What is the Fundamental Theorem of line integrals? Explain how it relates to the Fundamental Theorem of Calculus. • Specify three properties that are special about conservative fields. How can one tell when a field is conservative? • What is special about path-independent fields? • What is a potential function? Show by example how to find a potential function for a conservative field. • What is a differential form? What does it mean for such a form to be exact? How does one test for exactness? • What is Green's Theorem?	• To calculate the total mass of a wire lying along a curve in space, or to find work done by a variable force acting along such a curve, we can use a line integral. • To evaluate a line integral first find a smooth parametrization of the curve and then evaluate the integral. • Derive equations for finding the center of mass using integration. • A vector field is a function that assigns a vector to each point in its domain. • The gradient vector of a differentiable scalar-valued function at a point gives the direction of greatest increase of the function. • Use an integral to determine the flow or circulation of the vector field along the curve and also the work done in moving an object from the point A=r(a) to the point B=r(b). • Let C be a smooth curve joining points A and B in a plane or space that is parametrized by r(t). The Fundamental Theorem of Line Integrals says that if *f* is a differentiable function with a continuous gradient vector $F = \nabla f$ on a domain D containing C, then $\int_C F dr = f(B) - f(A)$. • A field is considered conservative when the principle of conservation of energy holds. • Path-independent fields are special fields where the integral's value is the same for all paths from A to B. • A field F is conservative if and only if it is the gradient field of a scalar function *f*. The function *f* is then called the potential function for F. • Any expression $M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$ is a differential form. • A differential form is considered exact if $Mdx + Ndy + Pdz = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy + \frac{\delta f}{\delta z}dz = df$. • The component test can be used to determine the exactness of a differential form. • Green's Theorem relates a line integral around a simple closed curve C to a double integral over the plane region D bounded by C.

• How can the area of a parametrized surface in space be calculated? • What is an oriented surface? What is the surface integral of a vector field in three-dimensional space over an oriented surface? How is it related to the net outward flux of the field? • What is the curl of a vector field? How can one interpret it? • What is Stokes' Theorem? • What is the Divergence Theorem? • How do Green's Theorem, Stokes' Theorem, and the Divergence Theorem relate to the Fundamental Theorem of Calculus for ordinary single integrals?	• To find the area of a parametrized surface in space, we must evaluate a double integral. • A curve C with a parametrization r(t) has a natural orientation, or direction, that comes from the direction of increasing t. There are two possible orientations for a curve, corresponding to whether we follow the direction of the tangent vector T at each point, or the direction of -T. • Let F be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors n orienting S. Then the surface integral of F over S is $\int\int F \cdot n\, d\sigma$. Which is also called the flux of the vector field. • The length of the curl vector measures the rate of rotation. • Stoke's Theorem states that the line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface. • The Divergence Theorem says that under suitable conditions, the outward flux of a vector field across a closed surface equals the triple integral of the divergence of the field over the three-dimensional region enclosed by the surface. • The Fundamental Theorem of Calculus, Green's Theorem, Stoke's Theorem, and Divergence Theorem all say the integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.

Evidence of Learning		
Formative & Alternative Assessments: • Classwork • Homework • Weekly topical quizzes • Desmos activity • Geogebra line integrals • Individual student check-ins with teacher	**Benchmark & Summative Assessments:** • Line Integral Quiz • Stoke's Theorem Quiz • Divergence Theorem Quiz • Tests	**Resources Needed:** • *Calculus Early Transcendentals* 14th Edition: By Thomas, Hass, Heil, Weir • Desmos • Geogebra

Section X: Unit Reflection

The *Multivariable Calculus* Instructional Team must confer upon the completion of each instructional unit in the *Multivariable Calculus* curriculum and rate the degrees to which the instructional units meet performance criteria established by the New Jersey Department of Education using the Unit Reflection Form. Completed unit reflection forms must be submitted to the Department Supervisor for approval upon completion of curriculum implementation with a complementing list of suggested modifications to the *Multivariable Calculus* curriculum.

Unit Reflection Form: *Multivariable Calculus*			
Lesson Activities:	**Strongly**	**Moderately**	**Weakly**
Foster student use of technology as a tool to develop critical thinking, creativity, and innovation skills;			
Are challenging and require higher-order thinking and problem-solving skills;			
Allow for student choice;			
Provide scaffolding for acquiring targeted knowledge/skills;			
Integrate modern, global perspectives, especially those regarding diversity, genocide, global issues, and historical ones regarding racial relations;			

Integrate 21st century skills;			
Provide opportunities for interdisciplinary connection and transfer of knowledge and skills;			
Are varied to address different student learning styles and preferences;			
Are differentiated based on student needs;			
Are student-centered with the teacher acting as a facilitator and co-learner during the teaching and learning process;			
Provide means for students to demonstrate knowledge and skills and progress in meeting learning goals and objectives;			
Provide opportunities for student reflection and self-assessment;			
Provide data to inform and adjust instruction to better meet the varying needs of learners.			

Appendix
Writing Instruction and the RFH Community

Writing instruction should happen across the RFH Community. Writing across the curriculum is a philosophy that advances the belief that writing is a method of learning. Since all departments are committed to helping students learn, writing must be used as a methodology to advance student learning.

Each academic discipline has its own unique conventions, formats and structures. It is the responsibility of each department to agree upon domain-specific writing praxes, model them for students, and require them to utilize them on a consistent basis. Students must understand that acceptable writing in one domain may not be acceptable writing in another area. The development of domain-specific writing skills supports the overall development of the student writer because all writing is grounded in the writing situation: audience, context, purpose, subject, and writer. Representatives from the academic disciplines must share their domain-specific writing praxes with each other, identify intersections, and determine how to address perceived gaps that limit student learning.

Students must experience writing situations that help them learn how to think creatively and critically and communicate effectively in the academic disciplines. Writing instruction, regardless of the academic discipline, must always reinforce student understanding of the writing situation. When students experience writing situations, they must study examples of domain-specific writing in order to understand how writers communicate in discipline-related contexts. This does not mean information embedded in textbooks. Domain-specific writing is writing that is used to inform and influence readers as it draws them into an established circle of discourse. Students must use these non-fiction texts to develop the close reading skills that will shape their own writing. Focused engagement with domain-specific writing should not be limited to basic reading comprehension and topical understanding. It must also include the analysis of the writing situation that is represented in the text: audience, context, purpose, subject, and writer. The close reading of well-written texts—regardless of the domain—will show students the importance of writing mechanics, diction, and syntax. The development of close reading skills will also help the students grow in terms of their ability to construct and advance independent and original claims that are well-supported by evidence. Domain-specific writing is grounded in positioning of claims and the effective use of evidence.

The final written product is important; nevertheless, the learning that results in this production must not be devalued. The writing process is not limited to the basic steps of planning, drafting, revising, and editing/proofreading. It is a complex sequence of critical and creative thinking and writing that leads to the production of a text that provides evidence of learning and understanding. Students must ultimately develop the ability to self-assess the effectiveness of their writing as a representation of the writing situation. Without the use of models that evidence learning and understanding, students will not develop the ability to self-assess their own work—the true outcome of the writing process.

What types of writing situations should RFH students engage in?

RFH students should engage in writing situations across the curriculum that require them to:

- write to improve mechanical proficiency, diction usage, and syntactical sophistication
- write to narrate, describe, and reflect
- write to summarize and report
- write to classify and define
- write to explain how process leads to an outcome
- write to compare, contrast and evaluate
- write to speculate on cause and effect
- write to propose solutions and solve problems
- write to analyze

These writing situations should be positioned in a coordinated, developmental sequence that extends across the academic disciplines.

Upon Completion of Grade 12, RFH students must be ready to transition to the following writing situations:

- write to analyze
- write to persuade (argument)

The core foci of first-year college writing courses are analysis and argument. These courses orient the students to the demands and expectations of writing for the academic culture of college. At colleges/universities with carefully coordinated writing programs, students must demonstrate proficiency in analysis and argument before they transition to upper level courses that require them to engage in the following writing situation:

- write to investigate (research)