Algebra II
Instructional Focus Documents

Introduction:

As districts adopt and implement high-quality instructional materials (HQIM) in mathematics, these Instructional Focus Documents (IFD) are intentionally designed to provide a lens into what effective mathematics instruction looks and sounds like in Tennessee classrooms. They are written to support all levels of leadership within a district and complement both the Math Implementation Framework and the Tennessee-specific Instructional Practice Guide (IPG). When used as a suite of resources, the IFDs, the Math Implementation Framework, and the IPG provide guidance and aligned measures with which to track and support district implementation of HQIM in mathematics.

Mathematical rigor does not simply mean increased difficulty or complexity of problems. Rigorous mathematical instruction and learning means deep thinking and exploring at a greater depth. The three aspects of rigor are Conceptual Understanding, Procedural Skill and Fluency, and Application. Each aspect is equally important and necessary for deep mathematical understanding and mastery. These aspects of rigor work in conjunction with the HQIM to provide a meaningful learning experience for students.

Aspects of Rigor:

Conceptual Understanding helps students understand the "how" and the "why" of mathematics. This aspect of rigor focuses on mathematical thinking and reasoning as opposed to answer-getting. Students should understand how and why the math works using mathematical models and manipulatives to aid in achieving conceptual understanding. Instruction should connect prior learning to new ideas and concepts. Opportunities for discussion and reflection may correct and unscramble common misconceptions. Flexible reasoning and fluency grow from conceptual understanding.

Procedural Skill and Fluency is the ability to apply mathematical knowledge accurately, flexibly, and efficiently. It is important to note that the phrase "procedural skill and fluency" is inclusive. The inclusive definition of procedural skill and fluency is *not* the rote use of an algorithm or the recall of facts, but a continuum of understanding. The continuum involves learning or developing algorithms and strategies, executing procedures accurately and efficiently, and learning how to use models and tools. Fluent mastery of a mathematical concept involves the ability to connect and use the Standards for Mathematical Practice while using algorithms and strategies for problem-solving. Students who have achieved fluency can link learned or developed algorithms and strategies to conceptual understanding to explain the "why" behind the procedures. Mathematically proficient students can understand the approaches to solving complex problems and identify correspondences between different approaches to select and use the most appropriate strategy to form an accurate solution path.

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Application refers to applying prior knowledge in new and unique situations, other subject areas, and mathematical and contextual problems. Application also includes intentionally integrated content that provides learning opportunities for students to apply and extend their knowledge of multiple standards, clusters, and/or domains within the grade level. The goal is for students to activate their prior knowledge in order to bring a sense of understanding to new mathematical and/or contextual situations.

Evidence of Learning Statements:
The evidence of learning statements provide guidance to connect the Tennessee Mathematics Standards with evidence of learning outcomes that can be collected through classroom activities, observations, or assessments, providing an indication of how students are tracking towards the grade-level expectations that are encompassed within the Tennessee Mathematics Standards. Within the evidence of learning statements, level 3 statements demonstrate on-grade level expectations for all Tennessee students.

The statements are designed to provide a continuum of concrete examples demonstrating what a student who has a particular level of conceptual understanding of the Tennessee mathematics standards will most likely be able to do in a classroom setting. Further, they provide a lens to help offer scaffolding to move a student with unfinished learning up to grade level expectations.

When used alongside high-quality instructional materials, these concrete examples serve to reinforce the grade level expectations and rigor that should be present within the materials and reinforce their inclusion within instruction, ensuring all students have access to on-grade level activities.

Instructional Focus Statements:
Instructional focus statements provide guidance to clarify the types of instruction that will help a student progress along a continuum of learning. These statements are written to provide strong guidance with a focus on Tier I, on-grade level instruction. Thus, the instructional focus statements are written for level 3 and 4.

When used in conjunction with HQIM, instructional focus statements support teacher understanding as they plan and implement HQIM to the depth and rigor of the Tennessee mathematics standards. Additionally, they serve as a benchmark for district and school leaders to use alongside the IPG as they are monitoring HQIM implementation.

Number and Quantity

The Real Number System (N.RN)

Standard A2.N.RN.A.1 **Cluster Heading: A. Extend the properties of exponents to rational exponents.**

Extend the properties of integer exponents to rational exponents.

A2.N.RN.A.1a Develop the meaning of rational exponents by applying the properties of integer exponents.

A2.N.RN.A.1b Explain why $x^{1/n}$ can be written as the n^{th} root of x.

A2.N.RN.A.1c Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Scope and Clarifications:

Part B: for example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5(^{1/3})^3$ to hold, so $(5^{1/3})^3$ must equal 5.

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Compute the square root and cube root of expressions. Identify equivalent forms of expressions involving integer exponents that include the product of powers or power of powers. Identify equivalent forms of expressions involving integer exponents that include the quotient of powers.	Use properties of integer exponents to generate equivalent numerical expressions. Identify patterns from properties of integer exponents and connect them to rational exponents. Identify the index and power of an expression. Understand the inverse relationship between radicals and exponents.	Explain why $x^{1/2}$ can be written as the square root of x. Explain why $x^{1/3}$ can be written as the cube root of x. Apply the properties of integer exponents to expressions involving rational exponents. Flexibly rewrite equivalent expressions using rational exponents or radical notation.	Construct a viable argument explaining the relationship between rational exponents and radical notation. Critique the reasoning of others by finding errors and justify changes that could create equivalent expressions involving radicals and rational exponents.

Instructional Focus Statements

Level 3:

In grade 8 (8.EE.A.1), students know and apply the properties of integer exponents to generate equivalent numerical expressions as well as evaluate square root and cube root expressions (8.EE.A.2). In Algebra II, students extend their knowledge of properties of exponents to include rational exponents. Instruction should include problems where students see a connection between inverse operations such as multiplication and division and how these inverse operations are expanded to radicals and exponents.

Begin instruction by giving students the opportunity to explore examples such as $(\sqrt{16})^2$ is an equivalent expression to 4^2 which are both equivalent to 16. By exploring this concept, students will understand how the square of a number is the inverse operation of taking the square root. Instruction should also lead students to the understanding that the ½ power represents a square root. Students should be expected to rewrite expression using factorizations and exponent rules to apply the meaning of these rational exponents. For example, given $(\sqrt{16})^3$, students will rewrite the example as $16^{3/2}$. This would help students recognize the square root of 16 is equal to 4, and then cube that number to reveal the answer of 64. Students should explore which patterns hold true to reinforce the order of operations and various exponent properties.

A common misconception for students is the confusion of the meaning of an exponent -2 with the meaning of ½ power. Students may confuse $(9^{-2})^2$ as the same as $(9^{1/2})^2$ rather than 9^{-4}. In addition, students often confuse rewriting expression in a simpler form by taking the square root of a number with finding the solution set of an equation. Discussion should address the difference between the $\sqrt{16}$ and $x^2 = 4$, using the principal square root function. With conceptual understanding of this standard, students should be able to explain why $(4^{1/2})^2 = 4$.

Students should be able to move fluently between radical and exponential form, meaning that they are flexible, efficient, and accurate. Instruction should expose students to a variety of problems that include radicals and exponents so that they realize that some problems are more efficient to work a certain form. By having students rely on their knowledge of exponent rules and meaning of radicals, students may approach rewriting equivalent expressions involving radicals or rational exponents using various solution paths. Facilitate discussion on different methods for the same type of problem and allow students to explain the various approaches. After students have practiced with multiple approaches, they should be able to choose which approach they prefer and defend their reasoning for choosing a particular method.

Level 4:

To deepen the level of understanding, instruction should provide the opportunity for students to engage in more complex problems involving rational exponents and the notation for radicals in terms of rational exponents. Students will apply properties of rational exponents in order to rewrite an expression and change parts written in radical form into exponential form. This process might make it easier for students to apply properties of

exponents. An example would be to rewrite $\sqrt[3]{x^2} * \left(\sqrt[6]{x^4}\right)^{-3}$ or $\frac{\sqrt[3]{x} * \sqrt{x^5}}{\sqrt{25x^{16}}}$. As students rewrite these expressions, they should justify their steps and explain their reasoning.

Additionally, focus should be placed on critiquing the reasoning of others. Instructions should include providing students with expressions that include rational exponents and radicals which have been rewritten. Students determine if the expressions are rewritten as equivalent expressions and if not, they should correct the mistakes and explain why it was incorrect.

Exposing students to more complex expressions, such as ones that include fractions with radicals in the denominator, will help deepen students understanding of this standard. While rationalizing denominators is not an expectation for this course, instruction should include providing students with simple problems where they are given a radical in the denominator. Challenging students to find another expression that, when multiplied, will produce a value that has an integer root will lead to discussions about repeated factors and various types of roots. For example, students should understand that when the $\sqrt[5]{4} * \sqrt[5]{8} = \sqrt[5]{32} = 2$ and would eliminate the radical.

As students become familiar with these types of expressions, instruction will extend to fractions and eventually, complex fractions. As students become proficient with rationalizing the denominator, they should understand why this process is important for eliminating the radical. Instruction should also include expressions where variables occur under the radical sign. Students should be introduced to the Absolute-Value-Square Root Theorem and in simple terms understand that if an odd power is removed from an even root, an absolute value sign will ensure variables have a positive value. In addition, students should be provided with radical and exponential expressions which have been simplified.

Number and Quantity

Quantities* (N.Q)

Standard A2.N.Q.A.1 **Cluster Heading: A. Reason quantitatively and use units to understand problems.**

Use units as a way to understand real-world problems.*

A2.N.Q.A.1a Choose and interpret the scale and the origin in graphs and data displays.*

A2.N.Q.A.1b Use appropriate quantities in formulas, converting units as necessary.*

A2.N.Q.A.1c Define and justify appropriate quantities within a context for the purpose of modeling.*

A2.N.Q.A.1d Choose an appropriate level of accuracy when reporting quantities.*

Scope and Clarifications: (Modeling Standard)

Apply this standard to any real-world problems studied within the scope of this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X	X*	X

**procedural skill and fluency only applies to part c.*

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify units appropriately when solving contextual problems. Choose a graph or data display, with appropriate scale and/or units, to represent a real-world problem. Identify individual quantities in context, label them with appropriate units, and connect the units to the values in the real-world problem. Explain the difference between precision and accuracy.	Choose units appropriately when solving contextual problems, including problems that contain real-world formulas. Choose an interpretation of a graph and/or data display, with appropriate scale and/or units, that represents a real-world problem. Identify and interpret necessary information, recognize any extraneous information, and attend to unit selection and/or conversions	Choose and interpret units, scale, and the origin in graphs and data displays that represents a real-world problem, including problems that contain real-world formulas. Determine the most appropriate data display based on the units given in a problem. Use appropriate quantities in formulas, converting units when necessary.	Explain if the information is represented appropriately using mathematical justification, given a numerical and/or a graphical representation of a real-world problem. Explain the interpretation of the units, scale, and origin with respect to the contextual situation using precise mathematical vocabulary. Create a real-world problem involving formulas and data, represented either in a table or

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
	in order to select a quantity that models a real-world situation. Choose a solution that is both accurate and reasonable with respect to the contextual situation. Describe how using inaccurate measurements when reporting quantities can affect the solution.	Define and justify with precision the appropriate quantities, and their units, within a context to model a real-world problem. Make observations about extraneous information embedded in context and explain why information is extraneous in a real-world problem. Choose an appropriate level of accuracy, including appropriate units, when reporting quantities in a real-world context. Explain the reasonableness of answers with respect to the context of the problem when reporting quantities as a result of solving the contextual problem. Describe the most common causes of inaccuracies in contextual problems (e.g., when using measurement tools).	graph, in which the data must be analyzed for appropriate units and scale. Flexibly represent quantities in modeling situations with data displays, numerically, algebraically, and graphically and explain relationships among representations. Describe the accuracy of a measurement embedded in a real-world context by stating the possible error when appropriate. Explain why it is important to choose an appropriate level of accuracy and what limitations exist on measurement when reporting quantities in contextual problems.

Instructional Focus Statements

Level 3:

In grades K-8, students developed an understanding of measuring, labeling values, and understanding how the value of a number relates to the described quantity. In the high school Numbers and Quantity (N) domain, students develop an understanding of reasoning quantitatively (MP 2) and solving

problems requiring the evaluation of the appropriateness of the form in which quantities are provided and reported. This standard is important throughout Algebra 1, Geometry, and Algebra 2 and should be integrated with a wide variety of standards throughout the course.

When choosing and interpreting the scale and the origin in graphs and data displays, instruction should focus on the importance of correctly interpreting units and representing contextual problems using appropriate graphs and data displays. Instruction should be centered around tasks that provide real-world, multi-step problems, where interpreting units appropriately is critical. Students should be able to identify and interpret necessary information, in order to select a quantity, with correct units, that models a real-world situation. Students should explore the importance of correctly interpreting units and designing displays of data to most appropriately represent a contextual problem. Students should understand that a key relationship exists between units and appropriate representation of units. Understanding this relationship is beneficial in developing students' conceptual understanding of units in contextual problems.

Students should be able to recognize the relationship between the units for all variables in a formula, use appropriate quantities in formulas, and convert units as necessary. Students should be engaged in classroom discourse (MP 8) that promotes explaining their reasoning and justifying quantities (MP 3) as a result of a solution pathway. Students should be given ample opportunity to work with data representations where students must think critically to set an appropriate scale that displays the data to show key features. It is imperative that students attend to precision (MP 6) in using, interpreting, and reporting units.

Students should extend their understanding of using appropriate quantities to defining and justifying appropriate quantities in modeling situations. Students should be able to select appropriate quantities embedded in real-world contextual problems, identify extraneous information, choose appropriate units, etc. Instruction should focus on providing opportunities for students to attend to precision (MP 6) by describing the quantities in modeling situations. Furthermore, students must be given opportunities to write and create appropriate labels for quantities and explain the meaning of the quantities in a context.

In middle grades, students have had experience interpreting and reporting quantities that involve area, volume, and rates. Students at this level are able to choose an appropriate level of accuracy when reporting quantities of a real-world problem. As this is a modeling standard, students should solve contextual problems and be able to choose a level of accuracy of measurement quantities that is reasonable and makes sense to the contextual situation. For example, when solving a multi-step problem or using graphing technology, students should determine when it is appropriate and not appropriate to use precise values (values that are not rounded or truncated), rounded values, or truncated values. Students should be able to justify their reasoning (MP 3) for using certain values and explain why their choice is important with respect to the context. Additionally, students should experience different solution paths that involve using different forms of values and explain how accuracy does or does not have an impact on the solution within the context of the problem.

This standard builds upon the opportunities students have been given to explain values of numbers in terms of units in previous grades. In middle school, students focused on ratios and proportional relationships and using quantities to describe data from a statistical lens. Both fields help prepare students for choosing a level of accuracy when reporting quantities. Instruction should focus on providing students a plethora of opportunities to use a variety of measurements including measuring tools and graphing technology (MP 5). Students should have ample time to explore traditional, physical tools as well as electronic, and digital tools. During this exploration, class discussion should focus on helping students ascertain the difference between precision and accuracy and when it is appropriate to apply each of them in certain problems. Furthermore, instruction should be infused with a broad spectrum of different types of units that describe tiny to very large quantities. This is a modeling standard and students should make connections to other disciplines such as science. Being able to identify, interpret, and justify quantities is a skill that will serve students well to have mastered during this course as this standard lays the foundation for using units to understand problems.

Level 4:

Students should be able to interpret, solve, and represent contextual information using appropriate units and attend to precision when interpreting and using units. One extension of this standard is for students to differentiate between data that is appropriately displayed versus data that is not appropriately displayed. Students can then analyze the difference between both displays, providing a critique of the representation. Students should use logical arguments, including explaining how to improve the representation so that all key features and units are displayed appropriately. Data should be represented graphically, numerically, algebraically, and verbally. Furthermore, students should be given opportunities to synthesize information from multiple sources and produce a model that represents the contextual situation.

Instruction should focus on providing opportunities for students to work with problems that have a variety of descriptors and units embedded in the context. Students should be asked to extend their knowledge of quantities by representing them in multiple formats such as a graphical representation of the given information, algebraic representation of the quantities, and multiple representations to predict or draw conclusions about the solution of the real-world problem. Instruction should provide opportunities for students to analyze and critique the interpretation of quantities in a modeling situation. Additionally, students should be given ample opportunities to design their own contextual problem in which they would have to use quantities appropriately in order to describe the modeled contextual situation.

Students have a great opportunity to support their understanding of this standard through the lens of a wide variety of other disciplines. Instruction should focus on providing students with experiences involving problem situations that interest them. Teachers should include opportunities for inquiry and experimentation with ample time for students to explore. For example, students could investigate repeated measurement by exploring the level of precision and/or accuracy provided by various measurement tools in science experiment to determine an acceptable level of accuracy when reporting quantities in real-world context (MP 6). Also, instruction should provide the opportunity for students to analyze and critique (MP 3) the level of accuracy chosen by others' reported quantities.

Number and Quantity

Matrices* (N.M)

Standard A2.N.M.A.1 **Cluster Heading: A. Perform operations on matrices and use matrices in applications.**

Use matrices to represent data in a real-world context. Interpret rows, columns, and dimensions of matrices in terms of the context.*

Scope and Clarifications: (Modeling Standard)

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify features of a matrix (rows, columns, dimensions) from a given matrix.	Identify a matrix that represents a real-world situation in a data display (chart, table, graph).	Represent data from a real-world context with a matrix. Interpret features of a matrix (rows, columns, dimensions) in terms of a real-world context.	Create a real-world context from a given matrix and explain how features of the matrix (rows, columns, dimensions, elements) represent the real-world context. Determine when a data display is best represented by a matrix and justify reasoning.

Instructional Focus Statements

Level 3:

A matrix is a rectangular array of numbers arranged in rows and columns. The dimension of a matrix, for example a *3x4* matrix, has three rows and four columns. A *3x4* matrix contains 12 elements (a_{11}, a_{12}, a_{31}, …a_{34}) where the first subscript represents the row location and the second subscript represents the column location. See Figure 1.

Figure1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

As this is students' first introduction to matrices, students will practice naming a matrix by its dimensions (*number of rows x number of columns),* and identifying features of the matrix (rows, columns, elements) absent of a real-world context. See Figure 2.

Figure 2

$$\begin{bmatrix} 1 & 2 & -4 \\ -1 & -5 & 3 \end{bmatrix}$$

# rows = 2
# columns = 3
dimensions = 2x3
element a_{13} = - 4

However, instruction for this standard should focus on students making connections among various data displays (charts, tables, graphs, etc.) and interpreting features of a matrix in a real-world context. Students should understand that matrices are just one way of organizing data/information from real-world situations. For example, students may be provided the following information in a table and be asked to create a matrix and interpret what the data represents. See Figure 3, a table representing five soil samples containing benzene (B), toluene (T), ethylbenzene (E), and xylenes (X) in (mg/kg).

Figure 3

Sample	B	T	E	X
1	0.06	0.95	0.90	18.50
2	0.06	1.05	0.73	13.50
3	0.35	6.00	5.60	49.00
4	0.22	0.19	2.00	19.50
5	0.11	0.82	2.50	26.00

$$\begin{bmatrix} 0.06 & 0.95 & 0.90 & 18.50 \\ 0.06 & 1.05 & 0.73 & 13.50 \\ 0.35 & 6.00 & 5.60 & 49.00 \\ 0.22 & 0.19 & 2.00 & 19.50 \\ 0.11 & 0.82 & 2.50 & 26.00 \end{bmatrix}$$

Consider connections to A2.N.Q.A.1a, where students are asked to choose and interpret the scale and origin of graphs and data displays.

Level 4:

As students become more flexible with matrices as data displays, students should be able to create a real-world context from a given matrix and explain how various features of the matrix (dimensions, rows, columns, elements) relate to the contextual situation. At this level, students should be able to interpret a contextual situation, determine which data display best represents the situation and justify their choice. Classroom discussions where students share solution paths and justify data display selections will strengthen students understanding of this concept.

Introducing students to matrix entry features using technology is also appropriate to preparation for matrix operations (A2.N.M.A.2), but is not an assessment expectation for this standard.

Number and Quantity

Matrices* (N.M)

Standard A2.N.M.A.2 **Cluster Heading: A. Perform operations on matrices and use matrices in applications.**

Perform operations on matrices in a real-world context.*

A2.N.M.A.2a Multiply a matrix by a scalar to produce a new matrix.*

A2.N.M.A.2b Add and/or subtract matrices by hand and using technology.*

A2.N.M.A.2c Multiply matrices of appropriate dimensions, by hand in simple cases and using technology for more complicated cases.*

A2.N.M.A.2d Describe the roles that zero matrices and identity matrices play in matrix addition and multiplication, recognizing that they are similar to the roles of 0 and 1 in the real number system.*

Scope and Clarifications: (Modeling Standard)

Part c: each matrix used as a factor is limited to no more than six elements when multiplying by hand.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X	X*	X

**procedural skill and fluency applies to parts a, b, and c.*

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify the scale used to produce a new matrix when a matrix is multiplied by a scalar. Add and/or subtract matrices with whole number elements and whole number solutions by hand and identify when the sum or difference is not possible. Multiply two matrices with whole number elements and whole number solutions by hand (limit	Multiply a matrix by a scalar to produce a new matrix. Add and/or subtract matrices with integer elements and integer solutions by hand and using technology and identify when the sum or difference is not possible. Multiply two matrices with integer elements and integer solutions by hand (limit hand calculation with no more than six elements) and with	Multiply a matrix by a scalar to produce a new matrix in a real-world context. Add and/or subtract matrices with rational elements and rational solutions by hand and using technology in a real-world context and identify when the sum or difference is not possible. Multiply two matrices with rational elements and rational solutions by	Choose to use scalar matrix multiplication to solve a non-routine problem in a real-world context (e.g., dilation of a geometric figure). Determine or create a real-world situation that can be solved with matrix addition or subtraction. Determine or create a real-world situation that can be solved with matrix multiplication.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
hand calculation with no more than four elements) and identify when the product of two matrices is not possible. Use properties of the real number system (additive identity) to identify an appropriate zero matrix to a given matrix.	the use of technology and identify when the product of two matrices is not possible. Use properties of the real number system (multiplicative identity) to identify an appropriate identity matrix to a given matrix.	hand (limit hand calculation with no more than six elements) and with the use of technology in a real-world context and identify when the product of two matrices is not possible. Describe the role of the zero matrix in matrix addition and the identity matrix in matrix multiplication in a real-world context.	Determine when a zero matrix or identity matrix is needed to solve a problem in a real-world context.

Instructional Focus Statements

Level 3:

Algebra 2 provides students their first exposure to matrices. Matrices, which are an ACT College and Career Readiness Standard math expectation, was not included in the previous iteration of TN Standards for Mathematics. In this new cluster, students will use matrices to organize and represent data, perform operations on matrices, and use matrices in applications. It is the expectation that all elements of matrices in Algebra 2 are limited to numerical entries. Matrix equations and matrices containing algebraic expressions are reserved for fourth year math courses.

In A2.N.M.A.1, students were introduced to matrices as a method of organizing and displaying data. In A2.N.M.A.2, instruction should build upon students' previous experiences with the real number system and properties of operations (additive identity, multiplicative identity, distributive property) and geometric concepts (dilations, scale, congruent and similar figures, coordinate graphing) to make connections to new learning with matrices. For example, previous learning on using and applying the distributive property (7.NS.A.2, 8.EE.C.7, A1.A.APR.A.1) supports students' understanding of multiplying a matrix by a scale factor. Further, connections to past learning around congruent and similar geometric figures, dilations and scale drawings of geometric figures, and representing dilations of geometric figures in the coordinate plane (7.G.A.1, 8.G.A.1, G.SRT.A.1) provide opportunities to situate new learning in real-world contexts and make the math of scalar matrix multiplication more visible for students. While performing scalar multiplication on a matrix to form a new matrix can easily be done by hand, teachers might also accompany instruction with the use of technology, such as introducing students to various menus and keystrokes needed for matrix entry and operations. Note: the terms "element/elements" and "entry/entries" are interchangeable with entry/entries typically used with calculator instructions.

Scalar multiplication can be performed on matrices of any dimension. However, two matrices must have equal dimensions to perform matrix addition and/or subtraction. Additionally, a new matrix, representing the sum or difference of a matrix operation, will also have the same dimension of the original matrices. Students may be asked if matrix addition and/or subtraction is possible and/or to identify when a sum or difference is not possible (i.e. ; the matrices do not have the same dimensions). Students meeting course-level expectations perform matrix addition and subtraction fluently by hand and with the use of technology. Students meeting course-level expectations can also identify when matrix addition and/or subtraction is not possible and can read and interpret technology errors accurately. Students should also be able to perform matrix addition and subtraction in a real-world context.

Students will multiply matrices of appropriate dimensions by hand in simple cases and using technology for more complicated cases. Students meeting course-level expectations perform matrix multiplication fluently by hand (limit hand calculation to no more than six elements) and with the use of technology. Students meeting course-level expectations can also identify when matrix multiplication is not possible and can read and interpret technology errors accurately. Students should also be able to perform matrix multiplication in a real-world context.

A common misconception for students in matrix multiplication is that, like in addition and subtraction, the two matrices must have the same dimensions. Thus, students may need additional support mastering a matrix multiplication algorithm and understanding how the associative and commutative properties apply to matrix multiplication. Connections to previous multiplication algorithms, such as multiplying multi-digit numbers or the product of two binomials, will help students make connections to using both multiplication and addition within the same problem to find a product. Tools and drawings, such as colored pencils, ink pens, highlights, and drawing arrows, will be helpful in assisting students in organizing their work. Student explorations of applying the associative and/or commutative properties to matrices of varying dimensions will support students' conceptual understanding rather than simply stating a rule for students to memorize.

To understand the roles that zero matrices and identity matrices play in matrix addition and multiplication, instruction should build upon students' past experiences with the properties of the real number system. For example, students will make connections between 0 and the zero matrix by exploring properties of the additive identity. Additionally, students will make connections to 1 and the identity matrix by exploring properties of the multiplicative identity. Students should have opportunities to discover, by hand and with the use of technology, how the zero and identity matrix behave, including identifying characteristics of needed dimensions and why the identity matrix does not consist of elements all equal to 1.

Level 4:

Students with a level 4 understanding become more flexible with matrix operations, knowing which operation to use when, as well as when to do calculations by hand verses with use of technology for increased accuracy or efficiency. Students are also comfortable interpreting real-world situations and knowing which matrix operation(s) are needed to solve problems. Students can also interpret sums, differences, and products of matrices in real-world contexts.

When presented with a problem in a real-world context, students recognize when a matrix can be used to represent and solve the problem. For example, students might be presented with a pre-image (e.g., geometric figure, real-world image) to identify/create a dilation or scale drawing. These students would determine that using matrices to organize ordered pairs and perform scalar multiplication is a useful solution path. Students with an advanced understanding of matrices for this course can determine or create real-world situations that can be solved with matrix addition, subtraction, or multiplication. This type of reasoning with matrix operations can help students build flexibility when solving non-routine problems.

Number and Quantity

Matrices* (N.M)

Standard A2.N.M.A.3 **Cluster Heading: A. Perform operations on matrices and use matrices in applications.** Create and use augmented matrices to solve systems of linear equations in real-world contexts, by hand and using technology. *

Scope and Clarifications: (Modeling Standard)

When solving by hand, limit system size to at most two unknowns, and when solving by technology, limit system size to at most three unknowns.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Rewrite a system of linear equations where each equation is written in standard form *(*A*x* + B*y* = *C;* A*x* + B*y* +*Cz* = D). Use elimination to solve a system of two linear equations with two unknowns by hand. Identify an augmented matrix that represents a system of two linear equations.	Create an augmented matrix that represents a system of linear equations. Perform operations by hand on rows of a matrix (at most two unknowns) to represent steps of solving a system of linear equations by elimination.	Create an augmented matrix that represents a system of linear equations (at most two unknowns) in a real-world context and solve by hand. Create an augmented matrix that represents a system of linear equations (at most three unknowns) in a real-world context and solve with technology. Interpret solutions of an augmented matrix, which represents a system of linear equations, as viable or non-viable within a context.	Analyze the results of a reduced augmented matrix to explain inconsistent and dependent systems of linear equations. Move flexibly between solution strategies (by hand, graphing, algebraic, matrices, technology), often combining more than one method, to determine the most efficient strategy/combination of strategies to solve, confirm, or explain the solution to a system of linear equations.

TN Department of Education

Instructional Focus Statements

Level 3:

In grade 8 (8.EE.C.8) students analyze and solve systems of two linear equations graphically. In high school, students write and solve a system of linear equations in a real-world context (A1.A.REI.A.1) and solve graphically, algebraically, and with technology (A2.A.REI.B.4). In A2.N.M.A.3, students learn a matrix can also be used to represent and solve a system of linear equations.

Given an *m*-by-*n* matrix, with *m* rows and *n* columns, equations written in standard form can be rewritten as a matrix where each row corresponds to an equation, each column corresponds to variable coefficients, and the final column corresponds to the equations' solutions. A matrix which contains the coefficients and constants from a system of equations is called an augmented matrix.

For example, consider the following system of equations.

$$\begin{array}{rcl} & 2x+y+3z=20 & \\ \textit{The system} & 5x+2y+z=21 & \text{is expressed as} \\ & 3x-2y+7z=9 & \end{array} \qquad \begin{array}{cccc} 2 & 1 & 3 & 20 \\ 5 & 2 & 1 & 21 \\ 3 & -2 & 7 & 9 \end{array}$$

Note, should an equation contain a missing term, an entry of zero is used in the augmented matrix.

It is important for students to understand a matrix as one way of organizing information or representing a real-world situation. Instruction should make explicit connections among representations (graphs, tables, equations, matrices, technology) to give students opportunities to see the same information represented in a variety of ways.

Students will be asked to solve a system of equations with at most two variables and two unknowns by performing elementary row operations by hand on an augmented matrix. Students will be asked to solve a system of equations with at most three variables and three unknowns with use of technology. Just as in graphical or algebraic solution strategies, students should be able to identify, interpret, and discuss solutions of systems of linear equations as viable or non-viable in a real-world context.

Level 4:

Students at this level should be fluid in matrix operations by hand and with the use of technology as well as solving systems of equations with graphing and algebraic methods. With access to a variety to tools and strategies, students become more sophisticated in determining which strategy to use when; often combining multiple strategies within one problem. For example, students may recognize an easy substitution to reduce a 3x4 matrix down to a 2x3 matrix and complete the rest of the calculations by hand or by entering less data into a calculator to save time.

When solving a system of equations with use of an augmented matrix, student and identify when solutions yield an inconsistent or dependent system. Students can also make connections and explain inconsistent and dependent systems in terms of no solutions or infinitely many solutions by using graphs or algebraic representations.

Algebra

Seeing Structure in Expressions* (A.SSE)

Standard A2.A.SSE.A.1 **Cluster Heading: A. Interpret the structure of expressions.**

Interpret expressions that represent a quantity in terms of its context.*

A1.A.SSE.A.1a Interpret parts of an expression, such as terms, factors, and coefficients.*

A1.A.SSE.A.1b Interpret complicated expressions by viewing one or more of their parts as a single entity.*

Scope and Clarifications: (Modeling Standard)

For example, interpret $P(1 + r)^n$ as the product of the initial value P and the growth rate after the first n years. View (1000 – 70x) (0.5 + 0.1x) as the product of the number of items sold and the cost of each item, which produces the profit, where x is the number of 10-cent price increases. View x(100 – 2x)(30 – 2x) as the product of the length, width, and height, which produces the volume of an open box made from a 100 by 30 rectangle with an x by x square cut out of each corner.

Tasks are limited to exponential, quadratic and cubic expressions.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify parts of an expression (i.e., factor, coefficient, term). Define the formal definition of the terms: *factor, coefficient, and term*. Define the formal definition of the term *expression*. Label the single entities in an expression.	Recognize arithmetic operations in an expression in order to see the structure of the expression. Understand and use the definitions of *terms, factors, coefficients, and like terms* in order to describe the structure of the individual parts of the expression. Identify parts of an expression as a single entity. Recognize that individual parts of	Interpret parts of an expression (i.e., term, factor, coefficient) embedded in a real-world situation and explain each part in terms of the context. Interpret parts of an expression (i.e., term, factor, and coefficient) and explain each part in terms of the function the expression defines. Explain the structure of an expression and how each term is related to the other terms by interpreting the arithmetic meaning	Interpret expressions in a variety of forms by explaining the relationship between the terms and the structure of the expression. Interpret parts of complex expressions with varying combinations of arithmetic operations and exponents by viewing one or more of their parts as a single entity. Write and interpret expressions that represent a real-world context and

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
	an expression affect the whole expression. State arithmetic operations performed within an expression.	of each term in the expression and recognizing when combining like terms is appropriate. Interpret an expression by describing each individual term as a single entity and the relationship to the expression.	use the expressions to solve contextual problems. Write expressions in a wide variety of formats and then for each describe the effects each term has considering them first individually and then considering them as a part of the expression. Identify and explain structure in patterns represented pictorially or graphically and write an algebraic expression to represent the pattern.

Instructional Focus Statements

Level 3:

Seeing structure in expressions is the connecting bridge between arithmetic operations in grades K-8 and algebraic thinking in high school. Instruction should build on students understanding of the relationship between arithmetic operations in expressions and equations. Students should explore a variety of expressions in equivalent forms to see and evaluate the structure in each form. Students should be exposed to exponents of varying degree. This allows them to recognize the attributes of a term in order to combine it appropriately with other like terms. Instruction should expose students to a variety of multiple representations and require students to interpret and explain the relationship between the representations. Students should be challenged with complex, multi-variable expressions to interpret.

Furthermore, students must be able to explain individual terms and interpret that term as a single entity and as a whole expression. Instruction should focus on using the structure of the expression to uncover the attributes of the function it defines. Students should also be able to use precise language to explain the relationship between a verbal description and an algebraic representation. Particular focus needs to be placed on translating words into mathematical expressions and vice versa.

This standard appears in both Algebra 1 and Algebra 2. Tasks are limited to linear, quadratic and exponential in Algebra 1. In Algebra 2, students will extend this understanding with exponentials, quadratics and cubics. In future courses, students will experience this standard with radical and trigonometric expressions, solidifying students' comprehension of the structure of expressions and interpreting the meaning of terms as single entities is imperative.

In Algebra 2 the focus is on exponential, quadratic and cubic expressions and functions. For the exponential function $f(x) = 3(1.5)^x$, students should recognize that the y-intercept is 3 and this function represents exponential growth with a growth factor of 50%, and $f(x) = 3(0.85)^x$ represents exponential decay with a decay factor of 15%. Instruction should include real-world situations where students can relate the structure of the function to the situation it is modeling.

Level 4:

Students need to be presented with complex expressions that include a combination of different arithmetic operations and interpret in terms of a real-world context. The pinnacle of level 4 understanding is being able understand, interpret, and explain the relationship between equivalent representations of an expression. Students should be able to explain not only the expression in terms of a contextual situation, but also how each term within the expression connects back to the contextual situation.

Additionally, instruction should focus on relating expressions to real-world contexts. For example, students should be given problems that describe contextual situations from multiple perspectives. Students should interpret the contextual situation for each individual perspective and write an expression that represents the context for each. Students should be challenged to interpret the meaning of the expressions created and use them to predict outcomes and solve problems. Instruction should expose students to multiple representations of the expressions by making connections between the equivalent expressions, which will in turn help students recognize the most useful form of an expression depending on context. Students should be challenged to justify why other formats are equivalent and which format is most relevant given the context of the problem.

Algebra

Arithmetic with Polynomials and Rational Expressions (A.APR)

Standard A2.A.APR.A.1 **Cluster Heading: A. Understand the relationship between zeros and factors of polynomials.**

Know and apply the Factor Theorem: For a polynomial *p(x)* and a number *a, p(a)* = 0 if and only if (*x* – *a)* is a factor of *p(x)*.

Scope and Clarifications:

Polynomials are limited to degree 3 or less.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Define factor. Given *p(x)* and a number *a*, find *p(a).*	Factor a polynomial of degree 3 or less. Determine if the graph of a polynomial *p(x)* has an intersection with the x-axis. Choose the remaining factor(s) of a polynomial *p(x)* when given a factor *x - a*. Verify by multiplication. Polynomials are limited to degree 3 or less.	Understand the relationships between the factors of a polynomial *p(x)* and the *x*-intercepts of the graph of *p(x).* Determine if *x-a* is a factor by applying the Factor Theorem for a polynomial *p(x)* and a number *a.* Polynomials are limited to degree 3 or less. Identify all factors of a polynomial *p(x)* by applying the Factor Theorem. Polynomials are limited to degree 3 or less.	Explain the Factor Theorem using appropriate mathematical vocabulary in both verbal and written form. Explain the equivalence between the linear factors *x-a* of a polynomial *p(x)* and the zeros *p(a)=0* of the function. Identify all possible factors of a polynomial *p(x)* using flexible solution paths. Polynomials are limited to degree 3 or less.

TN Department of Education

Instructional Focus Statements

Level 3:

Standard A2.A.APR.A.1 extends students' learning in Algebra 1 (A1.A.REI.B.3), where they find the zero(s)/solution(s)/root(s)/x-intercept(s) of quadratic functions with strategies such as graphing, taking the square root, factoring, and using the quadratic formula. In Algebra 2, students build on this foundation to factor cubic functions. The Factor Theorem is used to remove known zeros from a polynomial while leaving all unknown zeros intact, thus producing a lower degree polynomial whose zeros may be easier to find. Understanding the application and use of this theorem allows students to acquire a new strategy for factoring polynomials of degree 3 or less. Instruction should allow students the opportunity to explore how the Factor Theorem connects to their previously learned solution strategies.

To apply the Factor Theorem effectively and efficiently, students should develop both conceptual understanding and procedural skill and fluency. To build students' conceptual understanding, instruction should provide students multiple opportunities to make connections between the factors of the polynomial and zeros shown on a graph or in a table. To build students' procedural skill and fluency, students need opportunities to not only find accurate answers efficiently but must also have opportunities to explore connections to models and operations with real numbers. Students who employ flexibility with their procedural skill and fluency can apply their understanding of operations with real numbers to operations with polynomials. This is supported using concrete objects (e.g., algebra tiles), representational models (e.g., area models), and the relationships between a point on the graph and the zero of the function.

Students should be given opportunities to apply the Factor Theorem to linear, quadratic, and cubic functions. Discussion should lead students to also see the relationships between the factors of a linear, quadratic, and cubic function and the zeros of those functions (A2.A.APR.A.2).

Level 4:

Students at this level should be able to explain the Factor Theorem and its application using appropriate mathematical vocabulary in both verbal and written form. Instruction should include opportunities for students to both identify and explain how and when it should be used. Students with a deep conceptual understanding of the Factor Theorem can also explain the equivalence between linear factors and zeros: the fact that $p(a)=0$ if and only if $x-a$ is a factor of $p(x)$ and therefore deduce that if $x-a$ is a factor then $p(a)=0$.

When finding all possible zeros of the function $p(x)$, with degree 3 or less, instruction should give students the opportunity to make connections among multiple strategies and solution paths. For example, students at a level 4 understanding can explain how components within one strategy connect to components within another strategy (e.g., connecting the divisor $x-a$ in synthetic division to the zero a in the Factor Theorem). Students who are fluent in finding all zeros of polynomials of degree 3 or less draw upon multiple strategies for finding factors of the polynomial, can identify when one strategy is more efficient than another, and are comfortable combining multiple strategies along one solution path.

Algebra

Arithmetic with Polynomials and Rational Expressions (A.APR)

Standard A2.A.APR.A.2 **Cluster Heading: A. Understand the relationship between zeros and factors of polynomials.**

Identify zeros of polynomials when suitable factorizations are available and use the zeros to construct a rough graph of the function defined by the polynomial.

Scope and Clarification (Modeling Standard):

Polynomials are limited to degree 3 or less.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Factor a quadratic polynomial with a lead coefficient of 1. Choose the zeros for a given quadratic polynomial with a lead coefficient of 1. Choose a graph to represent a given quadratic polynomial in factored form.	Factor a quadratic polynomial with a lead coefficient of 1, identify the zeroes, and construct a rough graph of the function defined by the polynomial. Explain the mathematical term *zero* in the context of "zero of a polynomial" using appropriate mathematical vocabulary in both verbal and written form. Choose a graph to represent a given cubic polynomial presented in factored form. Generate a rough graph to represent a given quadratic	Factor a quadratic or cubic function, identify the zeroes and construct a rough graph of the function defined by the function. Generate a rough graph to represent a given non-quadratic polynomial function presented in factored form.	Explain the process for generating a rough sketch of any factorable polynomial function using accurate mathematical language in both written and verbal form.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
	polynomial presented in factored form.		

Instructional Focus Statements

Level 3:

Polynomial functions are, on the one hand, very elementary, in that they are built up out of the basic operations of arithmetic. On the other hand, they turn out to be amazingly flexible, and can be used to approximate more advanced functions such as trigonometric and exponential functions in later courses. Experience with constructing polynomial functions satisfying given conditions is useful preparation not only for calculus, but for understanding the mathematics behind curve-fitting methods used in applications to statistics and computer graphics.

The first step in developing this understanding is to construct a rough graph for polynomial functions by using their zeros. Eventually, this progression will lead to constructing polynomial functions whose graphs pass through any specified set of points in the plane.

It is important that students in this early stage continue to develop an understanding of the connection that exists between the graphical and algebraic representation of zeroes and that they are not simply following a rote procedure but are providing evidence of an understanding of this connection. Students should understand that the zero(s) of a polynomial is the same as the x-intercept(s), solution or the root(s). Students should be able to move flexibly between representations. Instruction should include opportunities to discover how to recognize the zeros of a quadratic and a cubic from various representations. They should discover that given a graph, the zeros are the x-intercepts, and given a table, the zeros are the x-values where the y-values are equal to zero, in a factored form of a function the zeros are the values of x that make each factor equal zero.

In Algebra II, students are focusing on quadratic and cubic polynomials when factors are not provided. Quadratic polynomials were also a focus for Algebra I. Thus, in Algebra II, when quadratics are the focus, they should be of appropriate difficulty.

Level 4:

At this level of understanding, students should be demonstrating strong command of the relationship that exists between an algebraic representation that elicits zeroes of a polynomial function and the graphical representation of zeros moving fluidly between the two. Additionally, they should be able to provide a mathematical explanation of the relationship between algebraic and graphical representations of zeroes.

Algebra

Creating Equations* (A.CED)

Standard A2.A.CED.A.1 **Cluster Heading: A. Create equations that describe numbers or relationships.**
Create equations and inequalities in one variable and use them to solve problems in a real-world context.*

Scope and Clarifications: (Modeling Standard)
Tasks are limited to quadratic, cubic, square root, cube root, or exponential equations and inequalities.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Choose a linear equation or inequality in one variable that represents a real-world situation. Solve a one-variable linear or quadratic equation or inequality. Represent real-world problems using equations in the form $px + q = r$ or $p(x + q) = r$.	Create and solve a one variable linear equation or inequality that represents a real-world situation. Choose a quadratic, cubic, square root, cube root or exponential equation or inequality to represent a real-world situation. Solve a one-variable quadratic, cubic, square root, cube root or exponential equation or inequality.	Create and solve a one-variable quadratic, cubic, square root, cube root, or exponential equation that represents a real-world situation. Create and solve a one-variable quadratic, cubic, square root, cube root or exponential inequality that represents a simple real-world situation.	Create a real-world situational problem to represent a given quadratic, cubic, square root, cube root, and exponential equation or inequality. Interpret and validate results by comparing with the situation and improve upon when necessary.

Instructional Focus Statements

Level 3:

In grade 7 (7.EE.B.4), students used variables to represent quantities in a real-world or mathematical problem and construct simple equations and inequalities to solve problems by reasoning about the quantities. Students continued to analyze and solve linear equations and inequalities in grade 8 (8.EE.C). In Algebra 1, students create equations and inequalities in one variable, limited to linear, quadratic, and absolute value equations and inequalities,

and use then to solve problems (A1.A.CED.A.1). In Algebra 2, students expand on this understanding to include quadratic, cubic, square root, cube root, and exponential equations and inequalities.

Students should write and solve equations and inequalities in one variable to answer questions about a real-world situation. Students should understand that equations and inequalities are examples of mathematical models. They are mathematical representations of a situation in life that can be used to make sense of problems and solve them (MP1). Facilitate classroom discussion where students make meaning of the context of a real-world problem in order to connect it to the structure of a mathematical equation. A common mistake is that students look only at the numbers and focus on key words that may change from one situation to another rather than the meaning of the given context. Provide opportunities for students to discover if a real-world situation can be represented by a linear, quadratic, or absolute value equation or inequality. They consider not only whether the equation or inequality appropriately models the situation, but also reflect on whether the solution or solution sets make sense in context. Students should practice reasoning quantitatively and abstractly (MP2) to make sense of quantities and their relationships in problem situations. Students should also engage in aspects of mathematical modeling (MP4) to be able to identify important quantities in a practical situation, solve problems arising in everyday life, and interpret the results in the context of the situation.

As this is a modeling standard, students need to encounter equations and inequalities in one variable that evolve from real-world situations. Students should be formulating equations and inequalities, computing solutions, interpreting findings, and validating their thinking and the reasonableness of attained solutions to justify solutions to real-world problems. Real-world situations should elicit equations and inequalities from situations which are quadratic, cubic, square root, cube root, and exponential in nature. It is imperative that students have the opportunity to work with each of these equation types equally.

Level 4:

When given an equation or inequality in one variable, students can generate a real-world situation that could be solved by a provided equation or inequality demonstrating a deep understanding of the interplay that exists between the situation and the equation or inequality used to solve the problem.

Additionally, students should continue to encounter real-world problems that are increasingly more complex. Students should be using the modeling cycle to solve real-world problems. The modeling cycle involves identifying variables in a situation, model by creating an algebraic representation, performing operations, interpreting the results in terms of the context, validating conclusions by comparing them with the original situation and then either improving upon the model or reporting on the reasoning behind the conclusions.

Algebra

Creating Equations* (A.CED)

Standard A2.A.CED.A.2 **Cluster Heading: A. Create equations that describe numbers or relationships.**

Create equations and inequalities in two variables to represent relationships between quantities and use them to solve problems in a real-world context. Graph equations with two variables on coordinate axes with labels and scales and use the graphs to make predictions.*

Scope and Clarifications: (Modeling Standard)

Tasks are limited to linear, quadratic, cubic, square root, cube root, exponential and absolute value equations and inequalities.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Choose a linear equation or graph that represents a real-world situation. Solve a two-variable linear, quadratic, or absolute value equation or inequality. Identify appropriate units of measure within a given real-world context.	Create and solve a two-variable linear, quadratic, or absolute value equation that represents a real-world situation. Graph a two-variable linear, quadratic, or absolute value equation that represents a real-world situation. Create a graph with appropriate scales and labels for a given real-world context.	Create and solve a two-variable linear, quadratic, cubic, square root, cube root, exponential or absolute value equation or inequality that represents a real-world context. Graph a two-variable linear, quadratic, cubic, square root, cube root, exponential or absolute value equation or inequality that represents a real-world context. Make predictions using the graph of a two-variable linear, quadratic, cubic, square root, cube root, exponential or absolute value equation or inequality that represents a real-world context.	Create a real-world situational problem to represent a given linear, quadratic cubic, square root, cube root, exponential or absolute value equation or inequality. Interpret and validate results by comparing with the situation and improve upon when necessary.

Instructional Focus Statements

Level 3:

In grade 8 (8.F.B.4), students constructed linear functions to model the relationship between two quantities. In Algebra I (A1.REI.D.5), students understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane. Students create and graph equations in two variables and use them to solve problems, limited to linear, quadratic and exponential in Algebra I (A1.CED.A2) and expand on this to also include cubic, square root, cube root, exponential equations and inequalities in Algebra II.

This standard is closely related to the interpreting functions standards (A2.F.IF), since a two-variable equation can be meaningfully reinterpreted as a function with a focus on the relationship between the variables. Students should graph equations and inequalities on coordinate axes and clearly label and define the values on the axes and the unit of measure. Students should also select intervals for the scale that are appropriate for the context and display adequate information about the relationship. Facilitate classroom discussion where students interpret the context and choose appropriate minimum and maximum values for a graph. To make predictions from graphs, students can read the graph to follow trends that appear in the data and use those trends to make predictions.

Students should write and solve equations and inequalities in two variables to answer questions about a real-world situation with attention paid to the relationships between the quantities. Students should understand that equations and inequalities are a type of mathematical model. They are mathematical representations of a situation in life that can be used to make sense of problems and solve them (MP1). Provide opportunities for students to analyze if a real-world situation should be represented by a linear, quadratic, cubic, square root, cube root, exponential or absolute value equation or inequality. They consider not only whether the equation or inequality appropriately models the situation, but also reflect on whether the solution or solution sets make sense in context. Students should practice reasoning quantitatively and abstractly (MP2) to make sense of quantities and their relationships in problem situations. Students should also engage in aspects of mathematical modeling (MP4) to be able to identify important quantities in a practical situation, solve problems arising in everyday life and interpret the results in the context of the situation.

As this is a modeling standard, students need to encounter equations and inequalities in two variables that evolve from real-world situations. Students should be formulating equations and inequalities, computing solutions, interpreting findings, and validating their thinking and the reasonableness of attained solutions to justify solutions to real-world problems. Real-world situations should elicit equations and inequalities from situations which are linear, quadratic, cubic, square root, cube root, exponential and absolute value in nature. It is imperative that students have the opportunity to work with each of these equation types equally.

Level 4:

One of the most natural situations for students to create an equation, inequality, or graph from is a real-world situation. Students need to be exposed to a variety of real-world situations that elicit the function types embedded within the Algebra II course. Students should encounter real-world problems that are increasingly more complex over time. They should be using the modeling cycle to develop and provide justifications for their solutions. The modeling cycle involves identifying variables in a situation, model by creating an algebraic representation, performing operations, interpreting the results in terms of the context, validating conclusions by comparing them with the original situation and then either improving upon the model or reporting on the reasoning behind the conclusions.

Additionally, students should be asked to generate a real-world situation that could be solved by a provided equation or inequality. Students with this capability are demonstrating a deep understanding of the interplay that exists between the situation and the equation or inequality used to solve the problem.

Algebra

Creating Equations* (A.CED)

Standard A2.A.CED.A.3 **Cluster Heading: A. Create equations that describe numbers or relationships.**

Rearrange formulas to isolate a quantity of interest using algebraic reasoning.*

Scope and Clarifications: (Modeling Standard)

Tasks are limited to quadratic, square root, cubic, cube root, exponential or logarithmic equations. For example, rearrange the formula for the area of a circle to isolate the radius. Rearrange the formula for the volume of a cube to isolate the side length.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X	X	X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Solve linear or quadratic equations with rational coefficients. Solve equations that require finding the square root of a number.	Choose equivalent forms of a given linear or quadratic real-world formula. Solve quadratic, square root, cubic, cube root, exponential or logarithmic equations and explain the solution path.	Describe the properties that allow an equation to be written in an equivalent form. Rearrange real-world quadratic, square root, cubic, cube root, exponential or logarithmic formulas to highlight a quantity of interest.	Rearrange real-world quadratic, square root, cubic, cube root, exponential or logarithmic formulas and explain the benefit of solving the formula for the various variables.

Instructional Focus Statements

Level 3:

Instruction for this standard should be integrated with A2.A.REI.A.1 where students understand solving equations as a process of reasoning and explain the reasoning. The properties used to solve equations apply regardless of how many variables appear in an equation or formula. Variables are placeholders for numbers and as such have the same properties.

Students understand that a relationship between two or more quantities can be expressed in multiple ways by writing equivalent equations. Students should discover that depending on the quantity in which they are interested, one form of equation might be more useful than others. To highlight a quantity of interest may mean manipulating or rearranging a given equation. The rearrangement may involve solving for a variable - isolating it and defining it in terms of the other variables. Students should notice that solving for a variable can be an efficient way to solve problems. Students should reason repeatedly and look for regularity as they manipulate equations and solve for a variable (MP8).
In Algebra 2, focus on real mathematical and scientific formulas including geometric formulas for area or volume or scientific formulas for energy or temperature. For example, The radius can be isolated in the formula for the area of a circle $A = \pi r^2$ by diving both sides of the equation by π, then taking the square root of both sides of the equation to generate the rearranged formula $r = \sqrt{A/\pi}$. As another example, the side length of a cube can be isolate in the formula for the volume of a cube $V = l^3$ by taking the cube root of both sides of the equation to generate the rearranged formula $l = \sqrt[3]{V}$.

Facilitate classroom discussion around the multiple different ways that the formula can be viewed as defining one quantity as a function of the other. Students should develop conceptual understanding of why they might need to write formulas in different ways and what the benefit would be to these various representations of the same real-world formula. This may be a good opportunity to collaborate with science teachers and ask them for formulas that they use often.

Note: Students are NOT expected to expand and condense logarithms.

Level 4:

Students should be exposed to a wide variety of real-world formulas increasing in complexity over time. Additionally, it is imperative that they can explain why formulas might need to be expressed in different ways and the benefit that each form provides. At this level, facilitate discussion where students to put into words what the original equation and rearranged equations say in relation to the given context. Students should construct a viable argument as to why they might be interested in highlighting a particular quantity of interest and how this is beneficial within the context of the formula (MP3).

Algebra

Reasoning with Equations and Inequalities (A.REI)

Standard A2.A.REI.A.1 **Cluster Heading: A. Understand solving equations as a process of reasoning and explain the reasoning.**
Understand solving equations as a process of reasoning and explain the reasoning. Construct a viable argument to justify a solution method.

Scope and Clarifications:
Tasks are limited to quadratic, radical, exponential, and logarithmic equations.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Choose the inverse operations used in solving a given quadratic, radical, exponential, and logarithmic equation and a list of steps for the solution path. Choose a possible next step to solve the equation, given a quadratic, radical, exponential, and logarithmic equation and a partial list of steps for the solution path. Arrange steps in the order they should be applied, given a quadratic, radical, exponential, and logarithmic equation and a list of unordered steps for the solution path.	Explain the reasoning for each step, given a quadratic equation and a list of steps for the solution method. Explain the reasoning for each step, given a radical equation and a list of steps for the solution method. Explain the reasoning for each step, given an exponential equation and a list of steps for the solution method. Explain the reasoning for each step, given a logarithmic equation and a list of steps for the solution method.	Solve quadratic, radical, exponential, and logarithmic equations using multiple solution strategies and explain the solution path. Construct a viable argument to justify a chosen solution path used to solve a quadratic, radical, exponential, or logarithmic equation. Determine an efficient solution path, given a quadratic, radical, exponential, or logarithmic equation with multiple solution paths.	Solve the problem and justify the solution path chosen, given a real-world problem and an equation that represents the contextual situation. Compare and contrast two given solution paths to a contextual problem and construct a viable argument on which method is most efficient. Correct the mistakes in the solution path and provide an explanation of the misconception using precise mathematical vocabulary, given a solution for a quadratic, radical, exponential, or logarithmic equation.

Instructional Focus Statements

Level 3:

In Algebra 1 (A1.A.REI.A.1), students understand solving linear, quadratic, and absolute value equations as a process of reasoning and explain the reasoning. In Algebra II, students should be exposed to quadratic, radical, exponential, and logarithmic equations. Students should develop a conceptual understanding of solving equations as a reasoning process to determine a solution that satisfies the equation rather than a procedural list of steps. Instruction should focus on students creating and determining solution paths for each unique equation and providing a viable argument to justify the chosen solution path (MP3). Students should also be able to explain how, when, and why equations have no solution or infinitely many solutions. To help give meaning to these solution types, discussion should focus on the solution being a value of the variable that makes the equation true. This will help students make the connections that an equation has no solution because there is no value that can maintain equivalency and an equation has infinitely many solutions because all values used for the variable create a true equivalency statement.

Students should understand that a problem can have multiple entry points and instruction should be focused on solving equations using a reasoning process centered around inverse operations, order of operations and properties of operations. Students develop a conceptual understanding of operations in previous grades, and they should deepen their understanding of the interplay that exists between the operations. Properties of operations can be used to change expressions on either side of the equation to equivalent expressions. To illustrate maintaining equivalency, a visual and/or concrete model of a balance scale can be used to aid students in understanding that the same inverse operations are being applied to each side of an equation to maintain equivalency. Emphasizing equivalency is vital in developing a conceptual understanding of solving equations and preventing common misconceptions.

A common misconception is applying an exponent to each term individually instead of applying the power to the entire side of an equation as a quantity. For example, when solving $\sqrt{x-1} = x+2$, students may make the common misconception of raising each individual term to the second power, resulting in the inaccurate next step of $x - 1 = x^2 + 4$. Classroom discussion should address the importance of using grouping symbols, when necessary, for example $\left(\sqrt{x-1}\right)^2 = (x+2)^2$, and applying the properties of exponents appropriately as students use inverse operations to solve equations. When solving equations, students should understand that inverse operations should be applied to the left side of the equation as one quantity and the right side of the equation as one quantity, not to each term individually.

Students should understand that the solution path they choose to solve any equation must result in a series of equivalent equations, all of which have the same solution set. As students apply mathematical properties and inverse operations to solve equations, they should be able to explain why equality holds true when performing the selected operation to both sides of the equation. In this course students should be exposed to quadratic, radical, exponential, and logarithmic equations.

Level 4:

As students develop a deeper understanding of solving equations and explaining their solution methods, instruction can be integrated with the application in contextual situation. Students should be able to construct equations that represent a contextual situation as well as create contextual situations to represent a given equation. As students develop a deep understanding of the relationship that exists between the type of equation and the context, they can be given equations embedded in real-world situations. When they are given a contextual situation and an equation, students should be able to determine what each part of the equation represents as it relates to the context. They should also be able to solve the equation and create a viable argument to justify their solution path. Students should understand that there are various ways to solve problems and justifying their steps will help them solidify their understanding of solving equations as well as the most efficient solution path. This standard pairs nicely with A2.A.CED.A.1 as it supports the idea of making connections between an equation and its context.

To challenge students to follow a thought process other than their own, they can be asked to critique or correct the solution paths of others. Students will develop a deeper level of understanding if they are given solution paths with incorrect steps in the process or invalid justifications and asked to correct the process or write justifications and defend them.

Algebra

Reasoning with Equations and Inequalities (A.REI)

Standard A2.A.REI.A.2 **Cluster Heading: A. Understand solving equations as a process of reasoning and explain the reasoning.**
Solve radical equations in one variable and identify extraneous solutions when they exist.

Scope and Clarifications:
Limit radicand to a linear or quadratic expression. Limit the index to a value of 2 or 3. Tasks may or may not have a real-world context.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify the relationship between squaring a number and taking the square root of a number as inverse operations. Identify the relationship between cubing a number and taking the cube root of a number as inverse operations.	Flexibly rewrite equivalent expressions using rational exponents or radical notation. Solve equations that require finding the square or cube root of a number.	Solve radical equations in one variable with an index value of two or three. Identify when extraneous solutions occur algebraically. Solve radical equations within a real-world context.	Critique the reasoning of others by finding errors and justify changes that could be made to correct mistakes, given a solution path to solve a radical equation. Explain why extraneous solutions exist.

Instructional Focus Statements

Level 3:

In grade 8, students solve equations of the form $x^2 = p$ and $x^3 = p$ (8.EE.A.2) and in Algebra I (A1.REI.B.3), students solve quadratic equations in one variable. Building on their understanding of solving an equation with a variable taken to a power, as well as previous work with square roots and cube roots, Algebra II students solve radical equations in one variable with unknown values under the radical, limited to index values of two or three and linear or quadratic radicands. Students should continue to apply the conceptual understanding developed in A2.A.REI.A.1 that solving an equation is a process of reasoning.

When solving radical equations, students should connect to their learning about rational exponents (A2.N.RN.A.1). Students should be able to flexibly rewrite equivalent expressions using rational exponents or radical notation. For example, students understand that the radical expression $\sqrt[3]{5x}$ can also be written using a rational exponent as the expression $(5x)^{1/3}$. Students may choose to approach solving radical equations by leaving expressions in radical form or rewriting an equivalent expression using a rational exponent.

Allow students the opportunity to discover that squaring each side of an equation doesn't necessarily preserve solutions, therefore introducing the idea of extraneous solutions, a new concept for students in Algebra II. Instruction should include opportunities for students to examine the case in which an equation has no solutions because squaring each side of the equation results in a new equation that does have a solution. A common mistake is to assume that solutions obtained after squaring each side are solutions to the original equation. For example, squaring both sides of the equation $\sqrt{x-1} = -3$ results in $x - 1 = 9$. Students should be able to identify that 10 is an extraneous solution by substituting the value for the variable in the original equation and identify that this generates a false statement. The solution to $x - 1 = 9$ is not a solution to $\sqrt{x-1} = -3$ because $\sqrt{9} = -3$ is false. Students should be able to explain why extraneous solutions arise and why they are not correct solutions. This can be supported by exposing students to the graphs of radical equations and connecting their understanding of solutions as x-intercepts of the function. Allow students to see what an extraneous solution looks like in the graph of a radical equation. This understanding is connected to A2.F.IF.A.1 where students further explore interpreting key features of graphs of various function types, including square and cube root functions. Students should also be able to explain the meaning of the solutions if working within a real-world context and consider whether the solutions are reasonable for the given situation.

A common misconception when solving square and cube root equations is to not isolate the radical expression on one side of the equation before solving. For example, when solving $\sqrt[3]{x+4} - 2 = 1$, students should first apply an inverse operation by adding 2 to each side of the equation, resulting in $\sqrt[3]{x+4} = 3$. The fewer terms there are before cubing both sides of the equation, the fewer additional terms will be generated by the process of cubing. When applying an exponent to an equation, classroom discussion should also address the importance of using grouping symbols, when necessary, in order to apply the power to the entire side of an equation as a quantity rather than the common misconception of raising each individual term to a power.

Students should explore how solving square root and cube root equations are similar and different. Solving cube root equations is different from solving square root equations because cube roots aren't always positive, so the extra step to check for an extraneous solution isn't necessary. Solving square root and cube root equations are similar because both require isolating the radical expression and then using exponents to generate a new equation without radicals. In this course, radicands should be limited to a linear or quadratic expression.

Level 4:

To deepen the level of understanding, instruction could include providing opportunities for students to follow a given solution method on a radical equation. By presenting partially solved problems and asking students to finish the problem to reveal the solution, students will have to follow the reasoning of others and justify their reasoning. To enforce this same concept, students can be provided radical equations that have been completed solved. Students could then be asked to critique the reasoning of others as they examine the problems and determine if mistakes are made. They should be expected to make corrections and justify their reasoning.

Algebra

Reasoning with Equations and Inequalities (A.REI)

Standard A2.A.REI.B.3 **Cluster Heading: B. Solve systems of equations.**

Write and solve a system of linear equations in a real-world context.*

Scope and Clarifications: (Modeling Standard)

When solving algebraically and graphically, tasks are limited to systems of at most two equations and two variables. When solving using technology, tasks are limited to systems of at most three equations and three variables. For example, use systems of equations to find the vertices of a triangle defined by three lines.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X	X	X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify the solution to a system of linear equations in two variables given a graphical representation of the system. Identify the solution to a system of linear equations in two variables given a tabular representation of the system. Use substitution to determine if a given solution satisfies a given system of linear equations.	Solve a system of linear equations by graphing. Solve a system of equations in two variables algebraically through substitution or elimination. Identify the solution to a system of equations in three variables from a table. Analyze a system of linear equations to determine if there is one solution, no solution or infinitely many solutions.	Write and solve a system of equations in two or three variables, from a real-world situation. Interpret the solution of a system of equations in two or three variables in a real-world context. Solve a system of equations in three variables using technology. Justify why a system of two or three linear equations may have one solution, no solutions, or infinitely many solutions.	Create a real-world scenario to represent a system of equations in three variables. Determine if the solution to a system of equations is reasonable in relationship to a real-world context. Interpret the differences between one solution, no solutions, or infinitely many solutions, given the graph of a system of equations in three variables.

TN Department of Education

Instructional Focus Statements

Level 3:

In Algebra I (A1.A.REI.C.4), students write and solve systems of two linear equations graphically and algebraically. In Algebra II, instruction should continue to use multiple representations including graphs and tables to help students visualize the solution to a system of equations and support their ability to solve algebraically. When solving a system of two equations graphically, students may approximate this intersection with technology.

In Algebra II, students will expand their understanding of systems to work with systems of equations in three variables. Students have experience with the process of substitution and elimination from Algebra I, so discussion should help lead students to recognize how these algebraic methods could be used to solve a system of equations in three variables. As the discussion focuses on elimination, students should be led to discover that the process should begin by rewriting the linear system in three variables as a pair of two equations. The solving process will have more layers to solve for more variables and will require students to have the ability to recognize the limitation of only solving for one unknown value at a time. Support student understanding of how to combine the value of one variable and continued use of the elimination method to solve for the other variables. It's important for students to understand why there is a need for multiple approaches to solving a system of equations. Encourage students to look for and assess the structure of the system of equations to determine whether using the substitution method, elimination method or solving graphically might be the most efficient and justify their reasoning.

The use of technology is also important when solving a system of three equations. Since the calculator cannot be used to graph the system as it is with a two-variable system of equations, there is an opportunity for matrices to be introduced as a tool for solving systems. In Algebra II, students learn to perform operations on matrices and use matrices in applications within the A2.N.M (Matrices) cluster of standards. Instruction should focus on helping students arrange the equations so they can be entered into a matrix, emphasizing the structure and importance of the coefficients of like variables arranged together. This would allow students to enter coefficients and constants into a matrix and then use operations of matrices in the graphing calculator to solve the system of equations.

Instruction should include engaging students in real-world problems where they write and solve equations from context. Students should be expected to define variables and write linear equations to represent a given context. In some problems, students must determine if an approximation of a solution or an exact solution is most appropriate for the problem and be able to interpret the meaning of these solutions in terms of the context. Additionally, students should be expected to differentiate among problems where there is one solution, no solutions, or infinitely many solutions and justify the meaning or reason for these results. Students should understand that one solution is represented by graphs of lines that intersect at one point, no solution is represented by graphs of lines that are parallel and never intersect, and infinitely many solutions is represented by graphs that lie on the same line and are infinitely intersecting.

Level 4:

As students build a strong understanding of how to solve a system of equations in three variables, they should be challenged to extend their understanding of these equations in context by being asked to write a real-world scenario to represent a system. They will continue to demonstrate understanding by choosing the most appropriate method for solving a system and justify their reasoning throughout the solution process. Although instruction may not include graphing a system of equations in three variables, discussion on the x, y, and z planes and how they are related can help students visualize the system. This experience can help students understand the visual representation of one solution, no solutions, or infinitely many solutions.

Algebra

Reasoning with Equations and Inequalities (A.REI)

Standard A2.A.REI.B.4 **Cluster Heading: B. Solve systems of equations.**

Solve a system consisting of a linear equation and a quadratic equation in two variables algebraically, graphically, and using technology.

Scope and Clarifications:

Tasks may or may not have a real-world context.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X	X	X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify the approximate solution to a system of linear equations from a graph. Identify the solution to a system of linear equations from a table. Use substitution to determine if a given solution is correct for a system of linear equations.	Solve a system of linear equations. Graph a quadratic equation given in any form. Solve a quadratic equation algebraically. Graph a quadratic equation and linear equation on the same coordinate plane to approximate the solution. Identify whether a system consisting of a linear equation and a quadratic equation would have no solution, one solution, or two solutions, given a graph.	Solve a system of a linear equation and a quadratic equation algebraically. Solve a system of a linear equation and a quadratic equation graphically. Solve a system of a linear equation and a quadratic equation using technology. Solve a system of a linear equation and a quadratic equation within a real-world context. Justify why a system consisting of a linear equation and a quadratic equation may have no solution, one solution, or two solutions.	Justify the solution to a system consisting of a linear equation and a quadratic equation using multiple representations. Write and solve a real-world scenario to represent the system, given a system of one linear equation and one quadratic equation. Construct an argument to explain if the solution to a system with one linear and one quadratic equation is reasonable within a given context.

TN Department of Education

Instructional Focus Statements

Level 3:

Students solve systems of linear equations graphically in grade 8 (8.EE.C.8) and algebraically in Algebra 1(A1.REI.C.4). Solving a system of one linear equation and one quadratic equation is a natural extension of prior learning about systems of linear equations. Instruction should continue to reinforce the use of multiple representations including graphs and tables to help students visualize the solutions and support solving algebraically. When solving a system of two equations graphically, students may approximate this intersection with technology. Algebraically, systems of one linear and one quadratic equation can be solved using substitution. Students have experience using the substitution method with linear equations in Algebra I, so building on this understanding, students should revisit how to substitute an expression for the value of a variable. When solving each equation for y to graph using technology, discussion should lead students to recognize the equivalency when both equations are equal to y and help them recognize how to set the equations equal to each other and solve for the remaining variable.

Previously, students discovered patterns that cause linear systems to have no solution, one solution, or infinitely many solutions. To build on this understanding, students should be provided graphs of a system of linear equations and a system with a linear and a quadratic equation and asked to compare the possible numbers of solutions for these systems. These discussions should help students recognize how the different number of solutions exist; no solution because the graphs do not intersect, one solution when the line is tangent to the quadratic, or two solutions when the line intersects the quadratic twice. Students can also discover why a system consisting of a linear and a quadratic equation cannot have infinitely many solutions. Although this is one of the first times students have seen systems of equations that can have two solutions, students have experience solving quadratic equations (A1.A.REI.B.3) that may result in two solutions. When solving a quadratic equation in one variable, students made the connection that the solution(s) to the equation are the x-intercepts on the graph. Students can connect this concept to finding the solution(s) to a quadratic equation as a system with the linear equation y=0. Students should recognize that solution(s) to a system are now the intersection points of the graph of the quadratic equation with the line created by the linear equation as opposed to the x-axis.

Students should have opportunities to work with equations and context that include rational numbers in a real-world situation. Students should be expected to define variables and write linear equation and quadratic equations to represent a given context. Students should have opportunities to engage in real-world problems in which they must determine if an approximation of a solution using a graph or an exact solution using other methods is most appropriate for the problem. Students should be encouraged to find exact solutions both with and without technology. By attending to precision, students will leave irrational solutions in simplified radical form, when the situation arises.

Note: Students are NOT expected to find complex solutions, but they are expected to recognize when non-real solutions exist.

Level 4:

Instruction can extend learning by challenging students to justify solutions to a system of one linear equation and one quadratic equation using different representations of equations. When given a linear and quadratic system represented one way, students would be expected to justify their solution using multiple representations. Options would include substitution of the solution, solving algebraically, using a table of values, graphically or using technology.

Once students have a solid algebraic understanding of systems of this type, they could be challenged to write a real-world problem representing the system, construct a table which represents both equations, find the solution and graph the system either by hand or using technology. Students would then need to be able to explain the meaning and justify the reasonableness of the solution in context.

Functions

Interpreting Functions* (F.IF)

Standard A2.F.IF.A.1 **Cluster Heading: A. Interpret functions that arise in applications in terms of the context.**

For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.*

Scope and Clarifications: (Modeling Standard)

Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and/or asymptotes where appropriate.

Tasks are limited to linear, quadratic, cubic, square root, cube root, exponential, logarithmic, and piecewise (including absolute value functions).

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify intercepts, maximums and minimums when provided a graphical representation of the function. Identify intervals where a given function is increasing, decreasing, positive or negative when provided a graphical representation of the function.	Identify key features of the graph or table of values and interpret the meaning of the key features in relationship to the context of the problem, given a graph or table of values representing a linear or quadratic function embedded in a real-world context. Identify all evident intercepts, maximums and minimums when provided a table of values representing an exponential function with domain in the integers.	Identify all evident key features when provided a table of values representing a linear, quadratic, cubic, square root, cube root, exponential, piecewise or logarithmic function. Identify key features of the graph or table and interpret the meaning of the key features in relationship to the context of the problem, given a graph or table of values representing a square root function embedded in a real-world context. Identify key features of the graph or table and interpret the meaning of	Graph the function, identify key features of the graph, and interpret the meaning of the key features in relationship to the context of the problem, given a square root function embedded in a real-world context. Graph the function, identify key features of the graph, and interpret the meaning of the key features in relationship to the context of the problem, given a cube root function embedded in a real-world context. Graph the function, identify key features of the graph, and interpret

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
	Identify key features of the graph and interpret the meaning of the key features in relationship to the context of the problem, given a graph of an exponential function with domain in the integers embedded in a real-world context, Identify evident intercepts, maximums and minimums when provided a table of values representing a square root, cube root, polynomial, exponential, piecewise or logarithmic function. Sketch a graph of the function, given a verbal description of the key features of a quadratic function.	the key features in relationship to the context of the problem, given a graph or table of values representing a cube root function embedded in a real-world context, Identify key features of the graph or table and interpret the meaning of the key features in relationship to the context of the problem, given a graph or table of values representing an exponential function embedded in a real-world context. Identify key features of the graph or table and interpret the meaning of the key features in relationship to the context of the problem, given a graph or table of values representing a polynomial function embedded in a real-world context. Identify key features of the graph or table and interpret the meaning of the key features in relationship to the context of the problem, given a graph or table of values representing a logarithmic function embedded in a real-world context.	the meaning of the key features in relationship to the context of the problem, given an exponential function embedded in a real-world context. Graph the function, identify key features of the graph, and interpret the meaning of the key features in relationship to the context of the problem, given a polynomial function embedded in a real-world context. Graph the function, identify key features of the graph and interpret the meaning of the key features in relationship to the context of the problem, given a logarithmic function embedded in a real-world context. Create a real-world context that would generate a function with the provided attributes, given key features of a linear, quadratic, cubic, square root, cube root, exponential, piecewise or logarithmic function.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
		Identify key features of the graph or table and interpret the meaning of the key features in relationship to the context of the problem, given a graph or table of values representing a piecewise function embedded in a real-world context. Sketch a graph of the function, given a verbal description of the key features of a linear, quadratic, cubic, square root, cube root, exponential, piecewise or logarithmic function.	

Instructional Focus Statements

Level 3:

Functions are often described and understood in terms of their key features and behaviors. Instruction for this standard should in part focus on building on the knowledge gained in Algebra I around identifying key features and behaviors from both graphs and tables and extend this understanding to new function families. In Algebra I, students focused on linear, quadratic, absolute value, and exponential functions with domain in the integers. The new part of this standard for students will be in the function families as opposed to the types of key features/behaviors. Thus, it is important to note that the overarching concept of key features and behaviors is not new to students.

As in Algebra I, instruction should extend beyond simple identification from isolated graphs and tables. As this is a modeling standard, students need opportunities to develop an understanding of the relationship between key features/behaviors and the real-world situation that the function models. The focus should be on developing a strong understanding of the relationship between key features/behaviors and their meaning within real-world situations. Additionally, instruction should provide students with an opportunity to develop an understanding of not only how to identify key features/behavior in graphs and tables, but also how to generate a graph when provided with the key features/behaviors.

Instruction can be very nicely paired with standard A2.F.IF.B.4 where students generate linear, quadratic, cubic, square root, cubed root, exponential, logarithmic, and piecewise graphs from real-world situations. This pairing allows students the opportunity to generate a graph from a real-world situation, identify key features/behaviors, and then discuss their meaning as related to the real-world situation.

Level 4:

As students develop a deep understanding of this standard, they should be exposed to increasingly more complex real-world situations. Students should begin to create their own real-world scenarios that generate functions with a pre-determined list of key features/behaviors. Additionally, students with a deep understanding of this standard can interpret key features/behaviors from non-traditional square root, cube root, exponential, polynomial, and logarithmic functions embedded in real-world situations.

Functions

Interpreting Functions* (F.IF)

Standard A2.F.IF.A.2 **Cluster Heading: A. Interpret functions that arise in applications in terms of the context.**

Calculate and interpret the average rate of change of a function (presented algebraically or as a table) over a specified interval. Estimate and interpret the rate of change from a graph.*

Scope and Clarifications: (Modeling Standard)

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Choose the average rate of change for an exponential function when given a symbolic representation, table, or graph. Choose the estimated rate of change when given a graph of an exponential function.	Calculate the average rate of change of an exponential function when given a graph. Interpret the rate of change for an exponential function in terms of a real-world context. Choose the estimated rate of change for a specific interval when given an exponential function.	Calculate average rate of change when given an equation or table of cubic, exponential, or logarithmic functions. Interpret the average rate of change of cubic, exponential, or logarithmic functions. Estimate and interpret the average rate of change for a specific interval of cubic, exponential, or logarithmic functions when given a graph.	Identify the average rate of change for specific intervals of a function as being greater or less than other intervals of the same function. Compare the average rate of change of multiple intervals of the same function and make connections to the real-world situation. Create a contextual situation and identify and interpret the average rate of change with a specific interval.

TN Department of Education

Instructional Focus Statements

Level 3:

In grades 6 (6.RP.A.2) and 7 (7.RP.A.2.b), students began developing the understanding of ratios and proportional relationships. Their understanding of rate of change involved both ratios and proportions using similar triangles to show the additive and multiplicative conceptual underpinnings of the concept. In grade 8 (8.F.A.2), students extended this understanding to functions by examining rate of change in linear functions. In high school, students should solidify this understanding of linear functions and generalize this concept to applying to additional function types. Students should make the connection that the rate of change is the ratio of the change between the dependent and independent variable. For linear functions, students have discovered that this ratio of change is constant between any two points on the line. Students should now make the connection that, for non-linear functions, the ratio of change is not constant due to the curvature of the function. This results in the ability to calculate the average rate of change over a specified interval. For example, for the cubic function $f(x) = x^3$, the average rate of change from x = 1 to x = 4 is $\frac{f(4)-f(1)}{4-1} = \frac{64-1}{4-1} = \frac{63}{3} = 21$. This is the slope of the line from (1, 1) to (4, 64) on the graph *f*. If *f* is interpreted as the volume of a cube of side length x, then this calculation means that over this interval the volume changes, on average, 21 square units for each unit increase in the side length of the cube.

It is imperative that students gain a conceptual understanding of the average rate of change for a specified interval for non-linear functions. To grasp this idea, students should draw illustrations of the graph and the secant line connecting the intended endpoints. Students should not only be able to calculate the average rate of change, but they should also be able to generate a visual representation and use the visual representation to estimate the average rate of change over a specified interval. Students will gain a deeper conceptual understanding when they compare their estimations to the actual average rate of change for a non-linear function. As students solidify their understanding, they should be able to explain what the average rate of change means in the context of a problem when given symbolic representations, tables, graphs, or contextual situations. As students use multiple representations to evaluate the average rate of change, they should be able to explain the relationship between the multiple representations using both appropriate mathematical language and appropriate justifications.

Level 4:

Students should extend their understanding of average rate of change by comparing the average rate of change of one interval to another interval of the same function. Students should also further their understanding by creating their own contextual situations and interpreting the average rate of change for a significant interval. Students should be intentional in determining which interval or intervals they select and explain the importance of the interval(s) with respect to the context using both precise mathematical vocabulary and precise justifications.

Functions

Interpreting Functions* (F.IF)

Standard A2.F.IF.A.3 **Cluster Heading: A. Interpret functions that arise in applications in terms of the context.**

Understand geometric formulas as functions.*

Scope and Clarifications: (Modeling Standard)

Limit to quadratic and cubic functions. For example, see geometric formulas such as area of a circle, area of a square, and surface area of a cube as quadratic functions. See the volume of a cube and volume of a sphere as cubic functions.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Understand the definition of *function*. Apply geometric formulas to find a given characteristic of a shape (such as the perimeter of a square, the circumference of a circle, or the measure of an interior angle).	Explain how the parts of a geometric formula relate to the shape that it is describing. Graph the formula related to the geometric formula. Interpret the parts of a quadratic function, such as y-intercept, zeros (x-intercepts), maximum/minimum, and the axis of symmetry. Interpret parts of a cubic function, such as the y-intercept, absolute maximum/minimum, and zeros (x-intercepts).	See a geometric formula as a function that has characteristics such as intercepts and maximums/minimums. Apply the characteristics of a quadratic function to geometric formulas. Apply the characteristics of a cubic function to geometric formulas. Explain how the parts of a geometric formula relates to the graph of the related function.	Create a real-world situation that can be solved using a geometric function.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
	Write a geometric formula as a function of the variable measure (i.e., the area of a circle is $A(r) = \pi r^2$).	Interpret the parts of a geometric formula as it relates to a real-world situation.	

Instructional Focus Statements

Level 3:

Students were introduced to linear geometric formulas as functions in Algebra 1(A1.F.IF.A.3). In Algebra 2 students will be expected to think about quadratic and cubic geometric formulas as functions. They will apply the characteristics of quadratic and cubic functions to geometric formulas. For example, students may be asked to create the graph of the area of a circle, $A(r) = \pi r^2$, where A is the area of a circle, r is the radius of the circle, and A(r) is the area of the circle as a function of the radius. Discussion should include finding the key characteristics of the area function and how they relate to the circle it describes. The y-intercept of 0 is reasonable, because if the radius is 0, there is no area.

It is imperative that the students build a conceptual understanding of functions. This standard affords an opportunity for students to gain that conceptual understanding. Instruction should focus on the idea that quadratic and cubic geometric formulas are functions. For example, students could be asked to consider the volume of a cube $V(s) = s^3$. Have students graph the function, then identify the key characteristics and how they relate to the cube. Students may also be to identify the domain in the context of the problem.

This standard may be paired with A2.F.IF.A.1.

Level 4:

Students who have a level 4 understanding of this standard will be able to create a real-world situation that can be solved using a geometric function. Students should be given opportunities to create problems that can be solved by thinking of quadratic and cubic geometric formulas as functions. For example, a company prints on cardboard boxes used for delivering merchandise. The boxes are different sizes, but they are all in the shape of a cube. To determine the amount of ink needed to print a box, they need to find the surface area. The goal here is not for students to find the surface area for lots of different size boxes. The goal is for students to demonstrate an understanding that the surface area is a function of the side length of the cube, and to use that relationship to solve problems.

TN Department of Education

Functions

Interpreting Functions* (F.IF)

Standard A2.F.IF.B.4 **Cluster Heading: B. Analyze functions using different representations.**

Graph functions expressed algebraically and show key features of the graph by hand and using technology.*

Scope and Clarification (Modeling Standard):

Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and/or asymptotes where appropriate.

Tasks are limited to linear, quadratic, cubic, square root, cube root, exponential, logarithmic, and piecewise functions (including absolute value functions).

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Use characteristics of the algebraic representation of a function to distinguish function type and behavior of the graph. Recognize the parent function from a graph of a transformed linear, quadratic, cubic, square root, cube root, exponential, logarithmic and absolute value curve. Identify key features, such as intercepts, extrema, and end behavior from a graph of a quadratic function.	Describe the behavior of the graph of the square root function by explaining the inverse relationship between squaring a number and taking the square root of a number. Describe the behavior of the graph of the cube root function by explaining the inverse relationship between cubing a number and taking the cube root of a number. Identify key features, such as shape, intercepts, extrema, and end behavior, from a graph of a square root and cube root function.	Graph a linear, quadratic, square root, cube root, cubic, piecewise, exponential, and logarithmic function by hand and using technology. Describe end behavior of a cubic function given in standard form and factored form. Attend to precision when illustrating intercepts, maxima, minima, and determine the domain, range, and end behavior of a function.	Explain the relationship that exists between a contextual problem and the key features of a graph for a linear, quadratic, square root, cube root, cubic, exponential, and logarithmic function. Critique graphs drawn by others to ensure key features are shown efficiently and appropriately. Write the corresponding function symbolically, given a graph. Explain restrictions on domain and range in context.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Graph a quadratic function by hand and using technology identifying intercepts maxima and minima.	Identify key features, such as extrema and intercepts, from a graph of a piecewise-defined function. Infer restrictions on the domain and range from a graph. Identify the asymptote, given a graph of an exponential or logarithmic curve.		

Instructional Focus Statements

Level 3:

In Algebra I, students use functions to model relationships between quantities (A1.F.IF.B.4) and construct a function to model a linear relationship (A1.F.BF.A.1). Students also determine if a function is linear or nonlinear, and they have experience interpreting and representing functions algebraically, numerically, graphically, and verbally. They learn to graph linear, quadratic, absolute value, and exponential functions by hand and using technology (A1.F.IF.C.7). In Algebra 2, students will be introduced to additional function families and more key features such as extrema and end behavior. Intercepts, shape, domain, and range take on greater meaning to students through the exploration of a variety of functions. Therefore, it is important for instruction to provide tasks that allow students to explore the behavior and varying parameters of functions. Providing students with models in context, such as connecting the cube root function to solving volume problems, will help develop the meaning of key features and identifying them from a graph, a table, or a verbal description. To meet the rigor of this standard, students should be given the opportunity to work with functions that vary in their algebraic representation such as standard form and factored form of quadratic and cubic functions. This will help students have access to the problem regardless of the algebraic representation, which is further developed in standard A2.F.IF.B.5 as students identify key features through algebraic manipulation.

Students should be able to graph functions by hand and with the use of technology. It is imperative for the teacher to model how to graph with graphing technology. Furthermore, ample time must be given for students to explore how a table of values can be helpful in identifying key features, domain, and range from a graph. The use of technology should allow students to explore problems whose key features are irrational values, which can be located with the use of technology. Students may have struggled with domain restrictions in previous classes, so continue supporting that understanding by integrating technology to explore domain restrictions so that students connect the relationship between the algebraic representation and the resulting domain. This exploration of the effect of transformations will support students' mastery of this standard to graph by hand and with technology.

In Algebra 1, students focused on parent functions of linear, quadratic, and absolute value and exponential functions (A1.F.IF.C.7). Instruction in Algebra 2 should build on the concept of parent functions. This will help students make the connection of how transformations affect the graph, equation, and table of a function, which is explored in standard A1.F.BF.B.2. Students should be presented with problem types whose algebraic representation varies and asked to identify the parent function and describe the transformation from its original, non-transformed graph. This will help students attend to precision as they graph functions of many types and use their understanding of transformations to support the reasonableness of their graph. Instruction should provide ample opportunity for students to compare and contrast the graphs of functions, and it should help them efficiently recognize a parent function when expressed algebraically and graph it fluently.

Additionally, standard A1.F.LE.A.2 introduced students to exponential functions. Instruction in Algebra 2 should build on that prior knowledge of geometric sequences in order to graph the exponential function to highlight key features. To continue tying prior knowledge with new knowledge of the logarithmic function, instruction should connect the inverse relationships that exist between the exponential function's initial value, y-intercept, end behavior, and horizontal asymptote to a logarithmic function's base value, x-intercept, end behavior, and vertical asymptote. Asking students to produce a table with these values will help them sketch a graph of a logarithmic curve.

Graphing cubic functions should be taught in tandem with standard A2.A.APR.A.2 where students have to identify zeros of polynomials (limited to polynomials of degree 3 or less) when suitable factorizations are available and use the zeros to construct a rough graph of the function defined by the polynomial. Instruction for this standard should allow students to discover the end behavior of polynomials through an exploration of varying leading coefficients as the degree of the polynomial increases.

Level 4:

Instruction at this level should provide opportunities for students to create a real-world problem that is modeled by a quadratic, square root, cube root, cubic, exponential, or logarithmic function. Students should be required to provide a graph, table, equation, and verbal representation of the problem. Instruction should include posing purposeful questions asking students to show and describe key features from their created problem in context. Instruction should also provide graphs drawn by others and require students to analyze and critique the graphs. Students should be given the opportunity to look at graphs drawn by others so they can analyze and critique their peers work. Through the analysis of many graphs, students should develop an understanding of when key features are efficiently and effectively represented, and, if not, provide a suggestion for representing them more appropriately.

Functions

Interpreting Functions* (F.IF)

Standard A2.F.IF.B.5 **Cluster Heading: B. Analyze functions using different representations.**

Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.*

A2.F.IF.B.5a Rewrite quadratic functions to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a real-world context.*

A2.F.IF.B.5b Know and use the properties of exponents to interpret expressions for exponential functions in terms of a real-world context.*

Scope and Clarifications: (Modeling Standard)

For example, the growth of bacteria can be modeled by either $f(t) = 3^{t+2}$ or $g(t) = 9(3^t)$ because the expression 3^{t+2} can be rewritten as $(3^t)(3^2) = 9(3^t)$.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Understand that changing the way that a function is represented (e.g., algebraically, with a graph, in words, or with a table) does not change the function, although different representations highlight different characteristics, and some may show only part of the function. Identify a quadratic function in standard form, factored form and vertex form, and identify key features within each form. Use the properties of exponents to rewrite exponential expressions.	Rewrite quadratic functions in standard form, factored form, and vertex form, and recognize which form of a quadratic function is most appropriate for revealing certain properties, when given a real-world problem. Interpret the meaning of zeros, extreme values, and symmetry of the graph in the context of a real-word problem. Identify an exponential function written in symbolic form as an exponential growth or decay.	Move fluently between different forms of a quadratic function in order to reveal specific key features of the function. Write a quadratic function that models a given real-world context. Within a given context interpret the key features of an exponential function. Rewrite an exponential function to reveal the percent rate of change. Rewrite an exponential function to reveal the y-intercept.	Create real-world problems that can be solved using a quadratic function. Identify reference points in non-standard forms of exponential functions.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
	Identify percent rate of change in an exponential function. Determine if the context represents exponential growth or exponential decay, given a real-world problem.	Write an exponential function given any point and the percent rate of change.	

Instructional Focus Statements

Level 3:

In Algebra I (A1.F.IF.C.8), students learned to rewrite quadratic functions in different forms to reveal key features. In Algebra 2 students will solidify their understanding of quadratics. The focus should be on being able to move fluently between different forms of a quadratic function in order to reveal specific key features of the function. For example, if a student is presented with a quadratic function in standard form and asked about the zeros, they should quickly realize they need the factored form. Students should be given opportunities to write a quadratic function given varied contexts. For example, Joe is launching a rocket from the ground into the air. He notes that the rocket hits the ground after 4 seconds. What was the maximum height of the rocket? Students should be able to write the quadratic model in factored form, then convert the model to vertex form to find the maximum. Students should also be asked to make connections between the quadratic function and its graph.

Students should also be able to explain the contextual meaning of the key features of the quadratic function when working within a real-world context. The typical key features of quadratic functions are: domain and range, y-intercept, x-intercepts (zeros), intervals of increasing and decreasing, intervals of positive and negative values, maximum/minimum, vertex, axis of symmetry and end behavior. Teachers should provide a variety of contexts in which the type of variables differ so that students can develop a deep understanding of how each component relates to the real-world context. Students can write quadratic functions in vertex form to reveal the vertex and axis of symmetry of the function. Visually, students see that the y-value of the vertex is the maximum or minimum of the function, which is important in a variety of contexts. For example, it can reveal the unit price that maximizes profit, the time when a projectile reaches its maximum height, or the length of a rectangle that maximizes the area. Teachers should provide a variety of contexts in which the type of variables differ so that students can develop a deep understanding of how each component relates to the real-world context (such as using the formula for compound interest).

In Algebra I (A1.A.SSE.A.1), students use the properties of exponents to rewrite exponential expressions, interpret key features of an exponential graph, and interpret the parameters in an exponential function in terms of a context. Students build upon this prior knowledge and combine these skills to rewrite exponential functions to reveal key features of the graph and interpret them in a context. Students should be able to identify the y-intercept and

percent rate of change (i.e., percent increase or decrease) by rewriting an exponential function into the form $y = ab^x$. Teachers should provide students with opportunities to use exponent properties to convert exponential functions into the form $y = ab^x$. For example, given the exponential function $y = (1/3)^{-x}$, students should write this as $y = ((1/3)^{-1})^x$ or $y = 3^x$. Therefore, students should identify the percent rate of change as a 200% increase. In other cases, students should identify the percent rate of change in a context. For example, students should be able to determine the percent rate of change in the population of a small town, when the population is represented by $p(t) = 15000(.978)^t$. A common misconception held by students is that it represents an increase of 97.8% rather than a decrease of 2.2%. Teachers can model how to write the function as $y = 15000(1 - .022)^t$ to help students see why it is a decrease of 2.2%.

Students can also use exponent properties to reveal the y-intercept of an exponential function. For instance, given the function $y = 2(3)^{x+1}$ students should be able to recognize that the function can be rewritten as $y = 2(3)^1 * (3)^x$, which is equivalent to $y = 6(3)^x$. Thus, the y-intercept is (0, 6). One useful strategy to support students in separating an exponential expression into the product of two exponential expressions is to first provide them with examples of combining the product of two expressions. By doing so, students can use the properties of exponents in ways they are accustomed to prior to reversing the thought process. Although separating the expressions can be useful, teachers should remind students that another strategy to find the y-intercept is to substitute 0 in for x, which is generalizable to all functions.

These concepts pairs with A2.F.BF.A.2, in which students write geometric sequences with an explicit formula and use them to model situations. Therefore, if students are given the ratio and a term in the sequence, they can use this concept to build the function without finding the first term. For example, given the 7th term is 30 and the common ratio is 2, students should be able to write the function $y = 30(2)^{x-7}$ because an input of 7 will result in an exponent of 0, which will produce an output of 30. This can help students work efficiently by making use of the structure of an exponential function.

Level 4:

As students become fluent with moving between equivalent forms of a quadratic function, they should be given opportunities to create real-world problems involving quadratic functions. Students may be asked to exchange their problems, then critique the reasoning of others. Students may be led to understand that their problem may be solved in different ways, thus challenging their ability to think fluently about quadratics.

As students develop a deep understanding of exponential functions, they should recognize reference points in non-standard forms of exponential functions. For example, given the function $y = 3(2)^{x-5}$, students should recognize that the function contains the point (5, 3). There are multiple ways that teachers can help students understand this principle. First, students can see the function $y = 3(2)^{x-5}$ as a horizontal translation of the function $y = 3(2)^x$, which has a y-intercept of (0, 3). Therefore, instead of having the point (0, 3), the function would contain the point (5, 3). Second, students should know that anything (except 0) raised to the 0 power is 1 and use this fact to see how making the exponent 0 simplifies the entire second factor to 1. Thus, by inspection, students should quickly see that an input of 5 produces an output of 3 and explain how this concept can be generalized to other exponential functions.

Functions

Interpreting Functions* (F.IF)

Standard A2.F.IF.B.6 **Cluster Heading: B. Analyze functions using different representations.**

Compare properties of functions represented algebraically, graphically, numerically in tables, or by verbal descriptions.*

A2.F.IF.B.6.a Compare properties of two different functions. Functions may be of different types and/or represented in different ways.*

A2.F.IF.B.6.b Compare properties of the same function on two different intervals or represented in two different ways.*

Scope and Clarifications: (Modeling Standard)

Functions may or may not have a real-world context.

Tasks are limited to linear, quadratic, cubic, square root, cube root, exponential, logarithmic, and piecewise functions (including absolute value functions).

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify the y-intercept of a function from multiple representations. Identify the slope of a linear function from multiple representations. Identify asymptotes of exponential functions from multiple representations. Describe connections among multiple representations of a linear function. Compare properties of two linear functions each represented in a	Identify the zeros of a function from multiple representations. Identify asymptotes of logarithmic and exponential functions from multiple representations. Identify the relative extrema and end behavior of a cubic function from multiple representations. Identify the percent rate of change of an exponential function from multiple representations. Describe connections among multiple representations of a linear,	Compare properties of two functions of the same type each represented in a different way, including linear, quadratic, cubic, square root, cube root, exponential, logarithmic, and piecewise functions (including absolute value functions). Compare properties of two functions from different function families each represented in a different way. Compare properties of the same function on two different intervals	Compare properties of two functions within a context. Use precise mathematical vocabulary to explain the relationships of the various representations of a function.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
different way. Compare properties of two quadratic functions each represented in a different way.	quadratic, cubic, square root, cube root, exponential, logarithmic, piecewise (include absolute value) functions. Move fluently among multiple representations of a function.	or represented in two different ways.	

Instructional Focus Statements

Level 3:

Prior to comparing properties of two functions represented in different ways, students need to first identify properties of functions and make connections between different representations of the same function. As students move fluently between representations, they must consider relationships among quantities and how each representation provides a unique perspective of the function. Teachers can foster this way of seeing mathematics by having students discuss the similarities among representations that reveal the key features of a function that persist regardless of the form. Through these discussions students can determine which representations are most appropriate for revealing certain key features of the function.

In grade 8 (8.F.A.2) and Algebra 1 (A1.F.IF.C.9), students compare properties of two linear, quadratic, exponential, and absolute value functions each represented in a different way. Once students have a strong understanding of the various representations of linear, quadratic, cubic, square root, cube root, exponential, logarithmic, and piecewise functions (including absolute value functions) in Algebra 2, they can begin to compare properties of two functions represented in different ways. For example, given a graph of one cubic function and a table of another, a student should be able to compare their y-intercepts. One strategy that can sometimes be useful is to convert one or both to a different form so that both functions are represented the same way. As students begin to grasp this concept, it is important that teachers provide students with examples that include each function type, with some situated within a context. Therefore, comparing properties in different representations further supports students' understanding of each function type, which means this standard can be paired nicely with other standards that focus on properties and graphs of linear, quadratic, cubic, square root, cube root, exponential, logarithmic, and piecewise functions (including absolute value functions), such as A2.F.IF.A.1 and A2.F.IF.B.4. As students recognize various function types in multiple representations, discussion should lead to the comparison of functions from different families represented in different ways. For example, compare y-intercepts given a table of values representing a cubic function and a verbal description of an exponential function. Instruction should support students in first recognizing the function family prior to comparing properties.

Students should be able to compare properties of the same function on two different intervals or represented in two different ways. For example, consider the functions $f(x) = x^2 + 7x + 12$ over the interval from $x = -4$ to $x = -2$ and $g(x) = (x+2)(x+4)$ over the interval from $x = -4$ to $x = -3$. Describe how the functions behave over these intervals. How is the behavior the same? How is the behavior different?

Level 4:

Students with a deep understanding of the various function types and representations should also be able to compare functions from different families represented in different ways. For example, compare y-intercepts given a table of values representing a cubic function and a verbal description of an exponential function. Instruction should support students in first recognizing the function family prior to comparing properties. Once conclusions are formed, teachers can ask further questions related to the context. For example, given a graph of a quadratic function and an algebraic representation of a cubic function each describing the position of a ball, decide which ball will have the maximum height. Students should be given the opportunity to describe how to identify function types and compare the properties of functions in various forms. At this level, teachers should expect students to use precise mathematical vocabulary to describe and justify these relationships and qualities.

Functions

Building Functions (F.BF)

Standard A2.F.BF.A.1 **Cluster Heading: A. Build a function that models a relationship between two quantities.**

Build a function that describes a relationship between two quantities.*

A2.F.BF.A.1.a Combine standard function types using arithmetic operations.*

A2.F.BF.A.1.b Combine standard function types using composition.*

Scope and Clarifications: (Modeling Standard)

Tasks are limited to linear, quadratic, square root, cubic, cube root, exponential, and logarithmic functions.

Part a: for example, if 1000 – 70x represents the number of items sold in a month and $0.5 + 0.1x$ represents the cost of each item, multiply $(1000 – 70x)(0.5 + 0.1x)$ to write the quadratic function representing the profit, where x is the number of 10-cent price increases.

Part b: for example, given a product originally priced at $\$x$ is $4 off, build a function that will calculate the final price including 10% sales tax (i.e., $f(g(x)) = 1.10 (x – 4)$).

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify the independent and dependent variable in a real-world context. Recognize when a quadratic function should be used to describe the given situation, given a real-world context. Recognize when an exponential function should be used to describe the given situation, given a real-world context.	Write a function defined by an expression to model a quadratic relationship, given a real-world context. Write a function defined by an expression to model an exponential relationship, given a real-world context.	Combine multiple functions of different types using arithmetic operations to write a function that describes a real-world situation with multiple steps. Combine functions using composition, including functions of the same type and functions of different types. Recognize that combining functions of different types may result in a	Explain the various ways a function can be defined and in what real-world situations they would be appropriate. Justify why specific types of functions should be used in combination to describe a given real-world context. Combine two functions using arithmetic operations, then relate

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Combine like terms of two functions of any type to build a new function. Use function notation to evaluate functions of any type at a given numeric value. Recognize the composition of functions notation. Know that $f \circ g\ (x)$ means $f(g(x))$.	Combine a linear function and quadratic function using arithmetic operations to build a new function. Compare key characteristics of real-world contexts that can described by various types of functions. Use function notation to evaluate functions of any type at a given value, including algebraic values. For example, if $f(x) = x^2 - 2$, find $f(3x+1)$. Combine two linear functions using composition.	function of greater or less degree. For example, the product of a linear function and a quadratic function is a cubic function. Solve real world problems involving the composition of two functions.	the graph of the resulting function to the graphs of the original functions. Create a real-world problem that can be solved using composition of functions.

Instructional Focus Statements

Level 3:

In Algebra 1 (A1.F.BF.A.1), students create linear, quadratic, and exponential functions from a real-world context defined by an explicit expression, recursive process, or steps for calculation. In Algebra 2, students should extend their understanding to combining function types using arithmetic operations and composition. Instruction for this standard should focus on building functions from a real-world context and recognizing when multiple functions need to be combined to describe multiple steps. Tasks should include multiple function types, including linear, quadratic, square root, cubic, cube root, exponential, and logarithmic functions. As students continue to work with combining functions, teachers should have students recognize that when combining functions using some operations, the original parent function is maintained, while other operations create new function types. For example, teachers should have students compare adding two quadratic functions with multiplying two quadratic functions. Students should be able to combine two different function types to build a new function. For example, they should recognize that the sum of a linear function and a quadratic function results in a quadratic function, while the product of a linear function and a quadratic function results in a cubic function, and usually the difference in two cubic functions will result in a cubic function but not always (i.e., if $f(x) = x^3 - 9x$ and $g(x) = x^3 + 3$, then $f(x) - g(x)$ will result in a linear function.)

If given a table of values describing a single function, students should first recognize which type of function the table of values represents. Teachers should focus students' attention on the relationship between consecutive points to see if there is a common first difference or constant additive change (linear function), a common second difference (quadratic function), or a common ratio or constant multiplicative change (exponential function). Once students identify the function type, teachers can then help students begin to write the function given the common first difference, second difference, or ratio and other information from the table.

In Algebra 1 (A1.F.IF.A.2), students learned to use function notation and to interpret functions written in function notation. In Algebra 2, students will extend this knowledge to composition of functions. While instruction should include understanding the notation of composition, f◦g (x) means f(g(x)), it is imperative that students experience using composite functions in real world situations. For example, Roy is buying new tires for his truck. The tax is 8.5%. He must also pay a $20 fee for having disposal of the old tires. If the cost of the new tires is represented by x, write a function, f(x), to model the cost of the tires including tax. [f(x) = 1.085x] Write a function, g(x) to model the cost of the tires plus the disposal fee. [g(x) = x + 20] Find f◦g (x). [f(g(x)) = 1.085(x+20)] What does this function represent in the context of this problem? [This represents the cost of the tires and the disposal fee if both are taxed.] Find g◦f (x). [g(f(x)) = 1.085x+20] What does this function represent in the context of this problem? [This represents the cost of the tires and the disposal fee if the fee is not taxed.] Which composite function should I use if the disposal fee is not taxed? Instruction should include using precise mathematical language to describe composition of functions.

Level 4:

As students develop a deep understanding of this standard, they should be able to create a real-world scenario given a function or combination of functions. Moreover, they should be able to describe which characteristics of their scenario correspond to each part of the given function. For example, given $y = 10 + 5(2)^{x-1}$ a student might create a scenario similar to the following: The results of a school election are shared with 15 people. 10 of those people keep the secret while the other 5 tell 2 people per day and everyone they tell also tells 2 people per day and those 2 people tell to 2 more and so on. In this function, y represents the number of people that know the results x days after the original 15 people were told. The student should also be able to explain how the $5(2)^{x-1}$ in the function relates to the continuation of everyone telling 2 people per day beginning with 5 of the original people and how the 10 in the function relates to the 10 people that remain constant at 10 because they keep the secret. Similar scenarios can be created to describe half-life or mortgage calculations.

Students with a level 4 understanding should be able to combine two functions using arithmetic operations, then relate the graph of the resulting function to the graphs of the original functions. For example, given $f(x) = x^2 - 3$ and $g(x) = 2x - 5$, find the product of f(x) and g(x). Graph f(x), g(x) and f(x)·g(x) on the same coordinate plane (using technology). Discuss how the graph of the product, f(x)·g(x), relates to the graphs of f(x) and g(x). What do you notice about the zeros and the y-intercepts?

Functions

Building Functions (F.BF)

Standard A2.F.BF.A.2 **Cluster Heading: A. Build a function that models a relationship between two quantities.**

Define sequences as functions, including recursive definitions, whose domain is a subset of the integers. Write explicit and recursive formulas for arithmetic and geometric sequences in context and connect them to linear and exponential functions.*

Scope and Clarifications: (Modeling Standard)

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Determine if a sequence of numbers is an arithmetic sequence, geometric sequence, or neither. Identify the common difference in an arithmetic sequence. Identify the common ratio in a geometric sequence.	Understand that a sequence is a function. Relate arithmetic sequences to linear functions. Relate geometric sequences to exponential functions. Write an arithmetic explicit formula to represent a relationship given by a sequence of numbers. Write an arithmetic recursive formula to represent a relationship given by a sequence of numbers.	Write an arithmetic explicit formula to model situations, given a real-world context. Write an arithmetic recursive formula to model situations, given a real-world context. Write a geometric explicit formula to model situations, given a real-world context. Write a geometric recursive formula to model situations, given a real-world context.	Create a real-world context that would generate the given function, given a function defined by an arithmetic recursive or explicit formula. Create a real-world context that would generate the given function, given a function defined by a geometric explicit or recursive formula. Justify why specific real-world contexts should be represented by arithmetic or geometric sequences.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
	Write a geometric explicit formula to represent a relationship given by a sequence of numbers. Write a geometric recursive formula to represent a relationship given by a sequence of numbers.		

Instructional Focus Statements

Level 3:

In Algebra I (A1.F.BF.A.1), students construct a function defined by an expression to model a linear or exponential relationship. In Algebra 2, students build on their knowledge of linear functions to form arithmetic explicit formulas from a real-world context. Instruction should provide students with choice on whether to use the zeroth term ($y = mx + b$), the first term ($a_n = a_1 + d(n-1)$) or others in developing their function. Students should recognize that these equations are equivalent and connect the concepts of slope and common difference. Instruction should include using the recursive formula to write an arithmetic function ($a_n = a_{n-1} + d$) when appropriate.

Students will make connections between geometric sequences and exponential functions created by explicit $a_n = a_1 r^{n-1}$ and recursive $a_n = r \cdot a_{n-1}$ formulas in a real-world context. Instruction should focus on making connections between an exponential function and the geometric sequence. Students should notice that each output value in the table of values can be rewritten as the initial value times a power of the common ratio. Then, relating each input value to the power of the common ratio will help students develop a function. Students should also be given choice as to which term to use as the coefficient in their formula. For example, if students are given the fourth term and the ratio, then $a_n = a_4 r^{n-4}$ might be a more accessible formula. To build coherence, it is important that students make connections between linear functions and arithmetic sequences and between exponential functions and geometric sequences.

Although many textbooks and examples will use the subscript notation as noted above, sequences can be written using function notation as well. For example, the geometric sequence 5, 15, 45, 135… has a common ratio of 3 with a starting value of 5. This can be expressed recursively as $f(x) = f(x-1) \cdot 3$ and explicitly as $f(x) = 5(3)^{x-1}$.

Level 4:

As students develop a deep understanding of this standard, they should be exposed to increasingly more complex real-world situations. Students should begin to create their own real-world scenarios that generate a given function and justify why their scenario represents that function type. Additionally, students with a deep understanding of this standard can create a function from a geometric pattern and describe how each component of their function relates to characteristics of figures in the pattern. For example, students should be able to build a function to represent the number of line segments used to form shapes in a series of shapes following a particular pattern. Students should be given opportunities to justify their reasoning and critique the reasoning of others. Instruction should support rich mathematical discourse.

Functions

Building Functions (F.BF)

Standard A2.F.BF.B.3 **Cluster Heading: B. Build new functions from existing functions.**

Identify the effect on the graph of replacing *f(x)* by *f(x) + k, k f(x), f(kx),* and *f(x + k)* for specific values of *k* (both positive and negative); find the value of *k* given the graphs.

Scope and Clarifications:

Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Describe transformations that would map a geometric figure to its image using precise mathematical vocabulary. Write the function defined by *f(x)+k*, *kf(x), f(x+k),* and *f(kx)* given the parent function and a positive value of *k*. Graph the parent functions for absolute value, quadratic, cubic, square root, cube root, exponential, and logarithmic functions using technology.	Know and graph the parent functions for absolute value, quadratic, cubic, square root, cube root, exponential, and logarithmic functions. Use function notation to identify the difference between *f(x) + k* and *f(x+k)* and the difference between *kf(x)* and *f(kx).*	Describe the effect on the graph for specific values of *k*, given two functions, *f(x)* and *f(x) + k*. Describe the effect on the graph for specific values of *k*, given two functions, *f(x)* and *f(x + k).* Describe the effect on the graph for specific values of *k*, given two functions, *f(x)* and *f(kx).* Describe the effect on the graph for specific values of *k*, given two functions, *f(x)* and *kf(x).* Describe the effects on a graph for more than one transformation using specific values of *a, h,* and *k*	Given a graph of a function, write the equation by identifying the transformation(s) to the parent function. Apply transformations to a function that has already been transformed. Explain why changes to the argument of *f(x)* affect the input values and changes outside the function affect the output values.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
		given two functions, *f(x)* and *af(x + h) + k.* Determine the value of *k* for vertical and horizontal translations, stretches, and compressions, given the graphs of the image and pre-image.	

Instructional Focus Statements

Level 3:

In grade 8 (8.G.A.1), students were introduced to the properties of rotations, reflections, translations, and dilations of geometric figures, this was further developed in Geometry (G.CO.A cluster). Students applied this concept in Algebra 1 (A1.F.BF.B.2) by exploring the effects of translations, stretches, and compressions on absolute value and quadratic functions and describing the transformations using function notation. In Algebra 2, students apply this same concept to additional function types including cubic, square root, cube root, exponential, and logarithmic functions. To master this standard, students need to understand function notation and how to write a function defined by function notation. For example, given $f(x) = x^3$, write the function $f(x + 3)$ *as* $f(x + 3) = (x + 3)^3$.

Students can use technology (i.e., graphing calculator or online graphing tool) to compare a parent function *f(x)* with each of the transformation types *f(x) + k, k f(x), f(kx),* and *f(x + k)*, to discover the effects of each transformation on the parent function's graph. As students vary one value of *k* at a time, they can begin to discern how each component affects the graph of the parent function, *f(x).* For example, students can graph $f(x) = x^3$ and then graph $f(x) = x^3 + 4$. By examining the graph, they can see that the + 4 translated the graph up 4 units. They can then predict what would happen if the equation was $f(x) = x^3 - 4$ and then graph it to see that the – 4 translated the graph down 4 units.

When exploring what happens to the parent function for *f(x + k)*, student discourse should also focus on why *k* translates the graph *–k* units horizontally. One strategy is to compare f(x) with *f(x − 5),* when *f(x)* is defined at *f*(3). In this example, an input of 3 into *f(x)* is equivalent to an input of 8 into *f(x − 5),* and 8 is 5 units to the right of 3, not left. Students can think about it as having to undo what has been done to *x* inside the argument, in this situation a translation 5 units to the right of the parent function will "undo" the –5. Likewise, when exploring the effects of *k* on *f(kx)*, students will need to see that if k>1 the graph will be compressed horizontally by 1/*k*, and if *k*<1, it will be stretched by 1/*k*.

After students have had the opportunity to explore the effects of each transformation visually using technology, instruction should guide students to examine the effects of each transformation on the coordinates. Instruction should connect to the concepts of transformation from grade 8 and Geometry to examine the effects of the transformation on each point in the parent function. For example, when exploring *f(kx)* with *k* = ½, upon closer examination of the coordinates, students will find that the x coordinates of the parent graph (input value) will be multiplied by $\frac{1}{\frac{1}{2}}$ or 2 in each corresponding point in the resulting image (output value).

Students should understand that since *f(x)* represents the output values, any operations applied to *f(x)* outside the argument only affect the y values, which results in vertical transformations. Any operations applied to *f(x)* inside the argument only affect the *x* values, which results in horizontal transformations. Once students understand the effects of each component individually, they should then be able to describe changes to a graph involving multiple transformations at once. Eventually, they should connect each transformation to how *a, b, h,* and *k* impact the graph of *f(x)* in *af(bx + h) + k*. Note that this is vertex form of a quadratic function since it reveals the vertex (h,k) of the parabola.

When working with multiple transformations, teachers should focus students' attention on the order of each transformation. For example, given *−f(x)* + 9, the graph is first reflected across the x-axis, then shifted up 9, rather than shifted up 9 then reflected across the x-axis, due to the order of operations. It is also important that teachers place an emphasis on factoring out b from inside the argument so that the horizontal shift can be found. For example, *f(x)* = $(2x-6)^2$ becomes *f(x)* = $(2(x-3))^2$, revealing a horizontal shift of 3 to the right, not 6.

Once students can describe the effects of transformations on the graph of a function, they should be able to graph a given absolute value or quadratic function by applying the concept of transformations. This strategy for graphing functions is applied in A2.F.IF.B.4. They should also be able to describe vertical and horizontal translations, stretches, and compressions and determine the value of *k* when given the graphs of the image and pre-image.

Level 4:

Students with a deep understanding of this standard should be able to write the equation of a function given the graph by identifying the parent function and any transformations that have been applied to it by using the format *f(x) = af(bx + h) + k.*

Teachers should help students understand that transformations can be applied to functions that have already been transformed. For example, given $g(x) = \sqrt{x+2}$, write an equation for *g(x−5)* and describe the overall change in the graph. The given function *g(x)* is simply translated to the right 5 units, but since it had already been translated to the left 2 from the parent function $f(x) = \sqrt{x}$, the transformed function will be $g(x-5) = \sqrt{x+2-5}$ or $\sqrt{x-3}$ and the initial point of the image will land at (3,0), which is a translation from the parent function to the right 3.

In addition, students at this level should be able to explain concepts such as why changes inside the argument of a function have the inverse effect on a graph. Taken collectively, students should understand a function as a process that generates output values from specific input values, which leads to a deeper understanding of functions.

Functions

Building Functions (F.BF)

Standard A2.F.BF.B.4 **Cluster Heading: B. Build new functions from existing functions.**

Find the inverse of a function.

A2.F.BF.B.4a Determine whether a function is one-to-one.

A2.F.BF.B.4b Find the inverse of a function on an appropriate domain.

A2.F.BF.B.4c Given an invertible function on an appropriate domain, identify the domain of the inverse function.

Scope and Clarifications: (Modeling Standard)

Tasks are limited to linear, quadratic, square root, cubic, cube root, exponential, and logarithmic functions.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Determine if a relation is a function. Find the output value given a function and an input value. Reflect a graph across horizontal, vertical, and diagonal lines. Identify the domain of the function, given a graph.	Determine if a given table, graph, equation, list of ordered pairs, or mapping is one-to-one. Find the inverse of a function from a table, graph, equation, list of ordered pairs, or mapping given the function is one-to-one. Find the inverse of a function that does not require domain restrictions. Understand independent and dependent variables, the relationship between each, and their relationship to the function.	Determine whether a function is one-to-one. Find the inverse of a function represented in a variety of ways (table, graph, equation, list of ordered pairs, mapping) flexibly given the function is one-to-one. Find the inverse of a given linear quadratic, square root, cubic, cube root, exponential, or logarithmic function and restrict the domain if necessary.	Determine if a function has an inverse that is a function. Understand and use the concept(s) of inverses flexibly. Explain how an inverse function undoes the operations of the original function. That is, $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$. Understand when the domain of a function needs to be restricted, so that it is one-to-one, in order to find its inverse.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
		Identify the domain of the inverse of an invertible function with an appropriate domain.	Explain why functions that are not one-to-one do not have an inverse that is a function. Explain when reversing the independent and dependent variables of a function in a context does not make sense.

Instructional Focus Statements

Level 3:

In elementary and middle grades, students develop understanding of inverse operations, such as adding and subtracting, multiplying and dividing, and squaring and taking the square root. As students' learning expands, they begin to develop the understanding that inverse functions undo each other. For example, in grade 8, students might be asked to find the side lengths of a cube with a volume of 512 cubic inches. In order to solve the problem, students would need to apply the inverse operation of cubing and apply the cube root function to the number 512 to arrive at side lengths of 8 inches.

In Algebra I, students use inverse operations to rearrange formulas to highlight a quantity of interest (A1.A.CED.A.4) and solve linear, quadratic, and absolute value equations (A1.REI.B.2). Students also begin to understand a function as a process that produces output values corresponding to certain input values.

In Algebra II, students continue to use inverse operations to find the inverse of a function, which produces the input values corresponding to particular output values. So that each output value produces one unique input value, the original function must be one-to-one. Otherwise, the inverse of the function would not be a function itself. Allowing students the opportunity to see the graph of functions and their inverses gives students the opportunity to make connections among key features of each graph, such as understand independent and dependent variables, the relationship between each, and their relationship to the function. Utilizing tools, like horizontal and vertical line tests, further solidify these connections.

At this level, students should be able to find the inverse of one-to-one functions such as linear, cube root, cubic, exponential, and logarithmic functions or restrict the domain of a function that is not one-to-one (e.g., quadratics). Students should have experiences working with functions represented in various forms (tables, graphs, list of ordered pairs, mappings, equations) and develop flexibility finding the inverse among representations. Instruction should provide students with opportunities for real-world connections that allow students to discover the need for an inverse function. For example, real-world

situation where if a set of output values are given, it would be tedious to solve for the corresponding input values individually. Consider, F = 9/5C +32, which is the formula to convert from Celsius to Fahrenheit. If given multiple Fahrenheit temperatures, it would be time consuming to solve for each of the corresponding Celsius temperatures individually using this function. Instead, the inverse function can be used to convert the values much quicker, and thus create a formula to convert any Fahrenheit temperature to Celsius (i.e., C=5/9(F−32). Other relationships that may be helpful examples include: the perimeter of a square and its side length (P= 4x), the area of a circle and its radius (A=πr^2), and the circumference of a circle and its radius (C= 2π).

Although switching x and y works algebraically, instruction should be carefully organized to ensure that students solve for the input variable to create a new function whose operations undo the original function's operations. This way, students understand that they are using inverse operations in the opposite order to solve for the independent variable (e.g., x), which addresses a common misconception of simply reciprocating numbers and changing operations. Encoding and decoding using functions is a great way for students to think about the undoing process. Also, allowing students to experience inverse functions through multiple representations can strengthen their understanding of this concept. For example, asking questions from a table of values requires new ways of thinking about an inverse function because it does not have an equation to manipulate.

Level 4:

As students develop a deep understanding of this standard, they should be able to explain inverse functions as an undoing process using the composition of functions. In addition, if the original function is not one-to-one (e.g., y=x^2), students at this level should restrict the domain so that every output value corresponds to a unique input value, which means its inverse is a function. Restricting the domain will be very useful in future courses when learning about inverse trigonometry functions.

Furthermore, students should understand the differences when finding the inverse of a function with a context (e.g., F= 9/5C+32). Students at this level should understand why switching the variables does not make sense in a real-world context and that they are simply changing which variable is the input and output. Therefore, graphing a function and its inverse on the same coordinate plane no longer makes sense because the axes need to be switched on a separate plane in order to graph the inverse function.

Functions

Linear, Quadratic, and Exponential Models* (F.LE)

Standard A2.F.LE.A.1 **Cluster Heading: A. Construct and compare linear, quadratic, and exponential models and solve problems.**

Know the relationship between exponential functions and logarithmic functions.*

A2.F.LE.A.1a Solve exponential equations using a variety of strategies, including logarithms.*

A2.F.LE.A.1b Understand that a logarithm is the solution to $ab^{ct} = d$, where a, b, c, and d are numbers.*

A2.F.LE.A.1c Evaluate logarithms using technology.*

Scope and Clarifications: (Modeling Standard)

Bases should include all numbers, including the natural base e.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X	X*	X

**procedural skill and fluency only applies to A2.F.LE.A.1c*

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Recognize an exponential function as one in which the base is a constant, while the exponent is the variable. Identify the components of an exponential function using correct math vocabulary.	Identify the components of a logarithmic function and explain how they relate to the components of an exponential function. Convert between exponential and logarithmic forms using appropriate notation. Identify and use common and natural logarithms. Solve exponential equations using inspection.	Solve exponential equations by rewriting one or more bases so that both sides have a common base. Use a logarithmic function (with a base of any positive number) to solve a contextual problem. Understand that the exact solution to an exponential equation can be written as a logarithm. Evaluate logarithms using technology to find exact or approximate solutions.	Create a real-world problem to represent a given exponential equation. Solve simple logarithmic equations using similar solution strategies.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
		Solve exponential equations, f(x) = g(x), by using technology to find the x-coordinates of the points where the graphs of f(x) and g(x) intersect.	

Instructional Focus Statements

Level 3:

Students need to have a good understanding of exponential functions before learning logarithms so they can make the necessary connections between these two functions. As students explore exponential and logarithmic functions, they should rely on their understanding of inverse functions from standard A2.F.BF.B.4 and recognize that exponential and logarithmic functions are inverse functions. Specifically, that $ab^{ct} = d$ can be rewritten as $log_b\left(\frac{d}{a}\right) = ct$, where a is the coefficient of the exponential, *b* is the base, *ct* is the exponent, and *d* is the argument.

It is important for students to attend to precision when writing the notation for a logarithm. Students often do not understand that the base is written as a subscript and therefore can get confused between the base and the argument. One way to help build understanding of these components is to have students write both the exponential and logarithmic functions side by side and identify all the components. Logarithms can be thought of as the "what power" function because it produces the power needed to raise a number to in order to get a given output. For example, $log_4 16$ asks, "what power is 4 raised to in order to get 16?" Thus, $log_4 16 = 2$.

Students should be able to solve exponential equations in a variety of ways, which includes but is not limited to inspection, change of base, logarithms, and technology. For example, $3(4)^x = 48$ could be solved using inspection, $3^{x+2} = 9^{2x-1}$ could be solved using change of base (by rewriting 9 and 3^2), and $4e^{5x} = 32$ could be solved using logarithms or technology. Students should understand that the exact solution to the last example can be written as $\frac{\ln(8)}{5}$ and an approximate solution can be found by graphing $y = 4e^{5x}$ and $y = 32$ and finding that they intersect at an x-value of 0.416. Students initially learned this strategy of finding solutions using technology in algebra I (A1.A.REI.D.6) and should revisit and apply it here.

Students will learn additional properties of logarithms in more advanced courses, including product, power, and quotient properties of logarithms.

Level 4:

When given an equation or inequality, students can generate a real-world situation that could be solved by a provided equation demonstrating a deep understanding of the interplay that exists between the situation and the equation used to solve the problem.

Students at this level should be provided opportunities to use the relationship between exponential and logarithmic functions to solve logarithmic equations. Students should be able to solve logarithmic equations in a variety of ways, which includes but is not limited to inspection, exponentials, and technology. For example, $4log_6(x) = 8$ could be solved using inspection and $log_5(x - 1) = 0.71$ could be solved using exponential form or technology.

Functions

Linear, Quadratic, and Exponential Models* (F.LE)

Standard A2.F.LE.A.2 **Cluster Heading: A. Construct and compare linear, quadratic, and exponential models and solve problems.**

Know that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or cubically.*

Scope and Clarifications: (Modeling Standard)

For example, illustrate using graphs and tables that $g(x) = 2^{1.6x}$ eventually exceeds $f(x) = 4x^3 + 18$.

Tasks are limited to linear, quadratic, cubic and exponential functions.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Understand what is meant by an increasing function. Describe numerically why an increasing graph visually rises as x increases. Identify the interval(s) where a function is increasing and decreasing given a graph or a table of values.	Calculate the average rate of change of linear, quadratic, cubic, and exponential functions over a given interval. Describe interval(s) where a function is increasing and decreasing using interval notation or inequality notation given a graph or table of values. Understand that an exponential function will eventually exceed a linear function.	Compare the end behavior of graphs of lines, quadratics, cubic, and exponentials to determine which increases faster. Find and compare the average rate of change of lines, quadratics, cubics, and exponentials over the same intervals and make conclusions. Defend why a quantity increasing exponentially will eventually exceed a linear, quadratic, or cubic function and justify the conclusion by testing values.	Verify and explain why a quantity in one function type will eventually exceed a quantity in another function type. Observe graphs that model real-world scenarios and explain in context the reasonableness of why one graph increases faster than the other. Find the exact quantity where an exponential function exceeds another using technology and explain what it means in context of the real-world situation.

Instructional Focus Statements

Level 3:

In algebra I, students were introduced to linear, quadratic, and exponential functions. In algebra II, students are introduced to cubic functions and are asked to observe that exponential functions will eventually exceed linear, quadratic, and cubic functions. Students should be provided examples of all these different types of functions in multiple representations to compare which type function will increase faster. It is important that students understand what is meant by "increasing faster". This will typically be easier for students to see in a table of values with the independent variable values listed in numerical order.

When analyzing graphs, students should be instructed to read the graph left to right (as x increases) and can consider end behavior to help them draw a conclusion. As this is a modeling standard, it is important that students interpret the graphs in context to understand the relationship between the variables.

Students have more experience with linear functions and should be able to tell if it is increasing or decreasing since the rate of change remains constant. To help students understand the rate at which a function is increasing for functions other than linear, they should compare average rates of change over equal intervals for the functions being compared. This concept was covered in algebra I through standard A1.F.IF.B.6 for linear, piecewise, quadratic, and exponential functions and expanded in algebra II in standard A2.F.IF.A.2 to include all function types. Therefore, this skill is a pre-requisite to this standard.

Comparing graphs of functions on the same coordinate plane will also help students see what "increasing faster" looks like. If students struggle with getting the two graphs confused when they are on the same coordinate plane, have them graph them with different colors, or put one on tracing paper so that it can be placed over the other one. This can also be alleviated by having students compare the graphs using technology.

Level 4:

At this level of understanding, students should be able to explain precisely how they know a quantity that increases exponentially will eventually exceed that of a quantity that increases linearly, quadratically, or cubically.

Students should be asked to explain in context the reasonableness of why a graph that models one situation would increase faster than a graph that models another situation. In algebra I, students learned to find the intersection of two functions and describe what the intersection means in context (A1.A.REI.D.6). Students can revisit and apply these skills when finding the exact quantity where an exponential function exceeds another.

Statistics and Probability

Interpreting Categorical and Quantitative Data* (S.ID)

Standard A2.S.ID.A.1 **Cluster Heading: A. Summarize, represent, and interpret data on a single count or measurement variable.**

Use statistics appropriate to the shape of the data distribution to compare center (median, mean, and/or mode) and spread (range, standard deviation) of two or more different data sets.*

Scope and Clarifications: (Modeling Standard)

Students will be given a visual and/or verbal description of a density curve. Shapes of distribution are limited to: uniform, symmetric, right skewed, and left skewed. Student will not have to calculate standard deviation.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Determine the mean, median, and mode of a single set of data from a table. Determine the range of a single set of data from a table. Define center and spread, when related to a data set.	Recognize and describe different distribution shapes using the appropriate mathematical terms: uniform, symmetric, right skewed, and left skewed. Find the mean, median, mode, range, interquartile range, and standard deviation of a data set using technology.	Given a visual and/or verbal depiction of a density curve, choose which measure of center and measure of spread are most appropriate to the shape of the data distribution. Determine within how many standard deviations above or below the mean a data value is. Explain similarities and differences using the mean, median, and/or mode and range and/or standard deviation of two or more data sets	Compare and contrast the shape and measures of center and spread of multiple data sets. Explain how standard deviations can be used to describe how much of the data is around the mean. Relate the shape and the measures of center and spread in a real-world context and write a conclusion using appropriate mathematical vocabulary.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
		and describe how they relate to the shape of the data distribution.	

Instructional Focus Statements

Level 3:

One of the major goals of statistics is to summarize, compare, and predict data. In Algebra 1, students compare the measures of center and spread of two or more different data sets. In Algebra 2, students learn an additional measure of spread (standard deviation) and focus primarily on comparing data sets represented as density curves rather than individual data points.

Students are introduced to standard deviation as a measure of spread in Algebra 2. However, they are not required to calculate it from a numerical data set by hand method. Students may be given the standard deviation, or use technology to find the standard deviation, or find it by examining a graphical representation with the standard deviation marked and use it to answer a question or relate it to the context. It is important that students develop a conceptual understanding of the use of standard deviation as a measure of spread and understand its relationship to the mean. This is further developed in A2.S.ID.A.2. When using technology to find the standard deviation teachers should help students to understand both population (σ_x) and sample (s_x) notation and when each is appropriate.

Students should be given the time to explore the measures of center and the measures of spread in differently shaped distributions and make connections between them. For example, allow students to explore what happens to the mean, median, and mode when the data set is uniform, symmetric, or skewed. Have them compare data sets that have equal measures of spread and different measures of center, or vice versa, and then draw conclusions about which measure of center or spread is most appropriate for the shape and context. While students should continue to practice creating the visual representations of the data by hand, having students use technology to create the graphs allows for a focus on the comparison, particularly between multiple representations such as comparing a box plot to a histogram.

When given a variety of graphical representations, students should be able to discuss which data set has a greater center and which data set has greater variability using appropriate measures of center and spread. Students should develop an understanding that the mean better represents the data when it is approximately uniform or symmetrical and does not contain any extreme values and that standard deviation is used to describe the variability around the mean of a data set. However, extreme values will pull on the mean, therefore the median better represents data that is strongly skewed. Likewise, the interquartile range is less affected by extreme data points. Using statistics appropriate to the shape of the data will allow students to represent their data sets accurately and make stronger comparisons between multiple data sets within given contexts.

Students should be expected to use precise mathematical vocabulary to describe the shape of the visual representation of the data distribution and the measures of center and spread.

Level 4:

Students should compare and contrast data sets based on real-world situations. Therefore, students should think about how the differences in shape, center, and spread relate to the real-world context accounting for the effects of any extreme data points. They should also be able to describe how the information can inform decision making. Teachers should guide students to write their conclusions using precise mathematical vocabulary.

Students at this level should be able to explain the relationship between the measure of spread and the measure of center recognizing that the measure of spread is described by its measure of center. Providing students multiple data sets to explore can help them realize this relationship, develop a deeper understanding of the connections between each measure of center and its appropriate measure of spread, and explain the relationship using precise mathematical vocabulary.

Statistics and Probability

Interpreting Categorical and Quantitative Data* (S.ID)

Standard A2.S.ID.A.2 **Cluster Heading: A. Summarize, represent, and interpret data on a single count or measurement variable.**

Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages using the Empirical Rule.*

Scope and Clarifications: (Modeling Standard)

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Recognize when a data set can be represented by a normal curve. Calculate the mean and standard deviation of a data set using technology. Understand the relationship between the mean and the standard deviation. Display data on a histogram.	Label a normal distribution curve with a given mean and standard deviation. Determine how many standard deviations above or below the mean a data value lies. Label a normal distribution curve with percentages from the Empirical Rule.	Fit a data set to a normal distribution using the mean and standard deviation. Use the Empirical Rule to estimate the percent of data above or below given values on the normal curve within a context. Use the Empirical Rule to estimate the percent of data between given values on the normal curve within a context.	Explain how the Empirical Rule can be used to inform decision making within a context.

Instructional Focus Statements

Level 3:

In Algebra I, students describe center and spread using median and interquartile range as well as mean and range. Standard deviation is new to Algebra 2 and is introduced in A2.S.ID.A.1. Instruction for this standard should extend students' understanding of mean and standard deviation to how they apply to

the normal curve as well as what types of situations normal curves are used. A normal curve occurs when data is approximately normally distributed, which is often referred to as “bell shaped” and symmetric about the mean. It is important for students to understand that the area under the curve represents 100% of the data. By locating a value along the horizontal axis of a normal curve, students can determine the percent of data that falls either above or below that value.

The Empirical Rule provides standardized percentages of the area under the curve that align to each whole number standard deviation. The percent of data that falls within one standard deviation of the mean is approximately 68%, within two standard deviations is approximately 95%, and within three standard deviations is approximately 99.7%, which, of course, means that .03% of the data can be expected to fall outside of 3 standard deviations of the mean. It is important that students understand that the Empirical Rule can only be applied with the distribution of the data is approximately normal. To help students understand the application of the Empirical Rule, provide them with a few different data sets, some that are normally distributed and some that are not. Then have them determine the percentages of data that fall within each standard deviation of the mean and compare their results to the 68, 95, and 99.7 percentages.

To develop an understanding of its usefulness, students should apply the Empirical Rule to approximately normal distributions. For example, given a normal distribution with a mean of 15 and a standard deviation of 5, what percent of the data is between 10 and 20? Teachers should then challenge students to use the Empirical Rule to investigate more complex problems such as: what percent of the data is below 30? Above 5? or between 10 and 25? Therefore, instruction should include using geometric concepts to estimate areas under the curve that are not directly provided by the empirical rule. For example, what percent of data is above two standard deviations to the right of the mean? Discussion and discovery should lead to the understanding that since 95% of the data falls within two standard deviations from the mean, 5% fall outside of these values and thus, half of that (2.5%) would fall above two standard deviations to the right of the mean. Students should be provided contextual problems to help them make meaning of the data and write conclusions that clearly indicate that understanding. For example, approximately 68% of women are within one standard deviation of the mean height or between 59.6 inches and 67.9 inches tall.

Level 4:

As students develop a deep understanding of this standard, they should be provided real-world examples and asked how the normal distribution and Empirical Rule can be used to inform decision making. Students may also be curious about how to find percentages on the normal curve that are not a multiple of the standard deviation from the mean. This concept is addressed in A2.S.ID.A.3 using z-scores.

Statistics and Probability

Interpreting Categorical and Quantitative Data* (S.ID)

Standard A2.S.ID.A.3 **Cluster Heading: A. Summarize, represent, and interpret data on a single count or measurement variable.**

Compute, interpret, and compare z-scores for normally distributed data in a real-world context.*

Scope and Clarifications: (Modeling Standard)

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X	X	X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Draw and label a normal distribution curve with a given mean and standard deviation.	Calculate a z-score using technology. Understand the relationship between a normal distribution and the Empirical Rule.	Compute z-scores for normally distributed data in a real-world context using technology and the formula. Understand why z-scores are useful in real-world applications. Interpret and compare z-scores for normally distributed data in a real-world context.	Explain why z-scores are less useful in data sets that do not fit a normal curve by comparing data distributions that have different shapes with those that fit a normal curve.

Instructional Focus Statements

Level 3:

Students compute and interpret measures of center and spread in Algebra 1. However, they are first introduced to standard deviation in Algebra 2. As students learn about the roles of mean and standard deviation in a normal distribution, they may be curious about how to find percentages of the population for values that fall between the whole number standard deviations. That is where z-scores come in. Therefore, this standard should be paired with A2.S.ID.A.2. Students should have a good understanding of mean and standard deviation prior to exploring the Empirical Rule and z-scores.

Instruction should develop a conceptual understanding of how a z-score is calculated using the formula $z = (x - \mu)/\sigma$, where z is the z-score, x is the data point, μ (mu) is the population mean, and σ (sigma) is the population standard deviation. Students should also know how to calculate a z-score using technology.

The use of standard deviation allows for comparison of two different distributions by considering where a given data point falls within a normal curve. Z-scores allow us to examine data points that lie between whole number standard deviations, which are explored in A2.S.ID.A.2 using the Empirical Rule. Applications can vary from considering the percent of the population that falls above or below a given data point (i.e., a student's score is above 82% of all test takers) to identifying values at a specific percentage (i.e., what data point marks the top 10%) all while allowing for comparison of distributions with different scales.

By having students compare data that is relevant to them, students will begin to better understand z-scores and why they are useful. For example, both the SAT and the ACT serve as college admissions exams, however, their scores look very different. Often students take both. Using z-scores allows for a meaningful comparison between the scores in relation to each exam's mean. For example, student A scores 1370 on the SAT. The mean SAT score is 1000 with a standard deviation of 200. Student B scores a 31 on the ACT. The ACT has a mean score of 21 and a standard deviation of 5. Who has the better score?

A common misconception that occurs is when students find a negative z-score and think that it indicates a negative value in the context. To avoid this misconception, it is essential that students understand that z-scores tell how far a given value is from the mean. If the z-score is between 2 and 3, then the value in context falls is between 2 and 3 standard deviations from the mean, which is indicated by 0, but above the average value in context. If the z-score falls between -2 and -3, then it also falls between 2 and 3 standard deviations of the mean but below the average value in context. A negative z-score simply indicates the value is below average, while a positive z-score indicates the value is above average.

Level 4:

It is important for all students to understand that z-scores are more useful if the data fits a normal distribution. At this level, students can explore why that is the case. Students can be challenged to explore data that is uniform or skewed with data that is normally distributed to develop a deeper understanding of a z-score and its applications in the real world. Differently shaped data can have the same mean and standard deviation. However, both mean and standard deviation, and thus a z-score, are sensitive to extreme values. Students should recognize and understand that median and interquartile range are better choices for the measure of center when data is skewed or contains extreme values.

Statistics and Probability

Interpreting Categorical and Quantitative Data* (S.ID)

Standard A2.S.ID.B.4 **Cluster Heading: B. Summarize, represent, and interpret data on two categorical and quantitative variables.**

Represent data from two quantitative variables on a scatter plot and describe how the variables are related. Fit a function to the data; use functions fitted to data to solve problems in the context of the data.*

Scope and Clarifications: (Modeling Standard)

Use given functions or choose a function suggested by the shape of the data.

Tasks are limited to linear, quadratic, cubic, logarithmic, and exponential functions.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Choose a linear function to fit a given data set. Informally decide if a given scatter plot is best represented by a linear or non-linear function.	Represent data from two quantitative variables on a scatter plot. Choose which function type (linear, quadratic, cubic, logarithmic, exponential) is suggested by the shape of the data. Informally fit a linear, exponential, or quadratic function to a given set of data represented in a scatter plot. Use a given linear, exponential, or quadratic function to solve a problem.	Represent data from two quantitative variables on a scatter plot and describe how the variables are related. Fit a function (linear, quadratic, cubic, logarithmic, exponential) to a given set of data. Use a function fitted to data to solve problems in the context of the data. Recognize that extrapolation may be unreliable for making predictions.	Create a contextual situation with an embedded data set derived from a given function. Explain the relationship between the function, data set, and the contextual situation using precise mathematical language and justifications. Use a given function or function type to explain the relationship between two quantities in context. Explain why extrapolation may be unreliable for making predictions.

Instructional Focus Statements

Level 3:

In Grade 8, students constructed and interpreted scatter plots and informally fit a line to the data in a scatterplot that suggested a linear relationship. In Algebra 1, students discovered that that non-linear functions sometimes provide a better fit to the data and found the line of best fit using technology. In Algebra 2, students will explore cubic and logarithmic functions as additional options for a fitted line. This broadens the options students have for modeling data sets, where data sets can be represented in tabular, graphical, or as a discrete set of points.

Students will explore real-world situations using the function types they are studying in Algebra 2 (linear, quadratic, cubic, logarithmic, and exponential) to choose an appropriate model that fits the data. They will use technology to find the equation for the function that best fits the data. As students progress through the course, they will learn to use these mathematical models to capture key elements of the relationship between the two variables and explain what the model reveals about the relationship.

Students should develop an understanding of the connection that exists between the model and the contextual situation that it represents and understand that this connection is essential in identifying and building appropriate models. To build this concept, students can start by choosing a function type suggested by the shape of the data in a scatter plot and informally finding the line of fit. Then they can find the actual line of best fit using technology. This should lead to an exploration around the similarities and differences between their informal line and the line of best fit. Discourse should focus on why the informal line was chosen and what data points may have affected the actual line of best fit, taking into consideration both the function type and the placement of the fitted line. Students can use various forms of technology to explore and represent scatterplots to enhance their ability to see the relationship that exists between the variables.

As students solidify their understanding, they should be able to describe how the variables are related within the context of the situation. For example, as a child ages (x-axis variable), she gets taller (y-axis variable) or the further a car travels (x-axis variable), the less gasoline is in the tank (y-axis variable). It is imperative that students discover that sometimes obvious patterns may not tell the whole story. Students should recognize that making predictions outside the given range of data points provided is referred to as extrapolation and can lead to false assumptions. For example, once a child reaches a certain age, she will stop getting taller. Therefore, the function that fits the data in this example cannot be used to predict the child's height at age 30.

Level 4:

As students extend their understanding, they should be able to create a contextual situation with an embedded data set derived from a given function. Students should also be able to explain and provide justifications for the relationships that exist between the function, data set, and the contextual situation using precise mathematical language. Particular attention should be placed on creating situations that differentiate between linear, quadratic,

cubic, logarithmic, and exponential functions. Students should be able to justify and explain why one function is more appropriate than another function for the contextual situation.

Students at this level should have a deeper understanding of why extrapolation can lead to false assumptions. As they apply models to make predictions about unknown situations, they should justify what domain is appropriate based on the given context. For example, if time is the independent variable, students should recognize that the domain should be restricted to values greater than or equal to zero. Teachers should also expect students to explain why extrapolation is not reliable in some situations and that it cannot be assumed that the existing trend will continue to unknown values beyond the given data.

TN Department of Education

Statistics and Probability

Making Inferences and Justifying Conclusions* (S.IC)

Standard A2.S.IC.A.1 Cluster Heading: A. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Recognize the purposes of and differences among sample surveys, experiments, and observational studies.*

Scope and Clarifications:

For example, in a given situation, is it more appropriate to use a sample survey, an experiment, or an observational study?

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify a study as a sample survey from a given verbal description of the context. Identify a study as an observational study from a given verbal description of the context. Identify a study as an experiment from a given verbal description of the context.	Determine if a sample survey, experiment, or observational study would be most appropriate, given a contextual situation,	Explain the differences among sample surveys, experiments, and observational studies. Describe what type of situations would be most appropriately studied with sample surveys, experiments, and observational studies. Explain the limitations of sample surveys, experiments, and observational studies.	Create a research question best answered through a sample survey and with a sound methodology that minimizes bias. Create a research question best answered through an experiment and with a sound methodology that minimizes bias. Create a research question best answered through an observational study and with a sound methodology that minimizes bias.

Instructional Focus Statements

Level 3:

In Algebra I, students learn that correlation does not imply causation (A1.S.ID.C.7). The way to determine causation is to conduct an experiment in which all other confounding factors are controlled. Teachers should emphasize the importance of randomization in selecting a sample for the study and random assignment to the control and treatment groups so that inferences can be made to the intended population. For example, if studying Tennessee males, randomly selecting 200 Tennessee males then randomly assigning 100 to the control group and 100 to the treatment group would suffice. If randomization is not met, the results will not be as useful, as the data may be biased (e.g., if only males from east Tennessee were selected). Randomization is also addressed in A2.S.IC.A.2 when identifying sources of bias. Although a simple random experiment is the best-case scenario, students should recognize it is often difficult to conduct due to ethical concerns, uncontrollable factors (e.g., weather), or financial barriers. In these situations, an observational study can be conducted instead. Teachers should lead conversations on various research questions and whether an experiment would be most appropriate.

Observational studies are best conducted when natural conditions allow for studying existing control and/or treatment groups (researchers do not intervene in any way). Due to using existing groups, observational studies are rarely, if ever, considered random. For example, an experiment researching the effect of spider bites on children would have major ethical concerns. Therefore, students should realize that researchers in this study would only be able to conduct an observational study with children that have already been bitten by a spider. The sample was not randomly selected; therefore, students should have some concerns about bias with this study. Observations studies may suggest or reject rationales for causation, but an experiment is necessary to prove causation.

The most common type of study for public use is the survey. Surveys occupy something of a gray area in the controlled experiment—observational study dichotomy. The researcher is far from a passive role, often expending great efforts in choosing and contacting participants. However, surveys usually ask about things that the participants have already done or intend to do or about characteristics that they already possess. Research surveys are designed to uncover associations between group membership and behavior. For this reason, surveys are often classified as observational studies, but they can be used as instruments in an experimental design as well depending on how participants are selected with regard to randomization and membership characteristics.

In a sample survey, participants are surveyed on the research topic. Again, randomly selecting participants from the population is best. If this is not possible, other methods can be used to select participants, such as stratified random sampling, systematic sampling, cluster sampling, or multi-stage sampling. A discussion about other sampling methods including convenience sampling and volunteer sampling should focus on how these can create bias. Bias is addressed more specifically in A2.S.IC.A.2.

Level 4:

As students develop a deep understanding of these types of studies, they should progress to designing their own studies. Beginning with a topic that interests them, students should create a research question, describe their methodology, and justify the chosen type of study. Other students should have opportunities to critique the shared methodology and describe its limitations. In future courses, students may learn about hypothesis tests that are used to analyze the data gathered through experiments, observational studies, and a sample survey.

Statistics and Probability

Making Inferences and Justifying Conclusions* (S.IC)

Standard A2.S.IC.A.2 **Cluster Heading: A. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.**

Identify potential sources of bias in statistical studies.*

Scope and Clarifications:

Sources of bias include but are not limited to: leading questions, lack of randomization, sampling bias, undercoverage, nonresponse, and/or small sample size.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Define various sources of bias in statistical studies.	Choose whether a given statistical study is biased or not.	Identify potential sources of bias in statistical studies. Explain how potential sources of bias impact the validity of statistical studies. Explain ways in which potential sources of bias in statistical studies could have been avoided.	Design studies in such a way as to limit bias.

Instructional Focus Statements

Level 3:

In 7th grade students learned that one can gain information about a population by examining a representative sample of that population (7.SP.A.1 and 7.SP.A.2). In Algebra 2 students will explore this idea more deeply. A statistical study shows bias if it consistently underestimates or overestimates the characteristic of interest of the population that is being studied. Conclusions based on biased samples are flawed, so sampling methods should be

specifically designed to avoid bias. Discourse should lead students to recognize samples that are and are not representative of the population. Teachers should provide students with ample opportunities to experience a variety of statistical studies with bias. This is a great opportunity to bring data from statistical studies relating to current events into the classroom. Instruction should include the analysis of statistical studies. Students should be asked to notice and wonder about the studies, focusing the conversation on critiquing studies to determine their validity and on exposing bias.

Some sources of bias include leading questions, lack of randomization, sampling bias, voluntary response, undercoverage, overcoverage, nonresponse, and/or small sample size. Students should learn to recognize bias in a given study and explain how the bias could have been avoided. Conclusions drawn from a biased sample is inherently flawed and are never reliable. It is important to note that unbiased samples can still produce estimates that differ from the true value simply by chance. These estimates will overestimate about half the time and underestimate about half the time. Therefore, repeated sampling is important.

Level 4:

Students at this level should be expected to redesign a biased study so as to avoid bias. They should also design studies in a way that limits bias. This is a good opportunity for students to reach beyond the classroom to conduct the study that they designed. These studies may also be utilized for extending learning for standard A2.S.IC.A.3 where students at this level are expected to write conclusions from well-collected data.

Statistics and Probability

Making Inferences and Justifying Conclusions* (S.IC)

Standard A2.S.IC.A.3 Cluster Heading: A. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Distinguish between a statistic and a parameter. Evaluate reports based on data and recognize when poor conclusions are drawn from well-collected data.

Scope and Clarifications:

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Explain the difference between a statistic and a parameter.	Identify whether a quantity is a statistic or parameter.	Evaluate statistical reports based on data. Identify appropriate conclusions from well-collected data. Recognize when poor conclusions are drawn from well-collected data.	Write appropriate conclusions from well-collected data. Critique and make improvements to a report with poorly drawn conclusions from well-collected data.

Instructional Focus Statements

Level 3:

Students will learn to recognize when a given value is a parameter (a number describing a whole population, such as the population mean) and when a given value is a statistic (a number describing a sample, such as the sample mean). A connection can be made between this expectation and the expectations of standard A2.S.IC.A.2. Discourse should lead students to understand the importance of unbiased sampling. The mean from one sample (statistic) may not be close to the true mean of the population (parameter), but with repeated sampling, the sample mean will be closer to the population mean.

Students are expected to evaluate reports based on data. Students should also be presented with opportunities to evaluate conclusions drawn from well-collected data. They should be able to identify appropriate conclusions and recognize poor conclusions from that well-collected data. There are many ways to misinterpret data, and, therefore, draw poor conclusions. Examples of poor conclusions can include assuming a cause-and-effect relationship, misleading data visualizations, misleading use of language, and omission of certain results. Students should be given opportunities to critique conclusions drawn from well-collected data. Instruction should focus student thinking on understanding that drawing poor conclusions from well-collected data is just as flawed as drawing conclusions from biased data. One issue with drawing conclusions from sampling is generalizing the results to a group larger that the population represented in the study. For example, a school board is considering changing the start of the school day for high school students to 7:00 am. They survey a simple random sample of high school students at every high school in the district. This sampling is repeated. The results determine that high school students do not agree with the earlier start time. The school board then announces that families in the district do not favor an earlier start time. The original population being studied was high school students in that district, but the results were extrapolated to the families of all students in the district. Data drawn from a sampling only applies to the population being sampled. That data cannot be generalized to a larger or different population.

Level 4:

At this level students should be able to write appropriate conclusions from well-collected data. They should be given opportunities to critique and make improvements to a report with poorly drawn conclusions from well-collected data. This is an opportunity to use the study that the student designed for standard A2.S.IC.A.2.

Statistics and Probability

Conditional Probability and the Rules of Probability* (S.CP)

Standard A2.S.CP.A.1 **Cluster Heading: A. Understand independence and conditional probability and use them to create visual representations of data.**

Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. Categorize events as independent or dependent.*

Scope and Clarifications: (Modeling Standard)

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Recognize concepts of conditional probability and/or independence from a given model (for example, Venn Diagram). Identify, without calculations, when contextual events are simple, compound or conditional.	Recognize concepts of conditional probability and independence from a verbal descriptions or a given model (for example, Venn Diagram). Identify two contextual events, without calculations, as independent or dependent.	Recognize and explain concepts of conditional probability and independence from a verbal description or a given model (for example, a Venn Diagram). Categorize, without calculations, contextual events as independent or dependent.	Create a real-world context of events that are independent. Challenge ideas of mathematical and statistical independence in real-world contexts.

Instructional Focus Statements

Level 3:

The domain Conditional Probability and Rules for Probability is first introduced in Geometry, where students use visual representations to understand and compute probabilities. Instruction for this standard should focus on developing students' conceptual understanding of conditional probability and independence. Examples should be given in context in everyday situations, paired with visual models, and supported with student discourse in everyday

language. It is not the expectation for students to know or use formulas or calculate probabilities in this standard. Students will use rules of probability and compute probabilities in Cluster C.

The probability event *A* occurs, given event *B* has already occurred, is called the conditional probability. The conditional probability of *A* given *B* is written in set notation *P(A|B)*. For example, students will consider if a student who earns a letter-grade B or greater in an Algebra 2 course has a greater chance of passing the Algebra 2 End of Course exam. In this example, *P(A)*, the probability of passing the Algebra 2 EOC exam, is being based on *P(B)*, the letter-grade in Algebra 2 course, *P(passing EOC|course grade B or greater)*. For this example, students will reason if event *A* and *B* are conditional and if so, are they independent.

Students can use conditional probabilities to tell if two events are independent or dependent. Two events are considered to be independent when the likelihood of one event occurring does not affect the probability of another event occurring. In everyday language, a student can explain a student who is enrolled in Algebra 2 does not impact on the probability of a student participating in after school activities. Thus, *P(participating in after school activities|enrolled in Algebra 2 course)* represents two independent events. Two events are dependent when one event alters the outcome of the other event. In everyday language, a student can explain the temperature being 32 degrees or lower has an impact on the probability of snow. Thus, *P(snowing|temperature 32 degrees or less)* represents two dependent events.

Level 4:

As students develop a deep understanding of conditional probability and independence, they can create real-world events that are independent and real-world events that are dependent. In addition, students can justify and explain why each situation is independent or dependent. Students are able to challenge ideas of mathematical and statistical independence in everyday language and real-world contexts.

Statistics and Probability

Conditional Probability and the Rules of Probability* (S.CP)

Standard A2.S.CP.B.2 **Cluster Heading: B. Understand and apply basic concepts of probability.**

Apply statistical counting techniques.*

A2.S.CP.B.2a Use the Fundamental Counting Principle to compute probabilities of compound events and solve problems.*

A2.S.CP.B.2b Use permutations and combinations to compute probabilities of compound events and solve problems.*

Scope and Clarifications: (Modeling Standard)

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Identify the number of outcomes from a given situation such as rolling a die or drawing a card from a deck. Count the number of ways a small number of objects can be arranged in a row.	Use a table, tree diagram, or other graphic organizer to determine the number of possible choices leading to the outcome of an event. Understand the factorial symbol "!" to use the permutation and combination formulas (where n! = n(n-1)(n-2)....(1)). Know that 0! = 1.	Understand the sample space of an event is all the possible outcomes for that event. Understand the fundamental counting principle states that if there are p ways to do one thing, and q ways to do another thing, then there are p×q ways to do both things. Extend the fundamental counting principle for k sets: If there are n_1 different objects in one set, n_2 different objects in a second set, and so on, for k sets, then the number of ways of choosing one	Design and analyze all possible outcomes of a real-world scenario where the counting principle can be extended to situations involving multiple stages and choices for each stage. Recognize the number of permutations of n objects with r identical objects is: $\frac{n!}{r!}$ Explain the difference between a permutation and a combination using real-world examples.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
		object from each set is $n_1 \times n_2 \times n_3 \times \ldots \times n_k$. Use the permutation formula: the number of permutations of n distinct objects taken r at a time is: ${}_nP_r = \frac{n!}{(n-r)!}$, n ≥ r, to compute probabilities of compound events and solve problems. Use the combination formula: the number of combinations of n distinct objects taken r at a time is: ${}_nC_r = \frac{n!}{r!(n-r)!}$, n ≥ r, to compute probabilities of compound events and solve problems.	

Instructional Focus Statements

Level 3:

Students can apply statistical counting techniques, such as the Fundamental Counting Principle and the use of permutations and combinations of compound events, to apply basic concepts of probability. Probabilities are characterized as parts of a whole. By using these counting techniques, students will be able to identify the size of the whole (i.e., the denominator of the probability ratio) and determine the appropriate probability representation (such as decimal form, fractional form, percentage, or ratio).

The focus of this standard should be on developing students' conceptual understanding of probabilities. Therefore, instruction and classroom level assessment should leverage the use of technology for calculations. It is not the expectation for students to memorize the formulas involving permutations and combinations, but rather to access these features within the technology tool.

Fundamental Counting Principle

The Fundamental Counting Principle (also called the counting rule) is a way to figure out the number of outcomes in a probability problem. Basically, you multiply the events together to get the total number of outcomes. The formula is:

*If you have an event with "p" outcomes and another event with "q" outcomes then the number of all the different outcomes consisting of one outcome from the first event and one outcome from the second event is p * q.*

For example: a fast-food restaurant has a meal special: $5 for a drink, sandwich, side item and dessert. The choices are:

- Sandwich: Grilled chicken, All Beef Patty, Veggie burger and Fish Filet.
- Side: Regular fries, Cheese Fries, Potato Wedges.
- Dessert: Chocolate Chip Cookie or Apple Pie.
- Drink: Fanta, Dr Pepper, Coke, Diet Coke and Sprite.

Q. How many meal combos are possible?

A. There are 4 stages:

1. Choose a sandwich.
2. Choose a side.
3. Choose a dessert.
4. Choose a drink.

There are 4 different types of sandwiches, 3 different types of sides, 2 different types of desserts and five different types of drinks.

The number of meal combos possible is 4 * 3 * 2 * 5 = 120.

To model this, you can make 4 lines, one for each choice that needs to be made. Fill in the number of ways you can make each choice, then multiply.

4	x	3	x	2	x	5	= 120
Choose sandwich		Choose side		Choose dessert		Choose drink	

In this example, 120 represents the total number of meal combo options. Thus, 120 represents the whole when considering the probability of any particular meal combination. For example, to calculate the probability an order containing a grilled chicken sandwich, regular fries, an apple pie, and a Fanta drink would be 1:120.

Another example, consider all combo meal that contain French fries as the choice of sides. To calculate the number of ways for combo meals to only have French fries as the choice of side is 4 * 1 * 2 * 5 = 40, where the underlined value of 1 represents the item choice of French fries. Therefore, the probability can be calculated as 40/120, or 1/3.

Permutations

An arrangement of a set of objects is called a permutation. The word permutation comes from the Latin words per + mutare that together mean "by change" or "through change." In a permutation, order matters; for example, counting all the ways in which the cars at the Indianapolis 500 could finish the race; all the ways books can be set on a shelf; and all the ways in which students line up to get lunch are permutations.

A permutation is any ordered arrangement of a given set of objects. Permutations that involve ordering *r* of *n* distinct objects obey the formula: ${}_nP_r = \frac{n!}{(n-r)!}$

Example: The letters S, T, A, R, P are placed in a hat. A letter is removed one at a time from the hat and is not replaced. This process is repeated until all of the letters have been removed. How many different ways can this be done? This problem can be solved using the permutation formula above as this is an ordered arrangement of 5 of 5 letters.

Solving using the permutation formula: *n* = 5 and *r* = 5; *n* represents the number of letters in the hat, and *r* represents the total number of letters being removed:

${}_nP_r = {}_5P_5 = \frac{5!}{(5-5)!} = \frac{5*4*3*2*1}{0!} = \frac{120}{1}$; answer: 120 ways

This problem can also be solved using the Counting Principle. There are 5 letters to choose for the first letter. Assuming that a letter can't be used more than once and increment down by 1 letter each time to show the number of choices available for each letter position:

5	x	4	x	3	x	2	x	1	= 120
first draw		second draw		third draw		fourth draw		fifth draw	

With permutations the purpose is to determine how many arrangements of objects from the same set can be made where the order of the objects matters.

Therefore, there are 120 ways the letters can be selected from the hat. Therefore, to calculate the probability of selecting a specific order of the five letters such as A,P,R,T,S is 1out of 120.

Another example: Kayla has 10 books in her room. She only has room for 5 of them on her shelf. Determine the number of ways that she can put the books on the shelf: n = 10 because she has 10 books; r = 5 because she is placing 5 of the books on the shelf; use ${}_nP_r = \frac{n!}{(n-r)!}$; ${}_{10}P_5 = \frac{10!}{(10-5)!}$ = 30,240; there are 30,240 ways she can arrange the books.

Using the Counting Principle, your solution would be:

10	x	9	x	8	x	7	x	6	= 30,240
first book		second book		third book		fourth book		fifth book	

Note that a pin code for a debit card, the digits in a telephone number, and the alphanumeric characters on a license place are also examples in which the order matters (e.g., mixing up the telephone number digits will cause you to call the wrong person), but using the permutation formula is not appropriate in these cases since your numbers and letters can be repeated. For these instances, use the Counting Principle.

Combinations

With combinations: the purpose is to determine how many groups of objects from the same set can be made where the order of the objects does not matter: ${}_nC_r = \frac{n!}{r!(n-r)!}$

Combinations count the number of selections (subsets) containing r objects that we can make from *n* objects. For example, counting the number of ways you can choose a committee of 3 members from a group of 10 people; choose 2 belts from a drawer containing 4 belts to pack for a trip; or choose 5 cards from a deck of 52 cards are all examples of combinations because it doesn't matter in which order you choose the items.

Example: Choosing 6 numbers, in any order, from the numbers 1 through 59 for a lottery game where the jackpot is won for all six numbers matching the winning numbers, ${}_{59}C_6 = \frac{59!}{6!(59!-6)!}$ = 45,057,474, there are 45,057,474 lottery jackpot combinations. The probability of selecting the winning number for the jackpot is odds of approximately forty-five million to one.

Combinations sound simpler than permutations, and they are. You have fewer combinations than permutations.

Level 4:

Students with a Level 4 understanding can adapt their understanding of permutations of distinct objects to calculate the number of permutations of a set of objects when some of the objects are not distinct. The number of permutations of n objects with r identical objects is: $\frac{n!}{r!}$; only the number of orderings change since some items are identical. Example: In how many ways can the letters in ALABAMA be arranged? You have 7 letters, but 4 of the letters are A's, so n = 7 and r = 4. The number of permutations is $\frac{7!}{4!}$ = 210.

Students will experience how Combinations can be used to expand a power of a binomial and to generate the terms in Pascal's triangle in further courses.

Statistics and Probability

Conditional Probability and the Rules of Probability* (S.CP)

Standard A2.S.CP.B.3 **Cluster Heading: B. Understand and apply basic concepts of probability.**

Use the Law of Large Numbers to assess the validity of a statistical claim.*

Scope and Clarifications: (Modeling Standard)

There are no assessment limits for this standard. The entire standard is assessed in this course.

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Understand a population is a group of objects about which inferences can be made. Understand a sample is a subset of the population. Understand a random sample of size *n* from the distribution of *X* is a collection of *n* independent random variables each with the same distribution as *X*. An experiment is a set of positive outcomes that can be repeated.	Recognize the term "law of large numbers" is sometimes used in a different sense with real-world applications to express the relationship between scale and growth rates. Recognize that as you perform more trials of an experiment, the relative frequency of an event approaches a single value. Understand the difference between empirical and theoretical probability.	Understand the law of large numbers states that an observed sample average from a large sample will be close to the true population average and that it will get closer the larger the sample. Understand the law of large numbers does not guarantee that a given sample, especially a small sample, will reflect the true population characteristics or that a sample which does not reflect the true population will be balanced by a subsequent sample. Understand an estimator is a statistic used to estimate a population parameter.	Design an experiment to show how the relative frequency converges to a single value with an increasing number of trials. Understand a small sample selected by using a good method can yield better results than a large sample selected by using a poor method.

Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
		Understand an unbiased estimator is a statistic that, on average, equals the value of the parameter.	

Instructional Focus Statements

Level 3:

There are two forms of the Law of Large Numbers, but the differences are primarily theoretical. The weak and strong laws of large numbers both apply to a sequence of values for independent and identically distributed (i.i.d.) random variables: X_1, X_2, ..., X_n.

The weak law of large numbers states that as *n* increases, the sample statistic of the sequence converges in probability to the population value. The weak law of large numbers is also known as Khinchin's law.

Here's what that means. Suppose you specify a nonzero difference between the theoretical value and the sample value. For example, you might define a difference between the theoretical probability for coin toss results (0.50) and the actual proportion you obtain over multiple trials. As the number of trials increases, the probability that the actual difference will be smaller than this predefined difference also increases. This probability converges on 1 as the sample size approaches infinity.

The strong law of large numbers describes how a sample statistic converges on the population value as the sample size or the number of trials increases. For example, the sample mean will converge on the population mean as the sample size increases. The strong law of large numbers is also known as Kolmogorov's strong law.

Both laws apply to various characteristics, ranging from the means for continuous variables to the proportions for Bernoulli trials.

For statistics, both laws of large numbers indicate that larger samples produce estimates that are consistently closer to the population value. These properties become important in inferential statistics, where you use samples to estimate the properties of populations.

A random variable, *X*, is a function that assigns to every outcome of an experiment, a real numerical value. If *X* can assume at most a finite or countably infinite number of possible values, *X* is said to be a discrete random variable. If *X* can assume any value in some interval, or intervals of real numbers and the probability that it assumes a specific given value is 0, then *X* is said to be a continuous random variable. Say, for example we are flipping a fair coin and are concerned with how many times the coin lands on heads. We can define the random variable *X* by

$$X = \begin{cases} 1, \text{ if the coin lands on heads} \\ 0, \text{ if the coin lands on tails} \end{cases}$$

This is a classic example of a random variable with a Bernoulli Distribution. As the number of trials grow, the probability of getting a head (or a tail) will approach 50% or $p = 0.5$.

The sample average of a population can differ greatly from the true population depending on the number of samples (or trials) taken. If a person wanted to determine the average value of a data set of 100 possible values, he is more likely to reach an accurate average by choosing 20 data points instead of relying on just two. For example, if the data set included all integers from one to 100, and sample-taker only drew two values, such as 95 and 40, he may determine the average to be approximately 67.5. If he continued to take random samplings up to 20 variables, the average should shift towards the true average as he considers more data points.

Another application example: two new companies are being assessed. Company A has a total market value of $60 million and company B has a total market value of $4 million. Which company is more likely to increase its total market value by 50% within the next few years?

Solution: in order for Company A to increase its total market value by 50% it requires an increase of $30 million whereas company B only requires a $2 million increase. The Law of Large Numbers implies that it is much more likely for company B to expand by 50% than company A. This makes sense because if company A was just as likely to expand by 50% as company B, company A could quickly have a higher market value than the entire economy.

Similarly, insurance providers use the LLN in order to determine effective prices for their clients. By analyzing and recording the number accidents among men aged 23-25, they can ascertain, with a high degree of accuracy, the probability of *X* amount of males aged 23 that will be the cause of an accident in any given year. This allows them to set an effective price for clients that fall into the range.

Misconception: what is known as the Law of Averages is a fallacy belief that the next occurrence in a trial will have different expected results following an occurrence of previous trial results. For example, if the results of a coin toss are H, H, & H believing that the result of the next trial is not a 50% chance of getting heads is wrong. The next trial result still has a 50% chance of resulting in heads. However, looking at the total outcome of getting the same result several times in a series of trials becomes a probabilistic event when grouping a series of trial outcomes. But, each trial maintains the expected outcome probability as a single event result.

Level 4:

A large sample size does not guarantee the accuracy of estimates made from that sample. In fact, statisticians assert that a large unrepresentative sample may do more damage than a small one because many people think that large sample are always better than small ones. The design of the sampling plan is far more important than the absolute size of the sample.

Students at this level are developing an understanding of the implications of sample size. Experiences in grades 6-8 mark the beginning of students' understanding of the role that sample size plays in inference, as students use data collected from random samples and simulations to make informal inference about populations. In high school, students' work in estimating population parameters makes the effect of sample size explicit. Student's informal ideas about the role of sample size often contain misconceptions, such as the mistaken idea that small samples are more effective with small populations than with large populations. Students entering high school often have an intuitive notion that "more is better," but through their experiences in high school grades they develop some sense of the costs of large sample sizes and the diminishing returns in increased precision. This concept is especially important for statistical literacy as students continue to develop statistical reasoning in later courses.

Statistics and Probability

Conditional Probability and the Rules of Probability* (S.CP)

Standard A2.S.CP.C.4 Cluster Heading: C. Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A* and interpret the answer in terms of the given context.*

Scope and Clarifications: (Modeling Standard)

Calculating conditional probability may be performed via use of a visual model (Venn diagrams, frequency tables, etc.), calculation/formula, or by using counting techniques.

For example, a teacher gave two exams. 75 percent passed the first exam and 25 percent passed both. What percent who passed the first exam also passed the second exam?

Aspect of Rigor Alignment

Conceptual Understanding	Procedural Skill and Fluency	Application
X		X

Evidence of Learning Statements

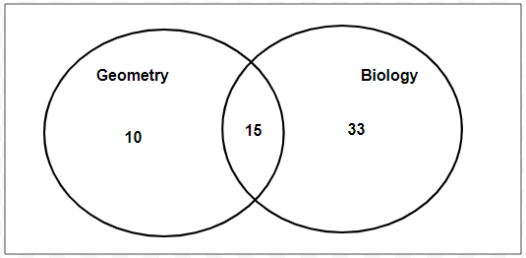
Students with a level 1 understanding of this standard will most likely be able to:	Students with a level 2 understanding of this standard will most likely be able to:	Students with a level 3 understanding of this standard will most likely be able to:	Students with a level 4 understanding of this standard will most likely be able to:
Recognize conditional probability situations in a real-world context. Perform numeric calculations with fractions (e.g., divide two fractions and put into simplest form) and percents (e.g., 5% is what percent of 20%).	Recognize conditional probabilities in real-world context and recognize the conditional whole may or may not equal the total. Find conditional probability in real-world context with a visual model (e.g., Venn diagram or frequency table) or counting technique that represents the relationship of two conditional events.	Interpret the conditional probability and the conditional whole of two events in terms of the given context. Find conditional probability in real-world context with a visual model (e.g., Venn diagram, frequency table, etc.), counting technique, or calculation/formula that represents the relationship of two conditional events.	Explain the conditional probability and the conditional whole of two events in terms of the regions of A and B in a Venn diagram. Explain the connections between the conditional probability formula and other representations (e.g., Venn diagram, frequency table, counting techniques, etc.).

Instructional Focus Statements

Level 3:

In standard G.S.CP.B.2, students use a visual model (e.g., Venn diagram, frequency table, etc.), that represents the relationship of two conditional events, to find conditional probability in a real-world context. Instruction in Geometry intentionally focuses on developing students' conceptual understanding of the conditional whole and population total, a common misconception. Teachers, in both Geometry and Algebra 2, should facilitate explicit connections to previous learning from Ratio and Proportional Relationships (RP) standards in middle grades, such as ratios, percent, and part-whole relationships, to allow student opportunities to reason about what constitutes the conditional whole in comparison to the population total in a given context. Students in Geometry do not yet use counting techniques or calculations/formulas to find conditional probabilities, which is an expectation for A2.S.CP.A.4.
In A.2.S.CP.A.1, students begin to recognize the probabilities of event *A*, *P(A),* and the probability of event *A* given event *B*, *P(A|B),* are different if the two events *A* and *B* are not independent. As students reason to find the conditional probability of two conditional events, problems should be contextual situations, have multiple entry points, and allow for multiple solution paths. An initial emphasis on finding conditional probability with visual models (Venn diagrams, frequency tables, etc.) or counting techniques will connect to previous learning, help students model real-world situations, and strengthen students' conceptual understanding of mathematical concepts.

Venn diagrams aid students in visualizing part-whole relationships by depicting the idea of conditional probability as the intersection of events *A* and *B*, *P(A ∩ B)*, as a part of the conditional whole. Understanding the population total in comparison to the conditional total is often a common misconception for students. In many cases, students are given the *P(A ∩ B)* and either *P(A)*, *P(B)*, or both and asked to calculate the conditional probability *P(B|A)* or *P(A|B).* Discussion during instruction would need to help students recognize that if they are given both *P(A)* and *P(B)*, one is superfluous information. For example, a Venn diagram can be used to model the number of students that are enrolled in Biology given they are also enrolled in Geometry, *P(enrolled in Biology | enrolled in Geometry)*. If 25 students are enrolled in Geometry (event A), 48 students are enrolled in Biology (event B), and 15 students are enrolled in both. Students may need support understanding the number of students enrolled in Geometry (25) becomes the conditional whole rather than the population total (58). Then, the conditional probability of students enrolled in Biology, given they are also enrolled in Geometry, can be expressed as a part-whole relationship using the conditional whole P(B given G)= *P(enrolled in Biology | enrolled in Geometry) = 15/25 or 60%.*



If more than two categories are used, teachers can provide or encourage students to use a table. For example, a frequency table can be used to compare the interest in sports across grade levels. If students are asked to find the probability a randomly selected student likes sports given they are in the 10th grade, *P(student likes sports | student in grade 10)*, students could be shown how to cover up the rest of the table to highlight only 10th graders. From there, students can use a part-whole relationship to express the conditional probability that a student likes sports as a fraction of the conditional whole. Explaining that the condition given narrowed down the subjects to only 10th graders helps students begin to focus on only the values involved in the problem. Instruction, which is first grounded in visual models, deepens student-made connections to calculations with formulas and strengthens their understanding calculating a percent of a conditional whole.

Students may also use formulas to calculate conditional probability. As teachers introduce students to conditional probability formulas, instruction should focus on helping students make connections to middle grade concepts of fractions, percents, ratio reasoning, visual models (Venn diagrams, frequency tables, etc.), and counting techniques, rather than instruction focused on rote memorization and procedural computation.

Level 4:

As students use the conditional probability formula and begin to see conditional probability as a percent of a percent, they should be expected to connect representations; explaining how the conditional probability formula can be modeled with a Venn diagram or table. Teachers should prompt students to discuss how the population whole is narrowed to a conditional whole and how that impacts the probability of a given event. Using a Venn diagram, students can shade in the area of A and B and show how it is a fraction of the whole (either A or B). Likewise, students could be asked to justify why conditional probability narrows a table to a row or column, in which the probability can then be found. Teachers can ask students to provide an example and write a response about their findings, explaining how the conditional probability was calculated and justifying their thinking. This activity would support the literacy standards in mathematics, as students would be discussing and articulating mathematical ideas while using correct mathematical vocabulary to build their argument.