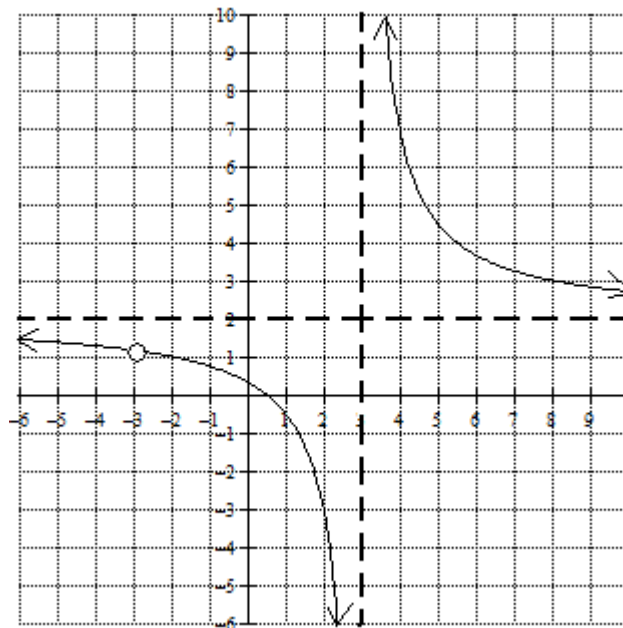


Understanding the Limit

A Numerical and Graphical Approach

The equation of the function graphed to the right

is $f(x) = \frac{2x^2 + 5x - 3}{x^2 - 9}$. The coordinates of the hole in the graph are $(-3, \frac{7}{6})$.



Pre-calculus Statements	Calculus Limit Notation
As $x \rightarrow -\infty$, the graph of $f(x) \rightarrow$ _____.	
As $x \rightarrow \infty$, the graph of $f(x) \rightarrow$ _____.	
As $x \rightarrow -3$ from the left, the graph of $f(x) \rightarrow$ _____.	
As $x \rightarrow -3$ from the right, the graph of $f(x) \rightarrow$ _____.	
As $x \rightarrow 3$ from the left, the graph of $f(x) \rightarrow$ _____.	
As $x \rightarrow 3$ from the right, the graph of $f(x) \rightarrow$ _____.	

Based on what you have just seen, how might you informally define what the value of a limit represents in terms of the graph?

A Numerical Analysis of Limits

- Now let's consider the function $f(x) = \frac{2x^2 + 5x - 3}{x^2 - 9}$
- Complete the table to below to perform a numerical analysis of the function as $x \rightarrow -\infty$ and ∞ .

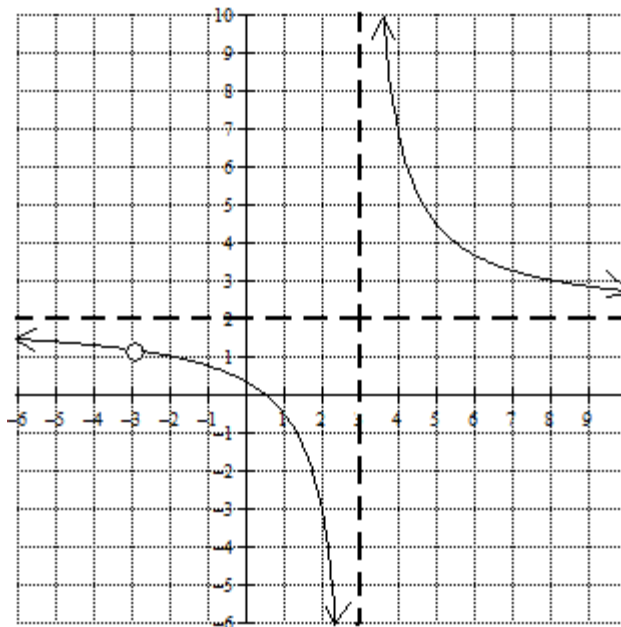
x	-1000	-500	-100	-50	50	100	500	1000
$f(x)$								

Based on the values in the table, what are the values of

$$\lim_{x \rightarrow -\infty} \frac{2x^2 + 5x - 3}{x^2 - 9}$$

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 5x - 3}{x^2 - 9}$$

How is the numerical analysis above related in the graph of the function pictured below?



A Numerical Analysis of Limits

- Now let's consider the function $f(x) = \frac{2x^2 + 5x - 3}{x^2 - 9}$
- Complete the table to below to perform a numerical analysis of the function as $x \rightarrow -3$ from the left and from the right.

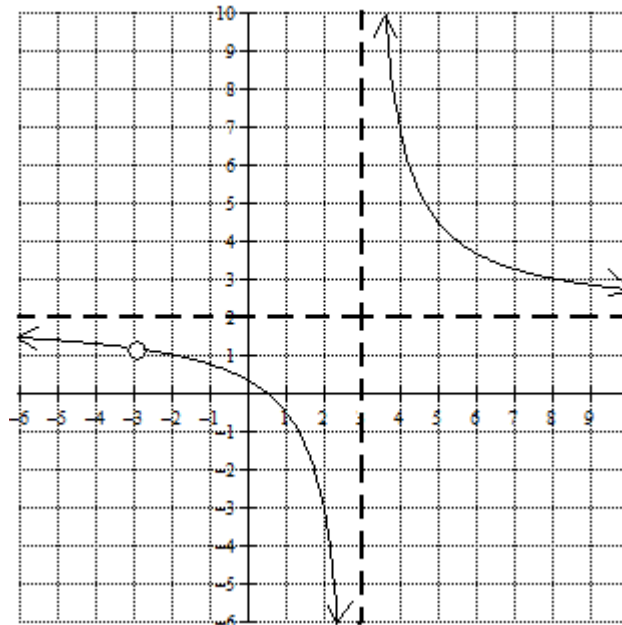
x	-3.75	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.75
$f(x)$								

Based on the values in the table, what are the values of

$$\lim_{x \rightarrow -3^-} \frac{2x^2 + 5x - 3}{x^2 - 9}$$

$$\lim_{x \rightarrow -3^+} \frac{2x^2 + 5x - 3}{x^2 - 9}$$

How is the numerical analysis above related in the graph of the function pictured below?



A Numerical Analysis of Limits

- Now let's consider the function $f(x) = \frac{2x^2 + 5x - 3}{x^2 - 9}$
- Complete the table to below to perform a numerical analysis of the function as $x \rightarrow 3$ from the left and from the right.

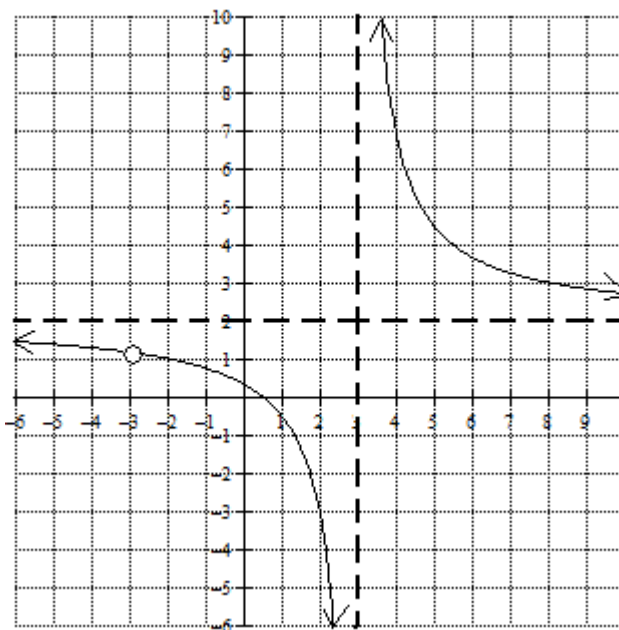
x	2.75	2.9	2.99	2.999	3.001	3.01	3.1	3.75
$f(x)$								

Based on the values in the table, what are the values of

$$\lim_{x \rightarrow 3^-} \frac{2x^2 + 5x - 3}{x^2 - 9}$$

$$\lim_{x \rightarrow 3^+} \frac{2x^2 + 5x - 3}{x^2 - 9}$$

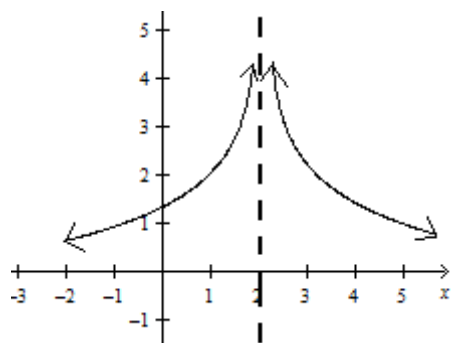
How is the numerical analysis above related in the graph of the function pictured below?



Limit Existence Theorem

Limits That Do Not Exist

Example #1



Find each of the following from the graph.

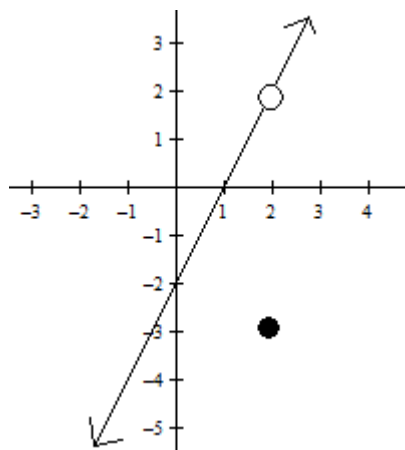
a) $\lim_{x \rightarrow 2^-} f(x) =$

b) $\lim_{x \rightarrow 2^+} f(x) =$

c) $f(2) =$

d) Does $\lim_{x \rightarrow 2} f(x)$ exist or not? Why or why not?

Example #2



Find each of the following from the graph.

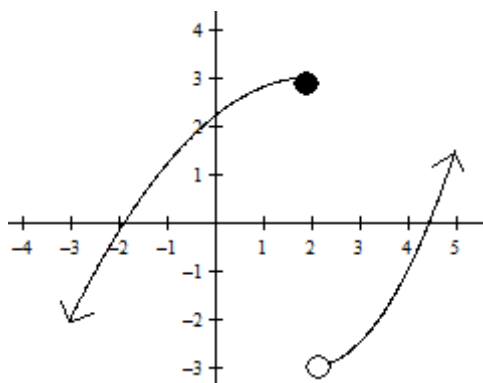
a) $\lim_{x \rightarrow 2^-} f(x) =$

b) $\lim_{x \rightarrow 2^+} f(x) =$

c) $f(2) =$

d) Does $\lim_{x \rightarrow 2} f(x)$ exist or not? Why or why not?

Example #3



Find each of the following from the graph.

a) $\lim_{x \rightarrow 2^-} f(x) =$

b) $\lim_{x \rightarrow 2^+} f(x) =$

c) $f(2) =$

d) Does $\lim_{x \rightarrow 2} f(x)$ exist or not? Why or why not?

Based on what you have seen so far, does $f(a)$ have to be defined in order for the $\lim_{x \rightarrow a} f(x)$ to exist? Draw and explain two different graphs to justify your reasoning. In both graphs, $f(a)$ should be undefined but in one graph, the limit should exist while in the second one, it should not exist.

Below is a table of values of an exponential function. Use the table to find the limits that follow.

x	-9	-5	-3	-1	1	3	9
$f(x)$	513	33	9	3	1.5	1.125	1.002

a) $\lim_{x \rightarrow -\infty} f(x)$

b) $\lim_{x \rightarrow -3} f(x)$

c) $\lim_{x \rightarrow 1} f(x)$

d) $\lim_{x \rightarrow \infty} f(x)$

Below is a table of values of a rational function. Use the table to find the limits that follow.

x	-1000	-1.001	-1	-0.999	0	1.999	2	2.001	1000
$F(x)$	1.002	2001	Undefined	-1999	-1	0.333	Undefined	0.334	0.998

a) $\lim_{x \rightarrow -\infty} f(x)$

b) $\lim_{x \rightarrow -1^-} f(x)$

c) $\lim_{x \rightarrow -1^+} f(x)$

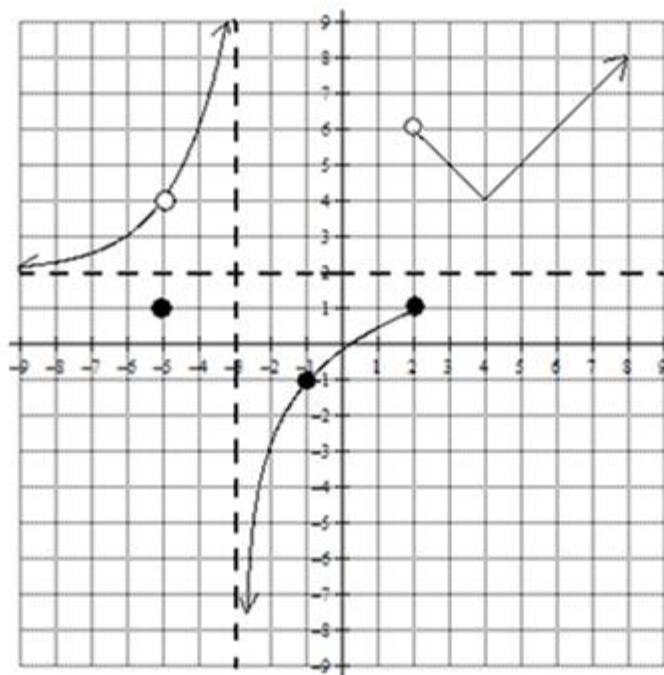
d) $\lim_{x \rightarrow 2} f(x)$

e) $\lim_{x \rightarrow \infty} f(x)$

f) $\lim_{x \rightarrow -1} f(x)$

A Graphical Analysis of Limits

- Consider the graph of the function, $f(x)$, graphed below.



A Graphical Analysis of Limits

Using the graph, find the value of each of the following limits. If a limit does not exist, state why.

a) $\lim_{x \rightarrow -3^-} f(x)$

b) $\lim_{x \rightarrow -5} f(x)$

c) $\lim_{x \rightarrow -1} f(x)$

c) $\lim_{x \rightarrow -3} f(x)$

d) $\lim_{x \rightarrow 2^-} f(x)$

e) $\lim_{x \rightarrow 2^+} f(x)$

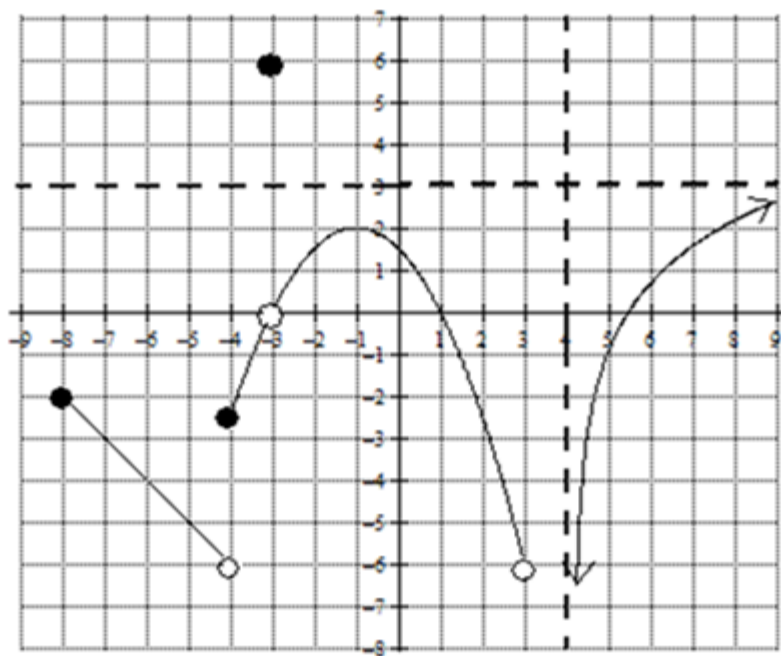
f) $\lim_{x \rightarrow 2} f(x)$

g) $\lim_{x \rightarrow -\infty} f(x)$

h) $\lim_{x \rightarrow \infty} f(x)$

A Graphical Analysis of Limits

Consider the graph of the function, $g(x)$, graphed below.



A Graphical Analysis of Limits

Find the value of each of the following limits using the graph of $g(x)$. If a limit does not exist, state why.

a) $\lim_{x \rightarrow -3^-} g(x)$

b) $\lim_{x \rightarrow -6} g(x)$

c) $\lim_{x \rightarrow -1^+} g(x)$

d) $\lim_{x \rightarrow -3^+} g(x)$

e) $\lim_{x \rightarrow 4^-} g(x)$

f) $\lim_{x \rightarrow 4^+} g(x)$

g) $\lim_{x \rightarrow 4} g(x)$

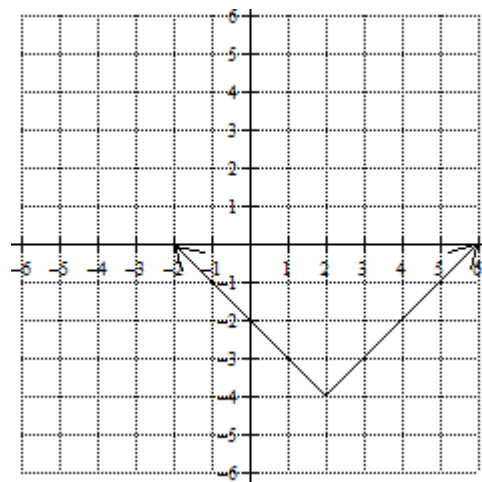
h) $\lim_{x \rightarrow -4^+} g(x)$

i) $\lim_{x \rightarrow -4^-} g(x)$

Understanding the Limit An Algebraic Approach

Consider the function, $f(x) = \frac{1}{2}|-2x + 4| - 4$, for a moment. The graph of $f(x)$ is pictured below. From the graph, determine the following limits.

$\lim_{x \rightarrow a} f(x)$	Find $f(a)$ using the equation.	Find $\lim_{x \rightarrow a} f(x)$ from the graph.
$\lim_{x \rightarrow 0} f(x)$		
$\lim_{x \rightarrow 2^+} f(x)$		
$\lim_{x \rightarrow 10} f(x)$		



When a function is defined and continuous at a value, $x = a$, how can $\lim_{x \rightarrow a} f(x)$ be found analytically?

Find each of the following limits analytically.

a) $\lim_{x \rightarrow 3} \frac{1}{2}x^2 - 2x + 3$	b) $\lim_{x \rightarrow 3} \frac{5x+2}{2x-3}$
c) $\lim_{x \rightarrow 2^-} \frac{\sqrt{x+2}-1}{x+1}$	d) $\lim_{\theta \rightarrow \frac{\pi}{2}} \sin 2\theta$
e) $\lim_{\theta \rightarrow \frac{2\pi}{3}} \frac{\cos \theta}{\theta}$	f) $\lim_{x \rightarrow 9} \log_8(11-x)$

Analytically Finding Limits of Functions at Undefined Values

What happens if we try to evaluate the limits below by the direct substitution method that was used in the previous six examples?

$$\lim_{x \rightarrow -3} \frac{x^2 + 4x + 3}{x^2 + 2x - 3}$$

$$\lim_{x \rightarrow 1^-} \frac{x^2 + 4x + 3}{x^2 + 2x - 3}$$

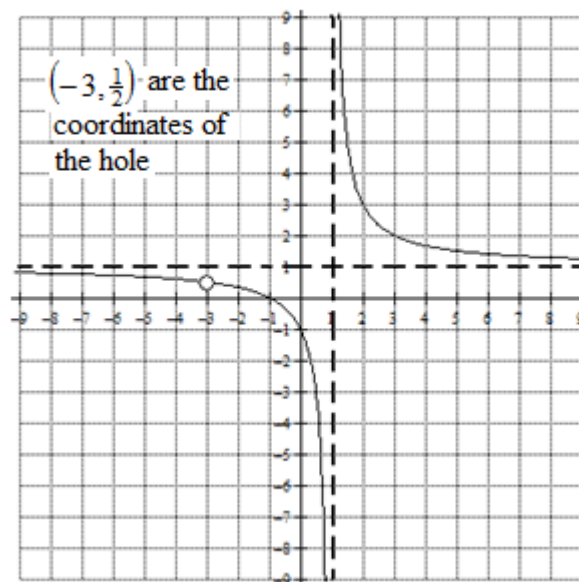
$$\lim_{x \rightarrow 1^+} \frac{x^2 + 4x + 3}{x^2 + 2x - 3}$$

Just because a function is undefined at a value of x does not mean that a conclusion cannot be reached about the limit. Consider the rational function above. From the graph of the function pictured to the right, what is the value of each limit below?

$$\lim_{x \rightarrow -3} \frac{x^2 + 4x + 3}{x^2 + 2x - 3} = \underline{\hspace{2cm}}$$

$$\lim_{x \rightarrow 1^-} \frac{x^2 + 4x + 3}{x^2 + 2x - 3} = \underline{\hspace{2cm}}$$

$$\lim_{x \rightarrow 1^+} \frac{x^2 + 4x + 3}{x^2 + 2x - 3} = \underline{\hspace{2cm}}$$



The task now is to determine how to find these limits analytically. How was it that we found the discontinuities of a rational function in pre-calculus?

We will perform the same algebraic analysis to find the limit of the removable, point discontinuities. Let's do this **Cancellation Process** below.

$$\lim_{x \rightarrow -3} \frac{x^2 + 4x + 3}{x^2 + 2x - 3}$$

Based on our knowledge from pre-calculus, we know that if a rational function has a non-removable infinite discontinuity, graphically a _____ exists. Since the y – values do not approach one specific value from both sides at a _____, then the limit does not exist. However, we can determine if the one sided limits approach $-\infty$ or ∞ . In order to do this analytically, we will marry the numerical, graphical, and algebraic approaches. For each limit below, determine the sign of the simplified function at the value to the right or the left of $x = 1$.

$$\lim_{x \rightarrow 1^-} \frac{x^2 + 4x + 3}{x^2 + 2x - 3}$$

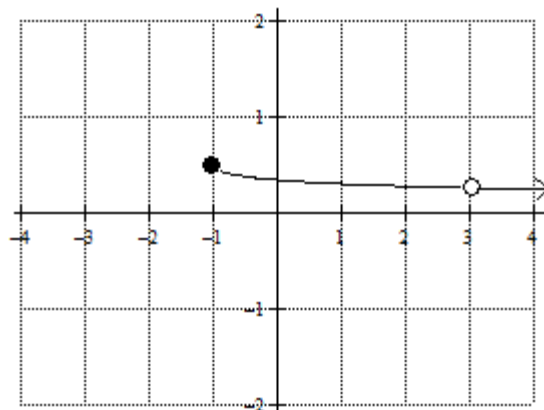
Value of x to the left of $x = 1$	Simplified function $\frac{x+1}{x-1}$
0.9	

$$\lim_{x \rightarrow 1^+} \frac{x^2 + 4x + 3}{x^2 + 2x - 3}$$

Value of x to the right of $x = 1$	Simplified function $\frac{x+1}{x-1}$
1.1	

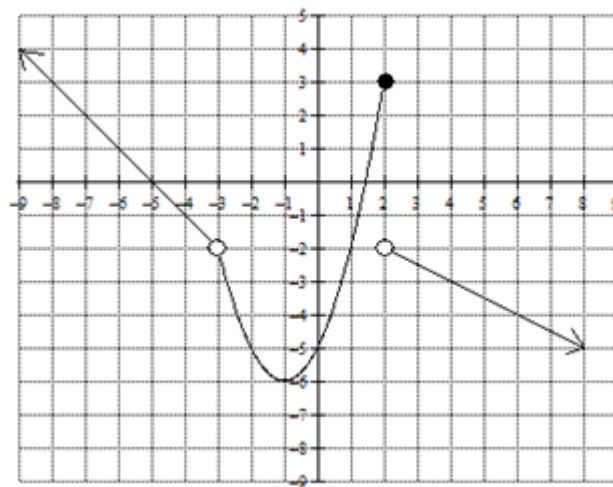
The graph of a function $g(x) = \frac{\sqrt{x+1}-2}{x-3}$ is pictured to the right. Often, **rationalization** can be used to evaluate a limit analytically. Find the following limit.

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3}$$



Write the equation of the piece-wise defined function pictured to the right.

	Equation of Each Piece	Constraint of Each Piece
$f(x) = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$		



Use the equation that you just wrote to find each of the following limits. Confirm your results based on the graph. If a limit does not exist, state why.

a) $\lim_{x \rightarrow 2^+} f(x)$

b) $\lim_{x \rightarrow 2} f(x)$

c) $\lim_{x \rightarrow -3^-} f(x)$

d) $\lim_{x \rightarrow -3^+} f(x)$

e) $\lim_{x \rightarrow -7} f(x)$

f) $\lim_{x \rightarrow -1} f(x)$

Consider the function below to find each limit. If a limit does not exist, state why.

$$G(x) = \begin{cases} 2x^2 + 3x, & x < -2 \\ -\frac{1}{2}x + 1, & x > -2 \end{cases}$$

a) $\lim_{x \rightarrow -2^-} G(x)$

b) $\lim_{x \rightarrow -2^+} G(x)$

c) $\lim_{x \rightarrow -2} G(x)$

Find each of the following limits analytically. Show your algebraic analysis.

a. $\lim_{x \rightarrow e} \frac{\ln x}{2x}$

b. $\lim_{x \rightarrow 5^-} \left(\frac{2}{5}x^2 + 2x \right)$

c. $\lim_{\theta \rightarrow \pi} (\sin^2 \theta + 2\cos \theta)$

d. $\lim_{\alpha \rightarrow \frac{5\pi}{3}} \frac{\tan \alpha}{\alpha^2}$

e. $\lim_{x \rightarrow -2} \frac{x^2 - x - 6}{2x + 4}$

f. $\lim_{x \rightarrow 3} \frac{x+5}{x^2-9}$

g. $\lim_{x \rightarrow \frac{3}{2}} \frac{8x^3-27}{2x-3}$

h. $\lim_{x \rightarrow -2} \frac{\sqrt{2x+5}-1}{x+2}$

i. $\lim_{x \rightarrow 1} \frac{1-\sqrt{2x^2-1}}{x-1}$

$$\text{j. } \lim_{x \rightarrow 0} \frac{\frac{1}{x+2} + \frac{1}{x}}{x}$$

$$\text{k. } \lim_{x \rightarrow 2^+} \frac{3x^2 + 7x + 2}{x^2 - 4}$$

$$\text{l. } \lim_{x \rightarrow 3^+} \frac{2x + 5}{x - 3}$$

$$\text{m. } \lim_{x \rightarrow 3^-} \frac{2x + 5}{x - 3}$$

If $f(x) = 2x^2 - 3x + 4$, find $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

Unit #1—Limits and Continuity
Notes



Properties of Limits

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Find each of the following limits in terms of L and M .

1. $\lim_{x \rightarrow a} [f(x) + g(x)]$

2. $\lim_{x \rightarrow a} [f(x) - g(x)]$

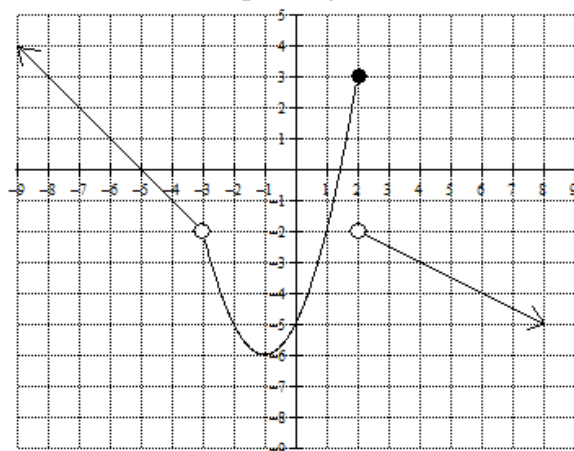
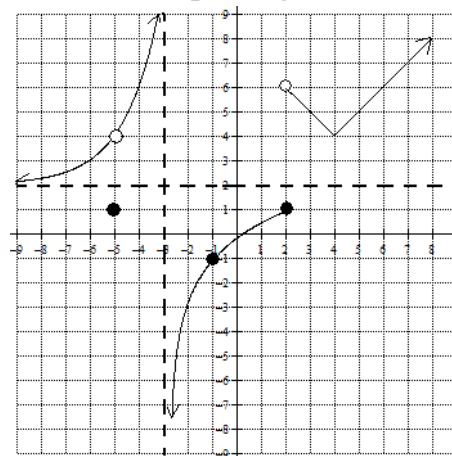
3. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$

4. $\lim_{x \rightarrow a} [f(x) \cdot g(x)]$

5. $\lim_{x \rightarrow a} [c \cdot f(x)]$

6. $\lim_{x \rightarrow a} [f(x)]^p$

7. $\lim_{x \rightarrow a} c$

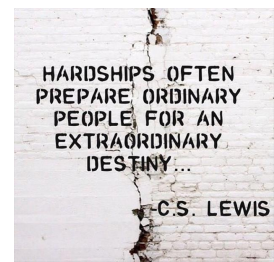
Graph of $f(x)$ Graph of $g(x)$ 

Find each of the following limits applying the properties of limits. If a limit does not exist, state why.

$\lim_{x \rightarrow 2^-} [f(x) + g(x)]$	$\lim_{x \rightarrow -1} [2f(x) - 3g(x)]$	$\lim_{x \rightarrow -3} [f(x) - g(x)]$
$\lim_{x \rightarrow 6} \frac{-2f(x)}{g(x)}$	$\lim_{x \rightarrow 4} 2[f(x)g(x)]$	$\lim_{x \rightarrow -2} [f(x)]^2$
$\lim_{x \rightarrow 2^+} \sqrt{2g(x)}$	$\lim_{x \rightarrow 2^+} \frac{f(x)}{g(x)}$	$\lim_{x \rightarrow -1} [f(x) - g(x)]$

Unit #1—Limits and Continuity

Notes

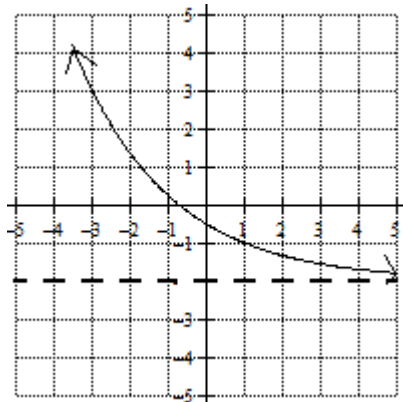


Limits of Exponential Functions

Graphical and Analytical Connections

Consider the four exponential functions graphed below. Find the indicated limits for each function based on the graph.

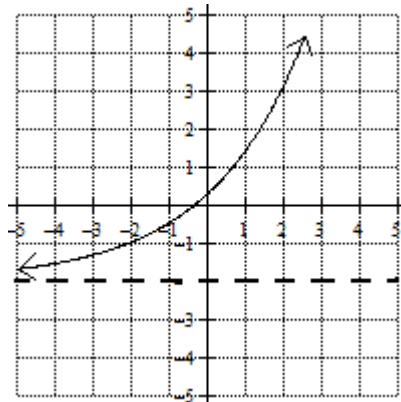
$$f(x) = \left(\frac{2}{3}\right)^{x-1} - 2$$



$$\lim_{x \rightarrow -\infty} f(x) =$$

$$\lim_{x \rightarrow \infty} f(x) =$$

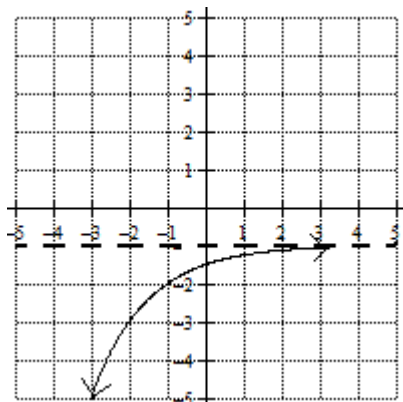
$$f(x) = \left(\frac{3}{2}\right)^{x+2} - 2$$



$$\lim_{x \rightarrow -\infty} f(x) =$$

$$\lim_{x \rightarrow \infty} f(x) =$$

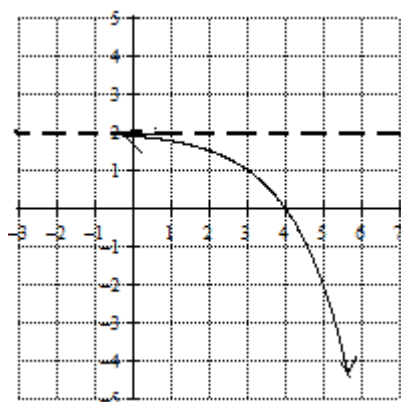
$$f(x) = -\left(\frac{1}{2}\right)^{x+1} - 1$$



$$\lim_{x \rightarrow -\infty} f(x) =$$

$$\lim_{x \rightarrow \infty} f(x) =$$

$$f(x) = -\left(\frac{1}{2}\right)^{-x+3} + 2$$



$$\lim_{x \rightarrow -\infty} f(x) =$$

$$\lim_{x \rightarrow \infty} f(x) =$$

In order to determine a limit as x approaches $-\infty$ or ∞ for an exponential function, you have to determine what the graph will look like. Based on what we have seen above, what are the three possible results of such a limit for an exponential function?

By studying the graphs above, remind yourself of the four rules determining if the function will be a growth or decay function.

1. _____

2. _____

3. _____

4. _____

Determine the limits of each of the following exponential functions.

1. $\lim_{x \rightarrow \infty} \left(\frac{2}{3}\right)^{-x-1} - 2$	2. $\lim_{x \rightarrow -\infty} -(0.4)^x - 4$
3. $\lim_{x \rightarrow \infty} -\left(\frac{2}{3}\right)^{-x+2} + 3$	4. $\lim_{x \rightarrow \infty} -2^{-x-1} + 2$
5. $\lim_{x \rightarrow -\infty} e^{-x-1} + 2$	6. $\lim_{x \rightarrow \infty} -(0.4)^x - 4$
7. $\lim_{x \rightarrow 3} (e^{2-x} + 2)$	8. $\lim_{x \rightarrow -2} \left[\left(\frac{1}{2}\right)^{-x-3} + 3 \right]$

Limits of Trigonometric Functions

An Analytical Approach

We have already looked at how to evaluate limits of trigonometric functions by direct substitution, provided that the function is defined and continuous at θ . Find each of the limits below.

$$\lim_{\theta \rightarrow \frac{2\pi}{3}} \frac{\sin 3\theta}{3\theta}$$

$$\lim_{\theta \rightarrow \pi} 2\cos^2 \theta$$

Each of the functions above was defined at the value that θ was approaching. However, we have seen that even in the algebraic world, not all functions are undefined at a value, but their limits do exist. The same is true in the trigonometric world.

Evaluating Trigonometric Limits by Rewriting the Function Using Identities

Let's consider for a moment the limit below. Try to evaluate this limit by direct substitution.

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin^2 \theta}$$

Again, this function is undefined at $\theta = 0$. However, that does not mean that the limit does not exist. In this case, we can often rewrite the function in terms of a single trig ratio using identities in hopes that the new form of the function is not undefined for the approached value of θ . Do this in the space below.

Find each of the following limits by rewriting the function in a form that is defined for the approached value of θ .

1. $\lim_{\theta \rightarrow \frac{3\pi}{2}} 3 \tan \theta \cos \theta$

$$2. \quad \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sec \theta \cos \theta}{4\theta}$$

$$3. \lim_{\theta \rightarrow \pi} \frac{\cos \theta \tan \theta}{\sin \theta}$$

It is important to note that as in all cases of evaluating limits, direct substitution should always be tried FIRST. If that does not yield a value, then a simplification of the function can be tried.

Occasionally, it is not even possible to rewrite the function so that it is undefined. There are two special trigonometric limits that can often be employed.

Use your graphing calculator to complete the table of values below for the function $f(\theta) = \frac{\sin \theta}{\theta}$.

θ	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\frac{\sin \theta}{\theta}$						

Based on the values in the table above, what do each of the limits below equal?

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = \underline{\hspace{2cm}}$$

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = \underline{\hspace{2cm}}$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \underline{\hspace{2cm}}$$

Use your graphing calculator to complete the table of values below for the function $f(\theta) = \frac{\sin 3\theta}{3\theta}$.

θ	- 0.01	- 0.001	- 0.0001	0.0001	0.001	0.01
$\frac{\sin 3\theta}{3\theta}$						

Based on the values in the table above, what do each of the limits below equal?

$$\lim_{\theta \rightarrow 0^-} \frac{\sin 3\theta}{3\theta} = \underline{\hspace{2cm}} \quad \lim_{\theta \rightarrow 0^+} \frac{\sin 3\theta}{3\theta} = \underline{\hspace{2cm}} \quad \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta} = \underline{\hspace{2cm}}$$

Based on what you have just observed, what inference can you make about the value of the limit

$$\lim_{\theta \rightarrow 0} \frac{\sin c\theta}{c\theta}, \text{ where } c \text{ is any constant?}$$

Now, in a similar fashion, use your graphing calculator to complete the table of values below for the function $f(\theta) = \frac{1 - \cos \theta}{\theta}$.

θ	- 0.01	- 0.001	- 0.0001	0.0001	0.001	0.01
$\frac{1 - \cos \theta}{\theta}$						

Based on the values in the table above, what do each of the limits below equal?

$$\lim_{\theta \rightarrow 0^-} \frac{1 - \cos \theta}{\theta} = \underline{\hspace{2cm}} \quad \lim_{\theta \rightarrow 0^+} \frac{1 - \cos \theta}{\theta} = \underline{\hspace{2cm}} \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = \underline{\hspace{2cm}}$$

These two special trigonometric functions derived above can often be used to find limits of trigonometric functions that cannot be evaluated by direct substitution nor by rewriting the function using identities.

Find each of the following limits.

1. $\lim_{x \rightarrow 0} \frac{e^x \cos x}{4}$

2. $\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\theta}$

3. $\lim_{x \rightarrow 0} \frac{\sin 2x}{3x}$

4. $\lim_{\theta \rightarrow 0} \frac{2 \sin 5\theta}{3\theta}$

5. $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta}$

$$5. \lim_{\theta \rightarrow 0} \frac{2 - 2\cos^2 \theta}{\theta}$$

$$6. \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta + \sin 2\theta}{\theta}$$

$$7. \lim_{\theta \rightarrow 0} \frac{\theta \csc \theta + 1}{\theta \csc \theta}$$

$$8. \lim_{x \rightarrow 0} \frac{\sin x - \sin x \cos x}{x^2}$$

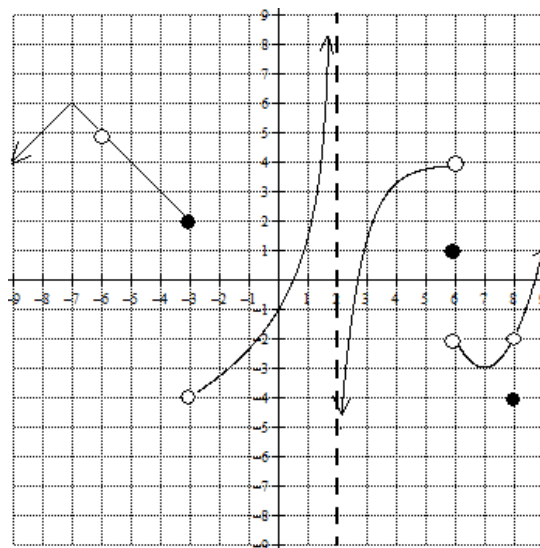
Limit-Based Continuity Graphical and Analytical Approaches

For the function graphed below, fill in the table with the given information. After filling in the table, write three pieces of information that must be true in order for a function, $G(x)$, to be continuous at $x = a$.

1.

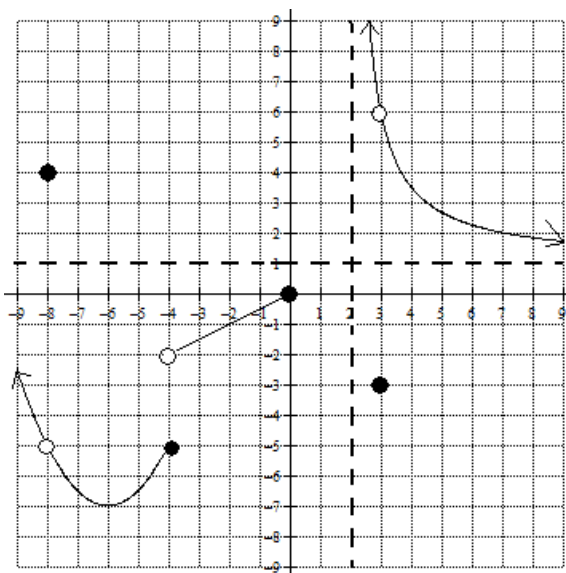
2.

3.



$x = a$	Is the function defined? If so, what is its value?	What is the value of $\lim_{x \rightarrow a^-} G(x)$?	What is the value of $\lim_{x \rightarrow a^+} G(x)$?	What is $\lim_{x \rightarrow a} G(x)$?	Is $G(x)$ continuous at $x = a$?
$x = -6$					
$x = -3$					
$x = 0$					
$x = 2$					
$x = 6$					
$x = 8$					

The graph of the function, $G(x)$, pictured to the right has several x – values at which the function is not continuous. For each of the following x – values, use the three part definition of continuity to determine if the function is continuous or not.



1. $x = -8$

2. $x = -6$

3. $x = -4$

Use the three part definition of continuity to determine if the given functions are continuous at the indicated values of x .

$$4. \quad f(x) = \begin{cases} -2\sqrt{x+6}, & x < -2 \\ 3x+2, & x = -2 \\ e^x + \cos(\pi x), & x > -2 \end{cases} \quad \text{at } x = -2$$

$$5. \quad g(x) = \begin{cases} e^x \cos x, & x < \pi \\ e^x \tan\left(\frac{3x}{4}\right), & x \geq \pi \end{cases} \quad \text{at } x = \pi$$

6. Consider the function, $f(x)$, to the right to answer the following questions.

$$f(x) = \begin{cases} 2, & x \leq -1 \\ mx+k, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$$

a. What two limits must equal in order for $f(x)$ to be continuous at $x = -1$?

b. What two limits must equal in order for $f(x)$ to be continuous at $x = 3$?

c. Determine the values of m and k so that the function is continuous everywhere.

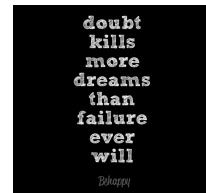
7. Consider the function, $g(x)$, to the right to answer the following questions.

$$g(x) = \begin{cases} kx^2 + m, & x < -2 \\ 4x + 1, & -2 \leq x \leq 3 \\ kx - m, & x > 3 \end{cases}$$

- a. What two limits must equal in order for $f(x)$ to be continuous at $x = -2$?
- b. What two limits must equal in order for $f(x)$ to be continuous at $x = 3$?
- c. Determine the values of m and k so that the function is continuous everywhere.

Unit #1—Limits and Continuity

Notes



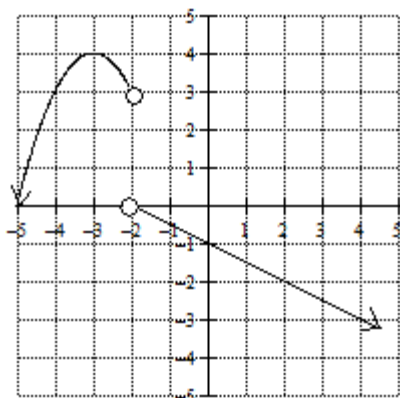
Intermediate Value Theorem

As we study calculus, we will study several different theorems. The first theorem of investigation is the Intermediate Value Theorem. Together, let's write the theorem.

Intermediate Value Theorem

Now, investigate the graphs below to determine if the theorem is applicable for these functions on the specified intervals for the values given.

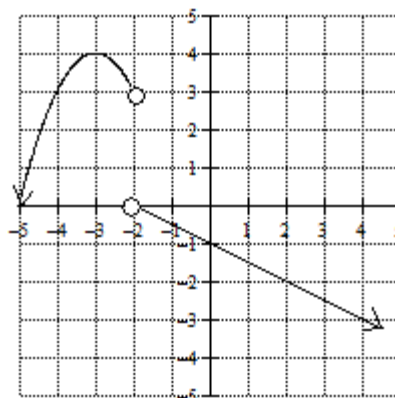
$$f(x) = \begin{cases} -(x+3)^2 + 4, & x < -2 \\ -\frac{1}{2}x - 1, & x > -2 \end{cases}$$



Is there a value of c on $[-5, 2]$ such that $f(c) = 2$?

Does the I.V.T. guarantee a value of c such that $f(c) = 2$ on the interval $[-5, 2]$? Why or why not?

$$f(x) = \begin{cases} -(x+3)^2 + 4, & x < -2 \\ -\frac{1}{2}x - 1, & x > -2 \end{cases}$$



Is there a value of c on $[-1, 5]$ such that $f(c) = 2$?

Does the I.V.T. guarantee a value of c such that $f(c) = 2$ on the interval $[-1, 5]$? Why or why not?

What two conditions must be true to verify the applicability of the Intermediate Value Theorem?

1. _____

2. _____

For each of the following functions, determine if the I.V.T. is applicable or not and state why or why not. Then, if it is applicable, find the value of c guaranteed to exist by the theorem.

1. $f(x) = \frac{x-3}{x+2}$ on the interval $[-1, 3]$ for $f(c) = \frac{2}{3}$

2. $f(x) = \frac{x-3}{x+2}$ on the interval $[-4, 1]$ for $f(c) = \frac{2}{3}$

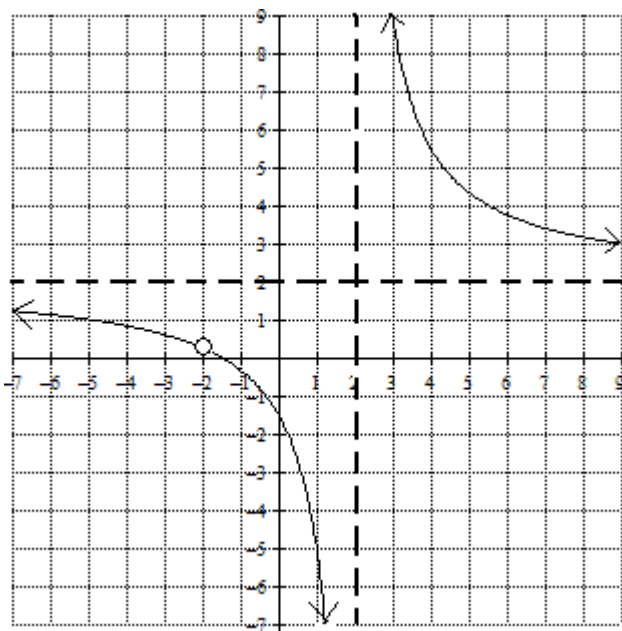
3. $f(x) = \frac{x}{x-2}$ on the interval $[-1, 1]$ for $f(c) = -\frac{1}{2}$

4. $f(x) = -\left(\frac{1}{2}\right)^{-x+3} - 2$ on the interval $[3, 5]$ for $f(c) = -4$

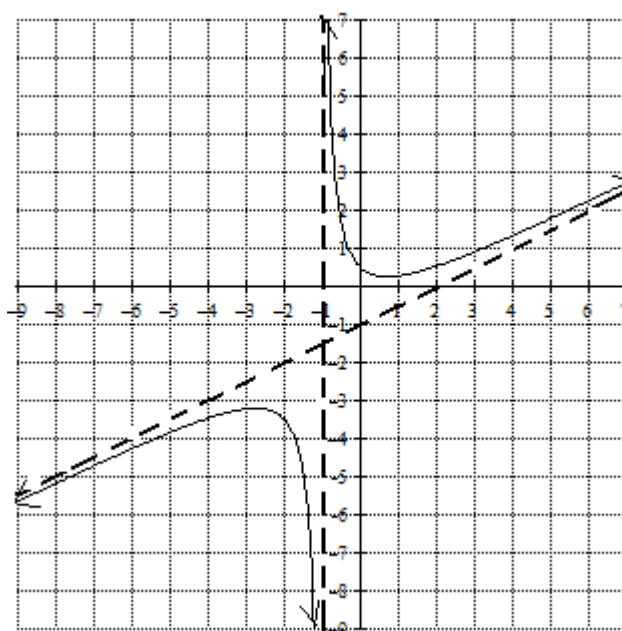
Infinite Limits and Limits at Infinity

In our graphical analysis of limits, we have already seen both an infinite limit and a limit at infinity. Let's consider the equations and the graphs of the two functions below to find the limits that follow.

$$f(x) = \frac{2x^2 + 7x + 6}{x^2 - 4}$$



$$g(x) = \frac{x^2 - x + 1}{2x + 2}$$



Infinite Limits	Limits at Infinity
$\lim_{x \rightarrow 2^-} f(x) =$	$\lim_{x \rightarrow \infty} f(x) =$
$\lim_{x \rightarrow 2^+} f(x) =$	$\lim_{x \rightarrow -\infty} f(x) =$
$\lim_{x \rightarrow -1^-} g(x) =$	$\lim_{x \rightarrow \infty} g(x) =$
$\lim_{x \rightarrow -1^+} g(x) =$	$\lim_{x \rightarrow -\infty} g(x) =$

We have already seen how to find infinite limits by marrying a numerical and analytical approach. For the function, $f(x)$ and $g(x)$, whose graphs appear on the previous page, find the infinite limits below.

$\lim_{x \rightarrow 2^-} \frac{2x^2 + 7x + 6}{x^2 - 4}$	$\lim_{x \rightarrow 2^+} \frac{2x^2 + 7x + 6}{x^2 - 4}$
$\lim_{x \rightarrow -1^+} \frac{x^2 - x + 1}{2x + 2}$	$\lim_{x \rightarrow -1^-} \frac{x^2 - x + 1}{2x + 2}$

Graphically, an infinite limit will always yield a _____.

In pre-calculus, we discovered through observation that such a graphical property existed when a factor in the equation would not _____. From this point forward, this is NOT a viable justification for the existence of a _____.

Justification of the Existence of a Vertical Asymptote Using Limits

For the function below, find any vertical asymptote(s) that exist. Justify your answer(s) using a limit(s).

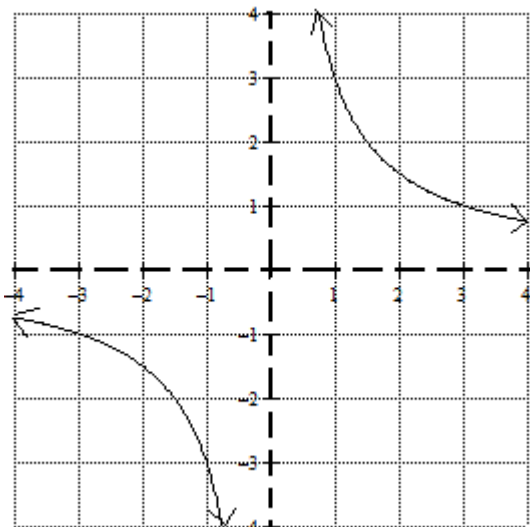
$$h(x) = \frac{2x^2 + 7x + 3}{x^2 + 2x - 3}$$

Now, we must develop an analytical procedure by which we can find limits at infinity. Basically, a limit at infinity describes the end behavior of a function. We have spent a great amount of time talking about end behavior of functions. Find each of the following limits at infinity. Give an explanation of your reasoning for each.

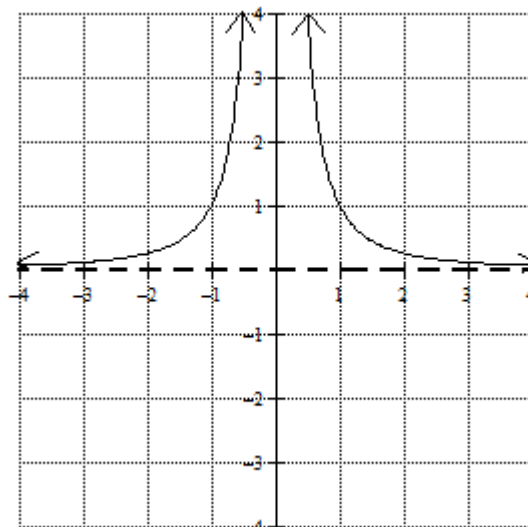
$\lim_{x \rightarrow \infty} -3x^3 + x - 4$	$\lim_{x \rightarrow -\infty} (4 - x)^2(x - 3)(x + 1)$	$\lim_{x \rightarrow \infty} \frac{3 - 2x}{x + 3}$
---	--	--

The third example, $\lim_{x \rightarrow \infty} \frac{3 - 2x}{x + 3}$ will provide us a basis for developing our analytical process by which we can find limits at infinity for all types of rational functions. Before we do that, investigate the two functions below both graphically and numerically.

$$f(x) = \frac{3}{x}$$



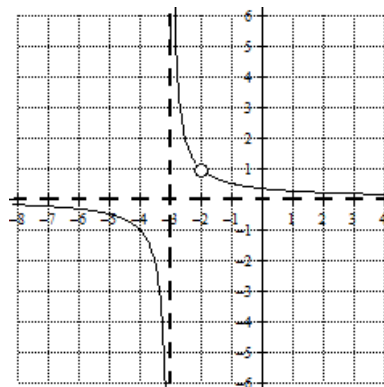
$$g(x) = \frac{1}{x^2}$$



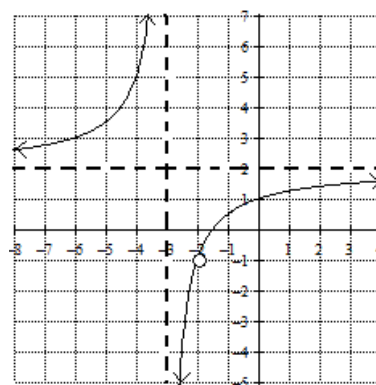
What does each of these functions have in common algebraically and what do they have in common graphically?

In pre-calculus, we learned three rules for determining the existence of horizontal asymptotes of rational functions. When a rational function had a horizontal asymptote, the end behavior was always such that as $x \rightarrow -\infty$ or ∞ , then the graph of $f(x) \rightarrow$ the horizontal asymptote. We learned three rules for determining the horizontal asymptote, if one existed, for rational functions. We are about to use the idea of a limit and calculus to find out why those rules are such as they are. For each function below, divide every term in both the numerator and the denominator by the highest power of x that appears in the denominator. Then, evaluate the indicated limit. Does the result of each limit make sense based on the graph that is pictured?

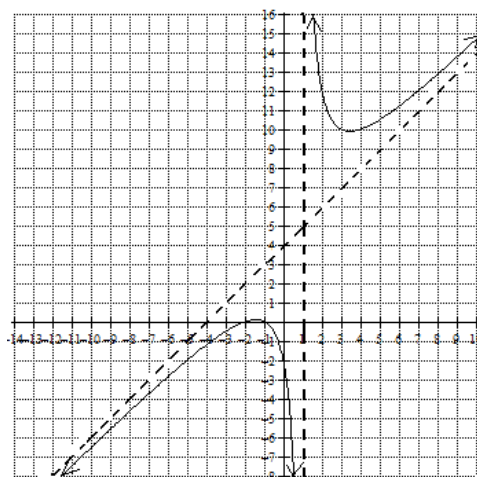
$$\lim_{x \rightarrow \infty} \frac{x+2}{x^2+5x+6}$$



$$\lim_{x \rightarrow \infty} \frac{2x^2+7x+6}{x^2+5x+6}$$

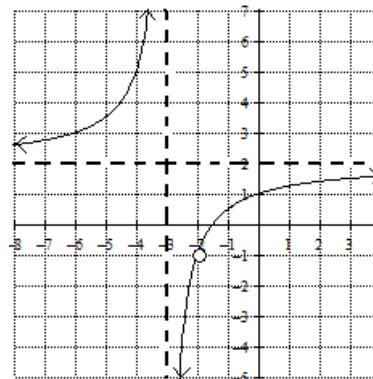


$$\lim_{x \rightarrow \infty} \frac{x^2+3x+2}{x-1}$$

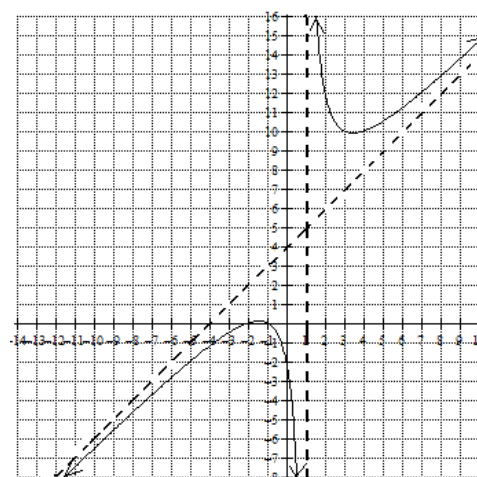


Let's see what would happen if our limit at infinity approached $-\infty$.

$$\lim_{x \rightarrow -\infty} \frac{2x^2 + 7x + 6}{x^2 + 5x + 6}$$



$$\lim_{x \rightarrow -\infty} \frac{x^2 + 3x + 2}{x - 1}$$



Based on what we have just seen and what we know graphically about the functions above, would does approaching $-\infty$ make a difference in the result of our limits?

Graphically, a limit at infinity can yield a _____. In pre-calculus, we discovered through observation that such a graphical property existed by comparing the _____.

From this point forward, this is NOT a viable justification for the existence of a _____.

Justification of the Existence of a Horizontal Asymptote Using Limits

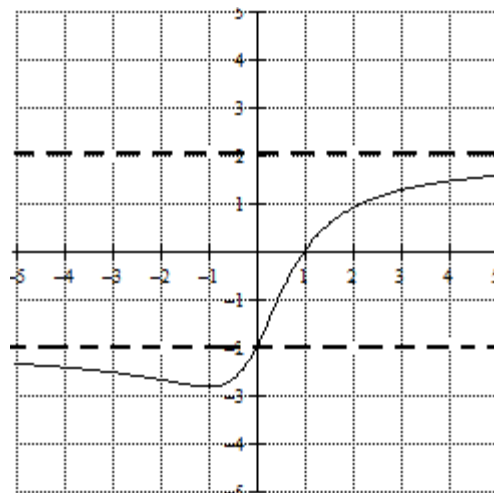
For the function below, find the horizontal asymptote if it exists. Justify your answer(s) using a limit(s).

$$h(x) = \frac{5 - 2x - 2x^2}{3x^2 + 2x - 3}$$

The algebraic analysis described above to evaluate a limit at infinity can be used to find limits at infinity for any type of rational function, even

$$f(x) = \frac{2x - 2}{\sqrt{x^2 + 1}}, \text{ whose graph is pictured to the right.}$$

What is the one thing that you notice is different about the graph of this rational function versus the others that we have investigated in the past?



Use the graph to find each of the following limits.

$$\lim_{x \rightarrow -\infty} \frac{2x - 2}{\sqrt{x^2 + 1}} = \underline{\hspace{2cm}}$$

$$\lim_{x \rightarrow \infty} \frac{2x - 2}{\sqrt{x^2 + 1}} = \underline{\hspace{2cm}}$$

Perform the same algebraic analysis that we did earlier to find the limits at infinity. The only problem that we will encounter is what to do when $x \rightarrow -\infty$.

$$\lim_{x \rightarrow \infty} \frac{2x - 2}{\sqrt{x^2 + 1}}$$

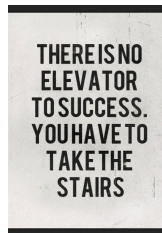
$$\lim_{x \rightarrow -\infty} \frac{2x - 2}{\sqrt{x^2 + 1}}$$

Look at the graph on the previous page to confirm these results. Then, find the limits below.

$$\lim_{x \rightarrow -\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$$

$$\lim_{x \rightarrow -\infty} \frac{x^2 + 2x}{\sqrt{3x^2 + 2}}$$

Unit #1—Limits and Continuity
Notes



AP Calculus Multiple Choice Practice
Graphing Calculator NOT Permitted – 20 minutes

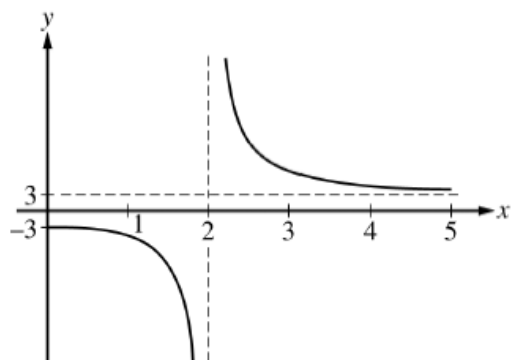
1. $\lim_{x \rightarrow 0} \frac{2x^6 + 6x^3}{4x^5 + 3x^3}$ is

- (A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) nonexistent

$$f(x) = \begin{cases} x^2 - 3x + 9 & \text{for } x \leq 2 \\ kx + 1 & \text{for } x > 2 \end{cases}$$

2. The function f is defined above. For what value of k , if any, is f continuous at $x = 2$?

- (A) 1
(B) 2
(C) 3
(D) 7
(E) No value of k will make f continuous at $x = 2$.



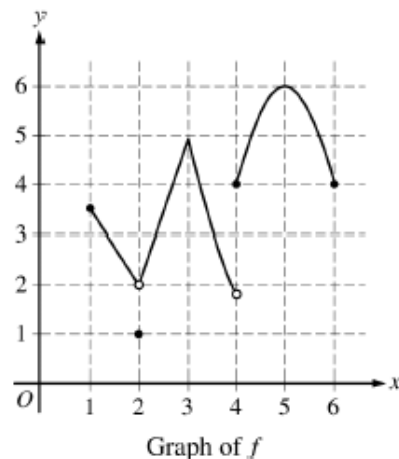
3. The function f is given by $f(x) = \frac{ax^2 + 12}{x^2 + b}$. The figure above shows a portion of the graph of f . Which of the following could be the values of the constants a and b ?
- (A) $a = -3, b = 2$
 (B) $a = 2, b = -3$
 (C) $a = 2, b = -2$
 (D) $a = 3, b = -4$
 (E) $a = 3, b = 4$

4. If $f(x) = 3x^2 + 2x$, then $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ is...

- (A) $6x + 2$
 (B) $6x$
 (C) 0
 (D) nonexistent
 (E) 2

5. The graph of the function f is shown above. Which of the following statements is false?

- (A) $\lim_{x \rightarrow 2} f(x)$ exists.
 (B) $\lim_{x \rightarrow 3} f(x)$ exists.
 (C) $\lim_{x \rightarrow 4} f(x)$ exists.
 (D) $\lim_{x \rightarrow 5} f(x)$ exists.
 (E) The function f is continuous at $x = 3$.



$$g(x) = \begin{cases} \sin \frac{x\pi}{4}, & x < 3 \\ x\sqrt{2}, & x = 3 \\ \frac{x\sqrt{2}}{4x-6}, & x > 3 \end{cases}$$

6. For the function above, which of the following would be the reason(s) why the function, $g(x)$, is not continuous at $x = 3$?

- I. $g(3)$ is undefined. II. $\lim_{x \rightarrow 3} g(x)$ does not exist. III. $\lim_{x \rightarrow 3} g(x) \neq g(3)$.

- (A) III only
(B) II only
(C) I and II only
(D) I only
(E) II and III only

7. $\lim_{x \rightarrow 2^+} \frac{3-2x}{x-2}$ is...

- (A) 0
(B) ∞
(C) $-\infty$
(D) 2
(E) $\frac{1}{2}$

8. Let f be a function that is continuous on the closed interval $[2, 4]$ with $f(2) = 10$ and $f(4) = 20$. Which of the following is guaranteed by the Intermediate Value Theorem?

- (A) $f(x) = 13$ has at least one solution in the open interval $(2, 4)$.
(B) $f(3) = 15$
(C) f attains a maximum on the open interval $(2, 4)$.
(D) $f(x) = 9$ has at least one solution in the open interval $(2, 4)$.
(E) $f(x)$ has at least one zero in the open interval $(2, 4)$.

AP Calculus Free Response Practice #1
Calculator Permitted

Consider the function $h(x) = \frac{-2x - \sin x}{x - 1}$ to answer the following questions.

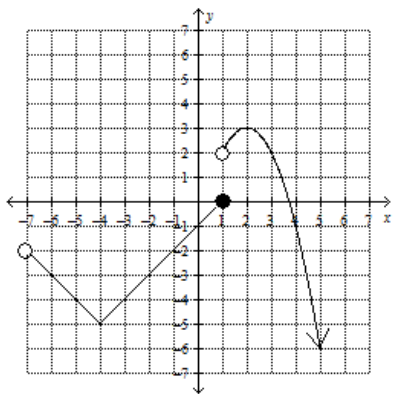
- a. Find $\lim_{x \rightarrow \frac{\pi}{2}} [h(x) \cdot (2x - 2)]$. Show your analysis.

- b. Identify the vertical asymptote(s), if any exist, of $h(x)$ and justify the existence by writing a limit.

c. Identify the horizontal asymptote(s), if any exist, of $h(x)$ and justify the existence by writing a limit.

d. Explain why the Intermediate Value Theorem guarantees a value of c on the interval $[1.5, 2.5]$ such that $h(c) = -4$. Then, find c .

AP Calculus Free Response Practice #2
Calculator NOT Permitted



Graph of $g(x)$

$$f(x) = \begin{cases} ax + 3, & x < -3 \\ x^2 - 3x, & -3 \leq x < 2 \\ bx - 5, & x \geq 2 \end{cases}$$

Equation of $f(x)$

Pictured above is the graph of a function $g(x)$ and the equation of a piece-wise defined function $f(x)$. Answer the following questions.

- a. Find $\lim_{x \rightarrow 1^+} [2g(x) - f(x) \cdot \cos \pi x]$. Show your work applying the properties of limits.

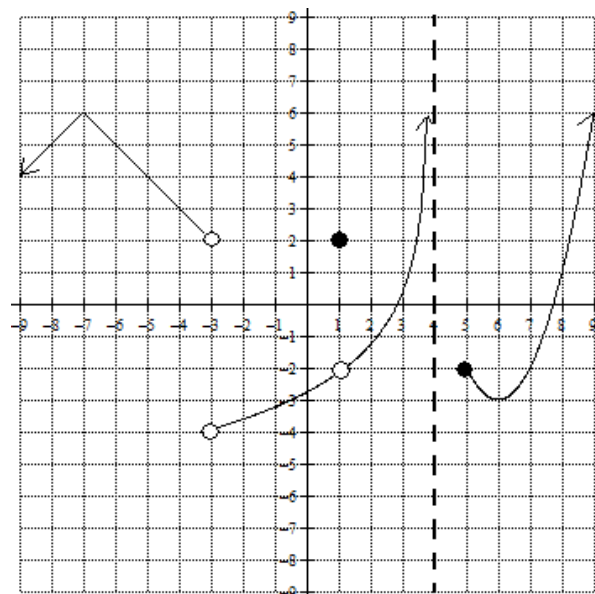
- b. On its domain, what is one value of x at which $g(x)$ is discontinuous? Use the three part definition of continuity to explain why $g(x)$ is discontinuous at this value.

- c. For what value(s) of a and b , if they exist, would the function $f(x)$ be continuous everywhere? Justify your answer.

AP Calculus
Extra Practice on Limits and Multiple Choice Practice

For questions 1 – 5, refer to the graph of $f(x)$ to the right. Find the value of each indicated limit. If a limit does not exist, give a reason.

1.	$\lim_{x \rightarrow -3^+} f(x) + \lim_{x \rightarrow 5^+} 3f(x)$	
2.	$\lim_{x \rightarrow 1} \left[\frac{1}{2} f(x) + \cos(\pi x) \right]$	
3.	$\lim_{x \rightarrow 4^-} f(x)$	
4.	$\lim_{x \rightarrow -\infty} f(x)$	
5.	$\lim_{x \rightarrow -3} f(x)$	



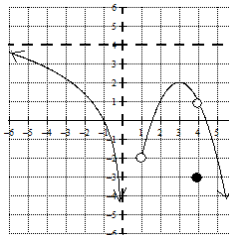
For questions 6 – 11, find the value of each limit analytically. If a limit does not exist, state why.

6. $\lim_{x \rightarrow 0} \frac{x^3 - 2x^2 + 3x}{x}$	7. $\lim_{x \rightarrow 0} \frac{3 \tan x}{x \sec x}$
8. $\lim_{x \rightarrow 3^+} \frac{x^2 - 4}{x^2 - x - 6}$	9. $\lim_{x \rightarrow 2^-} \ln(-x + 2)$
10. $\lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + 3x}{x^2 - 3x^3}$	11. $\lim_{x \rightarrow \infty} 5 + \frac{5}{x}$

For question 12 – 16, use the equation $g(x)$ below and the graph of the function $f(x)$.

$$g(x) = \begin{cases} 3|x+3|, & x < -2 \\ \cos\left(\frac{\pi x}{2}\right), & -2 \leq x < 2 \\ ax^2 + 2x, & x \geq 2 \end{cases}$$

Graph of $f(x)$



12. Is $g(x)$ continuous at $x = -2$. [Base your response on the three part definition of continuity.]

13. For what value(s) of a is $g(x)$ continuous at $x = 2$?

14. For what value(s) of b is the function $f(x)$ discontinuous? At which of these values does $\lim_{x \rightarrow b} f(x)$ exist? Explain your reasoning.

15. Find $\lim_{x \rightarrow 2^+} [g(x) + 2f(x)]$.

16. Which of the following limits do(es) not exist? Give a reason for your answers.

$\lim_{x \rightarrow 1} f(x)$	$\lim_{x \rightarrow 4} f(x)$	$\lim_{x \rightarrow 0^-} f(x)$

17. Find the values of k and m so that the function below is continuous on the interval $(-\infty, \infty)$.

$$f(x) = \begin{cases} x^2 - kx + 3, & x < -2 \\ 2x - 3, & -2 \leq x \leq 3 \\ 3 - 2m, & x > 3 \end{cases}$$

18. $\lim_{x \rightarrow 0} \frac{4x - 3}{7x + 1} =$

- A. ∞ B. $-\infty$ C. 0 D. $\frac{4}{7}$ E. -3

19. $\lim_{x \rightarrow \frac{1}{3}} \frac{9x^2 - 1}{3x - 1} =$

- A. ∞ B. $-\infty$ C. 0 D. 2 E. 3

20. $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4} =$

- A. 4 B. 0 C. 1 D. 3 E. 2

21. The function $G(x) = \begin{cases} x - 3, & x > 2 \\ -5, & x = 2 \\ 3x - 7, & x < 2 \end{cases}$ is not continuous at $x = 2$ because...

- A. $G(2)$ is not defined B. $\lim_{x \rightarrow 2} G(x)$ does not exist C. $\lim_{x \rightarrow 2} G(x) \neq G(2)$
D. Only reasons B and C E. All of the above reasons.

22. $\lim_{x \rightarrow \infty} \frac{-3x^2 + 7x^3 + 2}{2x^3 - 3x^2 + 5} =$

- A. ∞ B. $-\infty$ C. 1 D. $\frac{7}{2}$ E. $-\frac{3}{2}$

23. $\lim_{x \rightarrow -2} \frac{\sqrt{2x+5}-1}{x+2} =$

- A. 1 B. 0 C. ∞ D. $-\infty$ E. Does Not Exist
-

24. If $f(x) = 3x^2 - 5x$, then find $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

- A. $3x - 5$
B. $6x - 5$
C. $6x$
D. 0
E. Does not exist
-

25. $\lim_{x \rightarrow -\infty} \frac{2-5x}{\sqrt{x^2+2}} =$

- A. 5 B. -5 C. 0 D. $-\infty$ E. ∞
-

26. The function $f(x) = \frac{2x^2 + x - 3}{x^2 + 4x - 5}$ has a vertical asymptote at $x = -5$ because...

- A. $\lim_{x \rightarrow -5^+} f(x) = \infty$ B. $\lim_{x \rightarrow -5^-} f(x) = -\infty$
C. $\lim_{x \rightarrow -5^-} f(x) = \infty$ D. $\lim_{x \rightarrow \infty} f(x) = -5$
E. $f(x)$ does not have a vertical asymptote at $x = -5$
-

27. Consider the function $H(x) = \begin{cases} 3x-5, & x < 3 \\ x^2-2x, & x \geq 3 \end{cases}$. Which of the following statements is/are true?

- I. $\lim_{x \rightarrow 3^-} H(x) = 4$. II. $\lim_{x \rightarrow 3} H(x)$ exists. III. $H(x)$ is continuous at $x = 3$.
A. I only B. II only C. I and II only
D. I, II and III E. None of these statements is true
-

Unit #1 - Limits and Continuity

Practice Section

Below are tables of values for given types of functions. For each table, the type of function represented by the table is given. Use your knowledge of the numerical behavior of each type of function to find the indicated limits. For limits that do not exist, write D.N.E.

1. Exponential Function

x	-7	-4	-1	2	5	8	11
$H(x)$	-125	-13	1	2.75	2.969	2.996	2.999

a) $\lim_{x \rightarrow -\infty} H(x) =$

b) $\lim_{x \rightarrow -1} H(x) =$

c) $\lim_{x \rightarrow \infty} H(x) =$

2. Rational Function

x	-1000	-2.001	-2	-1.999	0.999	1	1.001	1000
$G(x)$	0.998	0.333	Undefined	0.333	-1999	Undefined	2001	1.002

a) $\lim_{x \rightarrow -\infty} G(x) =$

b) $\lim_{x \rightarrow -2^-} G(x) =$

c) $\lim_{x \rightarrow -2^+} G(x) =$

d) $\lim_{x \rightarrow -2} G(x) =$

e) $\lim_{x \rightarrow 1^-} G(x) =$

f) $\lim_{x \rightarrow 1^+} G(x) =$

g) $\lim_{x \rightarrow 1} G(x) =$

h) $\lim_{x \rightarrow \infty} G(x) =$

3. Rational Function

x	-10000	0.999	1	1.001	3.999	4	4.001	10000
$H(x)$	1.9999	-2.331	Undefined	-2.335	-12998	Undefined	13002	2.001

a) $\lim_{x \rightarrow \infty} H(x) =$

b) $\lim_{x \rightarrow 1} H(x) =$

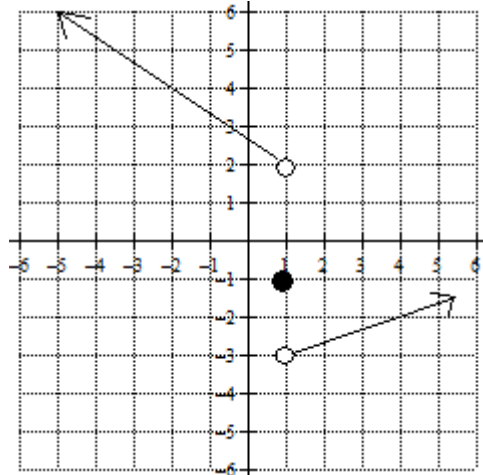
c) $\lim_{x \rightarrow 4^+} H(x) =$

d) $\lim_{x \rightarrow 4^-} G(x) =$

e) $\lim_{x \rightarrow 4} G(x) =$

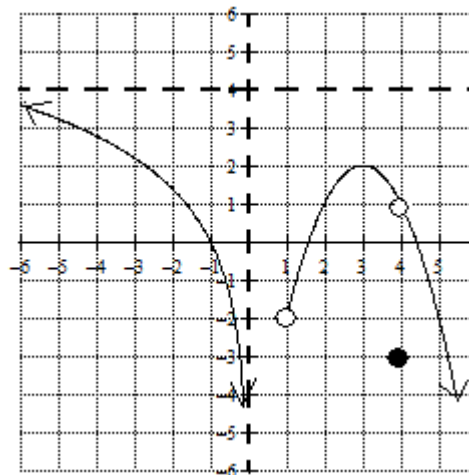
If they exist, determine the indicated values below each graph. For limits that do not exist, write D.N.E.

4. The graph of $h(x)$ is given.



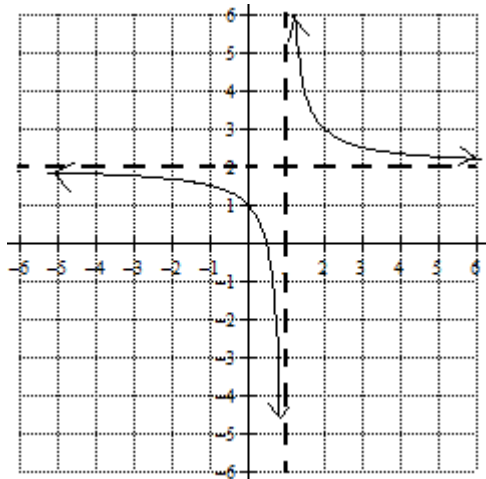
- a) $\lim_{x \rightarrow 1^-} h(x) =$ b) $\lim_{x \rightarrow 1^+} h(x) =$
c) $\lim_{x \rightarrow 1} h(x) =$ d) $h(1) =$
e) $h(-2) =$ f) $\lim_{x \rightarrow -2} h(x) =$

5. The graph of $g(x)$ is given.



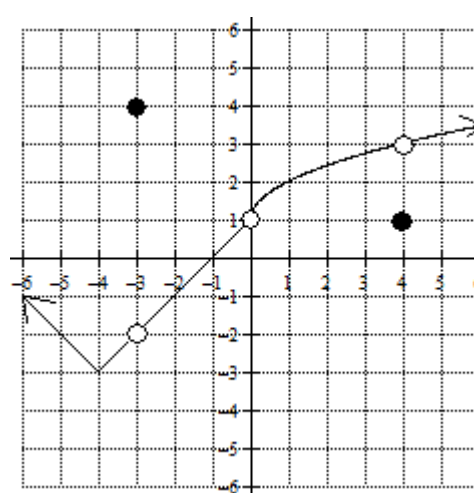
- a) $\lim_{x \rightarrow 0^-} g(x) =$ b) $\lim_{x \rightarrow 1^+} g(x) =$
c) $\lim_{x \rightarrow -\infty} g(x) =$ d) $\lim_{x \rightarrow 4} g(x) =$
e) $g(4) =$ f) $\lim_{x \rightarrow 3} g(x) =$

6. The graph of $f(x)$ is given.



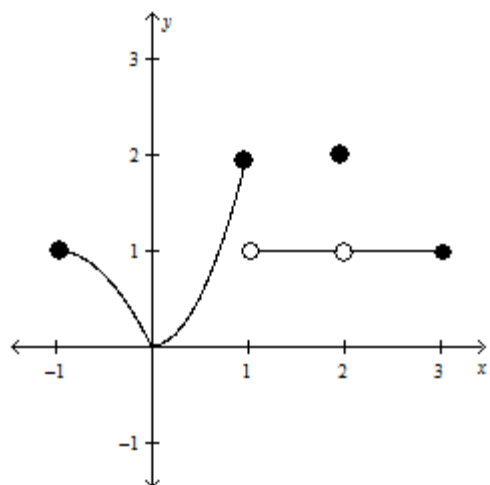
- a) $\lim_{x \rightarrow 0} f(x) =$ b) $\lim_{x \rightarrow -\infty} f(x) =$
c) $\lim_{x \rightarrow \infty} f(x) =$ d) $\lim_{x \rightarrow 1^+} f(x) =$
e) $\lim_{x \rightarrow 1^-} f(x) =$ f) $\lim_{x \rightarrow 1} f(x) =$

7. The graph of $q(x)$ is given.



- a) $\lim_{x \rightarrow 0} q(x) =$ b) $\lim_{x \rightarrow -3} q(x) =$
c) $\lim_{x \rightarrow 4} q(x) =$ d) $\lim_{x \rightarrow -4} q(x) =$
e) $q(-3) =$ f) $q(4) =$

Given the graph of the function, $g(x)$, below, determine if the statements are true or false. For statements that are false, explain why.



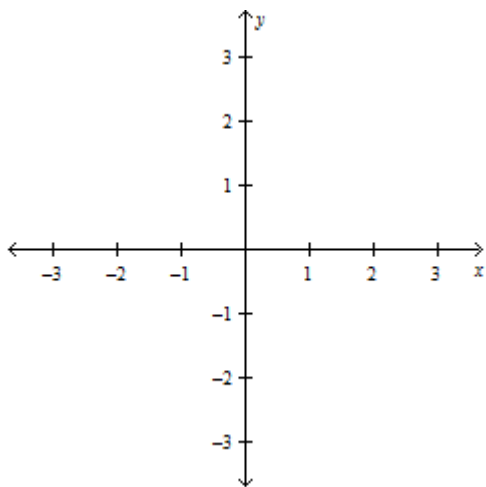
8. $\lim_{x \rightarrow 1} g(x) = 2$

9. $\lim_{x \rightarrow c} g(x)$ exists for every value of c on the interval $(-1, 1)$.

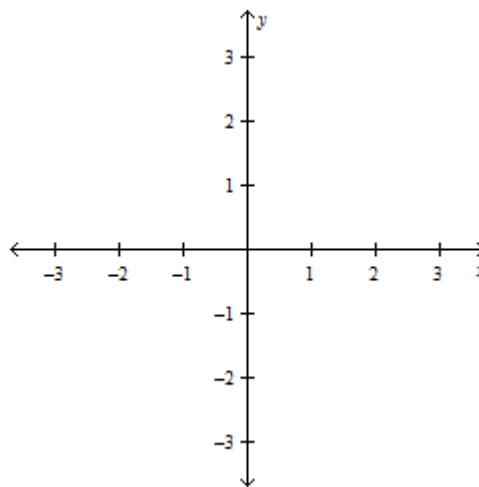
10. $\lim_{x \rightarrow 2} g(x)$ does not exist.

Sketch a graph of a function that fits the requirements described below.

11. $\lim_{x \rightarrow 1^-} f(x) = 3$ $\lim_{x \rightarrow 1^+} f(x) = -1$ $f(1) = 1$



12. $\lim_{x \rightarrow -2^-} f(x) = -\infty$ $\lim_{x \rightarrow -2^+} f(x) = \infty$
 $f(2)$ is undefined but $\lim_{x \rightarrow 2} f(x)$ exists.



13. In exercise 11, does $\lim_{x \rightarrow 1} f(x)$ exist? Explain why or why not.

Find the value of each limit. For a limit that does not exist, state why.

14. $\lim_{x \rightarrow -\frac{1}{2}} 3x^2(2x-1)$	15. $\lim_{x \rightarrow -1} x^3 + 2x^2 - 3x + 3$
16. $\lim_{x \rightarrow -2} (x-6)^{\frac{2}{3}}$	17. $\lim_{x \rightarrow 2} \frac{x^2+5x+6}{x+2}$
18. $\lim_{\theta \rightarrow \frac{\pi}{6}} \theta^2 \tan \theta$	19. $\lim_{x \rightarrow 0} \frac{(x+4)^2-16}{x}$
20. $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$	21. $\lim_{x \rightarrow 2} \frac{x^2-3x+2}{x^2-4}$

$$22. \lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2}$$

$$23. \lim_{x \rightarrow 0} \frac{\frac{1}{x+2} - \frac{1}{2}}{x}$$

$$24. \lim_{x \rightarrow 0} \frac{(2+x)^3 - 8}{x}$$

$$25. \lim_{h \rightarrow 0} \frac{(x+h)^2 + 2(x+h) - 3 - (x^2 + 2x - 3)}{h}$$

$$26. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ if } f(x) = 3x^2 - 2x$$

$$27. \lim_{x \rightarrow 2} f(x) \text{ if } f(x) = \begin{cases} 2x^2 - 4x, & x < 2 \\ 4\sin\left(\frac{\pi x}{4}\right), & x > 2 \end{cases}$$

$$28. \lim_{x \rightarrow 3} e^x \cos\left(\frac{\pi x}{3}\right)$$

$$29. \lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x-1}$$

$$30. \lim_{x \rightarrow 3^+} \frac{x+3}{x-3}$$

$$31. \lim_{x \rightarrow -3^+} \frac{2x^2-9x+9}{x^2-9}$$

$$32. \lim_{x \rightarrow 0} \frac{\frac{1}{x-2} + \frac{1}{2}}{x}$$

$$33. \lim_{x \rightarrow -2} \begin{cases} 2-x, & x < -2 \\ x^2-2x, & x > -2 \end{cases}$$

34. If $\lim_{x \rightarrow 3} f(x) = 2$ and $\lim_{x \rightarrow 3} g(x) = -4$, find each of the following limits. Show your analysis applying the properties of limits.		
a. $\lim_{x \rightarrow 3} \left[\frac{5f(x)}{g(x)} \right]$	b. $\lim_{x \rightarrow 3} [f(x) + 2g(x)]$	c. $\lim_{x \rightarrow 3} \sqrt{4f(x)}$
d. $\lim_{x \rightarrow 3} \frac{g(x)}{8}$	e. $\lim_{x \rightarrow 3} [3f(x) - g(x)]$	f. $\lim_{x \rightarrow 3} \left[\frac{f(x)g(x)}{12} \right]$

35. If $\lim_{x \rightarrow 4} f(x) = 0$ and $\lim_{x \rightarrow 4} g(x) = 3$, find each of the following limits. Show your analysis applying the properties of limits.	
a. $\lim_{x \rightarrow 4} \left[\frac{g(x)}{f(x)-1} \right]$	b. $\lim_{x \rightarrow 4} xf(x)$
c. $\lim_{x \rightarrow 4} [g(x) + 3]$	d. $\lim_{x \rightarrow 4} g^2(x)$

Find the limit of each of the following exponential functions. Sketch a graph of each function to aid in your determination of the limit.

36. $\lim_{x \rightarrow \infty} -(0.5)^{-x-2} + 3$

37. $\lim_{x \rightarrow \infty} 2^{-x-2} + 3$

38. $\lim_{x \rightarrow -\infty} -\left(\frac{1}{4}\right)^{-x-2} + 3$

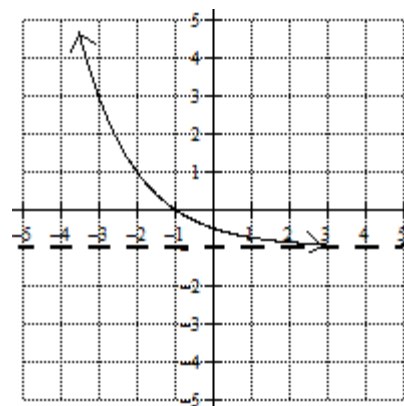
39. Using the graph of $g(x)$ pictured to the right, find each of the following limits.

a. $\lim_{x \rightarrow \infty} g(x) = \underline{\hspace{2cm}}$

b. $\lim_{x \rightarrow -\infty} g(x) = \underline{\hspace{2cm}}$

c. $\lim_{x \rightarrow -1} g(x) = \underline{\hspace{2cm}}$

d. $\lim_{x \rightarrow -3} g(x) = \underline{\hspace{2cm}}$



Find the value of each limit. For a limit that does not exist, state why.

40. $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\cos^2 \theta}{1 - \sin \theta}$

41. $\lim_{x \rightarrow 0} \frac{x + \sin x}{x}$

42. $\lim_{x \rightarrow 3} \begin{cases} 2x^2 - 3x, & x < 3 \\ 8 - \cos\left(\frac{\pi x}{3}\right), & x > 3 \end{cases}$	43. $\lim_{\theta \rightarrow 0} \frac{2 \sin 3\theta}{\theta}$
44. $\lim_{x \rightarrow 0} \frac{\sin x}{2x^2 - x}$	45. $\lim_{x \rightarrow 0} \frac{5x + \sin 3x}{x}$
46. $\lim_{x \rightarrow 0} \frac{\sin 2x}{6x}$	47. $\lim_{x \rightarrow 0} \frac{2 \sin 4x}{3x}$

$$48. \lim_{\theta \rightarrow 0} \frac{\cos \theta \tan \theta}{3\theta}$$

$$49. \lim_{\theta \rightarrow 0} \frac{3 - 3\cos \theta}{\theta}$$

$$50. \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\cos \theta}{\cot \theta}$$

$$51. \lim_{\theta \rightarrow 0} \frac{1 - \tan \theta}{\sin \theta - \cos \theta}$$

$$52. \lim_{c \rightarrow 3} \frac{c^3 - 27}{c - 3}$$

$$53. \lim_{x \rightarrow -1} \frac{(x+3)^3 - 8}{x+1}$$

Use the graph to the right to find each of the indicated values below. If a value does not exist, state why.

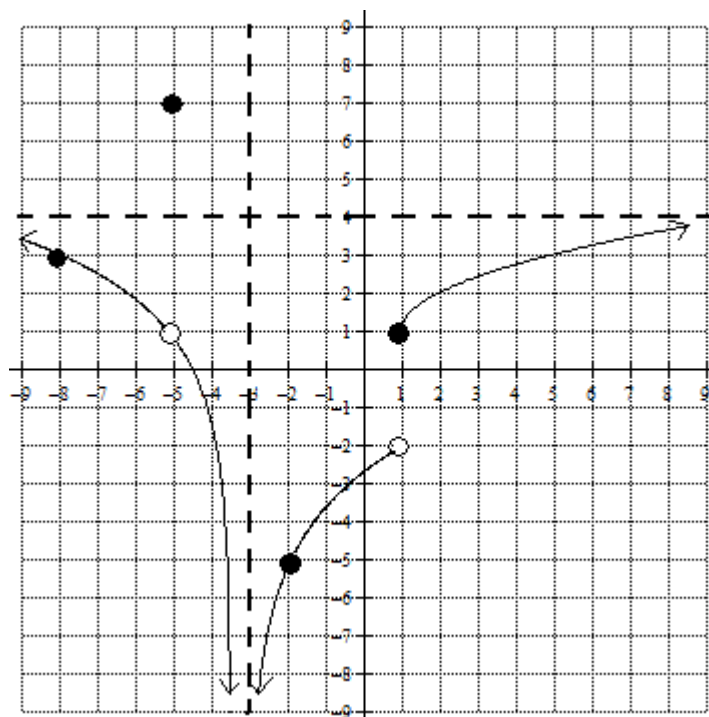
54. $\lim_{x \rightarrow \infty} F(x) =$ _____

55. $\lim_{x \rightarrow 2} F(x) =$ _____

56. $\lim_{x \rightarrow -2^-} F(x) =$ _____

57. $\lim_{x \rightarrow -2^+} F(x) =$ _____

58. $\lim_{x \rightarrow -2} F(x) =$ _____



59. $F(-4.5) =$ _____

60. $\lim_{x \rightarrow -5} F(x) =$ _____

61. $F(-5) =$ _____

62. $F(1) =$ _____

63. $\lim_{x \rightarrow 1^+} F(x) =$ _____

64. $\lim_{x \rightarrow 1^-} F(x) =$ _____

65. $\lim_{x \rightarrow 1} F(x) =$ _____

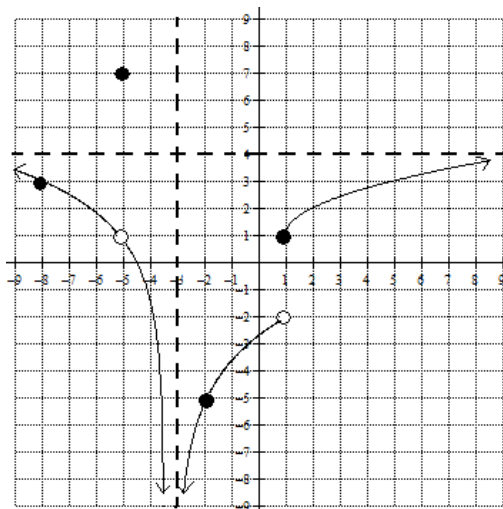
66. $\lim_{x \rightarrow -3^+} F(x) =$ _____

67. $\lim_{x \rightarrow -3^-} F(x) =$ _____

68. $F(-2) =$ _____

69. $F(-3) =$ _____

For exercises 70 – 72, determine if the function is continuous at each of the indicated values below. Use the three part definition of continuity to perform your analysis.

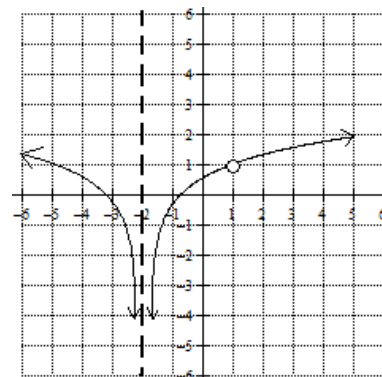


70. $x = -5$

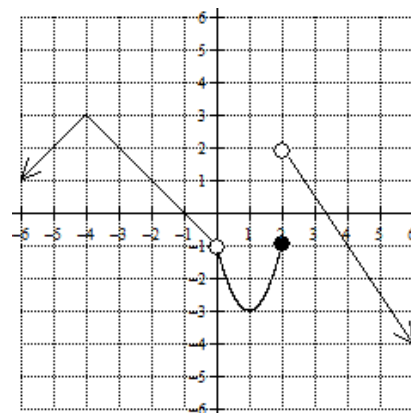
71. $x = 1$

72. $x = -2$

74. Write a discussion of continuity for each of the functions below. Be sure to include in your discussion where $g(x)$ is continuous and where $g(x)$ is discontinuous. For values at which the function is discontinuous, explain using the three part definition of continuity.



75. Find the equation of the function, $p(x)$, pictured to the right. Then, use the three part definition of continuity to analytically justify why $p(x)$ is discontinuous at $x = 0$ and $x = 2$.



Find the value of a that makes each of the functions below everywhere continuous. Write the two limits that must be equal in order for the function to be continuous.

76. $f(x) = \begin{cases} 4 - x^2, & x < -1 \\ ax^2 - 1, & x \geq -1 \end{cases}$

77. $f(x) = \begin{cases} x^2 + x + a, & x < 2 \\ ax^3 - x^2, & x \geq 2 \end{cases}$

78. For what values of k and m is the function $g(x)$ everywhere continuous? Use limits to set up your work.

$$g(x) = \begin{cases} kx^2 + m, & x < -2 \\ e^{\ln(2x+3)}, & -2 \leq x \leq 3 \\ kx - m, & x > 3 \end{cases}$$

79. Show, using the intermediate value theorem, that the function $F(x) = x^3 + 2x - 1$ has a zero on the interval $[0, 1]$.

80. Show, using the intermediate value theorem, whether or not the function $g(\theta) = \theta^2 - 2 - \cos\theta$ has a zero on the interval $[0, \pi]$.

For exercises 81 – 83, first, verify that the I.V.T. is applicable for the given function on the given interval. Then, if it is applicable, find the value of the indicated c , guaranteed by the theorem.

81. $f(x) = x^2 - 6x + 8$ $[0, 3]$ $f(c) = 0$

82. $g(x) = x^3 - x^2 + x - 2$ $[0, 3]$ $g(c) = 4$

83. $h(x) = \frac{x^2 + x}{x - 1}$ $\left[\frac{5}{2}, 4\right]$ $h(c) = 6$

For exercises 84 – 86, determine the vertical asymptotes of each function. Justify your answers by using limits, showing your numerical analysis.

84. $g(x) = \frac{x^2 - x - 6}{x + 5}$	85. $f(x) = \frac{x - 5}{x^2 + x - 2}$	86. $h(x) = \frac{2x^2 + x - 3}{x^2 - 3x + 2}$
--	--	--

Find each of the following limits at infinity. What do the results show about the existence of a horizontal asymptote? Justify your reasoning.

87. $\lim_{x \rightarrow -\infty} \frac{3x + 2 - 5x^2}{2x^2 - 3x - 1}$	88. $\lim_{x \rightarrow \infty} \frac{3x + 5}{2x^2 - 3x}$	89. $\lim_{x \rightarrow -\infty} \frac{-2x^2 + 5}{3x + 2}$
--	--	---

Find each of the following limits at infinity. What do the results show about the existence of a horizontal asymptote? Justify your reasoning.

90. $\lim_{x \rightarrow -\infty} \frac{2x + 1}{\sqrt{x^2 - x}}$	91. $\lim_{x \rightarrow \infty} \frac{-2x^2 + x}{\sqrt{2x^2 - 3}}$
--	---