Multivariable Calculus

MATHEMATICS KEY COMPETENCIES & COURSE STANDARDS

WITH LEARNING OBJECTIVES IN PROGRESSION ORDER

GEORGIA'S K-12 MATHEMATICS STANDARDS 2021

Governor Kemp and Superintendent Woods are committed to the best set of academic standards for Georgia's students – laying a strong foundation of the fundamentals, ensuring age- and developmentally appropriate concepts and content, providing instructional supports to set our teachers up for success, protecting and affirming local control and flexibility regarding the use of mathematical strategies and methods, and preparing students for life. These Georgia-owned and Georgia-grown standards leverage the insight, expertise, experience, and efforts of thousands of Georgians to deliver the very best educational experience for Georgia's 1.7 million students.

In August 2019, Governor Brian Kemp and State School Superintendent Richard Woods announced the review and revision of Georgia's K-12 mathematics standards. Georgians have been engaged throughout the standards review and revision process through public surveys and working groups. In addition to educator working groups, surveys, and the Academic Review Committee, Governor Kemp announced a new way for Georgians to provide input on the standards: the Citizens Review Committee, a group composed of students, parents, business and community leaders, and concerned citizens from across the state. Together, these efforts were undertaken to ensure Georgians will have buy-in and faith in the process and product.

The Citizens Review Committee provided a charge and recommendations to the working groups of educators who came together to craft the standards, ensuring the result would be usable and friendly for parents and students in addition to educators. More than 14,000 Georgians participated in the state's public survey from July through September 2019, providing additional feedback for educators to review. The process of writing the standards involved more than 200 mathematics educators -- from beginning to veteran teachers, representing rural, suburban, and metro areas of our state.

Grade-level teams of mathematics teachers engaged in deep discussions; analyzed stakeholder feedback; reviewed every single standard, concept, and skill; and provided draft recommendations. To support fellow mathematics teachers, they also developed learning progressions to show when key concepts were introduced and how they progressed across grade levels, provided examples, and defined age/developmentally appropriate expectations.

These teachers reinforced that strategies and methods for solving mathematical problems are classroom decisions -- not state decisions -- and should be made with the best interest of the individual child in mind. These recommended revisions have been shared with the Academic Review Committee, which is composed of postsecondary partners, age/development experts, and business leaders, as well as the Citizens Review Committee, for final input and feedback.

Based on the recommendation of Superintendent Woods, the State Board of Education will vote to post the draft K-12 mathematics standards for public comment. Following public comment, the standards will be recommended for adoption, followed by a year of teacher training and professional learning prior to implementation.

Multivariable Calculus

Overview

This document contains a draft of Georgia's 2021 K-12 Mathematics Standards for the High School Multivariable Calculus Course, which is a fourth mathematics course option in the high school course sequence.

The standards are organized into big ideas, course competencies/standards, and learning objectives/expectations. The grade level key competencies represent the standard expectation of learning for students in each grade level. The competencies/standards are each followed by more detailed learning objectives that further explain the expectations for learning in the specific grade levels.

New instructional supports are included, such as clarification of language and expectations, as well as detailed examples. These have been provided for teaching professionals and stakeholders through the Evidence of Student Learning Column that accompanies each learning objective.

Course Description:

Multivariable Calculus is a fourth-year mathematics course option for students who have completed AP Calculus BC. It includes three-dimensional coordinate geometry; matrices and determinants; eigenvalues and eigenvectors of matrices; limits and continuity of functions with two independent variables; partial differentiation; multiple integration; the gradient; the divergence; the curl; Theorems of Green, Stokes, and Gauss; line integrals; integrals independent of path; and linear first-order differential equations.

Prerequisite:

This course is designed for students who have successfully completed *Advanced Placement (AP) Calculus BC.*

Georgia's K-12 Mathematics Standards - 2021

Mathematics Big Ideas and Learning Progressions, High School

Mathematics Big Ideas, HS

HIGH SCHOOL
MATHEMATICAL PRACTICES (MP)
MATHEMATICAL MODELING (MM)
NUMERICAL (QUANTITATIVE) REASONING (NR)
PATTERNING & ALGEBRAIC REASONING (PAR)
FUNCTIONAL & GRAPHICAL REASONING (FGR)
GEOMETRIC & SPATIAL REASONING (GSR)
DATA & STATISTICAL REASONING (DSR)
PROBABILISTIC REASONING (PR)
ABSTRACT & QUANTATIVE REASONING (AQR)

The 8 Mathematical Practices and the Mathematical Modeling Framework are essential to the implementation of the content standards presented in this course. More details related to these concepts can be found in the links below and in the first two standards presented in this course:

Mathematical Practices

Mathematical Modeling Framework

Multivariable Calculus

The five course standards listed below are the key content competencies students will be expected to master in this course. Additional clarity and details are provided through the classroom-level learning objectives and evidence of student learning details for each course standard found on subsequent pages of this document.

COURSE STANDARDS
MVC.MP: Display perseverance and patience in problem-solving. Demonstrate skills and strategies needed to succeed in mathematics, including critical thinking, reasoning, and effective collaboration and expression. Seek help and apply feedback. Set and monitor goals.
MVC.MM.1: Apply mathematics to real-life situations; model real-life phenomena using mathematics.
MVC.PAR.2: Express spatial and functional relationships with vectors, functions, and analytic geometry in three dimensions, and use these relationships to solve contextual, mathematical problems.
MVC.AQR.3: Define, describe, and represent the differentiation of functions of two independent variables and differential vectors to solve contextual, mathematical problems and to explain real-life phenomena.
MVC.AQR.4: Interpret integrals of functions of two independent variables and of vector functions to solve contextual, mathematical problems and to explain real-life phenomena.

Multivariable Calculus

MATHEMATICAL MODELING		
MVC.MM.1: Apply mathematics to real-life situations; model real-life phenomena using mathematics.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
MVC.MM.1.1	Explain contextual, mathematical problems using a mathematical model.	***Fundamentals*** • Students should be provided with opportunities to learn mathematics in the context of real-life problems. • Contextual, mathematical problems are mathematical problems presented in context where the context makes sense, realistically and mathematically, and allows for students to make decisions about how to solve the problem (model with mathematics).
MVC.MM.1.2	Create mathematical models to explain phenomena that exist in the natural sciences, social sciences, liberal arts, fine and performing arts, and/or humanities contexts.	***Fundamentals*** • Students should be able to use the content learned in this course to create a mathematical model to explain real-life phenomena.
MVC.MM.1.3	Using abstract and quantitative reasoning, make decisions about information and data from a contextual situation.	
MVC.MM.1.4	Use various mathematical representations and structures with this information to represent and solve real-life problems.	

PATTERNING & ALGEBRAIC REASONING – *Vectors, Functions, and Analytic Geometry in Three Dimensions*		
MVC.PAR.2: Express spatial and functional relationships with vectors, functions, and analytic geometry in three dimensions, and use these relationships to solve contextual, mathematical problems.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
Express the relationships between points, lines, and planes in three dimensions.		
MVC.PAR.2.1	Represent equations of lines in space using vectors.	
MVC.PAR.2.2	Express the analytic geometry of three dimensions in terms of the dot product and cross product of vectors.	***Example*** • Equations of planes, parallelism, perpendicularity, and angles between lines and planes.

MVC.PAR.2.3	Use a linear system of equations to determine whether two planes intersect in a single point or a line, or whether they do not intersect at all.	
Explore functions of two independent variables of the form z = f(x, y) and implicit functions of the form f(x, y, z) = 0.		
MVC.PAR.2.4	Evaluate functions of two independent variables at a point in the plane.	
MVC.PAR.2.5	Graph the level curves of functions of two independent variables.	
MVC.PAR.2.6	Investigate the continuity of functions of two independent variables in terms of the limits of such functions as (*x, y*) approaches a given point in the plane.	
MVC.PAR.2.7	Determine points or regions of discontinuity of functions of two independent variables.	

ABSTRACT & QUANTITATIVE REASONING *– Partial Differentiation*		
MVC.AQR.3: Define, describe, and represent the differentiation of functions of two independent variables and differential vectors to solve contextual, mathematical problems and to explain real-life phenomena.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
MVC.AQR.3.1	Approximate the partial derivatives at a point of a function defined by a table of data.	
MVC.AQR.3.2	Find expressions for the first and second partial derivatives of a function.	***Fundamentals*** • Students should be able to extend the differentiation skills acquired in prerequisite courses.
MVC.AQR.3.3	Use the total differential to approximate mathematical models.	***Relevance and Application*** • Determine the sensitivity of the value of the dependent variable to small changes in one of the independent variables.
MVC.AQR.3.4	Represent the partial derivatives of a system of two functions in two variables using the Jacobian.	
MVC.AQR.3.5	Find the partial derivatives of the composition of functions using the general chain rule.	
MVC.AQR.3.6	Apply partial differentiation to problems of optimization, including problems requiring the use of the Lagrange multiplier.	

MVC.AQR.3.7	Find the family of solutions and the envelope of the family of solutions to differential equations, including Clairaut equations.	
MVC.AQR.3.8	Define and apply the gradient, the divergence, and the curl in terms of differential vector operations.	

ABSTRACT & QUANTITATIVE REASONING *– Integration*		
MVC.AQR.4: Interpret integrals of functions of two independent variables and of vector functions to solve contextual, mathematical problems and to explain real-life phenomena.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
Evaluate and apply double and triple integrals.		
MVC.AQR.4.1	Integrate functions of the form $z = f(x, y)$ or $w = f(x, y, z)$ through various techniques.	***Example*** • Integration methods including changing the order of integration, substituting variables with the Jacobian, or changing to polar coordinates.
MVC.AQR.4.2	Use, evaluate, and interpret double and triple integrals in terms of volume and mass.	
MVC.AQR.4.3	Represent and evaluate integrals of vector functions as double and triple integrals.	
Evaluate and interpret line and surface integrals.		
MVC.AQR.4.4	Apply line and surface integral to functions representing real-world phenomena.	
MVC.AQR.4.5	Solve first-order exact differential equations.	***Example*** • Solve $xy' + y = x^2$ by recognizing that $xy' + y$ is the exact differential of the product xy.
MVC.AQR.4.6	Use Green's Theorem to evaluate line integrals in the plane; use Stokes' Theorem to evaluate line integrals in space.	
MVC.AQR.4.7	Determine whether a line integral is independent of path and use line integrals in context.	***Examples*** • Interpret line integrals as a measure of work done to move a particle along a curve; interpret line integrals as a measure of the mass of a thin wire.
MVC.AQR.4.8	Use Gauss' Divergence Theorem to evaluate surface integrals.	
MVC.AQR.4.9	Define and apply the gradient, the divergence, and the curl in terms of integrals of vector functions.	

ESSENTIAL INSTRUCTIONAL GUIDANCE

MATHEMATICAL PRACTICES

The Mathematical Practices describe the reasoning behaviors students should develop as they build an understanding of mathematics – the "habits of mind" that help students become mathematical thinkers. There are eight standards, which apply to all grade levels and conceptual categories.

These mathematical practices describe how students should engage with the mathematics content for their grade level. Developing these habits of mind builds students' capacity to become mathematical thinkers. These practices can be applied individually or together in mathematics lessons, and no particular order is required. In well-designed lessons, there are often two or more Standards for Mathematical Practice present.

Mathematical Practices	
MVC.MP: Display perseverance and patience in problem-solving. Demonstrate skills and strategies needed to succeed in mathematics, including critical thinking, reasoning, and effective collaboration and expression. Seek help and apply feedback. Set and monitor goals.	
Code	**Expectation**
MVC.MP.1	Make sense of problems and persevere in solving them.
MVC.MP.2	Reason abstractly and quantitatively.
MVC.MP.3	Construct viable arguments and critique the reasoning of others.
MVC.MP.4	Model with mathematics.
MVC.MP.5	Use appropriate tools strategically.
MVC.MP.6	Attend to precision.
MVC.MP.7	Look for and make use of structure.
MVC.MP.8	Look for and express regularity in repeated reasoning.

MATHEMATICAL MODELING

Teaching students to model with mathematics is engaging, builds confidence and competence, and gives students the opportunity to collaborate and make sense of the world around them, the main reason for doing mathematics. For these reasons, mathematical modeling should be incorporated at every level of a student's education. This is important not only to develop a deep understanding of mathematics itself, but more importantly to give students the tools they need to make sense of the world around them. Students who engage in mathematical modeling will not only be prepared for their chosen career but will also learn to make informed daily life decisions based on data and the models they create.

The diagram below is a mathematical modeling framework depicting a cycle of how students can engage in mathematical modeling when solving a real-life problem or task.

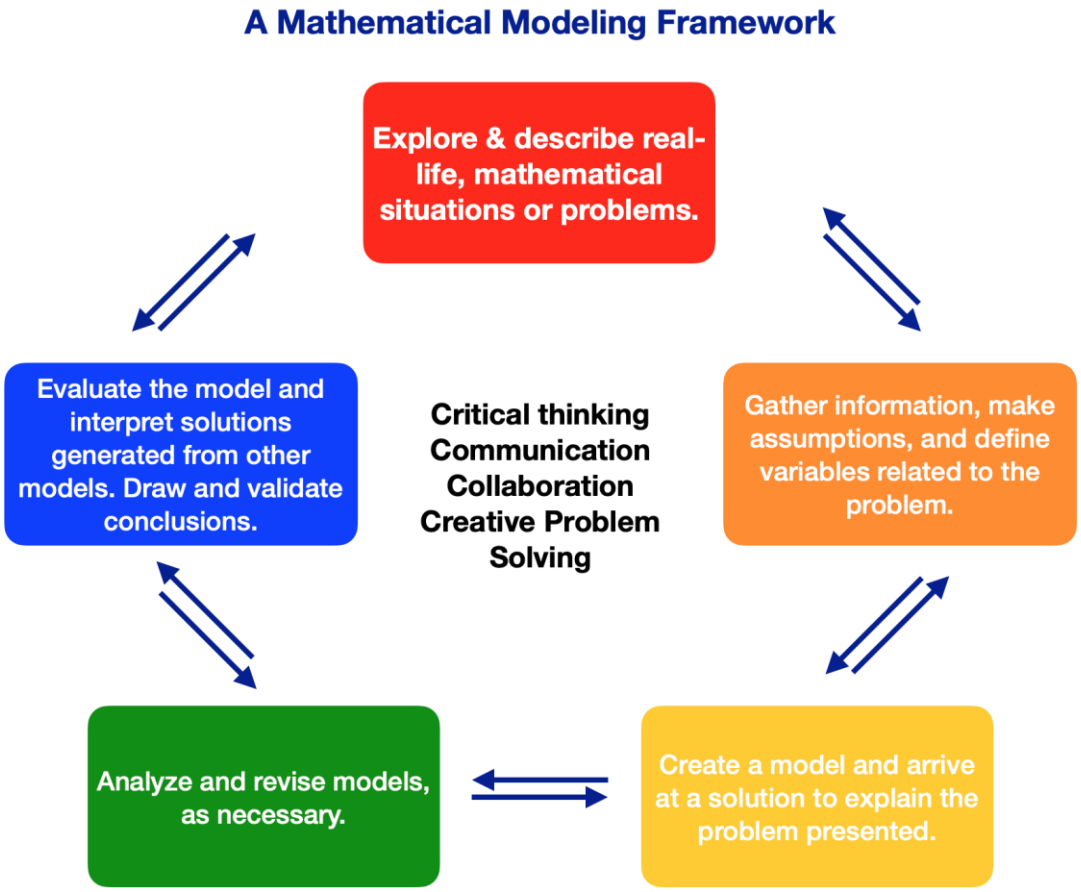

Image adapted from: Suh, Matson, Seshaiyer, 2017

FRAMEWORK FOR STATISTICAL REASONING

Statistical reasoning is important for learners to engage as citizens and professionals in a world that continues to change and evolve. Humans are naturally curious beings and statistics is a language that can be used to better answer questions about personal choices and/or make sense of naturally occurring phenomena. Statistics is a way to ask questions, explore, and make sense of the world around us.

The Framework for Statistical Reasoning should be used in all grade levels and courses to guide learners through the sense-making process, ultimately leading to the goal of statistical literacy in all grade levels and courses. Reasoning with statistics provides a context that necessitates the learning and application of a variety of mathematical concepts.

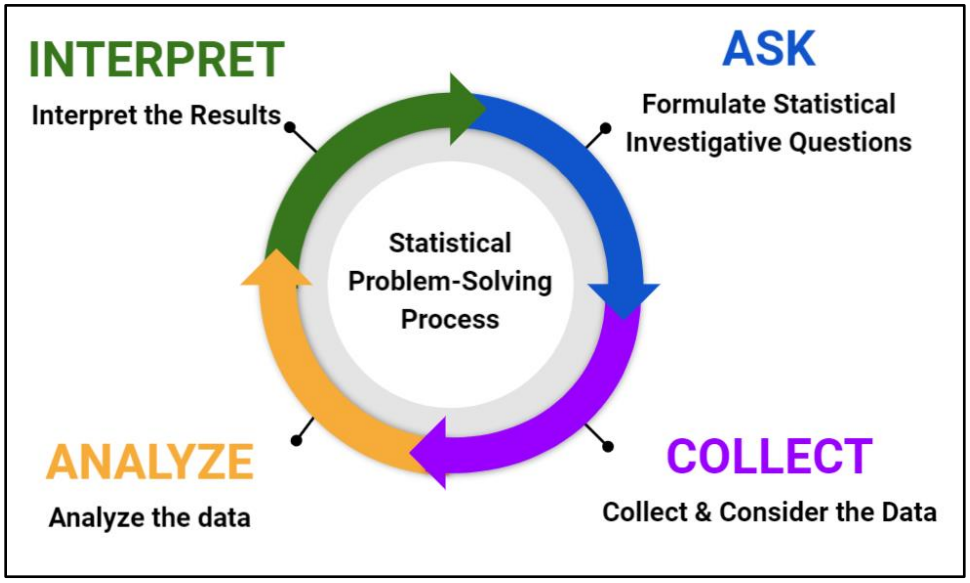

Figure 1: Georgia Framework for Statistical Reasoning

The following four-step statistical problem-solving process can be used throughout each grade level and course to help learners develop a solid foundation in statistical reasoning and literacy:

I. Formulate Statistical Investigative Questions
Ask questions that anticipate variability.

II. Collect & Consider the Data
Ensure that data collection designs acknowledge variability.

III. Analyze the Data
Make sense of data and communicate what the data mean using pictures (graphs) and words. Give an accounting of variability, as appropriate.

IV. Interpret the Results
Answer statistical investigative questions based on the collected data.