Calculus Unit 5 Differential Equations and Mathematical Modeling	25 to 30 days Established 14-15 Revised 20-21 Revised Nov 2021 Revised August 2023

Standard – NJSLS: A.APR, A.REI, F.IF, F.BF, F.LE, and F.TF
Differential Equations and Mathematical Modeling (Chapter 6)

Strand
A-APR: Algebra: Arithmetic with Polynomials and Rational Expressions
Perform arithmetic operations on polynomials
1. ***Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.***

A-REI: Algebra: Reasoning with Equations and Inequalities
Understand solving equations as a process of reasoning and explain the reasoning
2. ***Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.***

Solve equations and inequalities in one variable
3. ***Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.***
4. ***Solve quadratic equations in one variable.***

Represent and solve equations and inequalities graphically
11. ***Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.***

F-IF: Functions: Interpreting Functions
Analyze functions using different representations
7. **Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.**
    e. **Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.**

F -BF: Functions: Building Functions

Build new functions from existing functions

4. ***Find inverse functions.***
    a. ***Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse.***
    d. ***Produce an invertible function from a non-invertible function by restricting the domain.***

F-LE: Functions: ***Linear, Quadratic, and Exponential Models***

Construct and compare linear, quadratic, and exponential models and solve problems

1. ***Distinguish between situations that can be modeled with linear functions and with exponential functions.***
4. ***For exponential models, express as a logarithm the solution to ab to the ct power = d where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.***

F -TF: Functions: Trigonometric Functions

Model periodic phenomena with trigonometric functions

6. ***Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.***
7. ***Use functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.***

Prove and apply trigonometric identities

8. ***Prove the Pythagorean identity sin^2(theta) + cos^2(theta) = 1 and use it to find sin(theta), cos(theta), or tan(theta) given sin(theta), cos(theta), or tan(theta) and the quadrant of the angle.***

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Curriculum aligned with: 2009 New Jersey Core Curriculum Content Standards for 21st Century Skills (9.1 A-F)		
21st Century Theme: *Global Awareness ☐, Financial, economic, business and entrepreneurial literacy ☐, Civic literacy ☐, Health literacy ☐Environmental Literacy ☐,*		
21st Century Skills: *Critical Thinking & Problem Solving☐, Creativity and Innovation ☐, Collaboration, Teamwork and Leadership ☐, Cross-Cultural Understanding and Interpersonal Communications ☐, Communication and Media Fluency ☐, Accountability, Productivity and Ethics ☐*		
Interdisciplinary Connection: Math=MA, English=ELA, Science=SCI, Social Studies=SS, Physical Education=PE, Art=ART, Music=MU, Technology=TECH, World Language=WL, Business = BU		
Essential Questions	**Enduring Understandings**	**Activities, Investigation, and Student Experiences**

Calculus Unit 5 Differential Equations and Mathematical Modeling

25 to 30 days
Established 14-15
Revised 20-21
Revised Nov 2021
Revised August 2023

1. How are inverse functions related?
2. How does taking the derivative of a transcendental function compare to taking the derivative of an algebraic function?
3. How does finding the integral of a transcendental function compare to finding the integral of an algebraic function?
4. How are the three approaches for solving differential equations related?
5. Why is there a need for more than one method to solve differential equations?

- Just as subtraction can be used to undo addition, and division can be used to undo multiplication, the inverse function f^{-1} can be used to unto what has been done by function *f*.
- Transcendental functions are non-algebraic; thus, there are different properties for differentiation and integration compared to the differentiation and integration of algebraic functions.
- The natural logarithmic function is defined by
  $\ln x = \int_1^x 1/t\, dt$, $x > 0$.
  The domain of the natural logarithmic function is the set of all positive real numbers.
- The natural logarithmic function has the following properties:
  The domain is $(0, \infty)$ and the range $(-\infty, \infty)$.
  The function is continuous, increasing, and one –to – one.
  The graph is concave downward.
- If a and b are positive numbers and n is rational, then the following properties are true.
  $\ln(1) = 0$

Task 1:

Find the derivative of the function:

a) $g(x) = \ln \sqrt{x}$
b) $g(t) = t^2 e^t$
c) $g(x) = \log_3 \sqrt{1 - x}$
d) $y = ½ \arctan e^{2x}$
e) $y = x \tanh^{-1} 2x$

Answer:

Find the derivative of the function:

a) $g(x) = \ln \sqrt{x}$
$g(x) = ½ \ln x$
$g'(x) = 1/2x$
b) $g(t) = t^2 e^t$
$g'(t) = t^2e^t + 2te^t$
$= te^t (t + 2)$
c) $g(x) = \log_3 \sqrt{1 - x}$
$g(x) = ½ \log_3 (1 - x)$
d) $y = ½ \arctan e^{2x}$
$y' = ½ \left(\frac{1}{1 + e^{4x}}\right)(2e^{2x})$
$= \frac{e^{2x}}{1 + e^{4x}}$
e) $y = x \tanh^{-1} 2x$
$y' = x\left(\frac{2}{1 - 4x2}\right) + \tanh^{-1} 2x$
$= \frac{2x}{(1 - 4x2)} + \tanh^{-1} 2x$

Task 2:

Find the integral

a) $\int \left[\frac{\sin x}{(1 + \cos x)}\right] dx$
b) $\int [(e^{2x} - e^{-2x}) / (e^{2x} + e^{-2x})]\, dx$
c) $\int (x + 1)^{(5x + 1)\,2}\, dx$

$\ln(ab) = \ln a + \ln b$
$\ln(a^n) = n \ln a$
$\ln (a/b) = \ln a - \ln b$

- The letter *e* denotes the positive real number such that $\ln e = \int_1^e 1/t\, dt = 1$.
- Derivative of the Natural Logarithmic Function:
  Let u be a differentiable function of x
  $d/dx\, [\ln x] = 1/x,\ x>0$
  $d/dx\, [\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u},\ u>0$
- Derivative involving absolute value:
  If u is a differentiable function of x such that $u \neq 0$, then

  $d/dx\, [\ln|u|] = \frac{u'}{u}$
- Log Rule for Integration: Let u be a differentiable function of x.
  $\int(1/x)dx = \ln|x| + C$
  $\int(1/u)\, du = \ln|u| + C$
- Integrals of the Six Basic Trigonometric Functions:
  $\int \sin u\, du = -\cos u + C$
  $\int \cos u\, du = \sin u + C$
  $\int \tan u\, du = -\ln |\cos u| + C$
  $\int \cot u\, du = \ln |\sin u| + C$
  $\int \sec u\, du$

d) $\int {}^{1}/_{(16 + x2)}\, dx$
e) $\int {}^{x}/_{\sqrt{x4 - 1}}\, dx$

Answer:

Find the integral

a) $\int [{}^{\sin x}/_{(1+ \cos x)}]\, dx$

$$\int \frac{\sin x}{1+\cos x}\,dx = -\int \frac{-\sin x}{1+\cos x}\,dx$$
$$= -\ln|1+\cos x| + C$$

b) $\int [(e^{2x} - e^{-2x}) / (e^{2x} + e^{-2x})]\, dx$

Let $u = e^{2x} + e^{-2x}$, $du = (2e^{2x} - e^{-2x})\,dx$.
$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}}\,dx$$
$$= \frac{1}{2}\ln(e^{2x} + e^{-2x}) + C$$

c) $\int (x + 1)^{(5x + 1)\,2}\, dx$
$= (1/2)(1/\ln 5)\, 5_{(x+1)2} + C$

d) $\int {}^{1}/_{(16 + x2)}\, dx$
$= ¼ \arctan x/4 + C$

e) $\int {}^{x}/_{\sqrt{x4 - 1}}\, dx$

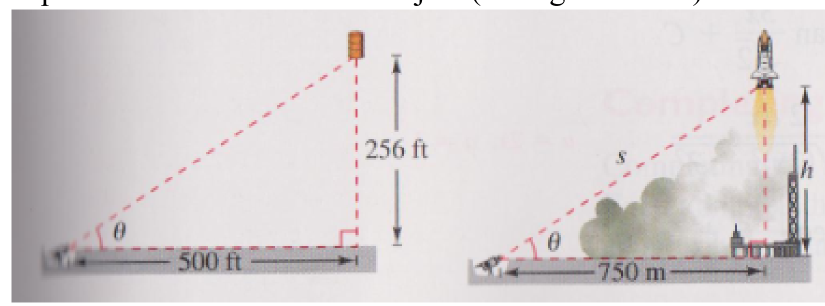
Let $u = x^2$, $du = 2x\,dx$.
$$\int \frac{x}{\sqrt{x^4-1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(x^2)^2-1}}(2x)\,dx = \frac{1}{2}\ln\left(x^2 + \sqrt{x^4-1}\right) + C$$

= ln |sec u + tan u| + C
∫csc u *du*
= -ln|csc u + cot u| + C

- A function g is the inverse function of the function f if f(g(x)) = x for each x in the domain of g and g(f(x)) = x for each x in the domain of f. The function g is denoted by f^{-1} (read "f inverse").
- Some important observations about inverse functions are:
  If g is the inverse function of f, then f is the inverse function of g.
  The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
  A function need not to have an inverse function, but if it does, the inverse function is unique.
- Whether a function has an inverse function.
- Reflective Property of Inverse Functions:
  The graph of f contains the point (a,b) if and only if the graph of f^{-1} contains the point (b,a).
- A function has an inverse function if and only if it is one –to – one.

***Task 3:**
Interdisciplinary SCI

In a free fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object (see figure below)



a) Find the position function giving the height of the object at time t assuming the object is released at time t =0. At what time will the object reach ground level?
b) Find the rates of change of the angle of elevation of the camera when t = 1 and t = 2.

Answer:

a) h(t) = -16t2 + 256
   $-16t^2 + 256 = 0$ when t = 4 sec
b) $\tan \theta = h/500$
   $= (-16t^2 + 256) / 500$

$\Theta = \arctan [\,(16/500)(-t^2 + 16)]$

$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2}$

$= \frac{-1000t}{\quad}$

	• If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function. • Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true: If f is continuous on its domain, then f^{-1} is continuous on its domain. If f is increasing on its domain, then f^{-1} is increasing on its domain. If f is decreasing on its domain, then f^{-1} is decreasing on its domain. If f is differentiable on an interval containing c and f'(c) ≠0, then f^{-1} is differentiable at f(c). • Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which f'(g(x)) ≠ 0. Moreover, $g'(x) = \frac{1}{f'(g(x))}$, f'(g(x)) ≠ 0 • The inverse function of the natural logarithmic function f(x) = ln x is called the natural	$15,625 + 16(16 - t^2)^2$ when t = 1, dθ/dt ≈ -0.0520 rad/sec when t = 2, dθ/dt ≈ -0.1116 rad/sec **<u>Task 4:</u>** Solve the differential equation. a) *dy/dx* = 6 – x b) *dy/dx* = y + 6 c) (2 + x)y' – xy = 0 **<u>Answer:</u>** Solve the differential equation. a) *dy/dx* = 6 – x $y = \int(6 - x)\,dx = 6x - (x^2/2) + c$ b) *dy/dx* = y + 6 $\int(dy / (y + 6)) = \int dx$ $= \ln\lvert y + 6\rvert = x + C_1$ $= \lvert y + 6\rvert = e^x + C_1 = Ce^x$ $y = -6 + Ce^x$ c) (2 + x)y' – xy = 0 (2 + x)*dy/dx* = xy (1/y) dy = (x / (2 + x)) dx (1/ y) dy = (1 – (2/(2 + x)) dx $\ln\lvert y\rvert = x - 2\ln\lvert 2 + x\rvert + C_1$ $y = Ce^x(2 + x)^{-2}$ $y = \frac{Ce^x}{(2 + X)^2}$ **<u>*Task 5:</u>** **Interdisciplinary BU**

	exponential function and is denoted by $f^{-1}(x) = e^x$. That is, $y = e^x$ if and only if $x = \ln y$. • Derivative of the Natural Exponential Function: Let u be a differentiable function of x $\frac{d}{dx}[e^x] = e^x$ $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$ • Integration Rules for Exponential Functions: Let u be a differentiable function of x. $\int e^x\,dx = e^x + C$ $\int e^u\,du = e^u + C$ • If a is a positive real number ($a \neq 1$) and x is any real number, then the exponential function to the base a is denoted by ax and is defined by $a^x = e^{(\ln a)x}$. If a = 1, then $y = 1^x = 1$ is a constant function. • Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x. $\frac{d}{dx}[a^x] = (\ln a)a^x$ $\frac{d}{dx}[a^u] = (\ln u)\, a^u \left(\frac{du}{dx}\right)$ $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)^x}$ $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)^u}\frac{du}{dx}$ • Occasionally, an integrand involves an exponential function	The sales S (in thousands of units) of a new product after it has been on the market for t years is given by: $S = 25(1 - e^{kt})$ a) Find S as a function of t if 4000 units have been sold after 1 year. b) How many units will saturate this market? c) How many units will have been sold after 5 years? **Answer:** The sales S (in thousands of units) of a new product after it has been on the market for t years is given by: $S = 25(1 - e^{kt})$ a) Find S as a function of t if 4000 units have been sold after 1 year. $4 = 25\,(1 - e^{k(1)})$ $4/25 = 1 - e^{k}$ $ek = 21/25$ $k = \ln(21/25) \approx -0.1744$ $S = 25\,(1 - e^{-0.1744t})$ b) How many units will saturate this market? 25,000 units c) How many units will have been sold after 5 years? When t = 5, S ≈ 14.545 which is 14,545 units **Task 6:** Solve the first – order linear differential equation. a) $y' - y = 8$ b) $e^x y' + 4e^x y = 1$ c) $dy = (y \tan x + 2e^x)dx$

	to a base other than e. When this occurs, there are two options: (1) convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or (2) integrate directly, using the integration formula $\int a^x\, dx = (1/\ln a)a^x + C$ ● Let u be a differentiable function of x, and let a > 0. $\int du/ (\sqrt{(a^2 – u^2)})$ $= \arcsin (u/a) + C$ $\int du/ (a^2 + u^2)$ $= (1/a) \arctan(u/a) + C$ $\int du/ (u\sqrt{(u^2 - a^2)})$ $= (1/a)\text{arcsec} (\lvert u\rvert/a) + C$ A function y = f(x) is called a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced by f(x) and its derivatives. ● Differentiation and substitution would show that $y = e^{-2x}$ is a solution of the differential equation $y' + 2y = 0$. It can be shown that every solution of this differential equation is of the form $y = Ce^{-2x}$ where C is any real number. This solution is called the general solution.	**Answer:** Solve the first – order linear differential equation. a) $y' – y = 8$ $P(x) = -1, Q(x) = 8$ $u(x) = e^{\int -dx} = e^{-x}$ $y = (1/e^{-x})\int 8e^{-x}dx$ $y = e^x[-8e^{-x} + C]$ $y = -8 + Ce^x$ b) $e^xy' + 4e^xy = 1$ $y' + 4y = e^{–x}$ $P(x) = 4, Q(x) = e^{–x}$ $U(x) = e^{\int 4dx} = e^{4x}$ $Y = (1/e^{4x}) \int e^{-x} e^{4x}\, dx$ $Y = e^{-4x} [1/3\, e^{3x} + C]$ $Y = 1/3\, e^{-x} + Ce^{-4x}$ c) $dy = (y \tan x + 2e^x)dx$ $dy/dx – (\tan x)y = 2e^x$ Integrating factor: $e^{-\int \tan x\, dx} = e^{\ln\lvert\cos x\rvert} = \cos x$ $Y\cos x = \int 2e^x \cos x\, dx = e^x (\cos x + \sin x) + C$ $Y = e^x (1 + \tan x) + C \sec x$ **Modifications and/or Accommodations:** ● **Special Education:** Utilize a multi-sensory (VAKT) approach during instruction, provide alternate presentations of skills by varying the method (repetition, simple explanations, additional examples, modeling, etc.), modify test content and/or format, allow students

	• Some differential equations have singular solutions that cannot be written as special cases of the general solution. • The order of a differential equation is determined by the highest-order derivative in the equation. • Geometrically, the general solution of a first-order differential equation represents a family of curves known as solution curves, one for each value assigned to the arbitrary constant. • Particular solutions of a differential equation are obtained from initial conditions that give the value of the dependent variable or one of its derivatives for a particular value of the independent variable. • A slope field, or a direction field, shows the general shape of all the solutions. • Euler's Method is a numerical approach to approximating the particular solution of the differential equation y'=F(x,y) that passes through the point (x_0,y_0).	to retake test for additional credit, provide additional times and preferential seating as needed, review, restate and repeat directions, provide study guides, and/or break assignments into segments of shorter tasks. • **English Language Learners:** Extend time requirements, preferred seating, positive reinforcement, check often for understanding/review, oral/visual directions/prompts when necessary, supplemental materials including use of online bilingual dictionary, and modified assessment and/or rubric. • **Students at Risk of School Failure:** Deliver instruction utilizing varied learning styles including audio, visual, and tactile/kinesthetic, provide individual instruction as needed, modify assessments and/or rubrics, repeat instructions as needed. • **Gifted Students:** Create an enhanced set of introductory activities, integrate active teaching/learning opportunities, incorporate authentic components, propose interest-based extension activities, and connect student to related talent development opportunities **Spot Light On:** *Show students the why behind how things are done when possible.* **Teacher Resources** online achievethecore resource online learnzillion resource

	• Consider a differential equation that can be written in the form M(x) + N(y)dy/dx = 0 where M is a continuous function of x alone and N is a continuous function of y alone. All x terms can be collected with dx and all y terms with dy, and a solution can be obtained by integration. Such equations are said to be separable, and the solution procedure is called separation of variables. • If y is a differentiable function of t such that y >0 and y'=ky, for some constant k, then $y = Ce^{kt}$. C is the initial value of y, and k is the proportionality constant. Exponential growth occurs when k>0 ,and exponential decay occurs when k<0. • A homogenous differential equation is an equation of the form M(x,y)dx + N(x,y)dy = 0 Where M and N are homogeneous functions of the same degree. • If M(x,y)dx + N(x,y)dy = 0 is homogeneous, then it can be	online khanacademy resource online desmos resource online ixl resource

	transformed into a differential equation whose variables are separable by the substitution y = vx where v is a differentiable function of x. ● A common problem in electrostatics, thermodynamics, and hydrodynamics involves finding a family of curves, each of which is orthogonal to all members of a given family of curves. ● Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the carrying capacity, which is the maximum population y(t) that can be sustained or supported as time t increases. ● A model that is often used for this type of growth is the logistic differential equation $dy/dt = k(1 - (y/L))$ ● A first-order linear differential equation is an equation of the form $dy/dx + P(x)y = Q(x)$	

	where P and Q are continuous functions of x. This is first-order linear differential equation is said to be in standard form. ● An integrating factor for the first-order linear differential equation $y' + P(x)y = Q(x)$ is $u(x) = e^{\int P(x)\,dx}$ $= \int Q(x)e^{\int P(x)dx}\,dx + C$ ● A well known nonlinear equation that reduces to a linear one with an appropriate substitution is the Benoulli equation, named after James Bernoulli. $y' + P(x)y = Q(x)y^n$	

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Content Statements	**Cumulative Progress Indicators**	
Students will know… ● How to use Slope Fields and Euler's Method ● How to use Differential Equations: Growth and Decay ● How to use Separation of Variables and the Logistic Equation ● How to use First-Order Linear Differential Equations ● How to apply the logarithmic properties ● How to evaluate natural logarithmic expressions ● How to use the number *e*. ● How to determine the derivative of the natural logarithmic function ● How to determine the derivative involving absolute value ● How to apply the log rule for integration ● How to find the integrals of trigonometric functions ● How to define and apply inverse functions, if it exists. ● How to find an inverse function	● Tests ● Quizzes ● Practice problems for homework ● Workbook pages ● Worksheets	

• How to find the derivative of an inverse function • How to solve exponential equations and logarithmic equations. • How to find derivatives of exponential functions • How to find the integrals of exponential functions • How to find the derivative and integral for bases other than *e*. • How to apply the Power Rule for real exponents. • How to use exponential functions using real world applications • How to evaluate the inverse trigonometric function • How to solve an equation using properties of inverse trigonometric functions • How to determine the derivatives of inverse trigonometric functions and apply it with real life applications. • How to summarize the basic differentiation rules for elementary functions. • How to integrate involving inverse trigonometric functions. • How to summarize basic integration rules. •		

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Desired Results	
Slope fields **Euler's Method** **Antidifferentiation by substitution** **Antidifferentiation by parts** **Exponential growth and decay** **Logistic growth**	
Standards for Mathematical Practices	

1. **Make sense of problems and persevere in solving them.**
2. **Reason abstractly and quantitatively.**
3. **Construct viable arguments and critique the reasoning of others**
4. **Model with mathematics**
5. **Use appropriate tools strategically.**
6. **Attend to precision.**
7. **Look for and make use of structure.**
8. **Look for and express regularity in repeated reasoning.**

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LGBT and Disabilities Law: *N.J.S.A. 18A:35-4.35*

Neil Devine - American Stellar and planetary astrophysicist whose work centered on the understanding of star formation.

The mission is to ensure that every student is able to see themselves in our rich and diverse history.

Social and Emotional Learning: *Competencies*	**Social and Emotional Learning: *Sub-Competencies***
Self-Awareness Social Awareness Self-Management Relationship Skills Responsible Decision-Making	• Recognizing the importance of self-confidence in handling daily tasks and challenges. • Demonstrate an awareness of the expectations for social interactions in a variety of ways. • Demonstrate an understanding of the need for mutual respect when viewpoints differ. • Recognize the skills needed to establish and achieve personal and educational goals. • Utilize positive communication and social skills to interact effectively with others. • Develop, implement, and model effective problem solving and critical thinking skills.

New Jersey Legislative Statutes and Administrative Code (place an "X" before each law/statute if/when present within the curriculum map)									
	Amistad Law: *N.J.S.A. 18A 52:16A-88*		Holocaust Law: *N.J.S.A. 18A:35-28*	**x**	LGBT and Disabilities Law: *N.J.S.A. 18A:35-4.35*	**x**	Diversity & Inclusion: *N.J.S.A. 18A:35-4.36a*		Standards in Action: *Climate Change*