# AP Physics C Summer Assignments

**Teacher:**

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**Welcome:**

Welcome to AP Physics C! I am excited that you have decided to take this course. It is one of my favorite courses to teach! It is a challenging course, but I will help you through it, and our AP scores are always outstanding. Because we have so much to cover and because we will need some advanced Calculus by the second week, I would like you , if possible, to complete the assignments listed below before our first class session. If you cannot, do not worry, I will let you catch up over the first few days of class.

**Assignments:**

1. Please complete the worksheets on Derivative, Integration, and Unit Vectors that follow this page. Hopefully they will be self explanatory, but please ask any questions that you would like via email.

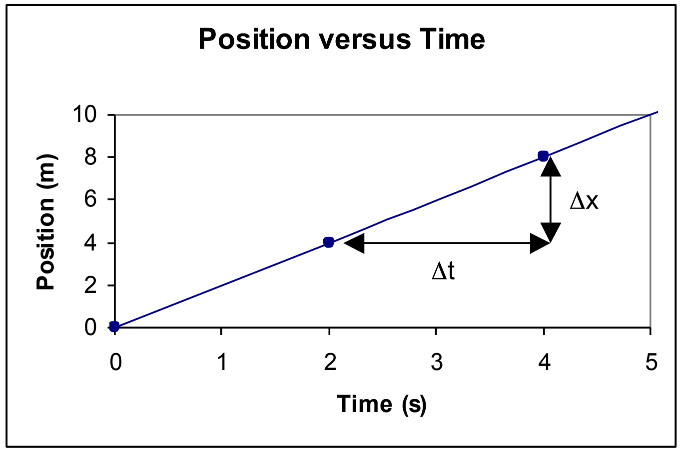
2. Once you've completed these worksheets, please complete the practice problems (link)

3. Please read Chapter 21 on Electric charge (link) and complete the following homework problems.
   p.575 Problems 11,13
   p.576 Problems 21,24,29
   p.579 Problem 66

4. Please read Chapter 22 on Electric Fields (link) and complete the following homework problems.
   p.596 Questions 1,2,9,10,11
   p.597 Questions 5,7,8
   p.598 Problems 1,2,8,11
   p.599 Problems 19,24,25
   p.600 Problem 26, 30, 38
   p.601 Problem 41

Thanks again and I look forward to meeting you on the first day of class!

# Introduction to Derivatives




### A Change in Notation:

You will remember that the slope of a ***position versus time*** graph (like the one shown to the right) will give you the velocity of an object. The slope of a graph is defined as the ***rise*** over ***run***, or the change in the ***vertical variable*** divided by the change in the ***horizontal variable***. In this particular graph, the vertical variable is the position and the horizontal variable is the time. Thus the slope of this graph can be defined as:

$$Slope = \frac{\Delta x}{\Delta t} = \frac{x_f - x_o}{t_f - t_o} = \frac{8m - 4m}{4s - 2s} = \frac{4m}{2s} = 2\,{}^{m}\!/\!{}_{s}$$

Thus the formula for the motion of this object could be described as: **x = 2t** [leaving out units (m/s) for clarity]

A ***derivative*** uses a slightly different notation, but with similar results. Instead of using the capital Greek letter delta (Δ), the derivative uses the lowercase Greek letter delta (δ) which is often written more simply as the letter *d*.

Thus the slope of this graph would be written as:

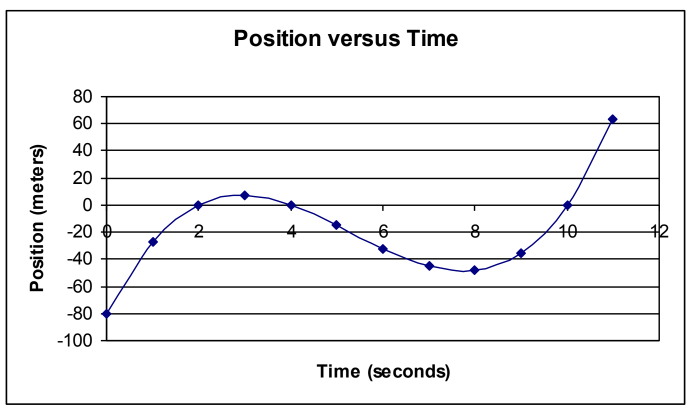
$$Slope = \frac{dx}{dt} = 2\,{}^{m}\!/\!{}_{s}$$

But this new form of the slope equation does not express the slope ***between*** two points; it actually expresses the slope at a ***single*** point. It does this by letting the change in time (Δt) approach zero – thus the change in notation to the lowercase delta. You could also find the slope of the curve at a single point manually by drawing a tangent to the curve, but as you see, derivatives offer a more elegant solution to this problem

Mathematically, the derivative of the graph above can be expressed as follows:

$$\frac{dx}{dt} = \frac{d(2t)}{dt} = 2$$

Which can expressed as the *change in the function* **2t** with respect to the *change in time* is 2. (Again leaving out the unit, m/s, for clarity)



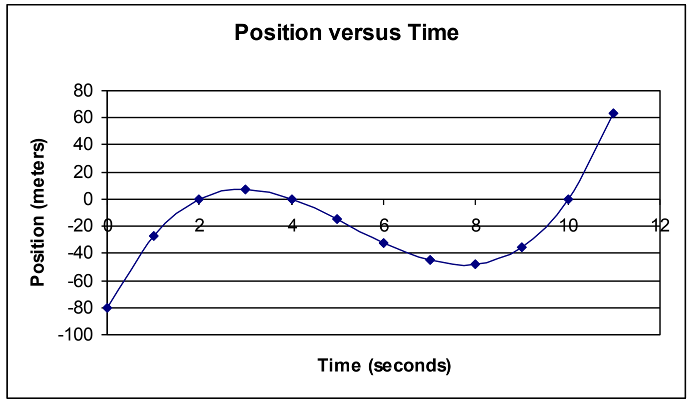

For the graph shown above, this change is not too interesting because the slope is the same (2 m/s) at ***every*** point, but it is especially useful in cases where the slope is ***not constant*** as in this next case:

Here the motion of the object is more complex, stopping and reversing direction for a portion of its journey. You can see that it stops briefly near 3 seconds and 8 seconds, but it would be nice to find the velocity at each point without resorting to drawing tangent lines at each point on the graph to find the velocity at that instant.

***Derivatives*** offer a solution to this problem as long as the function that describes the curve is ***known***.

The function that describes the position of this object is:

**$x = t^3 – 16t^2 + 68t - 80$**




To find the velocity (slope) of the graph at each point, we need to take the derivative of this function. The notation for this appears as follows:

$$\frac{dx}{dt} = \frac{d(\mathrm{t}^3 \text{ - } 16\mathrm{t}^2 \; + \; 68\mathrm{t} \text{ - } 80)}{dt}$$

This notation means that we are looking for the change in the function **$t^3 – 16t^2 + 68t - 80$** with respect to the change in time $\delta t$. The result of taking this derivative is:

$$\frac{d(\mathrm{t}^3 \text{ - } 16\mathrm{t}^2 \; + \; 68\mathrm{t} \text{ - } 80)}{dt} = 3t^2 - 32t + 68$$

This means that the slope (or velocity) of this graph can be found by plugging the time into the differentiated function **$3t^2 - 32t + 68$**. For example, we can find the slope at t = 8 seconds by substituting it into our new function:

Slope = **3(**8**)$^2$ – 32(**8**) + 68**
= 192 – 256 + 68
= 4

So at the instant that t = 8 seconds, the slope of the graph is 4 m/s. You could also use the same function to find times when the object was stopped (the slope = 0), or determine when the object has a positive or negative velocity. The question remains however, ***how do you find a derivative?***

## Calculating a derivative:

There is a simple rule to calculate the derivative of most functions. It simply requires multiplying the exponent of each term by the original coefficient and then reducing the exponent of each term by one. For example:

Taking the derivative of **$t^3$** with respect to the variable **t** becomes:

$$\frac{d(\mathrm{t}^3)}{dt} = (1*3)t^{3-1} = 3t^2$$

Or taking the derivative of **$-16t^2$** with respect to the variable **t** becomes:

$$\frac{d(16\mathrm{t}^2)}{dt} = (-16*2)t^{2-1} = -32t^1 = -32t$$

Or taking the derivative of **68t** with respect to the variable **t** becomes:

$$\frac{d(68\mathrm{t})}{dt} = (68*1)t^{1-1} = 68t^0 = 68$$

Finally, the ***derivative of a constant is always defined as zero***. This may seem strange at first, but remember that the derivative tells us the slope of a function; but when a function is constant (-80), the slope is zero. So taking the derivative of **-80** with respect to the variable **t** becomes:

$$\frac{d(\text{-}80)}{dt} = 0$$

So in our example, we found the derivative of our complex function (**$t^3$ – 16$t^2$ + 68t - 80**), by taking the derivative of each of its components:

$$\frac{d(\mathrm{t}^3 - 16\mathrm{t}^2 + 68\mathrm{t} - 80)}{dt} = \frac{d(\mathrm{t}^3)}{dt} + \frac{d(-16\mathrm{t}^2)}{dt} + \frac{d(68\mathrm{t})}{dt} + \frac{d(-80)}{dt} = 3t^2 - 32t + 68 - 0$$

This process will work with most functions. Try it with the following equations:

| | |
|---|---|
| $\frac{d(2t^2)}{dt} =$ | $\frac{d(2\pi t^2)}{dt} =$ |
| $\frac{d(2t^3)}{dt} =$ | $\frac{d(4t^{-1})}{dt} =$ |
| $\frac{d(3t^5 + 3t^2)}{dt} =$ | $\frac{d(2xt^2)}{dt} =$ |

***A note about the notation** dt*: The last problem you did included the variable **x**, but we are looking for changes in the function as **t** changes. Unless we know of another mathematical equation that describes how **x** changes as **t** changes, we assume that **x** is a ***constant*** in the equation just as $\pi$ was a constant in the fourth equation listed, and treat it as simply part of the coefficient.

## Multiple Order Derivatives:

Sometimes it is useful to take a **derivative of a derivative**. We know from the opening example on this worksheet that taking the derivative of **x** with respect to **t** (dx/dt) gives us the slope of the *position versus time* graph, which in Physics is defined as the velocity. You might remember that the slope of the *velocity versus time* graph is the acceleration; so if we take the derivative of **v** with respect to **t** (dv/dt), we can determine an object's acceleration at any time. Using our opening example:

$$x = \mathrm{t}^3 - 16\mathrm{t}^2 + 68\mathrm{t} - 80$$

$$v = \frac{d\mathrm{x}}{dt} = \frac{d(\mathrm{t}^3 - 16\mathrm{t}^2 + 68\mathrm{t} - 80)}{dt} = 3t^2 - 32t + 68 - 0$$

$$a = \frac{d\mathrm{v}}{dt} = \frac{d(3t^2 - 32t + 68)}{dt} = 6t - 32 + 0$$

The last derivative can also be written as the **second derivative** of **x** with respect to **t**. This notation appears as follows:

$$a = \frac{d^2\mathrm{x}}{dt^2} = \frac{d^2(\mathrm{t}^3 - 16\mathrm{t}^2 + 68\mathrm{t} - 80)}{dt^2} = 6t - 32 + 0$$

This notation means that you wish to take the derivative **twice** with respect to **t**. Notice that the result is the same as before. For the function **x = 2$t^4$ + 4$t^3$ + 12t – 50,** determine the following:

| | |
|---|---|
| **v =** | **a =** |

What is the object's position, velocity and acceleration at t = 3 seconds? x = v = a =
Is the object's acceleration increasing or decreasing at t = 3 seconds?

## Special Derivatives

There are four common derivatives used in Physics that do not fit the rule given above. They are the derivatives of sine, cosine and the natural logarithms. You will need to memorize these until your Calculus class explains in depth how their derivatives are taken. Their derivatives appear below:

$$\frac{d[\sin(t)]}{dt} = \cos(t) \qquad \frac{d[\cos(t)]}{dt} = -\sin(t) \qquad \frac{d[\ln(t)]}{dt} = \frac{1}{t} \qquad \frac{d[e^t]}{dt} = e^t$$

## The Product Rule:

If a function consists of two functions that are multiplied by each other, there is a simple rule to follow to find the resultant derivative:

$$\frac{d(function1 * function2)}{dt} = function1 * \frac{d(function2)}{dt} + function2 * \frac{d(function1)}{dt}$$

For example, consider the following complex function 5sin(t) that is the product of 5 and sin(t). It can be solved using the product rule as shown:

$$\frac{d[5\sin(t)]}{dt} = \frac{d[5 * \sin(t)]}{dt} = 5 * \frac{d[\sin(t)]}{dt} + \cos(t) * \frac{d[5]}{dt} = 5 * \cos(t) + \cos(t) * 0 = 5\cos(t)$$

$$\frac{d[5t * \sin(t)]}{dt} = \qquad \frac{d[2\pi f * \sin(t)]}{dt} =$$

$$\frac{d[\sin^2(t)]}{dt} = \frac{d[\sin(t) * \sin(t)]}{dt} = \qquad \frac{d(2t * \ln(t))}{dt} =$$

## The Quotient Rule:

Although it is not used commonly in Physics, there is a similar rule to use when you have a quotient of two functions:

$$\frac{d\left(\dfrac{function1}{function2}\right)}{dt} = \frac{function2 * \dfrac{d(function1)}{dt} - function1 * \dfrac{d(function2)}{dt}}{(function2)^2}$$

For example, consider the derivative of the following fraction. It can be solved using the quotient rule as shown:

$$\frac{d\left(\dfrac{t^2 - 1}{t^2 + 1}\right)}{dt} = \frac{(t^2 + 1) * 2t - (t^2 - 1) * 2t}{(t^2 + 1)^2} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2 + 1)^2} = \frac{4t}{(t^2 + 1)^2}$$

$$\frac{d[(t^2 - 1)/(t + 1)]}{dt} = \qquad \frac{d[2\pi ft / \sin(t)]}{dt} =$$

$$\frac{d[\tan(t)]}{dt} = \frac{d\left[\sin(t)/\cos(t)\right]}{dt} = \qquad \frac{d(2t / \ln(t))}{dt} =$$

## The Chain Rule:

You know how to take the derivative of $3t^2$ and sin(t), but how do you take the derivative of a composite function like $\sin(3t^2)$? The answer is the chain rule, probably the most commonly utilized differentiation rule used in Calculus and Physics.

The key part of this rule is to break the composite function back into two separate functions and then differentiate them together. The first step of this process is to choose the inner function and to define it temporary as **u** (u is the letter used most commonly in Calculus for this task). So for our example function, $\sin(3t^2)$, we would define the inner function, $3t^2$, as u. Now the original composite function becomes **sin(u),** where **u=3t²**. Now we can take the derivative of the new function **sin(u)** with respect to **u** and the inner function, **3t²**, with respect to **t** and multiply the result to find our answer.

The reason that this works is algebra. Instead of taking the original composite function's derivative with respect to t, we do it in two parts:

$$\frac{dx}{dt} = \frac{dx}{du} * \frac{du}{dt}$$

Notice that the product of the two derivatives on the right is the same (after you cancel the δu terms) as the original derivative.

Again remember that we set **u=3t²**, so the derivative of our composite function, $\sin(3t^2)$, becomes:

$$\frac{dx}{dt} = \frac{d[\sin(\mathrm{u})]}{du} * \frac{d(3t^2)}{dt}$$

The derivatives of each part are:

$$\frac{d[\sin(\mathrm{u})]}{du} = \cos(u)$$

$$\frac{d(3t^2)}{dt} = 6t$$

So the result is **6t*cos(u)**, or **6t*cos(3t²)** after placing our **u** back into the equation.

Here is another example: what is the derivative of $\sin^4(t)$? This may not appear as a composite function, but we can set **u=sin(t)** so that the original function becomes $u^4$. The result is as follows:

$$\frac{dx}{dt} = \frac{d(\mathrm{u}^4)}{du} * \frac{d[\sin(t)]}{dt} = 4u^3 * \cos(t) = 4[\sin(t)]^3 * \cos(t) = 4\sin^3(t)\cos(t)$$

| | |
|---|---|
| $\frac{d(2t+1)^5}{dt} =$ | $\frac{d[\sin^3(t)]}{dt} =$ |
| $\frac{d(e^{2t})}{dt} =$ | $\frac{d(4e^{-2t})}{dt} =$ |
| $\frac{d[\sin(2\pi ft)]}{dt} =$ | $\frac{d[2\pi f\sin(2\pi ft)]}{dt} =$ |

# An Introduction to Integration

**A Change in Direction:**

What if you knew a derivative, such as:

$$\frac{dx}{dt} = 3t^2 - 32t + 68$$

but you were seeking the original function of position, **x**, that produced the derivative. How could you find it? The answer would be to take the "anti-derivative" of the function shown.

After your experiences with derivatives you would probably quickly determine that the function that produces the derivative shown above would be **$x = t^3 - 16t^2 + 68t$**, but is this the ***only*** solution? Remember that the derivative of any constant is zero so the original function could have been **$x = t^3 - 16t^2 + 68t + 5$** or **$x = t^3 - 16t^2 + 68t - 24$** or even **$x = t^3 - 16t^2 + 68t + 4\pi$**. ***Any*** of these functions for **x** would produce the derivative shown above. In general, the "anti-derivative" of **$v = 3t^2 - 32t + 68$** can be expressed as **$x = t^3 - 16t^2 + 68t +$ *constant*.**

Find the "anti-derivatives" of the following functions adding constants as necessary:

| | |
|---|---|
| $\frac{dx}{dt} = 15t^4$ | $\frac{dx}{dt} = t^5$ |
| $\frac{dx}{dt} = -\sin(t)$ | $\frac{dx}{dt} = \frac{1}{t}$ |
| $\frac{dx}{dt} = -15t^{-4}$ | $\frac{dx}{dt} = 3\cos(3t)$ |

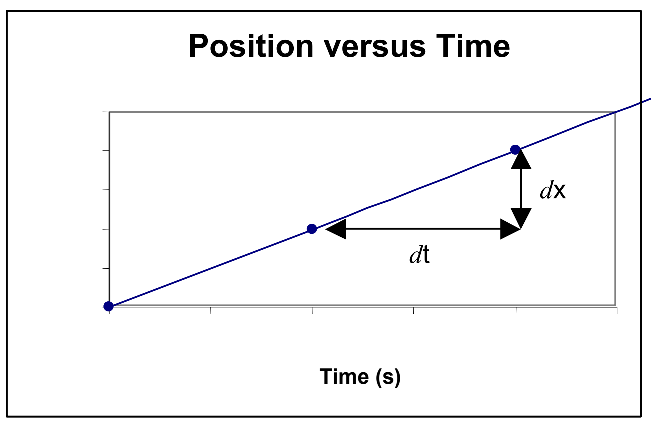
The process of taking the "anti-derivative" is called integration. It is convenient to have such a process to work "backwards" through the differentiation process, but it is also much more than this. It can be used to ***sum*** up a series of small changes to find the entire change in a variable.

To explain how this is possible will require us to take a closer look as how the integral operates. Remember that:

$$\frac{dx}{dt} = v$$

And like any other algebraic expression, it can be manipulated. For example, we could multiply both sides of the equation by $\delta t$:

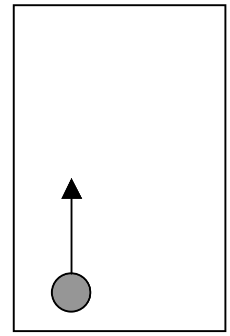
$$dx = v * dt$$




This means that the change in position ($d$x), is equal to the velocity (or slope) time the change in time ($d$t). **NOTE:** ***This will only give us the change in position, not the actual position***. To determine the actual position of the object, we would need to know its starting position at the beginning of the time period $d$t. In other words, we would need to add the ***appropriate constant*** to the equation to find the original formula. (Starting to sound familiar?)

## Integration and Motion:

Let us examine several cases where integration is useful in the study of motion. First, let's examine the case of a ball that is thrown upward. You already know that the acceleration near the Earth's surface is –9.81 m/s$^2$ (*if we define up as positive and down as negative*). So we can write:



$$a = \frac{dv}{dt} = -9.81$$

VELOCITY: We know that the derivative of ***v with respect to t*** gives us the acceleration, **a**. So if we take the anti-derivative (or integral) of the acceleration equation we should be able to determine an equation that will give us the ball's velocity at any instant. Without too much difficulty you should recognize that the integral of **9.81** with respect to **t** is:

$$v = -9.81t + constant$$

What is the constant in this case? In the first part of this worksheet we just left them as constants, but this is a formula that describes a real physical occurrence, the constant should have real meaning. You will notice that at t=0 seconds, **v** is equal to the ***constant***. We usual define this as ***initial velocity*** and represent it with the term $\mathbf{v_o}$. So we can rewrite our equation as:

$$v = -9.81t + v_o = v_o - 9.81t$$

If we happen to know that the initial velocity is +5 m/s, we could write our equation as:

$$v = 5 - 9.81t$$

POSITION: We also know that the derivative of ***x with respect to t*** gives us the acceleration, **v**. So if we take the integral of the velocity equation we should be able to determine an equation that will give us the ball's position at any instant. Without too much difficulty you should recognize that the integral of **v=5-9.81t** with respect to **t** is:

$$x = 5t - 4.91t^2 + constant$$

What is the constant in this case? You will notice that at t=0 seconds, **x** is equal to the ***constant***. We usual define this as ***initial position*** and represent it with the term $\mathbf{x_o}$. So we can rewrite our equation as:

$$x = 5t - 4.91t^2 + x_o = x_o + 5t - 4.91t^2$$

If we happen to know that the initial position is +1 m, we could write our equation as:

$$x = 1 + 5t - 4.91t^2$$

Try the following motion problems:

1. Given that the velocity of an object is given by v=4t-3t$^2$ and that at t=0 its position is –3 meters, write an equation for the object's position at any time.

2. Given that the acceleration of an object is given by a=12t find (a) the equation for its velocity at any time, (b) the equation for its position at any time, (c) given that at t=1 seconds x=-1 meters and at t=2 seconds x=15 meters find the constants ($v_o$ and $x_o$) for the position equation.

## The Integral Notation:

We saw on the first page of this worksheet that the derivative of position versus time to determine velocity can be rewritten as follows:

$$dx = v * dt$$

This equation will allow us to find the small change in position over some very small time interval *d*t. But what if we would like to determine the change in position across some larger time interval? To do this we would need to add up each very small change in position (*d*x) to find the total change in position. To do this we use a special notation. Previously in math courses you may have used the SIGMA notion (Σ) to note that you were adding up a sequence of factors, in this case we are going to use a special sigma notation (∫) to indicate that we are adding up a large (infinite) number of small changes.

What do we achieve by adding up these terms? Above we could calculate *d***x**, but the new notation (and more importantly, the <u>integration operation</u> that goes with it) allows us to find the ***entire*** change in position, **x**.

$$dx = v * dt$$

Adding the integral sign changes the result of the expression **v * δt** from **δx** to **x**:

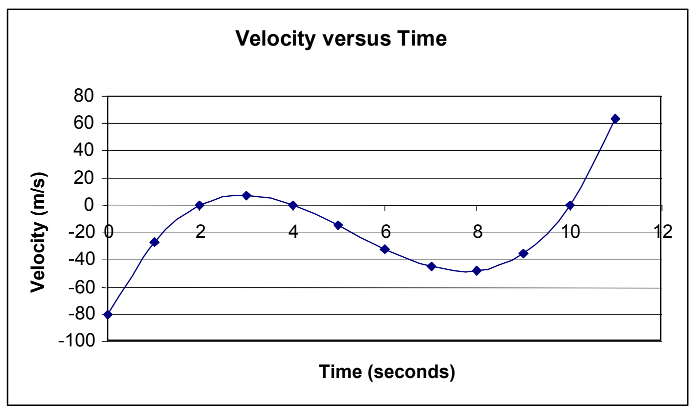
$$x = \int v * dt$$

| | |
|---|---|
| $\int a * dt =$ | $\int dt =$ |

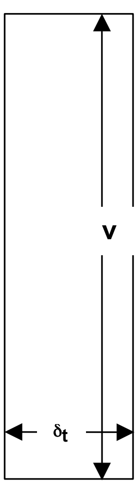
[The second integral may seem a little strange, but it is the sum of all δt's which by definition must be t. Another way to think about this is that it is the integral (or anti-derivative) of 1 with respect to t which again would have to be t.]

## Finding the Area under a Curve:

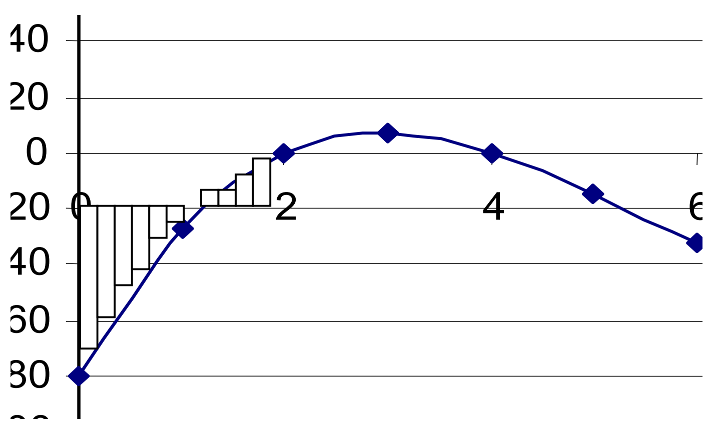
To the right is the graph of an object's velocity versus time where the velocity is given by the function **v = t$^3$ – 16t$^2$ + 68t – 80**.




What if we wanted to find the object's displacement at any time? We know that we can take the integral of this equation with respect to time and come up with an equation that will express its position. But why does this work?




You might remember from last year that the area under a velocity versus time graph would give you the displacement. But how do you find the area under a curve? One way of doing this would involve placing a large number of rectangles under the curve to approximate the entire area. The area of each of these rectangles would be the height (the velocity of the graph at that point) times the width (the change in time) as shown to the left. If we left the change in time get very small, the sum of the areas of the rectangles would approach the true area under the curve. That is exactly was the expression $x = \int v * dt$ does. It is "adding up" each rectangle whose individual area is v * *d*t.



In the same way,

$$v = \int a * dt$$

determines the velocity by finding the area under the *acceleration versus time* graph. The mathematical trick for this process you already know; it is the "anti-derivative" or integral.

## Common Integrations in Physics:

There are five common integral types that are used frequently in the study of Physics. They are all based on the Chain Rule, but using it from the reverse direction. Here are the four common integrations and their solutions:

$$\int ct^n dt = c\frac{t^{n+1}}{n+1} + \text{constant}$$

$$\int \cos(ct)dt = \frac{1}{c}\sin(ct) + \text{constant}$$

$$\int e^{ct} dt = \frac{e^{ct}}{c} + \text{constant}$$

$$\int \sin(ct)dt = -\frac{1}{c}\cos(ct) + \text{constant}$$

$$\int \frac{c}{t}dt = c * \ln(t) + \text{constant}$$

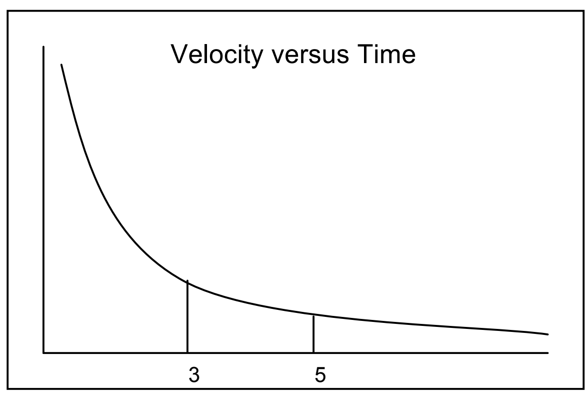
Find the integrals of the following functions adding constants as necessary:

| | |
|---|---|
| $\int 3e^{2t} dt =$ | $\int t^{1/2} dt =$ |
| $\int 2\sin(4t)dt =$ | $\int 1 - e^{-t} dt =$ |
| $\int t^2 dt =$ | $\int \frac{2}{t}dt =$ |

[HINT: Think about the last integral as 2 * 1/t]

## Definite Integrals:

Sometimes it is important to find the area under a curve between two distinct points. For example, the graph to the right shows an object that has a velocity given by v = 1/t. What if you wanted to know the object's displacement between t = 3 seconds and t = 5 seconds. How can you find only a section of the graph?




The answer lies in the definite integral. If we first solve the integral and find the ***area under the curve out to t = 5 seconds*** and then **subtract** the ***area under the curve out to t = 3 seconds***, the difference should equal the portion of the graph that we are interested in.

This is the basis for the *second fundamental theorem* of Calculus which states **the area under a function f'() between two points a and b is equal to f(b) – f(a), where f is the integral of f'().** Mathematically a definite integral is expressed as:

$$\int_a^b f'(t)dt = f(b) - f(a)$$

So in our example above the definite integral we are solving would appear as:

$$\int_{3}^{5} \frac{1}{t} dt =$$

We know that the integral of **v = 1/t** is **x = ln(t) + constant**, where the constant would be the initial position of the object. So we could write the result of our integration as **x = ln(t) +** $\mathbf{x_o}$**.**

So the answer would be:
**[ln(5) +** $\mathbf{x_o}$] – [**ln(3) +** $\mathbf{x_o}$] **= ln(5) – ln(3) = .511 meters**

Notice that our constant, which appears in both terms, "cancels out." This will always be true for all definite integrals; your answer will not need the addition of any constants.

The full notation for this example would be:

$$\int_{3}^{5} \frac{1}{t} dt = [\ln(t) + \text{constant}]\Big|_{3}^{5} = [\ln(5) + \text{constant}]\text{-}[\ln(3) + \text{constant}] = \ln(5)\text{-}\ln(3) = .511\text{m}$$

The special vertical line after the result of the integral indicates that we still need to take the difference in the values of the result:

$$[\ln(t) + \text{constant}]\Big|_{3}^{5}$$

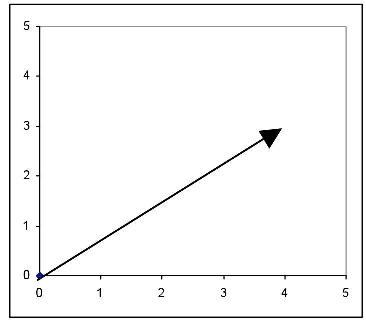
Find the definite integrals of the following functions:

| | |
|---|---|
| $\int_{0}^{\pi} \sin(t/2) dt =$ | $\int_{0}^{1} (1 + t + t^2) dt =$ |
| $\int_{-\pi}^{\pi} \sin(t) dt =$ | $\int_{-\pi}^{\pi} [\cos(t) + \frac{1}{2}] dt =$ |
| $\int_{0}^{1} 3e^{-3t} dt =$ | $\int_{0}^{\pi} [t^2 + \sin(t)] dt =$ |

Finally solve this motion problem:

As a racecar starts along a course, its acceleration is given by a(t) = $0.20t^3$. If the car starts from rest, how fast is it traveling after 5 seconds?

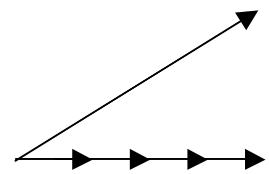
# Introduction to Unit Vectors



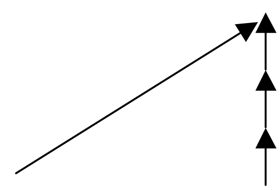

## The Unit Vectors *i* and *j*:

There are many ways we could draw a line pointing upwards and to the right. What can we say about this particular one that isn't true of any other? Think about the point at the top end of the line. We know that each point has a pair of coordinates (either Cartesian or polar) which specify its position uniquely. If we use Cartesians, we can say that the point at the end of the line has coordinates (4,3). So to get to that point, starting from the origin, we go **4 units** in **the x-direction** and **3 units** in the **y-direction**. Do you see what we just did in that last sentence? We added two directions!

It seems that the x-direction and y-direction are "special" directions, because we can describe **any** vector (that is, any line pointing out from the origin) by adding some number in the x-direction to some number in the y-direction.

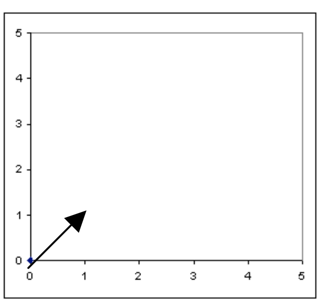


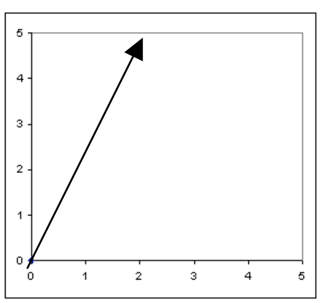
Above we said "we go 4 units in the x-direction...."; we could also say this as: go 1 unit in the x-direction, then another unit in the x-direction, then another and then another. So we have gone 1 unit in the x-direction four times and we end up at the same place as if we'd just gone 4 units. The vector which is ***1 unit in the x-direction*** is conventionally called **i.** What we have just found out is that the vector which is 4 units in the x-direction is 4**i**, since we could get there by moving 1 unit four times.

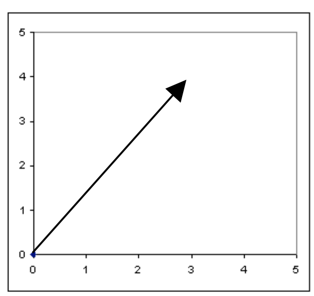


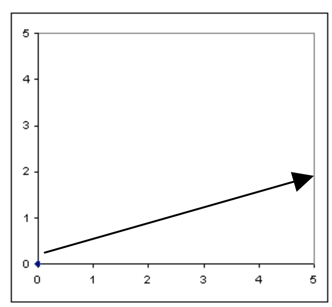
Now we're nearly there. The only other bit of information we need is that the vector which is ***1 unit in the y-direction*** is called **j**. So the vector "3 units in the y-direction" is 3**j.** The vectors **i** and **j** are called the "unit vectors". Now we can write down a symbolic version for our vector on the picture above. It's 4**i**+3**j**. This tells us exactly which line it is: there's no other line that leads from the origin to the point (4,3).

Take a look at the following vectors and write them in the appropriate **unit vector** notation:

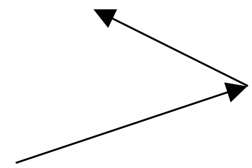




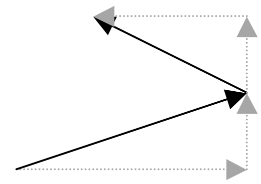




## So what is so useful about Unit Vectors?



Remember the method we used to use to add two vectors? First we broke each vector into its vertical and horizontal components, added each of the components in turn, found the resultant vertical and horizontal components, and finally added them back together to find the resulting vector.



Wouldn't it be easier to express each vector with its vertical and horizontal components to begin with? That is exactly what unit vector notion allows you to do!

The addition (and subtraction) of vectors becomes extremely easy if all of the vectors are expressed in unit vector notion; no trigonometry is required! To find the final vector all you need to do is to add all of the horizontal components together (all of the **i** coefficients) and then add all of the vertical components together (all of the **j** coefficients).

For example, if we wished to find the sum of two vectors **a** and **b**, where **a** = 5**i**+2**j** and **b** = -3**i**+2**j**, the resultant vector would be:

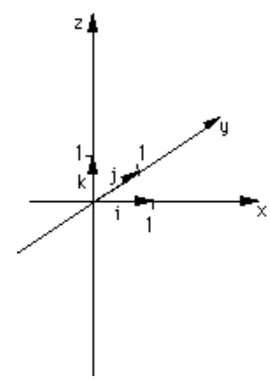
**a + b** = (5 + -3)**i** + (2 + 2)**j**
**a + b** = 2**i**+4**j**

Please find the sum (or resultant) of the following vectors, expressing the answer in unit vector notation:

| | |
|---|---|
| 1**i**+2**j** **+** 3**i**+4**j** **=** | 5**i**-2**j** **+** -5**i**+2**j** **=** |
| 5π**i**+2π**j** **+** -3π**i**+2π**j** **=** | 5**i**+2**j** **-** -3**i**+2**j** **=** |

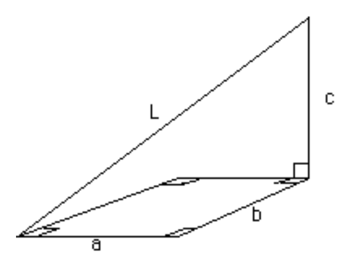
## Three-dimensional Vectors:

We can easily extend the ideas covered so far in two dimensions to vectors in three dimensions. When discussing vectors in component (**I j**) form, we need to introduce the third unit vector, called, unsurprisingly, **k**. Conventionally the three unit vectors are drawn as shown to the right:




Then any vector in three-dimensional space can be specified uniquely by giving its components in the three directions. So the vector from the origin to the point given in Cartesian coordinates by (2,3,4) is the vector 2**i**+3**j**+4**k**. We can add vectors in three dimensions in the same way as we did in two dimensions. In component form we can add the **i**, **j** and **k** components together, again keeping track of each direction separately. So the vector 2**i**+3**j**+4**k** added to the vector 1**i**-1**j**-1**k**, gives the resultant vector 3**i**+2**j**+3**k**.

To find the magnitude of a three-dimensional vector we need to use the three-dimensional version of Pythagoras' theorem: the square of the length of the longest side in the diagram below is still the sum of the squares of the other sides (but now there's three of them). Put more simply:




$L^2=a^2+b^2+c^2$.

Please find the sum (or resultant) of the following vectors, expressing the answer in unit vector notation:

| | |
|---|---|
| 1**i**+2**j**+3**k** **+** 3**i**+4**j**+5**k** **=** | 5**i**-2**j** **+** -5**i**+2**j**+3**k** **=** |
| 2**j**+3**k** **+** 3**i**+5**k** **=** | 5**i**+2**j** **-** -3**i**+3**k** **=** |

## Multiplying Unit Vectors I: The Dot (or Scalar) Product: [Video]

Multiplying vectors is not quite as straightforward as multiplying numbers. The first kind of product you can get by multiplying two vectors together is called the scalar product. This might seem a bad choice of name, since we're multiplying vectors, not scalars! In fact the scalar product is so-called because the **result** of the multiplication is a scalar. So we start with two vectors, multiply them together, and end up with a scalar. This type of product is used when we calculate Work. Remember that Work is defined as the product of Force and Displacement, both of which are vector quantities.

The quickest way to show how to calculate the scalar product of two vectors is to do an example. Suppose we have the two vectors **a** and **b**, where **a**=2**i**+3**j**+4**k** (the old familiar one!) and **b**=4**i**-2**j**+**k**. The scalar product of **a** and **b** is denoted **a•b** (hence the other name for it: the "dot product").

We calculate it by multiplying the **i** components together, the **j** components together and the **k** components together. (Notice the similarity with the way we added two such vectors). Then we add those three results together! So in this case we get:

**a•b** = (2**i**+3**j**+4**k**) • (4**i**-2**j**+**k**)
**a•b** = 2x4 + 3x(-2) + 4x1 = 8 - 6 + 4 = 6.
So **a•b**=6.
The result is a number, or scalar, as expected.
Here's another example: what is (3**i**-4**j**+5**k**) • (-2**i**+**j**+**k**) ?
Answer: -6-4+5=-5.

Now that we have got some idea of the procedure, let's see the general rule. For the two vectors (a**i**+b**j**+c**k**) and (d**i**+e**j**+f**k**), where a,b,c,d,e and f can be any numbers, we can form the scalar product:
(a**i**+b**j**+c**k**) • (d**i**+e**j**+f**k**) = ad + be + cf.

Please find the dot product of the following vectors:

| | |
|---|---|
| 1**i**+2**j**+3**k** • 3**i**+4**j**+5**k** = | 5**i**-2**j**+3**k** • -5**i**+2**j**+3**k** = |
| 2**j**+3**k** • 3**i**+5**k** = | 5**i**+2**j** • -3**i**+3**k** = |

***The geometric definition of the scalar product:***

Any vector has a direction and a magnitude. If we have two vectors with different directions there is an angle between those vectors. This angle is usually called q (the Greek letter "theta"). The definition of the scalar product is as follows:
**a•b** = |**a**||**b**| cos($\theta$),
where q is the angle between the vectors **a** and **b**, and |**a**| is the magnitude (or length) of the vector **a**. Hold on, you say, we have had a definition of the scalar product already, it's about multiplying the corresponding components together and adding them all up. Well the answer is, we can show that these two definitions are really the same, i.e. if we accept this definition involving cos($\theta$) then we end up with the same result as calculating using components as we did earlier. Recall that the unit vectors **i** and **j** and **k** are all at right angles to each other. So the angle between any two of them is $90^o$. Using the above definition, then we get that
**i•j**=|**i**||**j**|cos($90^o$).
Since the unit vectors have unit length, |**i**| is one, and so are |**j**| and |**k**|.

But cos($90^o$)=0. So we end up with
**i•j**=0

In fact the same will happen if we take the dot product of **any** two of the unit vectors, since they will all use cos($90^o$).
Now what happens if we take the dot product of **i** with itself? The angle between **i** and **i** is obviously $0^o$, so
**i•i** = |**i**||**i**| cos($0^o$) = 1.

So by using **a•b** = **|a||b|** cos($\theta$), we find that the dot product of any unit vector with itself is 1, and the dot product of any unit vector with another unit vector is zero. We can now use these results in our first example from earlier. We want to find the following dot product:
(2**i**+3**j**+4**k**) **•** (4**i**-2**j**+**k**)
If we just multiply out the brackets, as if we were multiplying numbers, we get nine terms:
2**i•**4**i** + 2**i•** (-2**j**) + 2**i•k** + 3**j•**4**i** + 3**j•** (-2**j**) + 3**j•k** +4**k•**4**i** + 4**k•** (-2**j**) + 4**k•k**
Now we use our results that the dot product of a unit vector with another unit vector is zero, and that the dot product of a unit vector with itself is 1, to simplify these nine terms. Six of them involve a dot product of two unit vectors, so those six terms disappear. We are left with the three terms that involved a dot product of a unit vector with itself:
2 x 4 + 3 x (-2) + 4 x 1.
As before we get the final result of 6.
So this way of calculating the dot product is really a consequence of the definition
**a•b** = **|a||b|** cos($\theta$).
From this definition, we can see that the scalar product of any two vectors at right angles to each other will be zero, since it will be some number multiplied by cos(90$^{o}$).
Similarly the scalar product of any parallel vectors will be simply the result of multiplying their lengths together, since cos(0$^{o}$)=1.

***Finding the angle between two vectors:***

If we are given two vectors **a** and **b** in their (**i-j**) component form, we can calculate their scalar product, **a•b**, as in the previous section. However, we also know that their scalar product is given by
**a•b** = **|a||b|** cos($\theta$).
Can we use this equation to find the angle q?
Well, we know the components of **a** and **b**, so we can calculate their magnitudes (by squaring and adding the three components and then taking the square root, as we've done previously). So we know their dot product, and we know their magnitudes. That means we can use the equation above to find the angle between them, which we often **don't** know.
We rearrange the equation to get the unknown on its own, as always:
cos($\theta$)=**a•b** / **|a||b|**
This tells us cos($\theta$) and hence $\theta$.
Let's use this result for our earlier dot-product example. We calculated that the dot product of the vector
2**i**+3**j**+4**k** and the vector 4**i**-2**j**+1**k** is 6.
If we calculate the magnitude of each of these vectors we can then use the above result to find the angle between the two vectors, which is not at all obvious!
The magnitude of 2**i**+3**j**+4**k** is the square root of (4+9+16), which is approximately 5.4.
The magnitude of 4**i**-2**j**+1**k** is the square root of (16+4+1), which is approximately 4.6.
The product of these two magnitudes is approximately 24.8.
The equation above therefore gives cos($\theta$)=6 / 24.8, which gives a $\theta$ of about 76$^{o}$.
So the angle between the vector 2**i**+3**j**+4**k** and the vector 4**i**-2**j**+1**k** is about 76 degrees.

Please find the angle between the following vectors in degrees:

| | |
|---|---|
| *For* 1**i**+2**j**+3**k** *and* 3**i**+4**j**+5**k**<br>\| 1**i**+2**j**+3**k** \| =<br>\| 3**i**+4**j**+5**k** \| =<br>the angle between the two vectors = | *For* 3**i**+4**j** *and* 4**i**+3**j**<br>\| 3**i**+4**j** \| =<br>\| 4**i**+3**j** \| =<br>the angle between the two vectors = |
| *For* 2**j**+3**k** *and* 3**i**+5**k**<br>\| 2**j**+3**k** \| =<br>\| 3**i**+5**k** \| =<br>the angle between the two vectors = | *For* 5**i**+2**j** *and* -3**i**+3**k**<br>\| 5**i**+2**j** \| =<br>\| -3**i**+3**k** \| =<br>the angle between the two vectors = |

## Multiplying Unit Vectors II: The Cross (or Vector) Product:

The second way to multiply two vectors together is by using the vector product. As you might expect, the result in this case is a vector. Also as you'd expect from the name, we denote the vector (or cross) product of two vectors **a** and **b** by **a x b**. We use this product in Physics when we are calculating Torque, which is defined as the cross product of the radius vector and the Force vector. The result, Torque, is itself a **new** vector.

The definition of the vector product of two vectors is in terms of the angle, $\theta$ (denoted by the Greek letter theta), between them, as with the scalar product. This time it's
**a x b** = **|a||b|** sin($\theta$) **n**

As we know, the result is supposed to be a vector, and it's this **n** that gives the direction of that vector. It doesn't contribute to the magnitude of the result, as **n** is a unit vector.

So how do we find what **n** is?
Well, **n** is defined as the direction which is at right angles to **both a** and **b**.
This gives two possible directions for **n**, so we have to specify that **n** is at right angles to **a** and **b** in a "right-handed sense". A way to think of the "right-handed sense" is to make the thumb of your **right** hand point in direction **a**, and your first finger in the direction of **b**, then stick your second finger out at right angles to both of them (it sounds painful but really it should be easy when you try it!). The way your second finger is pointing is the direction of **n**.

By the above definition,
**i x i = |i||i|** sin ($0^o$) **n**
But sin($0^o$) = 0, so this time it's the cross product of any unit vector with **itself** that's zero.

What about the cross product of a unit vector with one of the others?
**i x j** = **|i||j|** sin ($90^o$) **n.**
Well, the magnitudes are both 1, and sin($90^o$) is 1, so the magnitude of the result is 1. What is its direction? There are two directions which are at right-angles to both **i** and **j**, namely **k** and -**k**. The one which is in a right-handed sense is **k**. So we find that **i x j** = **k**.

Similarly, **j x k** = **i** and **k x i** = **j**. (One way to remember that these results are all positive, is that the letters i, j and k appear in a cycle of their original order i, j, k.)

Also, **j x i** =**|j||i|**sin($90^o$) **n**, where **n** is at right-angles to both **j** and **i**.
If you try the right-handed rule, starting with **j** and then moving to **i**, you find that the direction at right-angles to them both in a right-handed sense is -**k**.
So **j x i** = -**k**.

Similarly, **k x j** =-**i** and **i x k** = -**j**. (One way to remember that these are all negative is that the letters i, j and k are not in a cycle of their original order.)

***An example:***
We'll use the same two we had before: (2**i**+3**j**+4**k**) and (4**i**-2**j**+1**k**).

Now let's take the cross product:
(2**i**+3**j**+4**k**) **x** (4**i**-2**j**+1**k**)

If we just multiply out the brackets, as if we were multiplying numbers, we get nine terms:
<u>2**i x** 4**i**</u> + 2**i x** (-2**j**) + 2**i x** 1**k** + 3**j x** 4**i** + <u>3**j x** (-2**j**)</u> + 3**j x** 1**k** +4**k x** 4**i** + 4**k x** (-2**j**) + <u>4**k x** 1**k**</u>

Now we use our results that the cross product of a unit vector with itself is zero, and that the cross product of one unit vector with a second unit vector is either plus or minus the third unit vector, to simplify these nine terms. The underlined terms, which are cross products of a unit vector with itself, become zero. Six of them involve a cross product of two unit vectors, so we expect to end up with six terms.
So the result is:
2 x (-2) x **k** + 2 x 1 x (-**j**) + 3 x 4 x (-**k**) + 3 x 1 x **i** + 4 x 4 x **j** + 4 x (-2) x (-**i**).

We should have one term involving +**i**, one involving -**i**, one involving +**j**, one involving -**j** and one involving +**k** and one involving -**k**. This is a useful check.

The final result we get from adding up those six terms is therefore:
**i** x (3 x 1 - 4 x (-2)) + **j** x (4 x 4 - 2 x 1) + **k** x (2 x (-2) - 3 x 4) = 11**i** + 14**j** - 16**k**.

Please find the cross product of the following vectors:

| | |
|---|---|
| 1**i**+2**j**+3**k** **x** 3**i**+4**j**+5**k** = | 3**i**+4**j** **x** 4**i**+3**j** = |
| 1**i**+2**j**+3**k** **x** -3**I**-4**j**-5**k** = | 5**i**+2**j** **x** -3**i**+3**k** = |

***Is there an easier way to find a cross product? Finding a cross product using a determinant.***
Using our original cross product (2**i**+3**j**+4**k**) **x** (4**i**-2**j**+1**k**), we write down the following determinant:

$$\begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 4 & -2 & 1 \end{vmatrix}$$

You can see that we write the three unit vectors across the top row, and then in the next two rows we write the two vectors we want to multiply together. The first vector was (2**i**+3**j**+4**k**), so for the second row we write a 2 in the "**i**" column, a 3 in the "**j**" column and a 4 in the "**k**" column. Then we do the same to put the second vector in the bottom row.

The next (and final) stage is to evaluate the determinant. We do this as follows. For each of the unit vectors in turn we do a short calculation involving the numbers we put in.

First, for the **i**: we ignore the row and column that contain the **i** so we're just looking at the bottom right-hand square of four numbers.

$$\text{For i:} \quad \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 4 & -2 & 1 \end{vmatrix}$$

In this square we first multiply the top left-hand number by the bottom right-hand number (so we multiply the 3 by the 1) then we multiply the top right-hand number by the bottom left-hand number (so we multiply the 4 by the -2). Finally we take the second result away from the first, so we end up with 3 - (-8) = 11. This will be the number in front of the **i** in our final answer. It probably seems like a complicated process, but when you've done it a few times you'll find it's quite quick.

So let's move on to the **j**. Again we want to consider only a square of four numbers and we'll do exactly the same calculation with it. To form the square, again we ignore the row and column containing the **j**, but we have to copy the part of the first column over to make a sort of extra column at the end, as shown below.

For j: $\left|\begin{array}{ccc} i & j & k \\ 2 & 3 & 4 \\ 4 & -2 & 1 \end{array}\right| \begin{array}{c} \\ 2 \\ 4 \end{array}$

Then we can do the same calculation as before, so we multiply the 4 by the 4 and the 2 by the 1 and subtract, to get 16-2=14 for the number of **j**'s in our final result.

Now we come on to the **k** component. As always, we ignore the row and column containing the **k** and do our calculation on the remaining square of numbers as shown below.

For k: $\left|\begin{array}{ccc} i & j & k \\ 2 & 3 & 4 \\ 4 & -2 & 1 \end{array}\right|$

So for the **k** component we get 2x(-2) - 3x4 = -4 -12 = -16.

That's it! So our final result is:
**i** x (3 x 1 - 4 x (-2)) + **j** x (4 x 4 - 2 x 1) + **k** x (2 x (-2) - 3 x 4) = 11**i** + 14**j** - 16**k**.
This is the same result that we got the other way (as it must be of course!)

Please find the cross product of the following vectors, first writing the determinant and then solving:

| | |
|---|---|
| 1**i**+2**j**+3**k** **x** 3**i**+4**j**+5**k** = <br> $\left\lvert\begin{array}{ccc} i & j & k \\ & & \\ & & \end{array}\right\rvert$ | 3**i**+4**j** **x** 4**i**+3**j** = <br> $\left\lvert\begin{array}{ccc} i & j & k \\ & & \\ & & \end{array}\right\rvert$ |
| 1**i**+2**j**+3**k** **x** -3**I**-4**j**-5**k** = <br> $\left\lvert\begin{array}{ccc} i & j & k \\ & & \\ & & \end{array}\right\rvert$ | 5**i**+2**j** **x** -3**i**+3**k** = <br> $\left\lvert\begin{array}{ccc} i & j & k \\ & & \\ & & \end{array}\right\rvert$ |

***References:***

Portions of three-dimensional vectors and cross-products from:
*http://www.ucl.ac.uk/Mathematics/geomath/vecsnb/MHvecslnk6.html*