Diploma Programme

# Mathematics: applications and interpretation guide

First assessment 2021

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Diploma Programme

# Mathematics: applications and interpretation guide

First assessment 2021

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

**Diploma Programme**
**Mathematics: applications and interpretation guide**

Published February 2019
Updated August 2019, May 2020, August 2020, November 2020, November 2024

Published by the International Baccalaureate Organization, a not-for-profit educational foundation of Rue du Pré-de-la-Bichette 1, 1202 Genève, Switzerland.
Website: ibo.org

© International Baccalaureate Organization 2019

The International Baccalaureate Organization (known as the IB) offers four high-quality and challenging educational programmes for a worldwide community of schools, aiming to create a better, more peaceful world. This publication is one of a range of materials produced to support these programmes.

The IB may use a variety of sources in its work and check information to verify accuracy and authenticity, particularly when using community-based knowledge sources such as Wikipedia. The IB respects the principles of intellectual property and makes strenuous efforts to identify and obtain permission before publication from rights holders of all copyright material used. The IB is grateful for permissions received for material used in this publication and will be pleased to correct any errors or omissions at the earliest opportunity.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, without the IB's prior written permission, or as expressly permitted by the Rules for use of IB Intellectual Property.

IB merchandise and publications can be purchased through the IB Store (email: sales@ibo.org). Any commercial use of IB publications (whether fee-covered or commercial) by third parties acting in the IB's ecosystem without a formal relationship with the IB (including but not limited to tutoring organizations, professional development providers, educational publishers and operators of curriculum mapping or teacher resource digital platforms, etc.) is prohibited and requires a subsequent written licence from the IB. Licence requests should be sent to copyright@ibo.org. More information can be obtained on the IB public website.

**International Baccalaureate**, **Baccalauréat International**, **Bachillerato Internacional** and IB logos are registered trademarks of the International Baccalaureate Organization.

# IB mission statement

The International Baccalaureate aims to develop inquiring, knowledgeable and caring young people who help to create a better and more peaceful world through intercultural understanding and respect.

To this end the organization works with schools, governments and international organizations to develop challenging programmes of international education and rigorous assessment.

These programmes encourage students across the world to become active, compassionate and lifelong learners who understand that other people, with their differences, can also be right.

# IB learner profile

**The aim of all IB programmes is to develop internationally minded people who, recognizing their common humanity and shared guardianship of the planet, help to create a better and more peaceful world.**

## As IB learners we strive to be:

### INQUIRERS
We nurture our curiosity, developing skills for inquiry and research. We know how to learn independently and with others. We learn with enthusiasm and sustain our love of learning throughout life.

### KNOWLEDGEABLE
We develop and use conceptual understanding, exploring knowledge across a range of disciplines. We engage with issues and ideas that have local and global significance.

### THINKERS
We use critical and creative thinking skills to analyse and take responsible action on complex problems. We exercise initiative in making reasoned, ethical decisions.

### COMMUNICATORS
We express ourselves confidently and creatively in more than one language and in many ways. We collaborate effectively, listening carefully to the perspectives of other individuals and groups.

### PRINCIPLED
We act with integrity and honesty, with a strong sense of fairness and justice, and with respect for the dignity and rights of people everywhere. We take responsibility for our actions and their consequences.

### OPEN-MINDED
We critically appreciate our own cultures and personal histories, as well as the values and traditions of others. We seek and evaluate a range of points of view, and we are willing to grow from the experience.

### CARING
We show empathy, compassion and respect. We have a commitment to service, and we act to make a positive difference in the lives of others and in the world around us.

### RISK-TAKERS
We approach uncertainty with forethought and determination; we work independently and cooperatively to explore new ideas and innovative strategies. We are resourceful and resilient in the face of challenges and change.

### BALANCED
We understand the importance of balancing different aspects of our lives—intellectual, physical, and emotional—to achieve well-being for ourselves and others. We recognize our interdependence with other people and with the world in which we live.

### REFLECTIVE
We thoughtfully consider the world and our own ideas and experience. We work to understand our strengths and weaknesses in order to support our learning and personal development.

**The IB learner profile represents 10 attributes valued by IB World Schools. We believe these attributes, and others like them, can help individuals and groups become responsible members of local, national and global communities.**

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

© International Baccalaureate Organization 2017
International Baccalaureate® | Baccalauréat International® | Bachillerato Internacional®

# Contents


## Introduction 1

Purpose of this document 1
The Diploma Programme 2
Nature of Mathematics 6
Approaches to the teaching and learning of mathematics: applications and interpretation 13
Aims 20
Assessment objectives 21
Assessment objectives in practice 22

## Syllabus 23

Syllabus outline 23
Prior learning topics 24
Syllabus content 26

## Assessment 72

Assessment in the Diploma Programme 72
Assessment outline—SL 74
Assessment outline—HL 75
External assessment 76
Internal assessment 81

## Appendices 90

Glossary of command terms 90
Notation list 92
Updates to the publication 97

# Purpose of this document

This publication is intended to guide the planning, teaching and assessment of the subject in schools. Subject teachers are the primary audience, although it is expected that teachers will use the guide to inform students and parents about the subject.

This guide can be found on the subject page of the programme resource centre at resources.ibo.org, a password-protected IB website designed to support IB teachers. It can also be purchased from the IB store at store.ibo.org

## Additional resources

Additional publications such as specimen papers and markschemes, teacher support materials, subject reports and grade descriptors can also be found on the programme resource centre. Past examination papers as well as markschemes can be purchased from the IB store.

Teachers are encouraged to check the programme resource centre for additional resources created or used by other teachers. Teachers can provide details of useful resources, for example websites, books, videos, journals or teaching ideas.

### Acknowledgment

The IB wishes to thank the educators and associated schools for generously contributing time and resources to the production of this guide.

| First assessment 2021 |
| --- |

# The Diploma Programme

The Diploma Programme is a rigorous pre-university course of study designed for students in the 16-19 age range. It is a broad-based two-year course that aims to encourage students to be knowledgable and inquiring, but also caring and compassionate. There is a strong emphasis on encouraging students to develop intercultural understanding, open-mindedness, and the attitudes necessary for them to respect and evaluate a range of points of view.

## The Diploma Programme model

The course is presented as six academic areas enclosing a central core (see figure 1). It encourages the concurrent study of a broad range of academic areas. Students study two modern languages (or a modern language and a classical language), a humanities or social science subject, an experimental science, mathematics and one of the creative arts. It is this comprehensive range of subjects that makes the Diploma Programme a demanding course of study designed to prepare students effectively for university entrance. In each of the academic areas students have flexibility in making their choices, which means they can choose subjects that particularly interest them and that they may wish to study further at university.

*Figure 1*

*The Diploma Programme model*

# Choosing the right combination

Students are required to choose one subject from each of the six academic areas, although they can, instead of an arts subject, choose two subjects from another area. Normally, three subjects (and not more than four) are taken at higher level (HL), and the others are taken at standard level (SL). The IB recommends 240 teaching hours for HL subjects and 150 hours for SL. Subjects at HL are studied in greater depth and breadth than at SL.

At both levels many skills are developed, especially those of critical thinking and analysis. At the end of the course, students' abilities are measured by means of external assessment. Many subjects contain some element of coursework assessed by teachers.

# The core of the Diploma Programme model

All Diploma Programme (DP) students participate in the three course elements that make up the core of the model.

Theory of knowledge (TOK) is a course that is fundamentally about critical thinking and inquiry into the process of knowing rather than about learning a specific body of knowledge. The TOK course examines the nature of knowledge and how we know what we claim to know. It does this by encouraging students to analyse knowledge claims and explore questions about the construction of knowledge. The task of TOK is to emphasize connections between areas of shared knowledge and link them to personal knowledge in such a way that an individual becomes more aware of his or her own perspectives and how they might differ from others.

Creativity, activity, service (CAS) is at the heart of the Diploma Programme. The emphasis in CAS is on helping students to develop their own identities, in accordance with the ethical principles embodied in the IB mission statement and the IB learner profile. It involves students in a range of activities alongside their academic studies throughout the Diploma Programme. The three strands of CAS are creativity (arts, and other experiences that involve creative thinking), activity (physical exertion contributing to a healthy lifestyle) and service (an unpaid and voluntary exchange that has a learning benefit for the student). Possibly, more than any other component in the Diploma Programme, CAS contributes to the IB's mission to create a better and more peaceful world through intercultural understanding and respect.

The extended essay, including the world studies extended essay, offers the opportunity for IB students to investigate a topic of special interest, in the form of a 4,000-word piece of independent research. The area of research undertaken is chosen from one of the students' six Diploma Programme subjects, or in the case of the inter-disciplinary World Studies essay, two subjects, and acquaints them with the independent research and writing skills expected at university. This leads to a major piece of formally-presented, structured writing, in which ideas and findings are communicated in a reasoned and coherent manner, appropriate to the subject or subjects chosen. It is intended to promote high-level research and writing skills, intellectual discovery and creativity. An authentic learning experience, it provides students with an opportunity to engage in personal research on a topic of choice, under the guidance of a supervisor.

# Approaches to teaching and approaches to learning

Approaches to teaching and learning across the Diploma Programme refers to deliberate strategies, skills and attitudes which permeate the teaching and learning environment. These approaches and tools, intrinsically linked with the learner profile attributes, enhance student learning and assist student preparation for the Diploma Programme assessment and beyond. The aims of approaches to teaching and learning in the Diploma Programme are to:

- empower teachers as teachers of learners as well as teachers of content
- empower teachers to create clearer strategies for facilitating learning experiences in which students are more meaningfully engaged in structured inquiry and greater critical and creative thinking

- promote both the aims of individual subjects (making them more than course aspirations) and linking previously isolated knowledge (concurrency of learning)
- encourage students to develop an explicit variety of skills that will equip them to continue to be actively engaged in learning after they leave school, and to help them not only obtain university admission through better grades but also prepare for success during tertiary education and beyond
- enhance further the coherence and relevance of the students' Diploma Programme experience
- allow schools to identify the distinctive nature of an IB Diploma Programme education, with its blend of idealism and practicality.

The five approaches to learning (developing thinking skills, social skills, communication skills, self-management skills and research skills) along with the six approaches to teaching (teaching that is inquiry-based, conceptually-focused, contextualized, collaborative, differentiated and informed by assessment) encompass the key values and principles that underpin IB pedagogy.

# The IB mission statement and the IB learner profile

The Diploma Programme aims to develop in students the knowledge, skills and attitudes they will need to fulfill the aims of the IB, as expressed in the organization's mission statement and the learner profile. Teaching and learning in the Diploma Programme represent the reality in daily practice of the organization's educational philosophy.

# Academic honesty

Academic honesty in the Diploma Programme is a set of values and behaviours informed by the attributes of the learner profile. In teaching, learning and assessment, academic honesty serves to promote personal integrity, engender respect for the integrity of others and their work, and ensure that all students have an equal opportunity to demonstrate the knowledge and skills they acquire during their studies.

All coursework—including work submitted for assessment—is to be authentic, based on the student's individual and original ideas with the ideas and work of others fully acknowledged. Assessment tasks that require teachers to provide guidance to students or that require students to work collaboratively must be completed in full compliance with the detailed guidelines provided by the IB for the relevant subjects.

For further information on academic honesty in the IB and the Diploma Programme, please consult the IB publications *Academic honesty in the IB educational context*, *Effective citing and referencing*, *The Diploma Programme: From principles into practice* and *General regulations: Diploma Programme*. Specific information regarding academic honesty as it pertains to external and internal assessment components of this Diploma Programme subject can be found in this guide.

# Acknowledging the ideas or work of another person

Coordinators and teachers are reminded that candidates must acknowledge all sources used in work submitted for assessment. The following is intended as a clarification of this requirement.

Diploma Programme candidates submit work for assessment in a variety of media that may include audio-visual material, text, graphs, images and/or data published in print or electronic sources. If a candidate uses the work or ideas of another person, the candidate must acknowledge the source using a standard style of referencing in a consistent manner. A candidate's failure to acknowledge a source will be investigated by the IB as a potential breach of regulations that may result in a penalty imposed by the IB final award committee.

The IB does not prescribe which style(s) of referencing or in-text citation should be used by candidates; this is left to the discretion of appropriate faculty/staff in the candidate's school. The wide range of subjects, three response languages and the diversity of referencing styles make it impractical and restrictive to insist on particular styles. In practice, certain styles may prove most commonly used, but schools are free to

choose a style that is appropriate for the subject concerned and the language in which candidates' work is written. Regardless of the reference style adopted by the school for a given subject, it is expected that the minimum information given includes: name of author, date of publication, title of source, and page numbers as applicable.

Candidates are expected to use a standard style and use it consistently so that credit is given to all sources used, including sources that have been paraphrased or summarized. When writing text, candidates must clearly distinguish between their words and those of others by the use of quotation marks (or other method, such as indentation) followed by an appropriate citation that denotes an entry in the bibliography. If an electronic source is cited, the date of access must be indicated. Candidates are not expected to show faultless expertise in referencing, but are expected to demonstrate that all sources have been acknowledged. Candidates must be advised that audio-visual material, text, graphs, images and/or data published in print or in electronic sources that is not their own must also attribute the source. Again, an appropriate style of referencing/citation must be used.

# Learning diversity and learning support requirements

Schools must ensure that equal access arrangements and reasonable adjustments are provided to candidates with learning support requirements that are in line with the IB documents *Access and inclusion policy* and *Learning diversity and inclusion in IB programmes.*

The documents *Meeting student learning diversity in the classroom* and *The IB guide to inclusive education: a resource for whole school development* are available to support schools in the ongoing process of increasing access and engagement by removing barriers to learning.

# Nature of Mathematics

## Introduction

Mathematics has been described as the study of structure, order and relation that has evolved from the practices of counting, measuring and describing objects. Mathematics provides a unique language to describe, explore and communicate the nature of the world we live in as well as being a constantly building body of knowledge and truth in itself that is distinctive in its certainty. These two aspects of mathematics, a discipline that is studied for its intrinsic pleasure and a means to explore and understand the world we live in, are both separate yet closely linked.

Mathematics is driven by abstract concepts and generalization. This mathematics is drawn out of ideas, and develops through linking these ideas and developing new ones. These mathematical ideas may have no immediate practical application. Doing such mathematics is about digging deeper to increase mathematical knowledge and truth. The new knowledge is presented in the form of theorems that have been built from axioms and logical mathematical arguments and a theorem is only accepted as true when it has been proven. The body of knowledge that makes up mathematics is not fixed; it has grown during human history and is growing at an increasing rate.

The side of mathematics that is based on describing our world and solving practical problems is often carried out in the context of another area of study. Mathematics is used in a diverse range of disciplines as both a language and a tool to explore the universe; alongside this its applications include analyzing trends, making predictions, quantifying risk, exploring relationships and interdependence.

While these two different facets of mathematics may seem separate, they are often deeply connected. When mathematics is developed, history has taught us that a seemingly obscure, abstract mathematical theorem or fact may in time be highly significant. On the other hand, much mathematics is developed in response to the needs of other disciplines.

The two mathematics courses available to Diploma Programme (DP) students express both the differences that exist in mathematics described above and the connections between them. These two courses might approach mathematics from different perspectives, but they are connected by the same mathematical body of knowledge, ways of thinking and approaches to problems. The differences in the courses may also be related to the types of tools, for instance technology, that are used to solve abstract or practical problems. The next section will describe in more detail the two available courses.

## Summary of courses available

Individual students have different needs, aspirations, interests and abilities. For this reason there are two different subjects in mathematics, each available at SL and HL These courses are designed for different types of students: those who wish to study mathematics as a subject in its own right or to pursue their interests in areas related to mathematics, and those who wish to gain understanding and competence in how mathematics relates to the real world and to other subjects. Each course is designed to meet the needs of a particular group of students. Mathematics: analysis and approaches and Mathematics: applications and interpretation are both offered at SL and HL. Therefore, great care should be taken to select the course and level that is most appropriate for an individual student.

In making this selection, individual students should be advised to take into account the following factors:

- their own abilities in mathematics and the type of mathematics in which they can be successful
- their own interest in mathematics and those particular areas of the subject that may hold the most interest for them
- their other choices of subjects within the framework of the DP or Career-related Programme (CP)

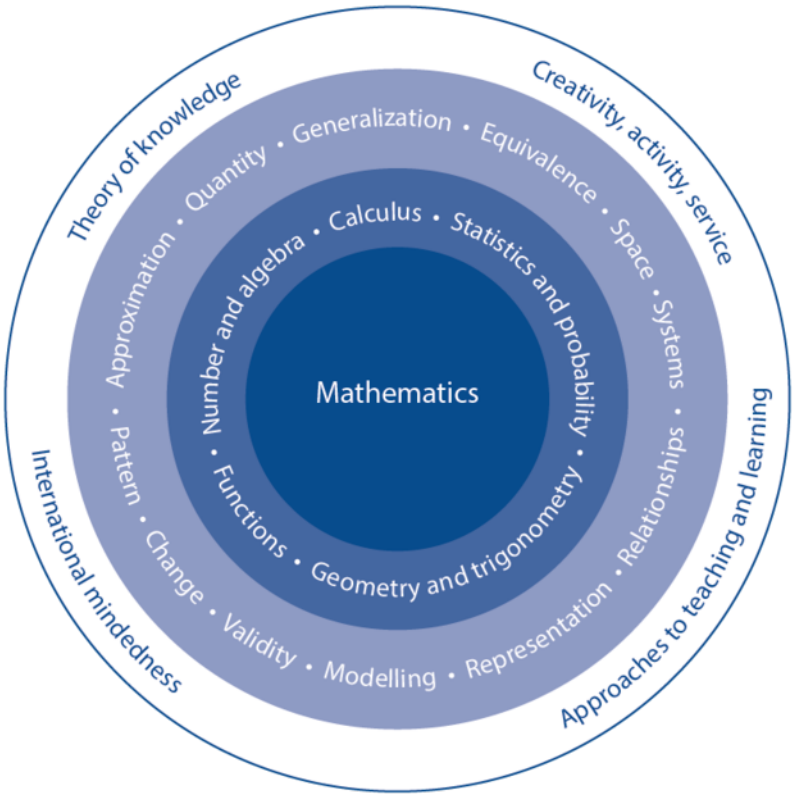
- their academic plans, in particular the subjects they wish to study in the future
- their choice of career.

Teachers are expected to assist with the selection process and to offer advice to students.

# The nature of IB mathematics courses

*Figure 2*

*The mathematics model*




The structure of IB DP mathematics courses, with two different routes to choose from, recognizes the two different aspects of mathematics discussed in the introduction.

Mathematics: analysis and approaches is for students who enjoy developing their mathematics to become fluent in the construction of mathematical arguments and develop strong skills in mathematical thinking. They will also be fascinated by exploring real and abstract applications of these ideas, with and without technology. Students who take Mathematics: analysis and approaches will be those who enjoy the thrill of mathematical problem solving and generalization.

Mathematics: applications and interpretation is for students who are interested in developing their mathematics for describing our world and solving practical problems. They will also be interested in harnessing the power of technology alongside exploring mathematical models. Students who take Mathematics: applications and interpretation will be those who enjoy mathematics best when seen in a practical context.

Both subjects are offered at higher level and SL. There are many elements common to both subjects although the approaches may be different. Both subjects will prepare students with the mathematics

needed for a range of further educational courses corresponding to the two approaches to mathematics set out above.

# Mathematics: analysis and approaches

This course recognizes the need for analytical expertise in a world where innovation is increasingly dependent on a deep understanding of mathematics. This course includes topics that are both traditionally part of a pre-university mathematics course (for example, functions, trigonometry, calculus) as well as topics that are amenable to investigation, conjecture and proof, for instance the study of sequences and series at both SL and HL, and proof by induction at HL.

The course allows the use of technology, as fluency in relevant mathematical software and hand-held technology is important regardless of choice of course. However, Mathematics: analysis and approaches has a strong emphasis on the ability to construct, communicate and justify correct mathematical arguments.

## Mathematics: analysis and approaches: Distinction between SL and HL

Students who choose Mathematics: analysis and approaches at SL or HL should be comfortable in the manipulation of algebraic expressions and enjoy the recognition of patterns and understand the mathematical generalization of these patterns. Students who wish to take Mathematics: analysis and approaches at higher level will have strong algebraic skills and the ability to understand simple proof. They will be students who enjoy spending time with problems and get pleasure and satisfaction from solving challenging problems.

# Mathematics: applications and interpretation

This course recognizes the increasing role that mathematics and technology play in a diverse range of fields in a data-rich world. As such, it emphasizes the meaning of mathematics in context by focusing on topics that are often used as applications or in mathematical modelling. To give this understanding a firm base, this course also includes topics that are traditionally part of a pre-university mathematics course such as calculus and statistics.

The course makes extensive use of technology to allow students to explore and construct mathematical models. Mathematics: applications and interpretation will develop mathematical thinking, often in the context of a practical problem and using technology to justify conjectures.

## Mathematics: applications and interpretation: Distinction between SL and HL

Students who choose Mathematics: applications and interpretation at SL or HL should enjoy seeing mathematics used in real-world contexts and to solve real-world problems. Students who wish to take Mathematics: applications and interpretation at higher level will have good algebraic skills and experience of solving real-world problems. They will be students who get pleasure and satisfaction when exploring challenging problems and who are comfortable to undertake this exploration using technology.

# Mathematics and theory of knowledge

The relationship between each subject and theory of knowledge (TOK) is important and fundamental to the DP. The theory of knowledge course provides an opportunity for students to reflect on questions about how knowledge is produced and shared, both in mathematics and also across different areas of knowledge. It encourages students to reflect on their assumptions and biases, helping them to become more aware of their own perspective and the perspectives of others and to become “inquiring, knowledgeable and caring young people” (IB mission statement).

As part of their theory of knowledge course, students are encouraged to explore tensions relating to knowledge in mathematics. As an area of knowledge, mathematics seems to supply a certainty perhaps impossible in other disciplines and in many instances provides us with tools to debate these certainties. This may be related to the "purity" of the subject, something that can sometimes make it seem divorced from reality. Yet mathematics has also provided important knowledge about the world and the use of mathematics in science and technology has been one of the driving forces for scientific advances.

Despite all its undoubted power for understanding and change, mathematics is in the end a puzzling phenomenon. A fundamental question for all knowers is whether mathematical knowledge really exists independently of our thinking about it. Is it there, "waiting to be discovered", or is it a human creation? Indeed, the philosophy of mathematics is an area of study in its own right.

Students' attention should be drawn to questions relating theory of knowledge (TOK) and mathematics, and they should also be encouraged to raise such questions themselves in both their mathematics and TOK classes. Examples of issues relating to TOK are given in the "Connections" sections of the syllabus. Further suggestions for making links to TOK can also be found in the mathematics section of the *Theory of knowledge guide*.

# Mathematics and international-mindedness

International-mindedness is a complex and multi-faceted concept that refers to a way of thinking, being and acting characterized by an openness to the world and a recognition of our deep interconnectedness to others.

Despite recent advances in the development of information and communication technologies, the global exchange of mathematical information and ideas is not a new phenomenon and has been essential to the progress of mathematics. Indeed, many of the foundations of modern mathematics were laid many centuries ago by diverse civilisations – Arabic, Greek, Indian and Chinese among others.

Mathematics can in some ways be seen as an international language and, apart from slightly differing notation, mathematicians from around the world can communicate effectively within their field. Mathematics can transcend politics, religion and nationality, and throughout history great civilizations have owed their success in part to their mathematicians being able to create and maintain complex social and architectural structures. Politics has dominated the development of mathematics, to develop ballistics, navigation and trade, and land ownership, often influenced by governments and leaders. Many early mathematicians were political and military advisers and today mathematicians are integral members of teams who advise governments on where money and resources should be allocated.

Science and technology are of significant importance in today's world. As the language of science, mathematics is an essential component of most technological innovation and underpins developments in science and technology, although the contribution of mathematics may not always be visible. Examples of this include the role of the binary number system, matrix algebra, network theory and probability theory in the digital revolution, or the use of mathematical simulations to predict future climate change or spread of disease. These examples highlight the key role mathematics can play in transforming the world around us.

One way of fostering international-mindedness is to provide opportunities for inquiry into a range of local and global issues and ideas. Many international organisations and bodies now exist to promote mathematics, and students are encouraged to access the resources and often-extensive websites of such mathematical organisations. This can enhance their appreciation of the international dimension of mathematics, as well as providing opportunities to engage with global issues surrounding the subject.

Examples of links relating to international-mindedness are given in the "Connections" sections of the syllabus.

# Mathematics and creativity, activity, service

CAS experiences can be associated with each of the subject groups of the DP.

CAS and mathematics can complement each other in a number of ways. Mathematical knowledge provides an important key to understanding the world in which we live, and the mathematical skills and techniques

students learn in the mathematics courses will allow them to evaluate the world around them which will help them to develop, plan and deliver CAS experiences or projects.

An important aspect of the mathematics courses is that students develop the ability to systematically analyse situations and can recognize the impact that mathematics can have on the world around them. An awareness of how mathematics can be used to represent the truth enables students to reflect critically on the information that societies are given or generate, and how this influences the allocation of resources or the choices that people make. This systematic analysis and critical reflection when problem solving may be inspiring springboards for CAS projects.

Students may also draw on their CAS experiences to enrich their involvement in mathematics both within and outside the classroom, and mathematics teachers can assist students in making links between their subjects and students' CAS experiences where appropriate. Purposeful discussion about real CAS experiences and projects will help students to make these links.

The challenge and enjoyment of CAS can often have a profound effect on mathematics students, who might choose, for example, to engage with CAS in the following ways:

- plan, write and implement a "mathematics scavenger hunt" where younger students tour the school answering interesting mathematics questions as part of their introduction to a new school
- as a CAS project students could plan and carry out a survey, create a database and analyse the results, and make suggestions to resolve a problem in the students' local area. This might be, for example, surveying the availability of fresh fruit and vegetables within a community, preparing an action plan with suggestions of how to increase availability or access, and presenting this to a local charity or community group
- taking an element of world culture that interests students and designing a miniature Earth (if the world were 100 people) to express the trend(s) numerically.

It is important to note that a CAS experience can be a single event or may be an extended series of events. However, CAS experiences must be distinct from, and may not be included or used in, the student's Diploma course requirements.

Additional suggestions on the links between DP subjects and CAS can be found in the *Creativity, activity, service teacher support material*.

# Prior learning

It is expected that most students embarking on a DP mathematics course will have studied mathematics for at least 10 years. There will be a great variety of topics studied, and differing approaches to teaching and learning. Thus, students will have a wide variety of skills and knowledge when they start their mathematics course. Most will have some background in arithmetic, algebra, geometry, trigonometry, probability and statistics. Some will be familiar with an inquiry approach, and may have had an opportunity to complete an extended piece of work in mathematics.

At the beginning of the syllabus section there is a list of topics that are considered to be prior learning for the mathematics courses. It is recognized that this may contain topics that are unfamiliar to some students, but it is anticipated that there may be other topics in the syllabus itself which these students have already encountered. Teachers should be informed by their assessment of students' prior learning to help plan their teaching so that topics mentioned that are unfamiliar to their students can be incorporated.

# Links to the Middle Years Programme

The MYP mathematics framework is designed to prepare students for the study of DP mathematics courses. As students progress from the MYP to the DP or the Career-related Programme (CP) they continue to develop their mathematics skills and knowledge, which will then allow them to go on to study a wide range of topics. Inquiry-based learning is central to mathematics courses in both MYP and the DP, providing students with opportunities to independently and collaboratively investigate, problem solve and communicate their mathematics with an increasing level of sophistication.

MYP mathematics courses are concept-driven, aimed at helping the learner to construct meaning through improved critical thinking and the transfer of knowledge. The MYP courses use a framework of key concepts with which the concepts in the DP mathematics courses are aligned. These concepts are broad, organizing, powerful ideas that have relevance within the subject but also transcend it, having relevance in other subject groups. The fundamental concepts of MYP mathematics provide a very useful foundation for students following DP mathematics courses.

The aims of the MYP mathematics courses align very closely with those of the DP mathematics courses. The prior learning topics for the DP mathematics courses have been written in conjunction with the MYP mathematics guide.

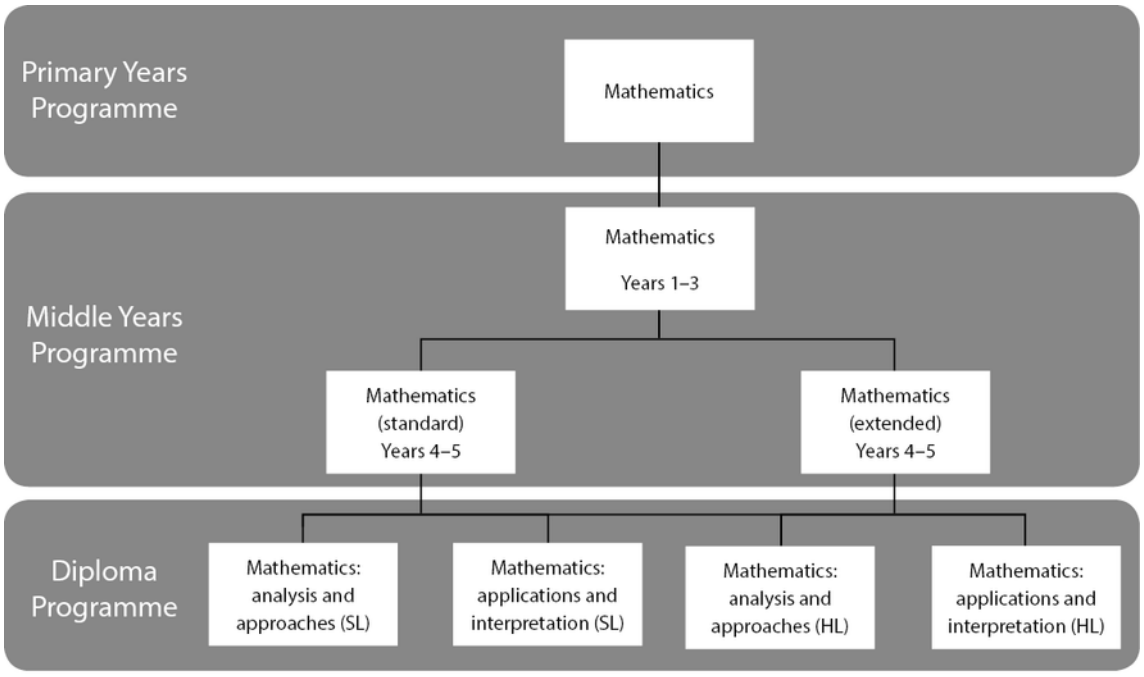
The MYP mathematics assessment objectives and criteria have been developed with both the internal and external assessment requirements of the DP in mind. MYP mathematics students are required to practise and develop their investigation skills, one of the MYP four assessment objectives, giving an important foundation for the internal assessment component of the DP mathematics courses. The MYP assessment objective of thinking critically also corresponds to the higher-order assessment objectives of communication and interpretation, and of reasoning that are expected from a DP mathematics student.

MYP and DP mathematics courses emphasized the use of technology as a powerful tool for learning, applying and communicating mathematics.

Where students in the MYP may select either standard or extended mathematics, at DP there are two mathematics subjects both available at SL and HL. MYP students enrolled in extended mathematics generally elect to take one of the HL mathematics courses in the DP. Students in MYP standard mathematics should seek the recommendation of their teacher when deciding which SL or HL course they are best suited to pursue.

*Figure 3*

*The IB continuum pathways to DP courses in mathematics*



# Links to the Career-related Programme

In the IB Career-related Programme (CP) students study at least two DP subjects, a core consisting of four components and a career-related study, which is determined by the local context and aligned with student needs. The CP has been designed to add value to the student's career-related studies. This provides the context for the choice of DP courses. Courses can be chosen from any group of the DP. It is also possible to study more than one course from the same group (for example, visual arts and film).

Mathematics may be a beneficial choice for CP students considering careers in, for example, finance, planning, healthcare systems or coding, tourism industries, the technology industry, social informatics, or urban planning. Mathematics helps students to understand the value of systematic approaches, how to analyse complex real-world contexts, how to communicate this concisely and precisely and understand the implications of conclusions.

Mathematics encourages the development of strong written, verbal, and graphical communication skills; critical and complex thinking; and moral and ethical considerations influenced by mathematics that will assist students in preparing for the future global workplace. This in turn fosters the IB learner profile attributes that are transferable to the entire CP, providing relevance and support for the student's learning.

For the CP students, DP courses can be studied at SL or HL. Schools can explore opportunities to integrate CP students with DP students.

# Approaches to the teaching and learning of mathematics: applications and interpretation

## Conceptual understanding

Concepts are broad, powerful, organizing ideas, the significance of which goes beyond particular origins, subject matter or place in time. Concepts represent the vehicle for students' inquiry into issues and ideas of personal, local and global significance, providing the means by which they can explore the essence of mathematics.

Concepts play an important role in mathematics, helping students and teachers to think with increasing complexity as they organize and relate facts and topics. Students use conceptual understandings as they solve problems, analyse issues and evaluate decisions that can have an impact on themselves, their communities and the wider world.

In DP mathematics courses, conceptual understandings are key to promoting deep learning. The course identifies twelve fundamental concepts which relate with varying emphasis to each of the five topics. Teachers may identify and develop additional concepts to meet local circumstances and national or state curriculum requirements. Teachers can use these concepts to develop connections throughout the curriculum.

Each topic in this guide begins by stating the essential understanding of the topic and highlighting relevant concepts fundamental to the topic. This is followed by suggested conceptual understandings relevant to the content within the topic, although this list is not intended to be prescriptive or exhaustive.

### The concepts

Concepts promote the development of a broad, balanced, conceptual and connected curriculum. They represent big ideas that are relevant and facilitate connections within topics, across topics and also to other subjects within the DP.

The twelve concepts identified below support conceptual understanding, can inform units of work and can help to organize teaching and learning. Explanations of each of these concepts in a mathematical context have also been provided.

| | |
|---|---|
| Approximation | This concept refers to a quantity or a representation which is nearly but not exactly correct. |
| Change | This concept refers to a variation in size, amount or behaviour. |
| Equivalence | This concept refers to the state of being identically equal or interchangeable, applied to statements, quantities or expressions. |
| Generalization | This concept refers to a general statement made on the basis of specific examples. |
| Modelling | This concept refers to the way in which mathematics can be used to represent the real world. |
| Patterns | This concept refers to the underlying order, regularity or predictability of the elements of a mathematical system. |
| Quantity | This concept refers to an amount or number. |
| Relationships | This concept refers to the connection between quantities, properties or concepts; these connections may be expressed as models, rules or statements. Relationships provide opportunities for students to explore patterns in the world around them. |

| | |
|---|---|
| Representation | This concept refers to using words, formulae, diagrams, tables, charts, graphs and models to represent mathematical information. |
| Space | This concept refers to the frame of geometrical dimensions describing an entity. |
| Systems | This concept refers to groups of interrelated elements. |
| Validity | This concept refers to using well-founded, logical mathematics to come to a true and accurate conclusion or a reasonable interpretation of results. |

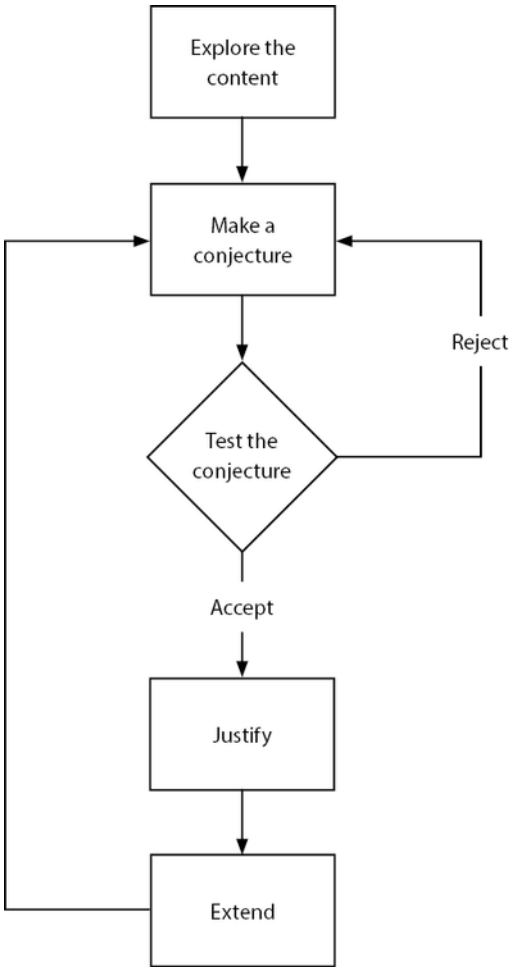
# Mathematical inquiry

Approaches to teaching and learning in the DP refer to deliberate strategies, skills and attitudes that permeate the teaching and learning environment. These approaches and tools are intrinsically linked to the IB learner profile, which encourages learning by experimentation, questioning and discovery.

In the IB classroom, students should regularly learn mathematics by being active participants in learning activities. Teachers should therefore provide students with regular opportunities to learn through mathematical inquiry, by making frequent use of strategies which stimulate students' critical thinking and problem-solving skills.

*Figure 4*

*The cycle of mathematical inquiry*



# Mathematical modelling

Mathematical modelling is an important technique used in problem solving, to make sense of the real world. It is often used to help us better understand a situation, to check the effects of change or to inform decision-making. Engaging students in the mathematical modelling process provides them with such opportunities. It is one of the most useful mathematical skills that students will need to be successful in many non-mathematical as well as mathematical courses and careers.

The mathematical modelling process begins with consideration of a situation that exists in the real world and is not usually artificially created. At this stage assumptions sometimes have to be made in order to simplify the situation to allow modelling to take place. There is often a fine balance needed between the simplicity and the accuracy of the model.

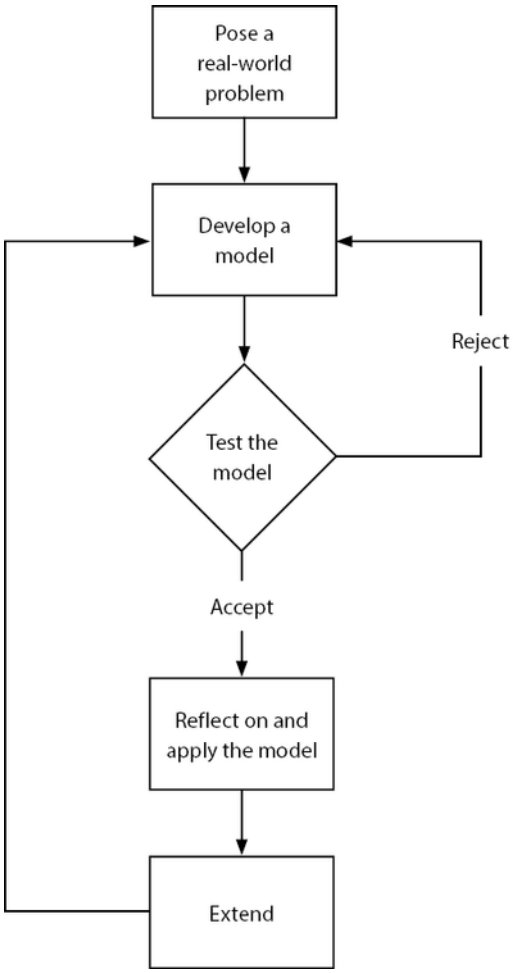
The first stage involves choosing or fitting a suitable mathematical representation to the context. This representation is tested to evaluate whether or not it returns expected results. The testing stage allows the results returned by the model to be reflected upon and adaptations to be made to the model if necessary. Once a satisfactory model is established it can be applied or used to explain a situation, check the effects of change or to inform decision-making.

The process of mathematical modelling requires critical reflection throughout the process.

More advice and guidance on the process of mathematical modelling is given in the teacher support material. The cycle of mathematical modelling is illustrated below.

*Figure 5*

*The cycle of mathematical modelling*



# Use of technology

The use of technology is an integral part of DP mathematics courses. Developing an appreciation of how developments in technology and mathematics have influenced each other is one of the aims of the courses and using technology accurately, appropriately and efficiently both to explore new ideas and to solve problems is one of the assessment objectives. Learning how to use different forms of technology is an important skill in mathematics and time has been allowed in each topic of the syllabus and through the "toolkit" in order to do this.

Technology is a powerful tool in mathematics and in recent years increased student and teacher access to this technology has supported and advanced the teaching and learning of mathematics. Discerning use of technology can make more mathematics accessible and motivating to a greater number of students.

Teachers can use technology to support and enhance student understanding in many ways including:

- to bring out teaching points
- to address misconceptions
- to aid visualization
- to enhance understanding of concepts that would otherwise be restricted by lengthy numerical calculations or algebraic manipulation
- to support students in making conjectures and checking generalizations
- to explicitly make the links between different mathematical representations or approaches.

Students can also use technology to engage with the learning process in many ways including the following:

- to develop and enhance their own personal conceptual understanding
- to search for patterns
- to test conjectures or generalizations
- to justify interpretations
- to collaborate on project based work
- to help organize and analyse data.

In the classroom teachers and students can use technology, working individually or collaboratively, to explore mathematical concepts. The key to successful learning of mathematics with technology is the fine balance between the teacher and student use of technology, with carefully chosen use of technology to support the understanding and the communication of the mathematics itself.

Many topics within the DP mathematics courses lend themselves to the use of technology. Graphical calculators, dynamic graphing software, spreadsheets, simulations, apps, dynamic geometry software and interactive whiteboard software are just a few of the many kinds of technology available to support the teaching and learning of mathematics.

Within the guide the term "technology" is used for any form of calculator, hardware or software that may be available in the classroom. The terms "analysis" and "analytic approach" are generally used in the guide to indicate an algebraic approach that may not require the use of technology. It is important to note there will be restrictions on which technology may be used in examinations, which will be detailed in relevant documents.

As teachers tie together the unifying themes of **mathematical inquiry**, **mathematical modelling** and the **use of technology**, they should begin by providing substantial guidance and then gradually encourage students to become more independent as inquirers and thinkers. DP students should learn to become strong communicators through the language of mathematics. Teachers should create a learning environment in which students are comfortable as risk takers and inquiry, conceptual understanding, collaboration and use of technology feature prominently.

For further information on "Approaches to teaching and learning in the Diploma Programme", please refer to the publication *The Diploma Programme: From principles into practice*. To support teachers, a variety of resources can be found on the programme resource centre and details of workshops for professional development are available on the public website.

# Format of the syllabus

The format of the syllabus section of the mathematics guides is the same for each subject and each level. This structure gives prominence and focus to the aspects of teaching and learning, including conceptual understandings, content and enrichment.

There are five topics and within these topics there are sub-topics. The five topics are:

- number and algebra
- functions
- geometry and trigonometry
- probability and statistics
- calculus

Each topic begins with a section on conceptual understandings. Details are given of which of the twelve key concepts could be used to relate to the topic. Essential understandings give details of the overall aims of the topic and then content-specific conceptual understandings give specific details of the aims and purpose of the topic and sub-topics.

Each topic begins with SL content which is common to both Mathematics: analysis and approaches and to Mathematics: applications and interpretation.

Each topic has SL content followed by additional higher level (AHL) content. Teachers should ensure that they cover all SL content with SL students, and all SL and AHL content with HL students. The topics are structured so that an informal introduction in the common content may be formalized in the SL content and then extended further in the AHL content. For example the extension of the set of numbers for which something is defined could be integers in the common content, positive real numbers in the SL content and all real and complex numbers in the AHL content.

The content of all five topics at the appropriate level must be taught, however not necessarily in the order in which they appear in the guide. Teachers are expected to construct a course of study that addresses the needs of their students and includes, where necessary, the topics noted in prior learning. Guidance on structuring a course is given in the teacher support material.

Each topic has three sections:

**Content:** The column on the left contains details of the sub-topics to be covered.

**Guidance, clarification and syllabus links:** The column on the right contains more detailed information on specific sub-topics listed in the content column. This clarifies the content for the examination and highlights where sub-topics relate to other sub-topics within the syllabus.

**Connections:** Each topic also contains a short section that provides suggestions for further discussion, including real-life examples and ideas for further investigation.

These suggestions are only a guide and are not exhaustive. A downloadable version of these sections is also available, so that additional connections can be added to the ones already suggested by the IB. Potential areas for connections include:

- **Other contexts: Real-life examples**
- **Links to other subjects:** Suggested links to other subjects within the DP. Note that these are correct for the current (2019) published versions of the guides
- **Aim:** Links to the aims of the course
- **International-mindedness:** Suggestions for discussions
- **TOK:** Suggestions for discussions
- **Links to TSM:** Links to the teacher support materials (TSM) in the "in practice" section of the website

- **Link to specimen paper:** Links to specific questions exemplifying how topics might be examined
- **Use of technology:** Suggestions as to how technology can be used in the classroom to enhance understanding
- **Links to websites:** Suggested websites for use in teaching or learning activities
- **Enrichment:** Suggestions for further discussions that may reinforce understanding.

# Planning your course

The syllabus as provided in the subject guide is not intended to be a teaching order. Instead it provides detail of what must be covered by the end of the course. A school should develop a scheme of work that best works for its own students. For example, the scheme of work could be developed to match available resources, to take into account prior learning and experience, or in conjunction with other local requirements.

HL teachers may choose to teach the SL content and AHL content at the same time or teach them in a spiral fashion, by teaching the SL content first and revisiting these topics through the delivery of the AHL topics after this.

However the course is planned, adequate time must be provided for examination revision. Time must also be given to students to reflect on their learning experience and their growth as learners.

# Time allocation

The recommended teaching time for HL courses is 240 hours and for SL courses is 150 hours. For mathematics courses at both SL and HL, it is expected that 30 hours will be spent on developing inquiry, modelling and investigation skills. This includes up to 15 hours for work on the internal assessment which is called the exploration. The time allocations given in this guide are approximate, and are intended to suggest how the remaining 210 hours for HL and 120 hours for SL allowed for the teaching of the syllabus might be allocated. The exact time spent on each topic depends on a number of factors, including the background knowledge and level of preparedness of each student. Teachers should therefore adjust these timings to correspond to the needs of their students.

# The Toolkit

Time has been allocated within the teaching hours for students to undertake the types of activities that mathematicians in the real world undertake and to allow students time to develop the skill of thinking like a mathematician–in other words providing students with a mathematical toolkit which will allow them to approach any type of mathematical problem. Underpinning this are the six pedagogical approaches to teaching and the five approaches to learning which support all IB programmes. This time gives students opportunities in the classroom for undertaking an inquiry-based approach and focusing on conceptual understanding of the content, developing their awareness of mathematics in local and global contexts, gives them opportunities for teamwork and collaboration as well as time to reflect upon their own learning of mathematics.

Students should be encouraged to actively identify skills that they might add to their personal mathematics toolkit. Teachers are encouraged to make explicit where these skills might transfer across areas of mathematics content and allow students to reflect upon where these skills transfer to other subjects the student is studying.

The teacher support material (TSM) contains a section referred to as the “toolkit”. This section contains ideas and resources that teachers can use with their students to encourage the development of mathematical thinking skills. These resources have been developed by teachers for use in their own classrooms and are not exhaustive.

# Formulae and the formula booklet

Formulae are only included in this guide document where there may be some ambiguity. All formulae required for the course are in the mathematics formula booklet.

From the beginning of the course it is recommended that teachers ensure students are familiar with the contents of the formula booklet by either giving students a printed copy or making an electronic copy available to them.

Each student is required to have access to a clean copy of the formula booklet during the examination. For each examination, it is the responsibility of the school to download a copy of the formula booklet from IBIS or the programme resource centre, check that there are no printing errors, and ensure that there are sufficient copies available for all students.

# Command terms and notation list

Teachers and students need to be familiar with the IB notation list and the command terms, as these will be used without explanation in the examination papers. The “Glossary of command terms” and “Notation list” appear as appendices in this guide.

# Aims

The aims of all DP mathematics courses are to enable students to:

1. develop a curiosity and enjoyment of mathematics, and appreciate its elegance and power
2. develop an understanding of the concepts, principles and nature of mathematics
3. communicate mathematics clearly, concisely and confidently in a variety of contexts
4. develop logical and creative thinking, and patience and persistence in problem solving to instil confidence in using mathematics
5. employ and refine their powers of abstraction and generalization
6. take action to apply and transfer skills to alternative situations, to other areas of knowledge and to future developments in their local and global communities
7. appreciate how developments in technology and mathematics influence each other
8. appreciate the moral, social and ethical questions arising from the work of mathematicians and the applications of mathematics
9. appreciate the universality of mathematics and its multicultural, international and historical perspectives
10. appreciate the contribution of mathematics to other disciplines, and as a particular "area of knowledge" in the TOK course
11. develop the ability to reflect critically upon their own work and the work of others
12. independently and collaboratively extend their understanding of mathematics.

# Assessment objectives

Problem solving is central to learning mathematics and involves the acquisition of mathematical skills and concepts in a wide range of situations, including non-routine, open-ended and real-world problems. Having followed a DP mathematics course, students will be expected to demonstrate the following:

1. **Knowledge and understanding:** Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of familiar and unfamiliar contexts.
2. **Problem solving:** Recall, select and use their knowledge of mathematical skills, results and models in both abstract and real-world contexts to solve problems.
3. **Communication and interpretation:** Transform common realistic contexts into mathematics; comment on the context; sketch or draw mathematical diagrams, graphs or constructions both on paper and using technology; record methods, solutions and conclusions using standardized notation; use appropriate notation and terminology.
4. **Technology:** Use technology accurately, appropriately and efficiently both to explore new ideas and to solve problems.
5. **Reasoning:** Construct mathematical arguments through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions.
6. **Inquiry approaches:** Investigate unfamiliar situations, both abstract and from the real world, involving organizing and analyzing information, making conjectures, drawing conclusions, and testing their validity.

# Assessment objectives in practice

| Assessment objectives | Paper 1 % | Paper 2 % | Paper 3 % HL only | Exploration % |
|---|---|---|---|---|
| Knowledge and understanding | 20-30 | 20-30 | 10-20 | 5-15 |
| Problem solving | 20-30 | 20-30 | 20-30 | 5-20 |
| Communication and interpretation | 20-30 | 20-30 | 20-30 | 15-25 |
| Technology | 20-35 | 20-35 | 10-30 | 10-20 |
| Reasoning | 5-15 | 10-20 | 10-20 | 5-25 |
| Inquiry approaches | 5-15 | 5-20 | 15-30 | 25-35 |

# Syllabus outline

| Syllabus component | Suggested teaching hours—SL | Suggested teaching hours—HL |
|---|---|---|
| **Topic 1—Number and algebra** | 16 | 29 |
| **Topic 2—Functions** | 31 | 42 |
| **Topic 3—Geometry and trigonometry** | 18 | 46 |
| **Topic 4—Statistics and probability** | 36 | 52 |
| **Topic 5—Calculus** | 19 | 41 |
| **The "toolkit" and Mathematical exploration**<br>Investigative, problem-solving and modelling skills development leading to an individual exploration. The exploration is a piece of written work that involves investigating an area of mathematics. | 30 | 30 |
| **Total teaching hours** | 150 | 240 |

All topics are compulsory. Students must study all the sub-topics in each of the topics in the syllabus as listed in this guide. Students are also required to be familiar with the topics listed as prior learning.

# Prior learning topics

Prior to starting a DP mathematics course students have extensive previous mathematical experiences, but these will vary. It is expected that mathematics students will be familiar with the following topics before they take the examinations because questions assume knowledge of them. Teachers must therefore ensure that any topics listed here that are unknown to their students at the start of the course are included at an early stage. Teachers should also take into account the existing mathematical knowledge of their students to design an appropriate course of study for mathematics. This table lists the knowledge, together with the syllabus content, that is essential for successful completion of the mathematics course.

## Number and algebra

- Number systems: natural numbers $\mathbb{N}$; integers, $\mathbb{Z}$; rationals, $\mathbb{Q}$, and irrationals; real numbers, $\mathbb{R}$
- SI (Système International) units for mass, time, length and their derived units, eg. speed, area and volume
- Rounding, decimal approximations and significant figures, including appreciation of errors
- Definition and elementary treatment of absolute value (modulus), $|a|$
- Use of addition, subtraction, multiplication and division using integers, decimals and fractions, including order of operations
- Prime numbers, factors (divisors) and multiples
- Greatest common factor (divisor) and least common multiples (HL only)
- Simple applications of ratio, percentage and proportion
- Manipulation of algebraic expressions, including factorization and expansion
- Rearranging formulae
- Calculating the numerical value of expressions by substitution
- Evaluating exponential expressions with simple positive exponents
- Evaluating exponential expressions with rational exponents (HL only)
- Use of inequalities, $<$ , $\leq$ , $>$ , $\geq$, intervals on the real number line
- Simplification of simple expressions involving roots (surds or radicals)
- Rationalising the denominator (HL only)
- Expression of numbers in the form $a \times 10^k$, $1 \leq a < 10$, $k \in \mathbb{Z}$
- Familiarity with commonly accepted world currencies
- Solution of linear equations and inequalities
- Solution of quadratic equations and inequalities with rational coefficients (HL only)
- Solving systems of linear equations in two variables
- Concept and basic notation of sets. Operations on sets: union and intersection
- Addition and subtraction of algebraic fractions (HL only).

## Functions

- Graphing linear and quadratic functions using technology
- Mappings of the elements of one set to another. Illustration by means of sets of ordered pairs, tables, diagrams and graphs.

# Geometry and trigonometry

- Pythagoras' theorem and its converse
- Mid-point of a line segment and the distance between two points in the Cartesian plane
- Geometric concepts: point, line, plane, angle
- Angle measurement in degrees, compass directions
- The triangle sum theorem
- Right-angle trigonometry, including simple applications for solving triangles
- Three-figure bearings
- Simple geometric transformations: translation, reflection, rotation, enlargement
- The circle, its centre and radius, area and circumference. The terms diameter, arc, sector, chord, tangent and segment
- Perimeter and area of plane figures. Properties of triangles and quadrilaterals, including parallelograms, rhombuses, rectangles, squares, kites and trapezoids; compound shapes
- Familiarity with three-dimensional shapes (prisms, pyramids, spheres, cylinders and cones)
- Volumes and surface areas of cuboids, prisms, cylinders, and compound three-dimensional shapes

# Statistics and probability

- The collection of data and its representation in bar charts, pie charts, pictograms, and line graphs
- Obtaining simple statistics from discrete data, including mean, median, mode, range
- Calculating probabilities of simple events
- Venn diagrams for sorting data
- Tree diagrams

# Calculus

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

# Syllabus content

## Topic 1: Number and algebra

### Concepts

#### Essential understandings

Number and algebra allow us to represent patterns, show equivalencies and make generalizations which enable us to model real-world situations. Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

#### Suggested concepts embedded in this topic

Generalization, representation, modelling, equivalence, approximation, quantity

**AHL:** Systems, relationships.

#### Content-specific conceptual understandings:

- Modelling real-life situations with the structure of arithmetic and geometric sequences and series allows for prediction, analysis and interpretation.
- Different representations of numbers enable quantities to be compared and used for computational purposes with ease and accuracy.
- Numbers and formulae can appear in different, but equivalent forms, or representations, which can help us to establish identities.
- Formulae are a generalization made on the basis of specific examples, which can then be extended to new examples
- Mathematical financial models such as compounded growth allow computation, evaluation and interpretation of debt and investment both approximately and accurately.
- Approximation of numbers adds uncertainty or inaccuracy to calculations, leading to potential errors but can be useful when handling extremely large or small quantities.
- Quantities and values can be used to describe key features and behaviours of functions and models, including quadratic functions.

***AHL***

- Utilizing complex numbers provides a system to efficiently simplify and solve problems.
- Matrices allow us to organize data so that they can be manipulated and relationships can be determined.
- Representing abstract quantities using complex numbers in different forms enables the solution of real-life problems.

### SL content

**Recommended teaching hours: 16**

The aim of the standard level (SL) content of the number and algebra topic is to introduce students to numerical concepts and techniques which combined with an introduction to arithmetic and geometric sequences and series can be used for financial and other applications.

Sections SL1.1 to SL1.5 are content common to Mathematics: analysis and approaches and Mathematics: applications and interpretation.

## SL 1.1

| Content | Guidance, clarification and syllabus links |
|---|---|
| Operations with numbers in the form $a \times 10^k$ where $1 \leq a < 10$ and $k$ is an integer. | Calculator or computer notation is **not** acceptable. For example, 5.2E30 is **not** acceptable and should be written as $5.2 \times 10^{30}$. |

***Connections***

**Other contexts:** Very large and very small numbers, for example astronomical distances, sub-atomic particles in physics, global financial figures

**Links to other subjects:** Chemistry (Avogadro's number); physics (order of magnitude); biology (microscopic measurements); sciences (uncertainty and precision of measurement)

**International-mindedness:** The history of number from Sumerians and its development to the present Arabic system

**TOK:** Do the names that we give things impact how we understand them? For instance, what is the impact of the fact that some large numbers are named, such as the googol and the googolplex, while others are represented in this form?

Download connections template

## SL 1.2

| Content | Guidance, clarification and syllabus links |
|---|---|
| Arithmetic sequences and series.<br>Use of the formulae for the *n*th term and the sum of the first *n* terms of the sequence.<br>Use of sigma notation for sums of arithmetic sequences. | Spreadsheets, GDCs and graphing software may be used to generate and display sequences in several ways.<br>If technology is used in examinations, students will be expected to identify the first term and the common difference. |
| Applications. | Examples include simple interest over a number of years. |
| Analysis, interpretation and prediction where a model is not perfectly arithmetic in real life. | Students will need to approximate common differences. |

***Connections***

**International-mindedness:** Aryabhatta is sometimes considered the "father of algebra"–compare with alKhawarizmi; the use of several alphabets in mathematical notation (for example the use of capital sigma for the sum).

**TOK:** Is all knowledge concerned with identification and use of patterns? Consider Fibonacci numbers and connections with the golden ratio.

Download connections template

## SL 1.3

| Content | Guidance, clarification and syllabus links |
|---|---|
| Geometric sequences and series.<br>Use of the formulae for the *n*th term and the sum of the first *n* terms of the sequence. | Spreadsheets, GDCs and graphing software may be used to generate and display sequences in several ways. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| Use of sigma notation for the sums of geometric sequences. | If technology is used in examinations, students will be expected to identify the first term and the ratio.<br>**Link to:** models/functions in topic 2 and regression in topic 4. |
| Applications. | Examples include the spread of disease, salary increase and decrease and population growth. |

***Connections***

**Links to other subjects:** Radioactive decay, nuclear physics, charging and discharging capacitors (physics).

**International-mindedness:** The chess legend (Sissa ibn Dahir)

**TOK:** How do mathematicians reconcile the fact that some conclusions seem to conflict with our intuitions? Consider for instance that a finite area can be bounded by an infinite perimeter.

Download connections template

## SL 1.4

| Content | Guidance, clarification and syllabus links |
|---|---|
| Financial applications of geometric sequences and series:<br>• compound interest<br>• annual depreciation. | Examination questions may require the use of technology, including built-in financial packages.<br>The concept of simple interest may be used as an introduction to compound interest.<br>Calculate the real value of an investment with an interest rate and an inflation rate.<br>In examinations, questions that ask students to derive the formula will not be set.<br>Compound interest can be calculated yearly, half-yearly, quarterly or monthly.<br>**Link to**: exponential models/functions in topic 2. |

***Connections***

**Other contexts:** Loans.

**Links to other subjects:** Loans and repayments (economics and business management).

**Aim 8:** Ethical perceptions of borrowing and lending money.

**International-mindedness:** Do all societies view investment and interest in the same way?

**TOK:** How have technological advances affected the nature and practice of mathematics? Consider the use of financial packages for instance.

**Enrichment:** The concept of e can be introduced through continuous compounding, $(1+\frac{1}{n})^n \to \mathrm{e}$, as $n \to \infty$, however this will not be examined.

Download connections template

## SL 1.5

| Content | Guidance, clarification and syllabus links |
|---|---|
| Laws of exponents with integer exponents. | **Examples:** |

| Content | Guidance, clarification and syllabus links |
|---|---|
| | $5^3 \times 5^{-6} = 5^{-3}$, $6^4 \div 6^3 = 6$, $\left(2^3\right)^{-4} = 2^{-12}$, $(2x)^4 = 16x^4$, $2x^{-3} = \frac{2}{x^3}$ |
| Introduction to logarithms with base 10 and e.<br>Numerical evaluation of logarithms using technology. | Awareness that $a^x = b$ is equivalent to $\log_a b = x$, that $b > 0$, and $\log_e x = \ln x$. |

***Connections***

**Other contexts:** Richter scale and decibel scale.

**Links to other subjects:** Calculation of pH and buffer solutions (chemistry)

**TOK**: Is mathematics invented or discovered? For instance, consider the number e or logarithms–did they already exist before man defined them? (This topic is an opportunity for teachers to generate reflection on "the nature of mathematics").

Download connections template

## SL 1.6

| Content | Guidance, clarification and syllabus links |
|---|---|
| Approximation: decimal places, significant figures. | Students should be able to choose an appropriate degree of accuracy based on given data. |
| Upper and lower bounds of rounded numbers. | If $x = 4.1$ to one decimal place, $4.05 \leq x < 4.15$. |
| Percentage errors. | Students should be aware of, and able to calculate, measurement errors (such as rounding errors or measurement limitations). For example finding the maximum percentage error in the area of a circle if the radius measured is 2.5 cm to one decimal place. |
| Estimation. | Students should be able to recognize whether the results of calculations are reasonable. For example lengths cannot be negative. |

***Connections***

**Other contexts:** Currency approximations are often to nearest whole number, for example peso, yen; to nearest cent, euro, dollar, pound; meteorology, alternative rounding methods.

**Links to other subjects:** Order of magnitudes (physics); uncertainty and precision of measurement (sciences).

**Aim 8:** Caring about approximations; ethical implications.

**TOK:** Is mathematical reasoning different from scientific reasoning, or reasoning in other areas of knowledge?

Download connections template

## SL 1.7

| Content | Guidance, clarification and syllabus links |
|---|---|
| Amortization and annuities using technology. | Technology includes the built-in financial packages of graphic display calculators, spreadsheets. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| | In examinations the payments will be made at the end of the period.<br>Knowledge of the annuity formula will enhance understanding but will not be examined.<br>**Link to**: exponential models (SL 2.5). |

***Connections***

**Other contexts:** Evaluating the real value of an investment when affected by interest rates and inflation rates. Credit card debt, student loans, retirement planning.

**Links to other subjects:** Exchange rates (economics), loans (business management).

**Aim 8:** Ethical perceptions of borrowing and lending money; short-term loans at high interest rates, how can knowledge of mathematics result in individuals being exploited or protected from extortion?

**International-mindedness**: Do all societies view investment and interest in the same way?

Download connections template

### SL 1.8

| Content | Guidance, clarification and syllabus links |
|---|---|
| Use technology to solve:<br>• Systems of linear equations in up to 3 variables<br>• Polynomial equations | In examinations, no specific method of solution will be required.<br>In examinations, there will always be a unique solution to a system of equations.<br>Standard terminology, such as zeros or roots, should be taught.<br>**Link to:** quadratic models (SL 2.5) |

***Connections***

**Links to other subjects:** Kirchhoff's laws (physics).

**TOK:** What role does language play in the accumulation and sharing of knowledge in mathematics? Consider for example that when mathematicians talk about "imaginary" or "real" solutions they are using precise technical terms that do not have the same meaning as the everyday terms.

Download connections template

## AHL content

**Recommended teaching hours: 13**

The aim of the AHL content in the number and algebra topic is to extend and build upon the aims, concepts and skills from the standard level content. It introduces students to the laws of logarithms, the important mathematical concepts of complex numbers and matrices, and their applications.

### AHL 1.9

| Content | Guidance, clarification and syllabus links |
|---|---|
| Laws of logarithms:<br>$\log_a xy = \log_a x + \log_a y$ | In examinations, $a$ will equal 10 or e.<br>**Link to:** scaling large and small numbers (AHL 2.10). |

| Content | Guidance, clarification and syllabus links |
|---|---|
| $\log_a \frac{x}{y} = \log_a x - \log_a y$<br>$\log_a x^m = m\log_a x$<br>for $a, \quad x, \quad y \quad > \quad 0$ | |

***Connections***

**Links to other subjects:** pH, buffer calculations and finding activation energy from experimental data (chemistry).

**TOK:** What is meant by the terms "law" and "theory" in mathematics. How does this compare to how these terms are used in different areas of knowledge?

Download connections template

## AHL 1.10

| Content | Guidance, clarification and syllabus links |
|---|---|
| Simplifying expressions, both numerically and algebraically, involving rational exponents. | **Examples:** $5^{\frac{1}{2}} \times 5^{\frac{1}{3}} = 5^{\frac{5}{6}}$, $6^{\frac{3}{4}} \div 6^{\frac{1}{2}} = 6^{\frac{1}{4}}$, $32^{\frac{3}{5}} = 8$, $x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$ |

***Connections***

Download connections template

## AHL 1.11

| Content | Guidance, clarification and syllabus links |
|---|---|
| The sum of infinite geometric sequences. | **Link to:** the concept of a limit (SL 5.1), fractals (AHL 3.9), and Markov chains (AHL 4.19). |

***Connections***

**Other contexts:** Total distance travelled by a bouncing ball.

**TOK:** Is it possible to know about things of which we can have no experience, such as infinity?

Download connections template

## AHL 1.12

| Content | Guidance, clarification and syllabus links |
|---|---|
| Complex numbers: the number i such that $\mathrm{i}^2 = -1$.<br>Cartesian form: $z = a + b\mathrm{i}$; the terms real part, imaginary part, conjugate, modulus and argument.<br>Calculate sums, differences, products, quotients, by hand and with technology. Calculating powers of complex numbers, in Cartesian form, with technology. | |

| Content | Guidance, clarification and syllabus links |
|---|---|
| The complex plane.<br>Complex numbers as solutions to quadratic equations of the form $ax^2 + bx + c = 0, a \neq 0$, with real coefficients where $b^2 - 4ac < 0$. | Use and draw Argand diagrams.<br>Quadratic formula and the link with the graph of $f(x) = ax^2 + bx + c$. |

***Connections***

**TOK:** How does language shape knowledge? For example do the words "imaginary" and "complex" make the concepts more difficult than if they had different names?

Download connections template

## AHL 1.13

| Content | Guidance, clarification and syllabus links |
|---|---|
| Modulus–argument (polar) form:<br>$z = r(\cos\theta + \mathrm{i}\sin\theta) = rcis\theta$. | |
| Exponential form:<br>$z = r\mathrm{e}^{\mathrm{i}\theta}$. | Exponential form is sometimes called the Euler form. |
| Conversion between Cartesian, polar and exponential forms, by hand and with technology. | |
| Calculate products, quotients and integer powers in polar or exponential forms. | In examinations students will not be required to find the roots of complex numbers. |
| Adding sinusoidal functions with the same frequencies but different phase shift angles. | Phase shift and voltage in circuits as complex quantities.<br>**Example:** Two AC voltage sources are connected in a circuit. If $V_1 = 10\cos(40t)$ and $V_2 = 20\cos(40t + 10)$ find an expression for the total voltage in the form $V = A\cos(40t + B)$. |
| Geometric interpretation of complex numbers. | Addition and subtraction of complex numbers can be represented as vector addition and subtraction. Multiplication of complex numbers can be represented as a rotation and a stretch in the Argand diagram. |

***Connections***

**TOK:** Why might it be said that $\mathrm{e}^{\mathrm{i}\pi}+1 = 0$ is beautiful? What is the place of beauty and elegance in mathematics? What about the place of creativity?

**Enrichment:** Solution of differential equations by separation of variables (AHL5.15), as both polar and exponential forms are solutions of $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \mathrm{i}y$.

Download connections template

## AHL 1.14

| Content | Guidance, clarification and syllabus links |
|---|---|
| Definition of a matrix: the terms element, row, column and order for $m \times n$ matrices. | |
| Algebra of matrices: equality; addition; subtraction; multiplication by a scalar for $m \times n$ matrices. | Including use of technology. |
| Multiplication of matrices.<br>Properties of matrix multiplication:<br>associativity, distributivity and non-commutativity. | Multiplying matrices to solve practical problems. |
| Identity and zero matrices.<br>Determinants and inverses of $n \times n$ matrices with technology, and by hand for $2 \times 2$ matrices. | Students should be familiar with the notation $\boldsymbol{I}$ and $\boldsymbol{0}$. |
| Awareness that a system of linear equations can be written in the form $\boldsymbol{A}x = b$. | In examinations $\boldsymbol{A}$ will always be an invertible matrix, except when solving for eigenvectors. |
| Solution of the systems of equations using inverse matrix. | Model and solve real-life problems including:<br>Coding and decoding messages<br>Solving systems of equations.<br>**Link to:** Markov chains (AHL 4.19), transition matrices (AHL 4.19) and phase portrait (AHL 5.17). |

***Connections***

**Other contexts:** Comparing sales/revenue/profit for multiple products over multiple weeks.

**TOK:** Given the many applications of matrices in this course, consider the fact that mathematicians marvel at some of the deep connections between disparate parts of their subject. Is this evidence for a simple underlying mathematical reality? Mathematics, sense, perception and reason–if we can find solutions of higher dimensions, can we reason that these spaces exist beyond our sense perception?

**External link:** Simulation for encoding and decoding messages using various methods including exercises:

| Download connections template |
|---|

## AHL 1.15

| Content | Guidance, clarification and syllabus links |
|---|---|
| Eigenvalues and eigenvectors.<br>Characteristic polynomial of $2 \times 2$ matrices.<br>Diagonalization of $2 \times 2$ matrices (restricted to the case where there are distinct real eigenvalues). | Students will only be expected to perform calculations by hand and with technology for $2 \times 2$ matrices. |
| Applications to powers of $2 \times 2$ matrices. | Applications, for example movement of population between two towns, predator/prey models.<br>$\boldsymbol{M}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors, and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues.<br>**Link to:** coupled differential equations (AHL 5.17). |

***Connections***

**Other contexts:** Invariant states; representation of conics.

**Links to other subjects:** Stochastic processes, stock market values and trends (business management).

**Aim 8:** Damping noise in car design, test for cracks in solid objects, oil exploration, the Google PageRank formula, and "the $25 billion dollar eigenvector"; the natural frequency of an object can be characterized by the eigenvalue of smallest magnitude (1940 Tacoma Narrows bridge disaster).

**TOK:** Mathematics can be used successfully to model real-world processes. Is this because mathematics was created to mirror the world or because the world is intrinsically mathematical?

**Enrichment:** Principal component and factor analysis. Link between discrete change and continuous change in dynamical systems (including why e is such an important number).

Download connections template

# Topic 2: Functions

## Concepts

### Essential understandings:

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variable. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and/or tables represents different ways to communicate mathematical ideas.

### Suggested concepts embedded in this topic:

Representation, relationships, space, modelling, change.

**AHL:** Generalization, validity.

### Content-specific conceptual understandings:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation may correspond to notable geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less of the function to best suit our needs.
- Changing the parameters of a trigonometric function changes the position, orientation and shape of the corresponding graph.
- Different representations facilitate modelling and interpretation of physical, social, economic and mathematical phenomena, which support solving real-life problems.
- Technology plays a key role in allowing humans to represent the real world as a model and to quantify the appropriateness of the model.

***AHL***

- Extending results from a specific case to a general form and making connections between related functions allows us to better understand physical phenomena.
- Generalization provides an insight into variation and allows us to access ideas such as half-life and scaling logarithmically to adapt theoretical models and solve complex real-life problems.
- Considering the reasonableness and validity of results helps us to make informed, unbiased decisions.

# SL content

**Recommended teaching hours: 31**

The aim of the standard level content in the functions topic is to introduce students to the important unifying theme of a function in mathematics and the skills needed to model and interpret practical situations with a variety of key functions.

Throughout this topic students should be given the opportunity to use technology such as graphing packages and graphing calculators to develop and apply their knowledge of functions, rather than using elaborate analytical techniques.

On examination papers:

- questions may be set requiring the graphing of functions that do not explicitly appear on the syllabus
- the domain will be the largest possible domain for which a function is defined unless otherwise stated; this will usually be the real numbers.

Sections SL2.1 to SL2.4 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

## SL 2.1

| Content | Guidance, clarification and syllabus links |
|---|---|
| Different forms of the equation of a straight line.<br>Gradient; intercepts.<br>Lines with gradients $m_1$ and $m_2$<br>Parallel lines $m_1 = m_2$.<br>Perpendicular lines $m_1 \times m_2 = -1$. | $y = mx + c$ (gradient-intercept form).<br>$ax + by + d = 0$ (general form).<br>$y - y_1 = m(x - x_1)$ (point-gradient form).<br>Calculate gradients of inclines such as mountain roads, bridges, etc. |

***Connections***

**Other contexts:** Gradients of mountain roads, gradients of access ramps.

**Links to other subjects:** Exchange rates and price and income elasticity, demand and supply curves (economics); graphical analysis in experimental work (sciences).

**TOK:** Descartes showed that geometric problems could be solved algebraically and vice versa. What does this tell us about mathematical representation and mathematical knowledge?

Download connections template

## SL 2.2

| Content | Guidance, clarification and syllabus links |
|---|---|
| Concept of a function, domain, range and graph.<br>Function notation, for example $f(x)$, $v(t)$, $C(n)$.<br>The concept of a function as a mathematical model. | **Example:** $f(x) = \sqrt{2 - x}$, the domain is $x \leq 2$, range is $f(x) \geq 0$.<br>A graph is helpful in visualizing the range. |
| Informal concept that an inverse function reverses or undoes the effect of a function.<br>Inverse function as a reflection in the line $y = x$, and the notation $f^{-1}(x)$. | **Example:** Solving $f(x) = 10$ is equivalent to finding $f^{-1}(10)$.<br>Students should be aware that inverse functions exist for one to one functions; the domain of $f^{-1}(x)$ is equal to the range of $f(x)$. |

***Connections***

**Other contexts:** Temperature and currency conversions.

**Links to other subjects:** Currency conversions and cost functions (economics and business management); projectile motion (physics).

**Aim 8:** What is the relationship between real-world problems and mathematical models?

**International-mindedness:** The development of functions by Rene Descartes (France), Gottfried Wilhelm Leibnitz (Germany) and Leonhard Euler (Switzerland); the notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries–how did the notation we use today become internationally accepted?

**TOK:** Do you think mathematics or logic should be classified as a language?

Download connections template

## SL 2.3

| Content | Guidance, clarification and syllabus links |
|---|---|
| The graph of a function; its equation $y = f(x)$. | Students should be aware of the difference between the command terms "draw" and "sketch". |
| Creating a sketch from information given or a context, including transferring a graph from screen to paper.<br>Using technology to graph functions including their sums and differences. | All axes and key features should be labelled.<br>This may include functions not specifically mentioned in topic 2. |

***Connections***

**Links to other subjects:** Sketching and interpreting graphs (sciences, geography, economics).

**TOK:** Does studying the graph of a function contain the same level of mathematical rigour as studying the function algebraically? What are the advantages and disadvantages of having different forms and symbolic language in mathematics?

Download connections template

## SL 2.4

| Content | Guidance, clarification and syllabus links |
|---|---|
| Determine key features of graphs. | Maximum and minimum values; intercepts; symmetry; vertex; zeros of functions or roots of equations; vertical and horizontal asymptotes using graphing technology. |
| Finding the points of intersection of two curves or lines using technology. | |

***Connections***

**Links to other subjects:** Identification and interpretation of key features of graphs (sciences, geography, economics); production possibilities curve model, market equilibrium (economics).

**International-mindedness:** Bourbaki group analytical approach versus the Mandelbrot visual approach.

**Use of technology:** Graphing technology with sliders to determine the effects of altering parameters and variables.

Download connections template

## SL 2.5

| Content | Guidance, clarification and syllabus links |
|---|---|
| Modelling with the following functions: | |
| Linear models.<br>$f(x) = mx + c$. | Including piecewise linear models, for example horizontal distances of an object to a wall, depth of a swimming pool, mobile phone charges.<br>**Link to:** equation of a straight line (SL 2.1) and arithmetic sequences (SL 1.2). |
| Quadratic models.<br>$f(x) = ax^2 + bx + c \quad ; \quad a \neq 0$. Axis of symmetry, vertex, zeros and roots, intercepts on the $x$-axis and $y$ -axis. | Technology can be used to find roots.<br>**Link to:** use of technology to solve quadratic equations (SL 1.8). |
| Exponential growth and decay models.<br>$f(x) = ka^x + c$<br>$f(x) = ka^{-x} + c \quad (\text{for } a > 0)$<br>$f(x) = ke^{rx} + c$<br>Equation of a horizontal asymptote. | **Link to:** compound interest (SL 1.4), geometric sequences and series (SL 1.3) and amortization (SL 1.7). |
| Direct/inverse variation:<br>$f(x) = ax^n, \quad n \in \mathbb{Z}$<br>The $y$-axis as a vertical asymptote when $n < 0$. | |
| Cubic models:<br>$f(x) = ax^3 + bx^2 + cx + d$. | |
| Sinusoidal models:<br>$f(x) = a\sin(bx) + d, \quad f(x) = a\cos(bx) + d$. | Students will **not** be expected to translate between $\sin x$ and $\cos x$, and will only be required to predict or find amplitude ($\lvert a \rvert$), period ($\frac{360°}{b}$), or equation of the principal axis ($y = d$). |

***Connections***

**Other contexts:**

*Linear models:* Conversion graphs, for example Fahrenheit to Celsius, currency. Cost of hiring an item at a daily rate with a fixed deposit.

*Quadratic models:* Cost functions, satellite dishes, bridges, projectile motion.

*Exponential models:* Population growth, radioactive decay, cooling of a liquid, spread of a virus, compound interest, depreciation and amortization.

*Direct/inverse variation*: Boyle's law and Charles's law of gases, laws of supply and demand.

*Cubic models:* The volume of a box with a fixed surface area, the amount of wasted space in a can of tennis balls, power produced by a wind turbine and wind speed.

*Sinusoidal models:* Periodic phenomena that give rise to sinusoidal models, for example tides, weather patterns, motion of ferris and bicycle wheels, annual temperatures.

**Links to other subjects:** Population growth, spread of a virus (biology); radioactive decay and half-life, X-ray attenuation, cooling of a liquid, kinematics, simple harmonic motion, projectile motion, inverse square law (physics); compound interest, depreciation (business management); the circular flow of income model (economics); the equilibrium law and rates of reaction (chemistry); opportunities to model as part of experimental work (science)

**Aim 8:** The phrase "exponential growth" is used popularly to describe a number of phenomena. Is this a misleading use of the mathematical term?

**International-mindedness:** The Babylonian method of multiplication: $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$. Sulba Sutras in ancient India and the Bakhshali Manuscript contained an algebraic formula for solving quadratic equations.

**TOK:** What role do models play in mathematics? Do they play a different role in mathematics compared to their role in other areas of knowledge?

**Use of technology:** Generating parabolas using dynamic geometry software.

**Enrichment:** Conics–how can a parabola be created by cutting a cone?

Download connections template

## SL 2.6

| Content | Guidance, clarification and syllabus links |
|---|---|
| Modelling skills:<br>Use the modelling process described in the "mathematical modelling" section to create, fit and use the theoretical models in section SL2.5 and their graphs. | Fitting models using regression is covered in topic 4.<br>**Link to**: theoretical models (SL 2.5) to be used to develop the modelling skills and, for HL students, (AHL 2.9). |
| Develop and fit the model:<br>Given a context recognize and choose an appropriate model and possible parameters.<br>Determine a reasonable domain for a model. | |
| Find the parameters of a model. | By setting up and solving equations simultaneously (using technology), by consideration of initial conditions or by substitution of points into a given function.<br>At SL, students will not be expected to perform non-linear regressions, but will be expected to set up and solve up to three linear equations in three variables using technology. |
| Test and reflect upon the model:<br>Comment on the appropriateness and reasonableness of a model.<br>Justify the choice of a particular model, based on the shape of the data, properties of the curve and/or on the context of the situation. | |
| Use the model:<br>Reading, interpreting and making predictions based on the model. | Students should be aware of the dangers of extrapolation. |

***Connections***

**Links to other subjects:** opportunities to model as part of experimental work (science).

**TOK:** What is it about models in mathematics that makes them effective? Is simplicity a desirable characteristic in models?

Download connections template

## AHL content

**Recommended teaching hours: 11**

The aim of the AHL functions topic is to extend the aims, concepts and skills from the standard level content. It introduces students to further numerical and graphical techniques and further key functions which can be used to model and interpret practical situations.

### AHL 2.7

| Content | Guidance, clarification and syllabus links |
|---|---|
| Composite functions in context.<br>The notation $(f \circ g)(x) = f(g(x))$.<br>Inverse function $f^{-1}$, including domain restriction.<br>Finding an inverse function. | $\left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$.<br>**Example:** $f(x) = (x-3)^2 - 2$ has an inverse if the domain is restricted to $x \geq 3$ or to $x \leq 3$. |

***Connections***

Download connections template

### AHL 2.8

| Content | Guidance, clarification and syllabus links |
|---|---|
| Transformations of graphs. | Students will be expected to be able to perform transformations on all functions from the SL and AHL section of this topic, and others in the context of modelling real-life situations. |
| Translations: $y = f(x) + b$ ;$y = f(x-a)$. | Translation by the vector $\binom{3}{-2}$ denotes horizontal translation of 3 units to the right, and vertical translation of 2 units down. |
| Reflections: in the $x$ axis $y = -f(x)$, and in the $y$ axis $y = f(-x)$. | |
| Vertical stretch with scale factor $p$: $y = pf(x)$.<br>Horizontal stretch with scale factor $\frac{1}{q}$: $y = f(qx)$ | $x$ and $y$ axes are invariant. |
| Composite transformations. | Students should be made aware of the significance of the order of transformations.<br>**Example:** $y = x^2$ used to obtain $y = 3x^2 + 2$ by a vertical stretch of scale factor 3 followed by a translation of $\binom{0}{2}$.<br>**Example:** $y = \sin x$ used to obtain $y = 4\sin 2x$ by a vertical stretch of scale factor 4 and a horizontal stretch of scale factor $\frac{1}{2}$. |

***Connections***

**Other contexts:** Translating curves to reduce rounding errors for large values.

**Links to other subjects:** Shifting of supply and demand curves (economics); electromagnetic induction (physics)

**TOK:** Is mathematics independent of culture? To what extent are we aware of the impact of culture on what we believe or know?

Download connections template

## AHL 2.9

| Content | Guidance, clarification and syllabus links |
|---|---|
| In addition to the models covered in the SL content the AHL content extends this to include modelling with the following functions:<br>Exponential models to calculate half-life. | **Link to:** modelling skills (SL2.6). |
| Natural logarithmic models:<br>$f(x) = a + b\ln x$ | |
| Sinusoidal models:<br>$f(x) = a\sin(b(x-c)) + d$ | Radian measure should be assumed unless otherwise indicated by the use of the degree symbol, for example with $f(x) = \sin x°$.<br>In radians, period is $\frac{2\pi}{b}$.<br>Students should be aware that a horizontal translation of $c$ can be referred to as a phase shift.<br>**Link to:** radian measure (AHL 3.7) |
| Logistic models:<br>$f(x) = \frac{L}{1 + Ce^{-kx}}; L, \quad C, \quad k > 0$ | The logistic function is used in situations where there is a restriction on the growth. For example population on an island, bacteria in a petri dish or the increase in height of a person or seedling.<br>Horizontal asymptote at $f(x) = L$ is often referred to as the carrying capacity. |
| Piecewise models. | In some cases, parameters may need to be found that ensure continuity of the function, for example find $a$ to make $f(x) = \begin{cases} 1 + x, & 0 \le x < 2 \\ ax^2 + x, & x \ge 2 \end{cases}$ continuous.<br>The formal definition of continuity is not required.<br>In examinations, students may be expected to interpret and use other models that are introduced in the question. |

***Connections***

**Other contexts:**

*Sinusoidal models*: Waxing and waning of the moon, rainfall patterns, temperature, movement of bridges and buildings, pH scale, Richter scale, sound intensity, brightness of stars.

*Piecewise models*: Income taxes and taxi fares, friction, mobile phone plans, depth of pool as function of distance from the deep end (piecewise linear), postage rates for letters, stock prices, parachuting before

and after the chute opens, Hooke's law, shapes of buildings, horizontal distance from a wall to a curved object.

**Links to other subjects:** Half-life (chemistry and physics); AC circuits and waves (physics); the Gini coefficient and the Lorenz curve, and progressive, regressive and proportional taxes, the J-curve (economics)

**TOK:** Is there a hierarchy of areas of knowledge in terms of their usefulness in solving problems?

**Enrichment:** For the population equation $\frac{dP}{dt} = kP(1 - \frac{P}{L})$ with $P = P_0$ when $t = 0$, the solution is the logistic equation: $P = \frac{L}{1 + Ce^{-kt}}$, with $C = \frac{L}{P_0} - 1$.

Download connections template

### AHL 2.10

| Content | Guidance, clarification and syllabus links |
|---|---|
| Scaling very large or small numbers using logarithms.<br>Linearizing data using logarithms to determine if the data has an exponential or a power relationship using best-fit straight lines to determine parameters. | Choosing a manageable scale, for example for data with a wide range of values in one, or both variables and/or where the emphasis of a graph is the rate of growth, rather than the absolute value.<br>**Link to:** laws of logarithms (AHL 1.9) and Pearson's product moment correlation coefficient (SL 4.4). |
| Interpretation of log-log and semi-log graphs. | In examinations, students will not be expected to draw or sketch these graphs. |

***Connections***

**Other contexts:** Growth of bacteria or traffic to websites/social media; exponential graphs that show alarming absolute figures, but reasonable rates of growth.

**Links to other subjects:** pH semi-log curves and finding activation energy from experimental data (chemistry); exponential decay (physics); experimental work (sciences).

**TOK:** Does the applicability of knowledge vary across the different areas of knowledge? What would the implications be if the value of all knowledge was measured solely in terms of its applicability?

**Links to websites:** Gapminder makes use of log-log graphs: www.gapminder.org

Download connections template

# Topic 3: Geometry and trigonometry

## Concepts

### Essential understandings

Geometry and trigonometry allow us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This branch provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

### Suggested concepts embedded in this topic

Generalization, space, relationships, systems, representations

**AHL:** Quantity, change.

### Content-specific conceptual understandings:

- The properties of shapes are highly dependent on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.
- Different representations of trigonometric expressions help to simplify calculations.
- Systems of equations often, but not always, lead to intersection points.
- In two dimensions, the Voronoi diagram allows us to navigate, path-find or establish an optimum position.

***AHL***

- Different measurement systems can be used for angles to facilitate ease of calculation.
- Vectors allow us to determine position, change of position (movement) and force in two and three-dimensional space.
- Graph theory algorithms allow us to represent networks and to model complex real-world problems.
- Matrices are a form of notation which allow us to show the parameters or quantities of several linear equations simultaneously.

## SL content

**Recommended teaching hours: 18**

The aim of the standard level content of the geometry and trigonometry topic is to introduce students to appropriate skills and techniques for practical problem solving in two and three dimensions.

Throughout this topic students should be given the opportunity to use technology such as graphing packages, graphing calculators and dynamic geometry software to develop and apply their knowledge of geometry and trigonometry.

Sections SL3.1 to SL3.3 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

### SL 3.1

| Content | Guidance, clarification and syllabus links |
|---|---|
| The distance between two points in three-dimensional space, and their midpoint.<br>Volume and surface area of three-dimensional solids including right-pyramid, right cone, sphere, hemisphere and combinations of these solids.<br>The size of an angle between two intersecting lines or between a line and a plane. | In SL examinations, only right-angled trigonometry questions will be set in reference to three-dimensional shapes.<br>In problems related to these topics, students should be able to identify relevant right-angled triangles in three-dimensional objects and use them to find unknown lengths and angles. |

***Connections***

**Other contexts:** Architecture and design.

**Links to other subjects:** Design technology; volumes of stars and inverse square law (physics).

**TOK:** What is an axiomatic system? Are axioms self evident to everybody?

Download connections template

## SL 3.2

| Content | Guidance, clarification and syllabus links |
|---|---|
| Use of sine, cosine and tangent ratios to find the sides and angles of right-angled triangles. | In all areas of this topic, students should be encouraged to sketch well-labelled diagrams to support their solutions.<br>**Link to:** inverse functions (SL2.2) when finding angles. |
| The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$<br>The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$<br>$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$<br>Area of a triangle as $\frac{1}{2}ab\sin C$. | This section does not include the ambiguous case of the sine rule. |

***Connections***

**Other contexts:** Triangulation, map-making.

**Links to other subjects:** Vectors (physics).

**International-mindedness:** Diagrams of Pythagoras' theorem occur in early Chinese and Indian manuscripts. The earliest references to trigonometry are in Indian mathematics; the use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

**TOK:** Is it ethical that Pythagoras gave his name to a theorem that may not have been his own creation? What criteria might we use to make such a judgment?

Download connections template

## SL 3.3

| Content | Guidance, clarification and syllabus links |
|---|---|
| Applications of right and non-right angled trigonometry, including Pythagoras' theorem.<br>Angles of elevation and depression.<br>Construction of labelled diagrams from written statements. | Contexts may include use of bearings. |

***Connections***

**Other contexts:** Triangulation, map-making, navigation and radio transmissions. Use of parallax for navigation.

**Links to other subjects:** Vectors, scalars, forces and dynamics (physics); field studies (sciences)

**Aim 8:** Who really invented Pythagoras' theorem?

**Aim 9:** In how many ways can you prove Pythagoras' theorem?

**International-mindedness:** The use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

**TOK:** If the angles of a triangle can add up to less than 180°, 180° or more than 180°, what does this tell us about the nature of mathematical knowledge?

Download connections template

## SL 3.4

| Content | Guidance, clarification and syllabus links |
|---|---|
| The circle: length of an arc; area of a sector. | Radians are not required at SL. |

***Connections***

**TOK:** Does personal experience play a role in the formation of knowledge claims in mathematics? Does it play a different role in mathematics compared to other areas of knowledge?

Download connections template

## SL 3.5

| Content | Guidance, clarification and syllabus links |
|---|---|
| Equations of perpendicular bisectors. | Given either two points, or the equation of a line segment and its midpoint.<br>**Link to:** equations of straight lines (SL 2.1). |

***Connections***

Download connections template

## SL 3.6

| Content | Guidance, clarification and syllabus links |
|---|---|
| Voronoi diagrams: sites, vertices, edges, cells.<br>Addition of a site to an existing Voronoi diagram.<br>Nearest neighbour interpolation.<br>Applications of the "toxic waste dump" problem. | In examinations, coordinates of sites for calculating the perpendicular bisector equations will be given. Students will not be required to construct perpendicular bisectors. Questions may include finding the equation of a boundary, identifying the site closest to a given point, or calculating the area of a region.<br>All points within a cell can be estimated to have the same value (e.g. rainfall) as the value of the site.<br>In examinations, the solution point will always be at an intersection of three edges.<br>**Contexts:** Urban planning, spread of diseases, ecology, meteorology, resource management. |

***Connections***

**Other contexts:** Applications in subjects including geography, economics, biology, and computer science. www.ics.uci.edu/~eppstein/gina/scot.drysdale.html

**TOK:** Is the division of knowledge into disciplines or areas of knowledge artificial?

**Link to TSM:** Incremental algorithm for constructing Voronoi diagrams.

**Enrichment:** Delaunay triangulations as the duals of Voronoi triangulations; self-driving cars; the art gallery problem. Natural neighbour interpolation. Manhattan metric.

Download connections template

## AHL Content

**Recommended teaching hours: 28**

The aim of the AHL content in the geometry and trigonometry topic is to extend and build upon the aims, concepts and skills from the standard level content. It introduces students to an alternative measurement system for angles and some important trigonometric identities, extends the application of matrices to transformations, and introduces students to vectors and their applications in kinematics. Graph theory is introduced to allow students to apply their knowledge of matrices and develop their knowledge of algorithms in practical contexts.

On HL examination papers radian measure should be assumed unless otherwise indicated.

### AHL 3.7

| Content | Guidance, clarification and syllabus links |
|---|---|
| The definition of a radian and conversion between degrees and radians.<br>Using radians to calculate area of sector, length of arc. | Radian measure may be expressed as exact multiples of $\pi$, or decimals.<br>**Link to**: trigonometric functions (AHL 2.9). |

***Connections***

**Links to other subjects:** Diffraction patterns and circular motion (physics)

**International-mindedness:** Seki Takakazu calculating $\pi$ to ten decimal places; Hipparchus, Menelaus and Ptolemy; why are there 360 degrees in a complete turn? Why do we use minutes and seconds for time?; Links to Babylonian mathematics.

**TOK:** Which is the better measure of an angle, degrees or radians? What criteria can/do/should mathematicians use to make such judgments?

Download connections template

### AHL 3.8

| Content | Guidance, clarification and syllabus links |
|---|---|
| The definitions of $\cos\theta$ and $\sin\theta$ in terms of the unit circle.<br>The Pythagorean identity: $\cos^2\theta + \sin^2\theta = 1$<br>Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$<br>Extension of the sine rule to the ambiguous case. | Students should understand how the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ can be constructed from the unit circle.<br>Knowledge of exact values of $\cos\theta$, $\sin\theta$, and $\tan\theta$ will not be assessed on examinations, but may aid student understanding of trigonometric functions. |
| Graphical methods of solving trigonometric equations in a finite interval. | **Link to:** sinusoidal models (SL2.5 and AHL2.9). |

***Connections***

**Other contexts:** Generation of sinusoidal voltage in electrical engineering.

**International-mindedness:** The origin of the word "sine"; trigonometry was developed by successive civilizations and cultures; how is mathematical knowledge considered from a sociocultural perspective?

**TOK:** To what extent is mathematical knowledge embedded in particular traditions or bound to particular cultures? How have key events in the history of mathematics shaped its current form and methods?

**Use of technology:** Animation applets that show the development of a trigometric function graph from the unit circle.

Download connections template

## AHL 3.9

| Content | Guidance, clarification and syllabus links |
|---|---|
| Geometric transformations of points in two dimensions using matrices: reflections, horizontal and vertical stretches, enlargements, translations and rotations. | Matrix transformations of the form: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$. **Link to:** matrices (AHL 1.14). |
| Compositions of the above transformations. | Iterative techniques to generate fractals. **Link to:** infinite geometric series (AHL 1.11) and Markov chains (AHL 4.19). |
| Geometric interpretation of the determinant of a transformation matrix. | Area of image $= \lvert \det \boldsymbol{A} \rvert \times \text{area of object}$. |

***Connections***

**Other contexts:** Fractals: "mutations" in biology–changing the probability with which different "matrix transformations" occur [or changing the initial value ($Z_0 = c$) in the Mandelbrot quadratic recurrence equation: $Z_{n+1} = Z_n^{\ 2} + c$] and then observing what part of the structure/form this affects.

Sierpinski's triangle and Koch's snowflake make a good introduction to algorithms that generate fractals.

**Aim 8:** Matrices used in computer graphics for three-dimensional modelling: how has this been used to advance diagnoses of health conditions?

**TOK:** When mathematicians and historians say that they have explained something, are they using the word "explain" in the same way?

**Website:** Useful examples of fractal applications: http://www.fractal.org/Bewustzijns-Besturings-Model/Fractals-Useful-Beauty.htm

**Enrichment:** Affine transformations and digital image processing.

Download connections template

## AHL 3.10

| Content | Guidance, clarification and syllabus links |
|---|---|
| Concept of a vector and a scalar.<br>Representation of vectors using directed line segments.<br>Unit vectors; base vectors $\boldsymbol{i}, \quad \boldsymbol{j}, \quad \boldsymbol{k}$.<br>Components of a vector; column representation;<br>$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$<br>The zero vector $\boldsymbol{0}$, the vector $-\boldsymbol{v}$. | Use algebraic and geometric approaches to calculate the sum and difference of two vectors, multiplication by a scalar, $k\boldsymbol{v}$ (parallel vectors), magnitude of a vector $\lvert \boldsymbol{v} \rvert$ from components.<br>The resultant as the sum of two or more vectors. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| Position vectors $\overrightarrow{OA} = a$. | |
| Rescaling and normalizing vectors. | $\frac{\boldsymbol{v}}{\lvert \boldsymbol{v} \rvert}$, the unit normal vector.<br>**Example:** Find the velocity of a particle with speed 7ms$^{-1}$ in the direction $3\boldsymbol{i} + 4\boldsymbol{j}$. |

***Connections***

**Links to other subjects:** Vector sums, differences and resultants (physics).

**Aims:** Vector theory is used for tracking displacement of objects, including peaceful and harmful purposes.

**TOK:** Vectors are used to solve many problems in position location. This can be used to save a lost sailor or destroy a building with a laser-guided bomb. To what extent does possession of knowledge carry with it an ethical obligation?

Download connections template

## AHL 3.11

| Content | Guidance, clarification and syllabus links |
|---|---|
| Vector equation of a line in two and three dimensions:<br>$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$, where $\boldsymbol{b}$ is a direction vector of the line. | Convert to parametric form:<br>$x = x_0 + \lambda l, y = y_0 + \lambda m, z = z_0 + \lambda n$. |

***Connections***

**TOK:** Mathematics and the knower: Why are symbolic representations of three-dimensional objects easier to deal with than visual representations? What does this tell us about our knowledge of mathematics in other dimensions?

Download connections template

## AHL 3.12

| Content | Guidance, clarification and syllabus links |
|---|---|
| Vector applications to kinematics.<br>Modelling linear motion with constant velocity in two and three dimensions. | Finding positions, intersections, describing paths, finding times and distances when two objects are closest to each other.<br>$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}t$.<br>Relative position of B from A is $\overrightarrow{AB}$. |
| Motion with variable velocity in two dimensions. | **For example:** $\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 7 \\ 6 - 4t \end{pmatrix}$.<br>Projectile motion and circular motion are special cases.<br>$f(t - a)$ to indicate a time-shift of $a$.<br>**Link to:** kinematics (AHL 5.13) and phase shift (AHL 1.13). |

***Connections***

Download connections template

## AHL 3.13

| Content | Guidance, clarification and syllabus links |
|---|---|
| Definition and calculation of the scalar product of two vectors.<br>The angle between two vectors; the acute angle between two lines. | Calculate the angle between two vectors using $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where $\theta$ is the angle between two non-zero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, and ascertain whether the vectors are perpendicular ($\boldsymbol{v} \cdot \boldsymbol{w} = 0$). |
| Definition and calculation of the vector product of two vectors. | $\boldsymbol{v} \times \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta \boldsymbol{n}$, where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ and $\boldsymbol{n}$ is the unit normal vector whose direction is given by the right-hand screw rule.<br>**Not required:** generalized properties and proofs of scalar and cross product. |
| Geometric interpretation of $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$. | Use of $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$ to find the area of a parallelogram (and hence a triangle). |
| Components of vectors. | The component of vector $\boldsymbol{a}$ acting in the direction of vector $\boldsymbol{b}$ is $\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\lvert \boldsymbol{b} \rvert} = \lvert \boldsymbol{a} \rvert \cos\theta$.<br>The component of a vector $\boldsymbol{a}$ acting perpendicular to vector $\boldsymbol{b}$, in the plane formed by the two vectors, is $\frac{\lvert \boldsymbol{a} \times \boldsymbol{b} \rvert}{\lvert \boldsymbol{b} \rvert} = \lvert \boldsymbol{a} \rvert \sin\theta$. |

***Connections***

**Other contexts:** *Computer graphics*: Lighting–normalize one vector onto another to determine the intensity of light on a surface. Perspective–project a three-dimensional vector onto a two-dimensional plane using the scalar product.

*Physics:* Torque– magnitude of the rotational force ('torque') applied to a point/object is equal to the magnitude of the cross product of the "length" of the lever and the "force" applied to that level. The direction of the torque, for example will the force twist the the nut "on" (tightening) or "off" (loosening) the bolt, is the cross product. Electro-magnetic forces and the right and left hand rules. $\mathrm{W} = \mathrm{F} \cdot \mathrm{d}$. Forces–what component of a vector's force is acting in the direction of another–useful for any "structural" analysis when working out the "strain" on each part of the structure that results from a given force.

**Links to other subjects:** Magnetic forces and fields, and dynamics (physics)

**TOK:** What counts as understanding in mathematics? Is it more than just getting the right answer?

Download connections template

## AHL 3.14

| Content | Guidance, clarification and syllabus links |
|---|---|
| Graph theory: Graphs, vertices, edges, adjacent vertices, adjacent edges. Degree of a vertex. | Students should be able to represent real-world structures (circuits, maps, etc) as graphs (weighted and unweighted). |
| Simple graphs; complete graphs; weighted graphs. | Knowledge of the terms connected and strongly connected. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| Directed graphs; in degree and out degree of a directed graph.<br>Subgraphs; trees. | **Link to**: matrices (AHL 1.14). |

***Connections***

**Aim 8:** The importance of symbolic maps, for example Metro and Underground maps, structural formulae in chemistry, electrical circuits.

**TOK:** Mathematics and knowledge claims. Proof of the four-colour theorem. If a theorem is proved by computer, how can we claim to know that it is true?

Download connections template

## AHL 3.15

| Content | Guidance, clarification and syllabus links |
|---|---|
| Adjacency matrices.<br>Walks.<br>Number of $k$ -length walks (or less than $k$ -length walks) between two vertices. | Given an adjacency matrix $A$, the $(i, \quad j)^{\text{th}}$ entry of $A^k$ gives the number of $k$ length walks from $i$ to $j$. |
| Weighted adjacency tables.<br>Construction of the transition matrix for a strongly-connected, undirected or directed graph. | Weights could be costs, distances, lengths of time for example.<br>Consideration of simple graphs, including the Google PageRank algorithm as an example of this.<br>**Link to:** transition matrices and Markov chains (AHL 4.19). |

***Connections***

**International-mindedness:** The "Bridges of Konigsberg" problem.

**Links to websites:** Adjacency matrix and airlines:

Page rank is one method used to determine the importance rank of a webpage. Simulation for PageRank: www.eprisner.de/MAT103/PageRank.html

Download connections template

## AHL 3.16

| Content | Guidance, clarification and syllabus links |
|---|---|
| Tree and cycle algorithms with undirected graphs.<br>Walks, trails, paths, circuits, cycles. | |
| Eulerian trails and circuits.<br>Hamiltonian paths and cycles.<br>Minimum spanning tree (MST) graph algorithms:<br>Kruskal's and Prim's algorithms for finding minimum spanning trees. | Determine whether an Eulerian trail or circuit exists.<br>Use of matrix method for Prim's algorithm. |
| Chinese postman problem and algorithm for solution, to determine the shortest route around a | Students should be able to explain why the algorithm for constructing the Chinese postman |

| Content | Guidance, clarification and syllabus links |
|---|---|
| weighted graph with up to four odd vertices, going along each edge at least once. | problem works, apply the algorithm and justify their choice of algorithm. |
| Travelling salesman problem to determine the Hamiltonian cycle of least weight in a weighted complete graph.<br>Nearest neighbour algorithm for determining an upper bound for the travelling salesman problem.<br>Deleted vertex algorithm for determining a lower bound for the travelling salesman problem. | Practical problems should be converted to the classical problem by completion of a table of least distances where necessary. |

***Connections***

**Other contexts:** Using GPS to find the shortest route home; describe current and voltage in circuits as cycles; vehicle routing problems.

**International-mindedness:** The "Bridges of Konigsberg" problem; the Chinese postman problem was first posed by the Chinese mathematician Kwan Mei-Ko in 1962.

**TOK:** What practical problems can or does mathematics try to solve? Why are problems such as the travelling salesman problem so enduring? What does it mean to say the travelling salesman problem is "NP hard"?

Download connections template

# Topic 4: Statistics and probability

## Concepts

### Essential understandings:

Statistics is concerned with the collection, analysis and interpretation of quantitative data and uses the theory of probability to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Probability enables us to quantify the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned, in detail, to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk and to make predictions about seemingly random events.

### Suggested concepts embedded in this topic:

Quantity, validity, approximation, modelling, relationships, patterns.

**AHL:** Systems, representation.

### Content-specific conceptual understandings:

- Organizing, representing, analysing and interpreting data, and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Approximation in data can approach the truth but may not always achieve it.
- Correlation and regression are powerful tools for identifying patterns and equivalence of systems.

- Modelling and finding structure in seemingly random events facilitates prediction.
- Different probability distributions provide a representation of the relationship between the theory and reality, allowing us to make predictions about what might happen.

***AHL***

- Statistical literacy involves identifying reliability and validity of samples and whole populations in a closed system.
- A systematic approach to hypothesis testing allows statistical inferences to be tested for validity.
- Representation of probabilities using transition matrices enables us to efficiently predict long-term behaviour and outcomes.

## SL content

**Recommended teaching hours: 36**

The aim of the standard level content in the statistics and probability topic is to introduce students to important concepts, techniques and representations used in statistics and probability and their meaningful application in the real world. Students should be given the opportunity to approach this topic in a practical way, to understand why certain techniques are used and to interpret the results. The use of technology such as simulations, spreadsheets, statistics software and statistics apps can greatly enhance this topic.

It is expected that most of the calculations required will be carried out using technology, but explanations of calculations by hand may enhance understanding. The emphasis is on choosing the most appropriate technique, and understanding and interpreting the results obtained in context.

In examinations students should be familiar with how to use the statistics functionality of allowed technology.

At SL the data set will be considered to be the population unless otherwise stated.

Sections SL4.1 to SL4.9 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

### SL 4.1

| Content | Guidance, clarification and syllabus links |
|---|---|
| Concepts of population, sample, random sample, discrete and continuous data. | This is designed to cover the key questions that students should ask when they see a data set/ analysis. |
| Reliability of data sources and bias in sampling. | Dealing with missing data, errors in the recording of data. |
| Interpretation of outliers. | Outlier is defined as a data item which is more than $1.5 \times$ interquartile range (IQR) from the nearest quartile.<br>Awareness that, in context, some outliers are a valid part of the sample but some outlying data items may be an error in the sample.<br>**Link to**: box and whisker diagrams (SL4.2) and measures of dispersion (SL4.3). |
| Sampling techniques and their effectiveness. | Simple random, convenience, systematic, quota and stratified sampling methods. |

***Connections***

**Links to other subjects:** Descriptive statistics and random samples (biology, psychology, sports exercise and health science, environmental systems and societies, geography, economics; business management); research methodologies (psychology).

**Aim 8:** Misleading statistics; examples of problems caused by absence of representative samples, for example Google flu predictor, US presidential elections in 1936, Literary Digest v George Gallup, Boston "pot-hole" app.

**International-mindedness:** The Kinsey report–famous sampling techniques.

**TOK:** Why have mathematics and statistics sometimes been treated as separate subjects? How easy is it to be misled by statistics? Is it ever justifiable to purposely use statistics to mislead others?

Download connections template

## SL 4.2

| Content | Guidance, clarification and syllabus links |
|---|---|
| Presentation of data (discrete and continuous): frequency distributions (tables). | Class intervals will be given as inequalities, without gaps. |
| Histograms.<br>Cumulative frequency; cumulative frequency graphs; use to find median, quartiles, percentiles, range and interquartile range (IQR). | Frequency histograms with equal class intervals.<br>**Not required:** Frequency density histograms. |
| Production and understanding of box and whisker diagrams. | Use of box and whisker diagrams to compare two distributions, using symmetry, median, interquartile range or range. Outliers should be indicated with a cross.<br>Determining whether the data may be normally distributed by consideration of the symmetry of the box and whiskers. |

***Connections***

**Links to other subjects:** Presentation of data (sciences, individuals and societies).

**International-mindedness:** Discussion of the different formulae for the same statistical measure (for example, variance).

**TOK:** What is the difference between information and data? Does "data" mean the same thing in different areas of knowledge?

Download connections template

## SL 4.3

| Content | Guidance, clarification and syllabus links |
|---|---|
| Measures of central tendency (mean, median and mode).<br>Estimation of mean from grouped data. | Calculation of mean using formula and technology.<br>Students should use mid-interval values to estimate the mean of grouped data. |
| Modal class. | For equal class intervals only. |
| Measures of dispersion (interquartile range, standard deviation and variance). | Calculation of standard deviation and variance of the data set using only technology, however hand calculations may enhance understanding. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| | Variance is the square of the standard deviation. |
| Effect of constant changes on the original data. | **Examples:** If three is subtracted from the data items, then the mean is decreased by three, but the standard deviation is unchanged.<br>If all the data items are doubled, the mean is doubled and the standard deviation is also doubled. |
| Quartiles of discrete data. | Using technology. Awareness that different methods for finding quartiles exist and therefore the values obtained using technology and by hand may differ. |

***Connections***

**Other contexts:** Comparing variation and spread in populations, human or natural, for example agricultural crop data, social indicators, reliability and maintenance.

**Links to other subjects:** Descriptive statistics (sciences and individuals and societies); consumer price index (economics).

**International-mindedness:** The benefits of sharing and analysing data from different countries; discussion of the different formulae for variance.

**TOK:** Could mathematics make alternative, equally true, formulae? What does this tell us about mathematical truths? Does the use of statistics lead to an over-emphasis on attributes that can be easily measured over those that cannot?

Download connections template

## SL 4.4

| Content | Guidance, clarification and syllabus links |
|---|---|
| Linear correlation of bivariate data.<br>Pearson's product-moment correlation coefficient, $r$. | Technology should be used to calculate $r$. However, hand calculations of $r$ may enhance understanding.<br>Critical values of $r$ will be given where appropriate.<br>Students should be aware that Pearson's product moment correlation coefficient ($r$) is only meaningful for linear relationships. |
| Scatter diagrams; lines of best fit, by eye, passing through the mean point. | Positive, zero, negative; strong, weak, no correlation.<br>Students should be able to make the distinction between correlation and causation and know that correlation does not imply causation. |
| Equation of the regression line of $y$ on $x$.<br>Use of the equation of the regression line for prediction purposes.<br>Interpret the meaning of the parameters, $a$ and $b$, in a linear regression $y = ax + b$. | Technology should be used to find the equation.<br>Students should be aware:<br>• of the dangers of extrapolation<br>• that they cannot always reliably make a prediction of $x$ from a value of $y$, when using a $y$ on $x$ line. |

***Connections***

**Other contexts:** Linear regressions where correlation exists between two variables. Exploring cause and dependence for categorical variables, for example, on what factors might political persuasion depend?

**Links to other subjects:** Curves of best fit, correlation and causation (sciences); scatter graphs (geography).

**Aim 8:** The correlation between smoking and lung cancer was "discovered" using mathematics. Science had to justify the cause.

**TOK:** Correlation and causation–can we have knowledge of cause and effect relationships given that we can only observe correlation? What factors affect the reliability and validity of mathematical models in describing real-life phenomena?

Download connections template

## SL 4.5

| Content | Guidance, clarification and syllabus links |
|---|---|
| Concepts of trial, outcome, equally likely outcomes, relative frequency, sample space ($U$) and event.<br>The probability of an event $A$ is $\mathrm{P}(A) = \frac{n(A)}{n(U)}$.<br>The complementary events $A$ and $A'$ (not $A$). | Sample spaces can be represented in many ways, for example as a table or a list.<br>Experiments using coins, dice, cards and so on, can enhance understanding of the distinction between experimental (relative frequency) and theoretical probability.<br>Simulations may be used to enhance this topic. |
| Expected number of occurrences. | **Example:** If there are 128 students in a class and the probability of being absent is 0.1, the expected number of absent students is 12.8. |

***Connections***

**Other contexts:** Actuarial studies and the link between probability of life spans and insurance premiums, government planning based on likely projected figures, Monte Carlo methods.

**Links to other subjects:** Theoretical genetics and Punnett squares (biology); the position of a particle (physics).

**Aim 8:** The ethics of gambling.

**International-mindedness:** The St Petersburg paradox; Chebyshev and Pavlovsky (Russian).

**TOK:** To what extent are theoretical and experimental probabilities linked? What is the role of emotion in our perception of risk, for example in business, medicine and travel safety?

**Use of technology:** Computer simulations may be useful to enhance this topic.

Download connections template

## SL 4.6

| Content | Guidance, clarification and syllabus links |
|---|---|
| Use of Venn diagrams, tree diagrams, sample space diagrams and tables of outcomes to calculate probabilities. | |
| Combined events:<br>$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.<br>Mutually exclusive events: $\mathrm{P}(A \cap B) = 0$. | The non-exclusivity of "or". |
| Conditional probability: $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$. | An alternate form of this is:<br>$\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A \mid B)$. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| | Problems can be solved with the aid of a Venn diagram, tree diagram, sample space diagram or table of outcomes without explicit use of formulae.<br>Probabilities with and without replacement. |
| Independent events: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$. | |

***Connections***

**Aim 8:** The gambling issue: use of probability in casinos. Could or should mathematics help increase incomes in gambling?

**TOK:** Can calculation of gambling probabilities be considered an ethical application of mathematics? Should mathematicians be held responsible for unethical applications of their work?

Download connections template

## SL 4.7

| Content | Guidance, clarification and syllabus links |
|---|---|
| Concept of discrete random variables and their probability distributions.<br>Expected value (mean), $\mathrm{E}(X)$ for discrete data.<br>Applications. | Probability distributions will be given in the following ways:<br>$\begin{array}{c\|c\|c\|c\|c\|c} X & 1 & 2 & 3 & 4 & 5 \\ \hline \mathrm{P}(X=x) & 0.1 & 0.2 & 0.15 & 0.05 & 0.5 \end{array}$<br>$\mathrm{P}(X = x) = \frac{1}{18}(4 + x)$ for $x \in \{1, 2, 3\}$;<br>$\mathrm{E}(X) = 0$ indicates a fair game where $X$ represents the gain of a player. |

***Connections***

**Other contexts:** Games of chance.

**Aim 8:** Why has it been argued that theories based on the calculable probabilities found in casinos are pernicious when applied to everyday life (for example, economics)?

**TOK:** What do we mean by a "fair" game? Is it fair that casinos should make a profit?

Download connections template

## SL 4.8

| Content | Guidance, clarification and syllabus links |
|---|---|
| Binomial distribution.<br>Mean and variance of the binomial distribution. | Situations where the binomial distribution is an appropriate model.<br>In examinations, binomial probabilities should be found using available technology.<br>**Not required:** Formal proof of mean and variance.<br>**Link to:** expected number of occurrences (SL4.5). |

***Connections***

**Aim 8:** Pascal's triangle, attributing the origin of a mathematical discovery to the wrong mathematician.

**International-mindedness:** The so-called "Pascal's triangle" was known to the Chinese mathematician Yang Hui much earlier than Pascal.

**TOK:** What criteria can we use to decide between different models?

**Enrichment:** Hypothesis testing using the binomial distribution.

Download connections template

## SL 4.9

| Content | Guidance, clarification and syllabus links |
|---|---|
| The normal distribution and curve.<br>Properties of the normal distribution.<br>Diagrammatic representation. | Awareness of the natural occurrence of the normal distribution.<br>Students should be aware that approximately 68% of the data lies between $\mu \pm \sigma$, 95% lies between $\mu \pm 2\sigma$ and 99.7% of the data lies between $\mu \pm 3\sigma$. |
| Normal probability calculations. | Probabilities and values of the variable must be found using technology. |
| Inverse normal calculations | For inverse normal calculations mean and standard deviation will be given.<br>This does not involve transformation to the standardized normal variable $z$. |

***Connections***

**Links to other subjects:** Normally distributed real-life measurements and descriptive statistics (sciences, psychology, environmental systems and societies)

**Aim 8:** Why might the misuse of the normal distribution lead to dangerous inferences and conclusions?

**International-mindedness:** De Moivre's derivation of the normal distribution and Quetelet's use of it to describe *l'homme moyen*.

**TOK:** To what extent can we trust mathematical models such as the normal distribution? How can we know what to include, and what to exclude, in a model?

Download connections template

## SL 4.10

| Content | Guidance, clarification and syllabus links |
|---|---|
| Spearman's rank correlation coefficient, $r_s$. | In examinations Spearman's rank correlation coefficient, $r_s$, should be found using technology.<br>If data items are equal, ranks should be averaged. |
| Awareness of the appropriateness and limitations of Pearson's product moment correlation coefficient and Spearman's rank correlation coefficient, and the effect of outliers on each. | Students should be aware that Pearson's product moment correlation coefficient is useful when testing for only linearity and Spearman's correlation coefficient for any monotonic relationship.<br>Spearman's correlation coefficient is less sensitive to outliers than Pearson's product moment correlation coefficient. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| | **Not required:** Derivation/proof of Pearson's product moment correlation coefficient and Spearman's rank correlation coefficient. |

***Connections***

**Links to other subjects:** Fieldwork (biology, psychology, environmental systems and societies, sports exercise and health science)

**Aim 8:** The physicist Frank Oppenheimer wrote: "Prediction is dependent only on the assumption that observed patterns will be repeated". This is the danger of extrapolation. There are many examples of its failure in the past, for example share prices, the spread of disease, climate change.

**TOK:** Does correlation imply causation? Mathematics and the world. Given that a set of data may be approximately fitted by a range of curves, where would a mathematician seek for knowledge of which equation is the "true" model?

**Links to websites:** www.wikihow.com/Calculate-Spearman%27s-Rank-Correlation-Coefficient

**External website:** Use of databases such as Gapminder.

Download connections template

## SL 4.11

| Content | Guidance, clarification and syllabus links |
|---|---|
| Formulation of null and alternative hypotheses, $H_0$ and $H_1$.<br>Significance levels.<br>$p$ -values. | Students should express $H_0$ and $H_1$ as an equation or inequality, or in words as appropriate. |
| Expected and observed frequencies.<br>The $\chi^2$ test for independence: contingency tables, degrees of freedom, critical value.<br>The $\chi^2$ goodness of fit test. | In examinations:<br>• the maximum number of rows or columns in a contingency table will be 4<br>• the degrees of freedom will always be greater than one. At SL the degrees of freedom for the goodness of fit test will always be $n-1$<br>• the $\chi^2$ critical value will be given if appropriate<br>• students will be expected to use technology to find a $p$ -value and the $\chi^2$ statistic<br>• only questions on upper tail tests with commonly-used significance levels (1%, 5%, 10%) will be set<br>• students will be expected to either compare a $p$ -value to the given significance level or compare the $\chi^2$ statistic to a given critical value<br>• expected frequencies will be greater than 5.<br>Hand calculations of the expected values or the $\chi^2$ statistic may enhance understanding.<br>If using $\chi^2$ tests in the IA, students should be aware of the limitations of the test for expected frequencies of 5 or less. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| The $t$ -test.<br>Use of the $p$ -value to compare the means of two populations.<br>Using one-tailed and two-tailed tests. | In examinations calculations will be made using technology.<br>At SL, samples will be unpaired, and population variance will always be unknown.<br>Students will be asked to interpret the results of a test.<br>Students should know that the underlying distribution of the variables must be normal for the $t$ -test to be applied. In examinations, students should assume that variance of the two groups is equal and therefore the pooled two-sample $t$ -test should be used. |

***Connections***

**Other contexts:** Psychology: A common test is the Mann-Whitney U test. When and why is this thought to be a more reliable test in psychology?

**Links to other subjects:** Fieldwork (biology, psychology, environmental systems and societies, sports exercise and health science, geography).

**TOK:** Why have some research journals "banned" $p$ -values from their articles because they deem them too misleading? In practical terms, is saying that a result is significant the same as saying it is true? How is the term "significant" used differently in different areas of knowledge?

**Use of technology:** Use of simulations to generate data.

**Enrichment:** When performing a $\chi^2$ test Yates continuity correction is often applied to small samples. Is it universally accepted as a valid method? In what situations would you use Yates and why? Are there other ways to deal with small sample sizes?

Download connections template

# AHL content

**Recommended teaching hours: 16**

The aim of the AHL content in the statistics and probability topic is to extend and build upon the aims, concepts and skills from the standard level content. It allows students to develop skills in the design of data collection methods taking consideration of validity and reliability, regression is extended to non-linear situations, concepts involving samples and populations are introduced and students will develop their skills in deciding which tests to use in context. Students will be introduced to transition matrices and establish the links between matrices, probability and eigenvalues.

It is expected that students will be able to choose appropriate techniques and interpret their results. Students are expected to set up a problem mathematically and then calculate the answers using technology. Technology-specific language should not be used within these explanations.

## AHL 4.12

| Content | Guidance, clarification and syllabus links |
|---|---|
| Design of valid data collection methods, such as surveys and questionnaires.<br>Selecting relevant variables from many variables.<br>Choosing relevant and appropriate data to analyse. | Biased and unbiased, personal, unstructured and structured (with consistent answer choices), and precise questioning. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| Categorizing numerical data in a $\chi^2$ table and justifying the choice of categorisation.<br>Choosing an appropriate number of degrees of freedom when estimating parameters from data when carrying out the $\chi^2$ goodness of fit test. | Appropriate categories should be chosen with expected frequencies greater than 5. |
| Definition of reliability and validity.<br>Reliability tests.<br>Validity tests. | Students should understand the difference between reliability and validity and be familiar with the following methods:<br>Reliability: Test-retest, parallel forms.<br>Validity: Content, criterion-related. |

***Connections***

**Other contexts:** Data from social media behaviour and algorithms.

**Links to other subjects:** Data collection in field work (biology, psychology, environmental systems and societies, sports exercise and health science, geography, business management and design technology); data from social media and marketing sources (business management)

**TOK:** What are the strengths and limitations of different methods of data collection, such as questionnaires?

Download connections template

## AHL 4.13

| Content | Guidance, clarification and syllabus links |
|---|---|
| Regression with non-linear functions. | **Link to:** geometric sequences and series (SL1.3). |
| Evaluation of least squares regression curves using technology. | In examinations, questions may be asked on linear, quadratic, cubic, exponential, power and sine functions. |
| Sum of square residuals ($SS_{res}$) as a measure of fit for a model. | |
| The coefficient of determination ($R^2$).<br>Evaluation of $R^2$ using technology. | $R^2$ gives the proportion of variability in the second variable accounted for by the chosen model.<br>Awareness that $R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$ and hence = 1 if $SS_{res} = 0$, may enhance understanding but will not be examined.<br>Awareness that many factors affect the validity of a model and the coefficient of determination, by itself, is not a good way to decide between different models.<br>The connection between the coefficient of determination and the Pearson's product moment correlation coefficient for linear models. |

***Connections***

**Links to other subjects:** Evaluation of $R^2$ in graphical analysis (sciences).

Download connections template

## AHL 4.14

| Content | Guidance, clarification and syllabus links |
|---|---|
| Linear transformation of a single random variable. | $\mathrm{Var}(X)$ is the variance of the random variable $X$. Variance formula will not be required in examinations.<br>$\mathrm{E}(aX+b) = a\mathrm{E}(X)+b$.<br>$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$. |
| Expected value of linear combinations of $n$ random variables.<br>Variance of linear combinations of $n$ independent random variables. | |
| $\bar{x}$ as an unbiased estimate of $\mu$. | $\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$. |
| $s_{n-1}^2$ as an unbiased estimate of $\sigma^2$. | $s_{n-1}^2 = \frac{n}{n-1}s_n^2 = \sum_{i=1}^{k} \frac{f_i(x_i-\bar{x})^2}{n-1}$, where $n = \sum_{i=1}^{k} f_i$.<br>Demonstration that $\mathrm{E}(\bar{X}) = \mu$ and $\mathrm{E}(s_{n-1}^2) = \sigma^2$ will not be examined, but may help understanding. |

***Connections***

**TOK:** Mathematics and the world: In the absence of knowing the value of a parameter, will an unbiased estimator always be better than a biased one?

Download connections template

## AHL 4.15

| Content | Guidance, clarification and syllabus links |
|---|---|
| A linear combination of $n$ independent normal random variables is normally distributed. In particular,<br>$X \sim \mathrm{N}(\mu, \sigma^2) \Rightarrow \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$. | |
| Central limit theorem. | In general, $\bar{X}$ approaches normality for large $n$, how large depends upon the distribution from which the sample is taken. In examinations, $n > 30$ will be considered sufficient.<br>Online simulations are useful for visualisation. |

***Connections***

**Links to other subjects:** Data from multiple samples in field studies (sciences, and individuals and societies).

**Aim 8:** Mathematics and the world. "Without the central limit theorem, there could be no statistics of any value within the human sciences".

**TOK:** The central limit theorem can be proved mathematically (formalism), but its truth can be confirmed by its applications (empiricism). What does this suggest about the nature and methods of mathematics?

**Enrichment:** For a normally distributed population of size N, how many random samples of size $n$ do you need to take in order to verify the central limit theorem?

Download connections template

## AHL 4.16

| Content | Guidance, clarification and syllabus links |
|---|---|
| Confidence intervals for the mean of a normal population. | Students should be able to interpret the meaning of their results in context.<br>Use of the normal distribution when $\sigma$ is known and the t-distribution when $\sigma$ is unknown, regardless of sample size. |

***Connections***

**Other contexts:** Forecasting–attaching value to claims and predictions.

**Links to other subjects:** Analysis of data from field studies (sciences and individuals and societies).

**TOK:** Mathematics and the world. Claiming brand A is "better" on average than brand B can mean very little if there is a large overlap between the confidence intervals of the two means.

Download connections template

## AHL 4.17

| Content | Guidance, clarification and syllabus links |
|---|---|
| Poisson distribution, its mean and variance.<br>Sum of two independent Poisson distributions has a Poisson distribution. | Situations in which it is appropriate to use a Poisson distribution as a model:<br>1. Events are independent<br>2. Events occur at a uniform average rate (during the period of interest).<br>Given a context, students should be able to select between the normal, the binomial and the Poisson distributions, recognizing where a particular distribution is appropriate.<br>**Not required:** Formal proof of means and variances for probability distributions. |

***Connections***

**Other contexts:** Telecommunications, call management, traffic management, biological mutations, emergency room admissions, typos in publications.

**TOK:** To what extent can mathematical models such as the Poisson distribution be trusted? What role do mathematical models play in other areas of knowledge?

Download connections template

## AHL 4.18

| Content | Guidance, clarification and syllabus links |
|---|---|
| Critical values and critical regions. | Students will not be expected to calculate critical regions for $t$-tests. |
| Test for population mean for normal distribution. | Use of the normal distribution when $\sigma$ is known and the $t$-distribution when $\sigma$ is unknown, regardless of sample size.<br>The case of matched pairs is to be treated as an example of a single sample technique. |
| Test for proportion using binomial distribution. | |
| Test for population mean using Poisson distribution. | Poisson and binomial tests will be one-tailed only. |
| Use of technology to test the hypothesis that the population product moment correlation coefficient ($\rho$) is 0 for bivariate normal distributions. | In examinations the data will be given. |
| Type I and II errors including calculations of their probabilities. | Applied to normal with known variance, Poisson and binomial distributions.<br>For discrete random variables, hypothesis tests and critical regions will only be required for one-tailed tests. The critical region will maximize the probability of a Type I error while keeping it less than the stated significance level. |

***Connections***

**Links to other subjects:** Field studies (sciences and individuals and societies).

**TOK:** Mathematics and the world. In practical terms, is saying that a result is significant the same as saying that it is true? Mathematics and the world. Does the ability to test only certain parameters in a population affect the way knowledge claims in the human sciences are valued? When is it more important not to make a Type I error and when is it more important not to make a Type II error?

Download connections template

## AHL 4.19

| Content | Guidance, clarification and syllabus links |
|---|---|
| Transition matrices.<br>Powers of transition matrices. | In general, the column state matrix ($\boldsymbol{s}_n$) after $n$ transitions is given by $\boldsymbol{s}_n = \boldsymbol{T}^n\boldsymbol{s}_0$, where $\boldsymbol{T}$ is the transition matrix, with $\boldsymbol{T}_{ij}$ representing the probability of moving from state $j$ to state $i$, and $\boldsymbol{s}_0$ is the initial state matrix.<br>Use of transition diagrams to represent transitions in discrete dynamical systems. |
| Regular Markov chains.<br>Initial state probability matrices. | |
| Calculation of steady state and long-term probabilities by repeated multiplication of the transition matrix or by solving a system of linear equations. | Examination questions will state when exact solutions obtained from solving equations are required. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| | Awareness that the solution is the eigenvector corresponding to the eigenvalue equal to 1 and whose entries have been scaled to sum to 1.<br>**Link to:** matrices (AHL1.14), eigenvalues (AHL1.15) and adjacency matrices (AHL3.15). |

***Connections***

**Other contexts:** Absorbing states for Markov chains, the gambler's ruin problem.

**Website:** Simulation for Markov chains: setosa.io/blog/2014/07/26/markov-chains/

**Enrichment:** Leslie matrices are used extensively in biology.

Download connections template

# Topic 5: Calculus

## Concepts

### Essential understandings:

Calculus describes rates of change between two variables and the accumulation of limiting areas. Understanding these rates of change allows us to model, interpret and analyze real-world problems and situations. Calculus helps us understand the behaviour of functions and allows us to interpret the features of their graphs.

### Suggested concepts embedded in this topic:

Change, patterns, relationships, approximation, space, generalization.

**AHL:** Systems, quantity.

### Content-specific conceptual understandings:

- Students will understand the links between the derivative and the rate of change and interpret the meaning of this in context.
- Students will understand the relationship between the integral and area and interpret the meaning of this in context.
- Finding patterns in the derivatives of polynomials and their behavior, such as increasing or decreasing, allows a deeper appreciation of the properties of the function at any given point or instant.
- Calculus is a concise form of communication used to approximate nature.
- Numerical integration can be used to approximate areas in the physical world.
- Optimization of a function allows us to find the largest or smallest value that a function can take in general and can be applied to a specific set of conditions to solve problems.
- Maximum and minimum points help to solve optimization problems.
- The area under a function on a graph has a meaning and has applications in space and time.

***AHL***

- Kinematics allows us to describe the motion and direction of objects in closed systems in terms of displacement, velocity, and acceleration.
- Many physical phenomena can be modelled using differential equations and analytic and numeric methods can be used to calculate optimum quantities.
- Phase portraits enable us to visualize the behavior of dynamic systems.

# SL content

**Recommended teaching hours: 19**

The aim of the standard level content in the calculus topic is to introduce students to the key concepts and techniques of differential and integral calculus and their use to approach practical problems.

Throughout this topic students should be given the opportunity to use technology such as graphing packages and graphing calculators to develop and apply their knowledge of calculus.

Sections SL5.1 to SL5.5 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

## SL 5.1

| Content | Guidance, clarification and syllabus links |
|---|---|
| Introduction to the concept of a limit. | Estimation of the value of a limit from a table or graph.<br>**Not required:** Formal analytic methods of calculating limits. |
| Derivative interpreted as gradient function and as rate of change. | Forms of notation: $\frac{\mathrm{d}y}{\mathrm{d}x}$, $f'(x)$, $\frac{\mathrm{d}V}{\mathrm{d}r}$ or $\frac{\mathrm{d}s}{\mathrm{d}t}$ for the first derivative.<br>Informal understanding of the gradient of a curve as a limit. |

***Connections***

**Links to other subjects:** Marginal cost, marginal revenue, marginal profit, market structures (economics); kinematics, induced emf and simple harmonic motion (physics); interpreting the gradient of a curve (chemistry)

**Aim 8:** The debate over whether Newton or Leibnitz discovered certain calculus concepts; how the Greeks' distrust of zero meant that Archimedes' work did not lead to calculus.

**International-mindedness:** Attempts by Indian mathematicians (500-1000 CE) to explain division by zero.

**TOK:** What value does the knowledge of limits have? Is infinitesimal behaviour applicable to real life? Is intuition a valid way of knowing in mathematics?

**Use of technology:** Spreadsheets, dynamic graphing software and GDC should be used to explore ideas of limits, numerically and graphically. Hypotheses can be formed and then tested using technology.

Download connections template

## SL 5.2

| Content | Guidance, clarification and syllabus links |
|---|---|
| Increasing and decreasing functions.<br>Graphical interpretation of $f'(x) > 0, f'(x) = 0, f'(x) < 0$. | Identifying intervals on which functions are increasing $(f'(x) > 0)$ or decreasing $(f'(x) < 0)$. |

***Connections***

Download connections template

## SL 5.3

| Content | Guidance, clarification and syllabus links |
|---|---|
| Derivative of $f(x) = ax^n$ is $f'(x) = anx^{n-1}, n \in \mathbb{Z}$<br>The derivative of functions of the form $f(x) = ax^n + bx^{n-1} + ...$ where all exponents are integers. | |

***Connections***

**TOK:** The seemingly abstract concept of calculus allows us to create mathematical models that permit human feats such as getting a man on the Moon. What does this tell us about the links between mathematical models and reality?

Download connections template

## SL 5.4

| Content | Guidance, clarification and syllabus links |
|---|---|
| Tangents and normals at a given point, and their equations. | Use of both analytic approaches and technology. |

***Connections***

**Links to other subjects:** Instantaneous velocity and optics, equipotential surfaces (physics); price elasticity (economics).

**TOK:** In what ways has technology impacted how knowledge is produced and shared in mathematics? Does technology simply allow us to arrange existing knowledge in new and different ways, or should this arrangement itself be considered knowledge?

Download connections template

## SL 5.5

| Content | Guidance, clarification and syllabus links |
|---|---|
| Introduction to integration as anti-differentiation of functions of the form $f(x) = ax^n + bx^{n-1} + ....$, where $n \in \mathbb{Z}, \quad n \neq -1$. | Students should be aware of the link between anti-derivatives, definite integrals and area. |
| Anti-differentiation with a boundary condition to determine the constant term. | **Example:** If $\frac{dy}{dx} = 3x^2 + x$ and $y = 10$ when $x = 1$, then $y = x^3 + \frac{1}{2}x^2 + 8.5$. |
| Definite integrals using technology.<br>Area of a region enclosed by a curve $y = f(x)$ and the $x$-axis, where $f(x) > 0$. | Students are expected to first write a correct expression before calculating the area, for example $\int_2^6 (3x^2 + 4)dx$.<br>The use of dynamic geometry or graphing software is encouraged in the development of this concept. |

***Connections***

**Other contexts:** Velocity-time graphs

**Links to other subjects:** Velocity-time and acceleration-time graphs (physics and sports exercise and health science)

**TOK:** Is it possible for an area of knowledge to describe the world without transforming it?

Download connections template

## SL 5.6

| Content | Guidance, clarification and syllabus links |
|---|---|
| Values of $x$ where the gradient of a curve is zero.<br>Solution of $f'(x) = 0$.<br>Local maximum and minimum points. | Students should be able to use technology to generate $f'(x)$ given $f(x)$, and find the solutions of $f'(x) = 0$.<br>Awareness that the local maximum/minimum will not necessarily be the greatest/least value of the function in the given domain. |

***Connections***

**Other contexts:** Profit, area, volume, cost.

**Links to other subjects:** Displacement-time and velocity-time graphs and simple harmonic motion graphs (physics).

**TOK:** Is it possible for an area of knowledge to describe the world without transforming it?

Download connections template

## SL 5.7

| Content | Guidance, clarification and syllabus links |
|---|---|
| Optimisation problems in context. | **Examples:** Maximizing profit, minimizing cost, maximizing volume for a given surface area.<br>In SL examinations, questions on kinematics will not be set. |

***Connections***

**Other contexts:** Efficient use of material in packaging.

**Links to other subjects:** Kinematics (physics); allocative efficiency (economics).

**TOK:** How can the rise in tax for plastic containers, for example plastic bags, plastic bottles etc be justified using optimization?

Download connections template

## SL 5.8

| Content | Guidance, clarification and syllabus links |
|---|---|
| Approximating areas using the trapezoidal rule. | Given a table of data or a function, make an estimate for the value of an area using the trapezoidal rule, with intervals of equal width.<br>**Link to:** upper and lower bounds (SL1.6) and areas under curves (SL5.5). |

***Connections***

**Other contexts:** Irregular areas that are not described by mathematical functions, for example lakes.

**Links to other subjects:** Kinematics (Physics).

**Use of technology:** Use dynamic graphing software to calculate the approximate area under a curve and interpret its meaning.

**Enrichment:** Exploring other numerical integration techniques such as Simpson's rule.

Download connections template

# AHL content

**Recommended teaching hours: 22 hours**

The aim of the AHL content in the calculus topic is to extend and build upon the aims, concepts and skills from the standard level content. Further differential and integral calculus techniques are introduced to enable students to model and interpret practical contexts.

## AHL 5.9

| Content | Guidance, clarification and syllabus links |
|---|---|
| The derivatives of $\sin x$, $\cos x$, $\tan x$, $e^x$, $\ln x$, $x^n$ where $n \in \mathbb{Q}$.<br>The chain rule, product rule and quotient rules.<br>Related rates of change. | **Link to**: maximum and minimum points (SL5.6) and optimisation (SL5.7). |

***Connections***

**Links to other subjects:** Uniform circular motion and induced emf (physics).

**TOK:** Euler was able to make important advances in mathematical analysis before calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work. What does this suggest about the nature of progress and development in mathematics? How might this be similar/different to the nature of progress and development in other areas of knowledge?

Download connections template

## AHL 5.10

| Content | Guidance, clarification and syllabus links |
|---|---|
| The second derivative. | Both forms of notation, $\frac{d^2y}{dx^2}$ and $f''(x)$ for the second derivative. |
| Use of second derivative test to distinguish between a maximum and a minimum point. | Awareness that a point of inflexion is a point at which the concavity changes and interpretation of this in context. |
| | Use of the terms "concave-up" for $f''(x) > 0$, and "concave-down" for $f''(x) < 0$.<br>**Link to:** kinematics (AHL5.13) and second order differential equations (AHL5.18). |

***Connections***

**Links to other subjects:** Simple harmonic motion (physics).

**TOK:** Music can be expressed using mathematics. Does this mean that music is mathematical/that mathematics is musical?

Download connections template

## AHL 5.11

| Content | Guidance, clarification and syllabus links |
|---|---|
| Definite and indefinite integration of $x^n$ where $n \in \mathbb{Q}$, including $n = -1$, $\sin x$, $\cos x$, $\frac{1}{\cos^2 x}$ and $e^x$. | |
| Integration by inspection, or substitution of the form $\int f(g(x))g'(x)\mathrm{d}x$. | **Examples:** $\int \sin(2x+5)\mathrm{d}x$, $\int \frac{1}{3x+2}\mathrm{d}x$, $\int 4x\sin x^2\,\mathrm{d}x$, $\int \frac{\sin x}{\cos x}\,\mathrm{d}x$. |

***Connections***

**International-mindedness:** The successful calculation of the volume of a pyramidal frustrum by ancient Egyptians (the Egyptian Moscow mathematical papyrus).

Download connections template

## AHL 5.12

| Content | Guidance, clarification and syllabus links |
|---|---|
| Area of the region enclosed by a curve and the $x$ or $y$-axes in a given interval. | Including negative integrals. |
| Volumes of revolution about the $x$- axis or $y$- axis. | $V = \int_a^b \pi y^2 \mathrm{d}x$ or $V = \int_a^b \pi x^2 \mathrm{d}y$ |

***Connections***

**Other contexts:** Industrial design; architecture.

**International-mindedness:** Accurate calculation of the volume of a cylinder by Chinese mathematician Liu Hui; use of infinitesimals by Greek geometers; Ibn Al Haytham, the first mathematician to calculate the integral of a function in order to find the volume of a paraboloid.

Download connections template

## AHL 5.13

| Content | Guidance, clarification and syllabus links |
|---|---|
| Kinematic problems involving displacement $s$, velocity $v$ and acceleration $a$. | $v = \frac{\mathrm{d}s}{\mathrm{d}t}$; $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}s}$.<br>Displacement $= \int_{t_1}^{t_2} v(t)\mathrm{d}t$.<br>Total distance travelled $= \int_{t_1}^{t_2} \lvert v(t)\rvert \mathrm{d}t$.<br>Speed is the magnitude of velocity. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| | Use of $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$. |

***Connections***

**Links to other subjects:** Kinematics (physics).

**International-mindedness:** Does the inclusion of kinematics as core mathematics reflect a particular cultural heritage? Who decides what is mathematics?

**TOK:** What is the role of convention in mathematics? Is this similar or different to the role of convention in other areas of knowledge?

Download connections template

## AHL 5.14

| Content | Guidance, clarification and syllabus links |
|---|---|
| Setting up a model/differential equation from a context. | **Example:** The growth of an algae $G$, at time $t$, is proportional to $\sqrt{G}$. |
| Solving by separation of variables. | **Example:** An exponential model as a solution of $\frac{dy}{dx} = ky$.<br>The term "general solution". |

***Connections***

Download connections template

## AHL 5.15

| Content | Guidance, clarification and syllabus links |
|---|---|
| Slope fields and their diagrams. | Students will be required to use and interpret slope fields. |

***Connections***

**TOK:** In what ways do values affect our representations of the world, for example in statistics, maps, visual images or diagrams?

Download connections template

## AHL 5.16

| Content | Guidance, clarification and syllabus links |
|---|---|
| Euler's method for finding the approximate solution to first order differential equations.<br>Numerical solution of $\frac{dy}{dx} = f(x, y)$. | Spreadsheets should be used to find approximate solutions to differential equations.<br>In examinations, values will be generated using permissible technology. |

| Content | Guidance, clarification and syllabus links |
|---|---|
| Numerical solution of the coupled system $\frac{dx}{dt} = f_1(x, y, t)$ and $\frac{dy}{dt} = f_2(x, y, t)$. | Contexts could include predator-prey models. |

***Connections***

**Other contexts:** The SIR model for infection as an extension of the method; Lotka-Volterra models.

**TOK:** To what extent is certainty attainable in mathematics? Is certainty attainable, or desirable, in other areas of knowledge?

**Enrichment:** Runge-Kutta methods.

Download connections template

## AHL 5.17

| Content | Guidance, clarification and syllabus links |
|---|---|
| Phase portrait for the solutions of coupled differential equations of the form:<br>$\frac{dx}{dt} = ax + by$<br>$\frac{dy}{dt} = cx + dy.$<br>Qualitative analysis of future paths for distinct, real, complex and imaginary eigenvalues.<br>Sketching trajectories and using phase portraits to identify key features such as equilibrium points, stable populations and saddle points. | Systems will have distinct, non-zero, eigenvalues.<br>If the eigenvalues are:<br>• Positive or complex with positive real part, all solutions move away from the origin<br>• Negative or complex with negative real part, all solutions move towards the origin<br>• Complex, the solutions form a spiral<br>• Imaginary, the solutions form a circle or ellipse<br>• Real with different signs (one positive, one negative) the origin is a saddle point.<br>Calculation of exact solutions is only required for the case of real distinct eigenvalues (link to AHL5.18).<br>**Link to:** eigenvectors and eigenvalues (AHL1.15). |

***Connections***

**Other contexts:** Jacobian matrix is used to investigate the stability of equilibrium states for non-linear differential equations.

Download connections template

## AHL 5.18

| Content | Guidance, clarification and syllabus links |
|---|---|
| Solutions of $\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt}, t)$ by Euler's method and by finding exact solutions (as AHL5.17). | Write as coupled first order equations $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = f(x, y, t)$.<br>Solutions of $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$, can also be investigated using the phase portrait method in AHL 5.17 above.<br>Understanding the occurrence of simple second order differential equations in physical phenomena |

| Content | Guidance, clarification and syllabus links |
|---|---|
| | would aid understanding but in examinations the equation will be given. |

***Connections***

**TOK:** How have notable individuals such as Euler shaped the development of mathematics as an area of knowledge?

**Use of technology:** Use of spreadsheets to generate values.

Download connections template

# Assessment in the Diploma Programme

## General

Assessment is an integral part of teaching and learning. The most important aim of assessment in the DP is that it should support curricular goals and encourage appropriate student learning. Both external and internal assessments are used in the DP. IB examiners mark work produced for external assessment, while work produced for internal assessment is marked by teachers and externally moderated by the IB.

There are two types of assessment identified by the IB.

Formative assessment informs both teaching and learning. It is concerned with providing accurate and helpful feedback to students and teachers on the kind of learning taking place and the nature of students' strengths and weaknesses in order to help develop students' understanding and capabilities. Formative assessment can also help to improve teaching quality, as it can provide information to monitor progress towards meeting the course aims and objectives.

Summative assessment gives an overview of previous learning and is concerned with measuring student achievement.

The DP primarily focuses on summative assessment designed to record student achievement at, or towards the end of, the course of study. However, many of the assessment instruments can also be used formatively during the course of teaching and learning, and teachers are encouraged to do this. A comprehensive assessment plan is viewed as being integral with teaching, learning and course organization. For further information, see the IB *Programme standards and practices* document.

The approach to assessment used by the IB is criterion-related, not norm-referenced. This approach to assessment judges students' work by their performance in relation to identified levels of attainment, and not in relation to the work of other students. For further information on assessment within the DP please refer to the publication *DP assessment: Principles and practice*.

To support teachers in the planning, delivery and assessment of the DP courses, a variety of resources can be found on the programme resource centre or purchased from the IB store (store.ibo.org). Additional publications such as specimen papers and markschemes, teacher support materials, subject reports and grade descriptors can also be found on the programme resource centre. Past examination papers as well as markschemes can be purchased from the IB store.

## Methods of assessment

The IB uses several methods to assess work produced by students.

### Assessment criteria

Assessment criteria are used when the assessment task is open-ended. Each criterion concentrates on a particular skill that students are expected to demonstrate. An assessment objective describes what students should be able to do, and assessment criteria describe how well they should be able to do it. Using assessment criteria allows discrimination between different answers and encourages a variety of responses. Each criterion comprises a set of hierarchically-ordered level descriptors. Each level descriptor is worth one or more marks. Each criterion is applied independently using a best-fit model. The maximum marks for each criterion may differ according to the criterion's importance. The marks awarded for each criterion are added together to give the total mark for the piece of work.

### Markbands

Markbands are a comprehensive statement of expected performance against which responses are judged. They represent a single holistic criterion divided into level descriptors. Each level descriptor corresponds to a range of marks to differentiate student performance. A best-fit approach is used to ascertain which particular mark to use from the possible range for each level descriptor.

### Analytic markschemes

Analytic markschemes are prepared for those examination questions that expect a particular kind of response and/or a given final answer from students. They give detailed instructions to examiners on how to break down the total mark for each question for different parts of the response.

### Marking notes

For some assessment components marked using assessment criteria, marking notes are provided. Marking notes give guidance on how to apply assessment criteria to the particular requirements of a question.

## Inclusive assessment arrangements

Inclusive assessment arrangements are available for candidates with assessment access requirements. These arrangements enable candidates with diverse needs to access the examinations and demonstrate their knowledge and understanding of the constructs being assessed.

The IB's *Access and inclusion policy* provides details on all the inclusive assessment arrangements available to candidates with learning support requirements. The IB document *Learning diversity and inclusion in the IB programmes* outlines the position of the IB with regard to candidates with diverse learning needs in the IB programmes. For candidates affected by adverse circumstances, the IB documents *General regulations: Diploma Programme* and Diploma Programme *Assessment procedures* provide details on access consideration.

## Responsibilities of the school

The school is required to ensure that that equal access arrangements and reasonable adjustments are provided to candidates with learning support requirements that are in line with the IB documents *Access and inclusion policy* and *Learning diversity and inclusion in the IB programmes.*

# Assessment outline—SL

First assessment 2021

| Assessment component | Weighting |
|---|---|
| **External assessment (3 hours)** | **80%** |
| **Paper 1 (90 minutes)**<br>Technology required. (80 marks)<br>Compulsory short-response questions based on the syllabus. (80 marks) | **40%** |
| **Paper 2 (90 minutes)**<br>Technology required. (80 marks)<br>Compulsory extended-response questions based on the syllabus. (80 marks) | **40%** |
| **Internal assessment**<br>This component is internally assessed by the teacher and externally moderated by the IB at the end of the course.<br>**Mathematical exploration**<br>Internal assessment in mathematics is an individual exploration. This is a piece of written work that involves investigating an area of mathematics. (20 marks) | **20%** |

# Assessment outline—HL

**First assessment 2021**

| Assessment component | Weighting |
|---|---|
| **External assessment (5 hours)** | **80%** |
| **Paper 1 (120 minutes)**<br>Technology required. (110 marks)<br>Compulsory short-response questions based on the syllabus. | **30%** |
| **Paper 2 (120 minutes)**<br>Technology required. (110 marks)<br>Compulsory extended-response questions based on the syllabus. | **30%** |
| **Paper 3 (75 minutes)**<br>Technology required. (55 marks)<br>Two compulsory extended response problem-solving questions. | **20%** |
| **Internal assessment**<br>This component is internally assessed by the teacher and externally moderated by the IB at the end of the course.<br>**Mathematical exploration**<br>Internal assessment in mathematics is an individual exploration. This is a piece of written work that involves investigating an area of mathematics. (20 marks) | **20%** |

# External assessment

## General

Markschemes are used to assess students in all papers. The markschemes are specific to each examination.

## External assessment details—SL

### General information

#### Paper 1 and paper 2

These papers are externally set and externally marked. Together, they contribute 80% of the final mark for the course. These papers are designed to allow students to demonstrate what they know and what they can do.

Papers 1 and 2 will contain some questions, or parts of questions, which are common with HL.

### Calculators

For both examination papers, students must have access to a graphic display calculator (GDC) at all times. Regulations covering the types of GDC allowed are provided in Diploma Programme *Assessment procedures*.

### Formula booklet

Each student must have access to a clean copy of the formula booklet during the examination. It is the responsibility of the school to download a copy from IBIS or the Programme Resource Centre and to ensure that there are sufficient copies available for all students.

### Awarding of marks

Marks are awarded for method, accuracy, answers and reasoning, including interpretation.

In paper 1 and paper 2, full marks are not necessarily awarded for a correct answer with no working. Where an answer is incorrect, some marks may be given for correct method, provided this is shown by written working. All students should therefore be advised to show their working.

### Paper 1

**Duration: 1 hour 30 minutes**

**Weighting: 40%**

- This paper consists of compulsory short-response questions.
- Questions on this paper will vary in terms of length and level of difficulty.
- A GDC is required for this paper, but not every question will necessarily require its use.
- Individual questions will not be worth the same number of marks. The marks allocated are indicated at the start of each question.

#### Syllabus coverage

- Knowledge of **all** topics is required for this paper. However, not all topics are necessarily assessed in every examination session.
- The intention of this paper is to test students' knowledge and understanding across the breadth of the syllabus. However, it should not be assumed that the separate topics are given equal emphasis.

### Mark allocation

- This paper is worth **80** marks, representing **40%** of the final mark.

### Question type

- Questions of varying levels of difficulty are set.
- One or more steps may be needed to solve each question.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

## Paper 2

**Duration: 1 hour 30 minutes**

**Weighting: 40%**

- This paper consists of compulsory extended-response questions.
- Questions on this paper will vary in terms of length and level of difficulty.
- A GDC is required for this paper, but not every question will necessarily require its use.
- Individual questions will not be worth the same number of marks. The marks allocated are indicated at the start of each question.

### Syllabus coverage

- Knowledge of **all** topics is required for this paper. However, not all topics are necessarily assessed in every examination session.
- The intention of this paper is to assess students' knowledge and understanding of the syllabus in depth. The range of syllabus topics tested in this paper may be narrower than that tested in paper 1.

### Mark allocation

- This paper is worth **80** marks, representing **40%** of the final mark.
- Questions of varying levels of difficulty and length are set. Therefore, individual questions may not necessarily each be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of the question.

### Question type

- Questions require extended responses.
- Individual questions may require knowledge of more than one topic.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline of difficulty, from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis is upon sustained reasoning.

# General

Markschemes are used to assess students in all papers. The markschemes are specific to each examination.

# External assessment details—HL

## General information

### Papers 1, 2 and 3

These papers are externally set and externally marked. Together, they contribute 80% of the final mark for the course. These papers are designed to allow students to demonstrate what they know and what they can do.

Papers 1 and 2 will contain some questions, or parts of questions, which are common with SL.

## Calculators

For all three examination papers, students must have access to a GDC at all times. Regulations covering the types of GDC allowed are provided in Diploma Programme *Assessment procedures*.

## Formula booklet

Each student must have access to a clean copy of the formula booklet during the examination. It is the responsibility of the school to download a copy from IBIS or the programme resource centre and to ensure that there are sufficient copies available for all students.

## Awarding of marks

Marks are awarded for method, accuracy, answers and reasoning, including interpretation.

In papers 1, 2 and 3, full marks are not necessarily awarded for a correct answer with no working. Where an answer is incorrect, some marks may be given for correct method, provided this is shown by written working. All students should therefore be advised to show their working.

## Paper 1

**Duration: 2 hours**

**Weighting: 30%**

- This paper consists of compulsory short-response questions.
- Questions on this paper will vary in terms of length and level of difficulty.
- A GDC is required for this paper, but not every question will necessarily require its use.
- Individual questions will not be worth the same number of marks. The marks allocated are indicated at the start of each question.

### Syllabus coverage

- Knowledge of **all** topics is required for this paper. However, not all topics are necessarily assessed in every examination session.
- The intention of this paper is to test students' knowledge and understanding across the breadth of the syllabus. However, it should not be assumed that the separate topics are given equal emphasis.

### Mark allocation

- This paper is worth **110** marks, representing **30%** of the final mark.

### Question type

- Questions of varying levels of difficulty are set.
- One or more steps may be needed to solve each question.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

## Paper 2

**Duration: 2 hours**

**Weighting: 30%**

- This paper consists of compulsory extended-response questions.
- Questions on this paper will vary in terms of length and level of difficulty.
- A GDC is required for this paper, but not every question will necessarily require its use.
- Individual questions will not be worth the same number of marks. The marks allocated are indicated at the start of each question.

### Syllabus coverage

- Knowledge of **all** topics is required for this paper. However, not all topics are necessarily assessed in every examination session.
- The intention of this paper is to assess students' knowledge and understanding of the syllabus in depth. The range of syllabus topics tested in this paper may be narrower than that tested in paper 1.

### Mark allocation

- This paper is worth **110** marks, representing **30%** of the final mark.
- Questions of varying levels of difficulty and length are set. Therefore, individual questions may not necessarily each be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of the question.

### Question type

- Questions require extended responses.
- Individual questions may require knowledge of more than one topic.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline of difficulty, from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis is upon sustained reasoning.

## Paper 3

**Duration: 1 hour and 15 minutes**

**Weighting: 20%**

- This paper consists of two compulsory extended-response problem-solving questions.
- A GDC is required for this paper, but not every question part will necessarily require its use.

### Syllabus coverage

- Where possible, the first part of each question will be on syllabus content leading to the problem-solving context. Therefore, knowledge of all syllabus topics is required for this paper.

### Mark allocation

- This paper is worth **55** marks, representing **20%** of the final mark.
- Questions may be unequal in terms of length and level of difficulty. Therefore, each question may not be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of each question.

### Question type

- Questions require extended responses involving sustained reasoning.

- Individual questions will develop from a single theme where the emphasis is on problem solving leading to a generalization or the interpretation of a context.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline in difficulty, from relatively easy at the start of a question to relatively difficult tasks at the end of the question. The emphasis is on problem solving.

# Internal assessment

## Purpose of internal assessment

Internal assessment is an integral part of the course and is compulsory for both SL and HL students. It enables students to demonstrate the application of their skills and knowledge and to pursue their personal interests without the time limitations and other constraints that are associated with written examinations. The internal assessment should, as far as possible, be woven into normal classroom teaching and not be a separate activity conducted after a course has been taught.

The internal assessment requirements at SL and at HL is an individual exploration. This is a piece of written work that involves investigating an area of mathematics. It is marked according to five assessment criteria.

## Guidance and authenticity

The exploration submitted for internal assessment must be the student's own work. However, it is not the intention that students should decide upon a title or topic and be left to work on the internal assessment component without any further support from the teacher. The teacher should play an important role during both the planning stage and the period when the student is working on the exploration.

It is the responsibility of the teacher to ensure that students are familiar with:

- the requirements of the type of work to be internally assessed
- the IB academic honesty policy available on the programme resource centre
- the assessment criteria; students must understand that the work submitted for assessment must address these criteria effectively.

Teachers and students must discuss the exploration. Students should be encouraged to initiate discussions with the teacher to obtain advice and information, and students must not be penalized for seeking guidance. As part of the learning process, teachers should read and give advice to students on one draft of the work. The teacher should provide oral or written advice on how the work could be improved, but not edit the draft. The next version handed to the teacher must be the final version for submission.

It is the responsibility of teachers to ensure that all students understand the basic meaning and significance of concepts that relate to academic honesty, especially authenticity and intellectual property. Teachers must ensure that all student work for assessment is prepared according to requirements and must explain clearly to students that the internally assessed work must be entirely their own.

All work submitted to the IB for moderation or assessment must be authenticated by a teacher, and must not include any known instances of suspected or confirmed malpractice. Each student must confirm that the work is his or her authentic work and constitutes the final version of that work. Once a student has officially submitted the final version of the work it cannot be retracted. The requirement to confirm the authenticity of work applies to the work of all students, not just the sample work that will be submitted to the IB for the purpose of moderation. For further details refer to the IB publication *Academic honesty in the IB educational context, The Diploma Programme: From principles into practice* and the relevant articles in *General regulations: Diploma Programme*.

Authenticity may be checked by discussion with the student on the content of the work, and scrutiny of one or more of the following:

- the student's initial proposal
- the draft of the written work
- the references cited
- the style of writing compared with work known to be that of the student

- the analysis of the work by a web-based plagiarism detection service such as www.turnitin.com.

The same piece of work cannot be submitted to meet the requirements of both the internal assessment and the extended essay.

# Collaboration and teamwork

Collaboration and teamwork are a key focus of the approaches to teaching in the DP. It is advisable that the teacher uses the available class time to manage student collaboration. While working on their exploration students should be encouraged to work collaboratively in the various phases of the process, for example:

- generating ideas
- selecting the topic for their exploration
- sharing research sources
- acquiring the necessary knowledge, skills and understanding
- seeking peer feedback on their writing.

The Approaches to teaching and learning (ATL) website on the programme resource centre provides an excellent source for developing collaborative skills in students.

While students should be encouraged to talk through their ideas with others, it is not appropriate to work together on a single exploration. It is important that students demonstrate how they incorporated sources and collaborative ideas into their work and that they always show their understanding and engagement in the work as described in the assessment criteria. Marks are awarded for the student's development and contribution to their exploration, not for work found in literature or carried out by others either individually or collaboratively.

It is imperative that students understand that the writing and calculations they do in their work must always be their own. This means that the argument they make and the ideas they rely on to make it, should either be their own or they should give credit to the source of those ideas. Any sources must be cited accordingly. This includes pictures, diagrams, graphs, formulae, etc.

In the specific cases of collecting information, data or measurements it is imperative that each student collects their own data, even in a case where collection of measurements is made from a group experiment. Group data or measurements can be combined to provide enough information for individual analysis and this should be clearly described in the written exploration.

# Time allocation

Internal assessment is an integral part of the mathematics courses, contributing 20% to the final assessment in the SL and the HL courses. This weighting should be reflected in the time that is allocated to teaching the knowledge, skills and understanding required to undertake the work, as well as the total time allocated to carry out the work.

It is recommended that a total of approximately 10-15 hours of teaching time should be allocated to the work. This should include:

- time for the teacher to explain to students the requirements of the exploration
- class time for students to work on the exploration and ask questions
- time for consultation between the teacher and each student
- time to review and monitor progress, and to check authenticity.

# Requirements and recommendations

Students can choose from a wide variety of activities: for example, modelling, investigations and applications of mathematics. To assist teachers and students in the choice of a topic, a list of stimuli is available in the teacher support material. However, students are not restricted to this list.

The exploration should be approximately 12-20 pages long with double line spacing, including diagrams and graphs, but excluding the bibliography. However, it is the quality of the mathematical writing that is important, not the length.

The teacher is expected to give appropriate guidance at all stages of the exploration by, for example, directing students into more productive routes of inquiry, making suggestions for suitable sources of information, and providing advice on the content and clarity of the exploration in the writing-up stage.

Teachers are responsible for indicating to students the existence of errors but should not explicitly correct these errors. It must be emphasized that students are expected to consult the teacher throughout the process.

All students should be familiar with the requirements of the exploration and the criteria by which it is assessed. Students need to start planning their explorations as early as possible in the course. Deadlines should be firmly established and adhered to. There should be a date for submission of the exploration topic and a brief outline description, a date for the submission of the draft and, of course, a date for completion.

In developing their explorations, students should aim to make use of mathematics learned as part of the course. The mathematics used should be commensurate with the level of the course–that is, it should be similar to that suggested in the syllabus. It is not expected that students produce work that is outside the syllabus–however, this will not be penalized.

Ethical guidelines should be adhered to throughout the planning and conducting of the exploration. Further details are given in the *Ethical practice in the Diploma Programme* poster on the programme resource centre.

## Presentation

The following details should be stated on the cover page of the exploration:

- title of the exploration
- number of pages.

The references are not assessed. However, if they are not included in the final report it may be flagged in terms of academic honesty.

## Using assessment criteria for internal assessment

For internal assessment, a number of assessment criteria have been identified. Each assessment criterion has level descriptors describing specific achievement levels, together with an appropriate range of marks. The level descriptors concentrate on positive achievement, although for the lower levels failure to achieve may be included in the description.

Teachers must judge the internally assessed work at SL and at HL against the criteria using the level descriptors.

The assessment criteria A to D are the same for both SL and HL. Criterion E "Use of mathematics" is different for SL and HL.

The aim is to find, for each criterion, the descriptor that conveys most accurately the level attained by the student, using the best-fit model. A best-fit approach means that compensation should be made when a piece of work matches different aspects of a criterion at different levels. The mark awarded should be one that most fairly reflects the balance of achievement against the criterion. It is not necessary for every single aspect of a level descriptor to be met for that mark to be awarded.

When assessing a student's work, teachers should read the level descriptors for each criterion until they reach a descriptor that most appropriately describes the level of the work being assessed. If a piece of work seems to fall between two descriptors, both descriptors should be read again and the one that more appropriately describes the student's work should be chosen.

Where there are two or more marks available within a level, teachers should award the upper marks if the student's work demonstrates the qualities described to a great extent; the work may be close to achieving

marks in the level above. Teachers should award the lower marks if the student's work demonstrates the qualities described to a lesser extent; the work may be close to achieving marks in the level below.

Only whole numbers should be recorded; partial marks, (fractions or decimals) are not acceptable.

Teachers should not think in terms of a pass or fail boundary, but should concentrate on identifying the appropriate descriptor for each assessment criterion.

The highest-level descriptors do not imply faultless performance but should be achievable by a student. Teachers should not hesitate to use the extremes if they are appropriate descriptions of the work being assessed.

A student who attains a high achievement level in relation to one criterion will not necessarily attain high achievement levels in relation to the other criteria. Similarly, a student who attains a low achievement level for one criterion will not necessarily attain low achievement levels for the other criteria. Teachers should not assume that the overall assessment of the students will produce any particular distribution of marks.

It is recommended that the assessment criteria be made available to students.

# Internal assessment details

## Mathematical exploration

**Duration: 10 to 15 hours**

**Weighting: 20%**

### Introduction

The internally-assessed component in this course is a mathematical exploration. This is a short report written by the student based on a topic chosen by him or her, and it should focus on the mathematics of that particular area. The emphasis is on mathematical communication (including formulae, diagrams, graphs, tables and so on), with his or her own focus, with the teacher providing feedback via, for example, discussion and interview. This will allow the students to develop areas of interest to them without a time constraint as in an examination, and allow all students to experience a feeling of success.

The final report should be approximately 12-20 pages long with double line spacing. It can be either word processed or handwritten. Students should be able to explain all stages of their work in such a way that demonstrates clear understanding. While there is no requirement that students present their work in class, it should be written in such a way that their peers would be able to follow it fairly easily. The report should include a detailed bibliography, and sources need to be referenced in line with the IB academic honesty policy. Direct quotes must be acknowledged.

### The purpose of the exploration

The aims of the Mathematics: analysis and approaches and Mathematics: applications and interpretation courses at both SL and HL are carried through into the objectives that are formally assessed as part of the course, through either written examination papers or the exploration, or both. In addition to testing the objectives of the course, the exploration is intended to provide students with opportunities to increase their understanding of mathematical concepts and processes, and to develop a wider appreciation of mathematics. These are noted in the aims of the course. It is intended that, by doing the exploration, students benefit from the mathematical activities undertaken and find them both stimulating and rewarding. It will enable students to acquire the attributes of the IB learner profile.

The specific purposes of the exploration are to:

- develop students' personal insight into the nature of mathematics and to develop their ability to ask their own questions about mathematics
- provide opportunities for students to complete a piece of mathematical work over an extended period of time
- enable students to experience the satisfaction of applying mathematical processes independently

- provide students with the opportunity to experience for themselves the beauty, power and usefulness of mathematics
- encourage students, where appropriate, to discover, use and appreciate the power of technology as a mathematical tool
- enable students to develop the qualities of patience and persistence, and to reflect on the significance of their work
- provide opportunities for students to show, with confidence, how they have developed mathematically.

### Management of the exploration

Work on the exploration should be incorporated into the course so that students are given the opportunity to learn the skills needed. Time in class can therefore be used for general discussion of areas of study, as well as familiarizing students with the criteria. Further details on the development of the exploration are included in the teacher support material.

# Internal assessment criteria—SL and HL

The exploration is internally assessed by the teacher and externally moderated by the IB using assessment criteria that relate to the objectives for mathematics.

Each exploration is assessed against the following five criteria. The final mark for each exploration is the sum of the scores for each criterion. The maximum possible final mark is 20.

**Students will not receive a grade for their mathematics course if they have not submitted an exploration.**

| | |
|---|---|
| **Criterion A** | Presentation |
| **Criterion B** | Mathematical communication |
| **Criterion C** | Personal engagement |
| **Criterion D** | Reflection |
| **Criterion E** | Use of mathematics |

## Criterion A: Presentation

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | The exploration has some coherence or some organization. |
| 2 | The exploration has some coherence and shows some organization. |
| 3 | The exploration is coherent and well organized. |
| 4 | The exploration is coherent, well organized, and concise. |

The “presentation” criterion assesses the organization and coherence of the exploration.

A **coherent** exploration is logically developed, easy to follow and meets its aim. This refers to the overall structure or framework, including introduction, body, conclusion and how well the different parts link to each other.

A **well-organized** exploration includes an introduction, describes the aim of the exploration and has a conclusion. Relevant graphs, tables and diagrams should accompany the work in the appropriate place and not be attached as appendices to the document. Appendices should be used to include information on large data sets, additional graphs, diagrams and tables.

A **concise** exploration does not show irrelevant or unnecessary repetitive calculations, graphs or descriptions.

The use of technology is not required but encouraged where appropriate. However, the use of analytic approaches rather than technological ones does not necessarily mean lack of conciseness, and should not be penalized. This does not mean that repetitive calculations are condoned.

## Criterion B: Mathematical communication

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | The exploration contains some relevant mathematical communication which is partially appropriate. |
| 2 | The exploration contains some relevant appropriate mathematical communication. |
| 3 | The mathematical communication is relevant, appropriate and is mostly consistent. |
| 4 | The mathematical communication is relevant, appropriate and consistent throughout. |

The "mathematical communication" criterion assesses to what extent the student has:

- used appropriate mathematical language (**notation, symbols, terminology**). Calculator and computer notation is acceptable only if it is software generated. Otherwise it is expected that students use appropriate mathematical notation in their work
- defined **key terms** and variables, where required
- used **multiple forms of mathematical representation**, such as formulae, diagrams, tables, charts, graphs and models, where appropriate
- used a **deductive method** and set out proofs logically where appropriate

Examples of level 1 can include graphs not being labelled, consistent use of computer notation with no other forms of correct mathematical communication.

Level 4 can be achieved by using only one form of mathematical representation as long as this is appropriate to the topic being explored. For level 4, any *minor* errors that do not impair clear communication should not be penalized.

## Criterion C: Personal engagement

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | There is evidence of some personal engagement. |
| 2 | There is evidence of significant personal engagement. |
| 3 | There is evidence of outstanding personal engagement. |

The "personal engagement" criterion assesses the extent to which the student engages with the topic by exploring the mathematics and making it their own. It is not a measure of effort.

Personal engagement may be recognized in different ways. These include thinking independently or creatively, presenting mathematical ideas in their own way, exploring the topic from different perspectives, making and testing predictions. Further (but not exhaustive) examples of personal engagement at different levels are given in the teacher support material (TSM).

There must be evidence of personal engagement demonstrated in the student's work. It is not sufficient that a teacher comments that a student was highly engaged.

Textbook style explorations or reproduction of readily available mathematics without the candidate's own perspective are unlikely to achieve the higher levels.

**Significant:** The student demonstrates authentic personal engagement in the exploration on a few occasions and it is evident that these drive the exploration forward and help the reader to better understand the writer's intentions.

**Outstanding:** The student demonstrates authentic personal engagement in the exploration in numerous instances and they are of a high quality. It is evident that these drive the exploration forward in a creative way. It leaves the impression that the student has developed, through their approach, a complete understanding of the context of the exploration topic and the reader better understands the writer's intentions.

## Criterion D: Reflection

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | There is evidence of limited reflection. |
| 2 | There is evidence of meaningful reflection. |
| 3 | There is substantial evidence of critical reflection. |

The "reflection" criterion assesses how the student reviews, analyses and evaluates the exploration. Although reflection may be seen in the conclusion to the exploration, it may also be found throughout the exploration.

Simply describing results represents **limited reflection**. Further consideration is required to achieve the higher levels.

Some ways of showing **meaningful reflection** are: linking to the aims of the exploration, commenting on what they have learned, considering some limitation or comparing different mathematical approaches.

**Critical reflection** is reflection that is crucial, deciding or deeply insightful. It will often develop the exploration by addressing the mathematical results and their impact on the student's understanding of the topic. Some ways of showing critical reflection are: considering what next, discussing implications of results, discussing strengths and weaknesses of approaches, and considering different perspectives.

**Substantial evidence** means that the critical reflection is present throughout the exploration. If it appears at the end of the exploration it must be of high quality and demonstrate how it developed the exploration in order to achieve a level 3.

Further (but not exhaustive) examples of reflection at different levels are given in the teacher support material (TSM).

## Criterion E: Use of mathematics—SL

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | Some relevant mathematics is used. |
| 2 | Some relevant mathematics is used. Limited understanding is demonstrated. |
| 3 | Relevant mathematics commensurate with the level of the course is used. Limited understanding is demonstrated. |

| Achievement level | Descriptor |
|---|---|
| 4 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is partially correct. Some knowledge and understanding are demonstrated. |
| 5 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is mostly correct. Good knowledge and understanding are demonstrated. |
| 6 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Thorough knowledge and understanding are demonstrated. |

The "Use of mathematics" SL criterion assesses to what extent students use mathematics that is **relevant** to the exploration.

**Relevant** refers to mathematics that supports the development of the exploration towards the completion of its aim. Overly complicated mathematics where simple mathematics would suffice is not relevant.

Students are expected to produce work that is **commensurate with the level** of the course, which means it should not be completely based on mathematics listed in the prior learning. The mathematics explored should either be part of the syllabus, or at a similar level.

A key word in the descriptor is **demonstrated**. The command term demonstrate means "to make clear by reasoning or evidence, illustrating with examples or practical application". Obtaining the correct answer is not sufficient to demonstrate understanding (even some understanding) in order to achieve level 2 or higher.

For knowledge and understanding to be **thorough** it must be demonstrated throughout.

The mathematics can be regarded as **correct** even if there are occasional minor errors as long as they do not detract from the flow of the mathematics or lead to an unreasonable outcome.

Students are encouraged to use technology to obtain results where appropriate, but **understanding must be demonstrated** in order for the student to achieve higher than level 1, for example merely substituting values into a formula does not necessarily demonstrate understanding of the results.

The mathematics only needs to be what is required to support the development of the exploration. This could be a few small elements of mathematics or even a single topic (or sub-topic) from the syllabus. It is better to do a few things well than a lot of things not so well. If the mathematics used is relevant to the topic being explored, commensurate with the level of the course and understood by the student, then it can achieve a high level in this criterion.

## Criterion E: Use of mathematics—HL

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | Some relevant mathematics is used. Limited understanding is demonstrated. |
| 2 | Some relevant mathematics is used. The mathematics explored is partially correct. Some knowledge and understanding is demonstrated. |
| 3 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Some knowledge and understanding are demonstrated. |

| Achievement level | Descriptor |
|---|---|
| 4 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Good knowledge and understanding are demonstrated. |
| 5 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct and demonstrates sophistication or rigour. Thorough knowledge and understanding are demonstrated. |
| 6 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is precise and demonstrates sophistication and rigour. Thorough knowledge and understanding are demonstrated. |

The "Use of mathematics" HL criterion assesses to what extent students use **relevant** mathematics in the exploration.

Students are expected to produce work that is **commensurate with the level** of the course, which means it should not be completely based on mathematics listed in the prior learning. The mathematics explored should either be part of the syllabus, at a similar level or slightly beyond. However, mathematics of a level slightly beyond the syllabus is **not** required to achieve the highest levels.

A key word in the descriptor is **demonstrated**. The command term demonstrate means to make clear by reasoning or evidence, illustrating with examples or practical application. Obtaining the correct answer is not sufficient to demonstrate understanding (even some understanding) in order to achieve level 2 or higher.

For knowledge and understanding to be thorough it must be demonstrated throughout. Lines of reasoning must be shown to justify steps in the mathematical development of the exploration.

**Relevant** refers to mathematics that supports the development of the exploration towards the completion of its aim. Overly complicated mathematics where simple mathematics would suffice is not relevant.

The mathematics can be regarded as **correct** even if there are occasional minor errors as long as they do not detract from the flow of the mathematics or lead to an unreasonable outcome. **Precise** mathematics is error-free and uses an appropriate level of accuracy at all times.

**Sophistication:** To be considered as sophisticated the mathematics used should be commensurate with the HL syllabus or, if contained in the SL syllabus, the mathematics has been used in a complex way that is beyond what could reasonably be expected of an SL student. Sophistication in mathematics may include understanding and using challenging mathematical concepts, looking at a problem from different perspectives and seeing underlying structures to link different areas of mathematics.

**Rigour** involves clarity of logic and language when making mathematical arguments and calculations. Mathematical claims relevant to the development of the exploration must be justified or proven.

Students are encouraged to use technology to obtain results where appropriate, but **understanding must be demonstrated** in order for the student to achieve level 1 or higher, for example merely substituting values into a formula does not necessarily demonstrate understanding of the results.

The mathematics only needs to be what is required to support the development of the exploration. This could be a few small elements of mathematics or even a single topic (or sub-topic) from the syllabus. It is better to do a few things well than a lot of things not so well. If the mathematics used is relevant to the topic being explored, commensurate with the level of the course and understood by the student, then it can achieve a high level in this criterion.

# Glossary of command terms

## Command terms for Mathematics: applications and interpretation

Students should be familiar with the following key terms and phrases used in examination questions, which are to be understood as described below. Although these terms will be used frequently in examination questions, other terms may be used to direct students to present an argument in a specific way.

| Command term | Definition |
|---|---|
| Calculate | Obtain a numerical answer showing the relevant stages in the working. |
| Comment | Give a judgment based on a given statement or result of a calculation. |
| Compare | Give an account of the similarities between two (or more) items or situations, referring to both (all) of them throughout. |
| Compare and contrast | Give an account of similarities and differences between two (or more) items or situations, referring to both (all) of them throughout. |
| Construct | Display information in a diagrammatic or logical form. |
| Contrast | Give an account of the differences between two (or more) items or situations, referring to both (all) of them throughout. |
| Deduce | Reach a conclusion from the information given. |
| Demonstrate | Make clear by reasoning or evidence, illustrating with examples or practical application. |
| Describe | Give a detailed account. |
| Determine | Obtain the only possible answer. |
| Differentiate | Obtain the derivative of a function. |
| Distinguish | Make clear the differences between two or more concepts or items. |
| Draw | Represent by means of a labelled, accurate diagram or graph, using a pencil. A ruler (straight edge) should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve. |
| Estimate | Obtain an approximate value. |
| Explain | Give a detailed account including reasons or causes. |
| Find | Obtain an answer showing relevant stages in the working. |
| Hence | Use the preceding work to obtain the required result. |
| Hence or otherwise | It is suggested that the preceding work is used, but other methods could also receive credit. |
| Identify | Provide an answer from a number of possibilities. |
| Integrate | Obtain the integral of a function. |

| Command term | Definition |
|---|---|
| Interpret | Use knowledge and understanding to recognize trends and draw conclusions from given information. |
| Investigate | Observe, study, or make a detailed and systematic examination, in order to establish facts and reach new conclusions. |
| Justify | Give valid reasons or evidence to support an answer or conclusion. |
| Label | Add labels to a diagram. |
| List | Give a sequence of brief answers with no explanation. |
| Plot | Mark the position of points on a diagram. |
| Predict | Give an expected result. |
| Prove | Use a sequence of logical steps to obtain the required result in a formal way. |
| Show | Give the steps in a calculation or derivation. |
| Show that | Obtain the required result (possibly using information given) without the formality of proof. “Show that” questions do not generally require the use of a calculator. |
| Sketch | Represent by means of a diagram or graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship, and should include relevant features. |
| Solve | Obtain the answer(s) using algebraic and/or numerical and/or graphical methods. |
| State | Give a specific name, value or other brief answer without explanation or calculation. |
| Suggest | Propose a solution, hypothesis or other possible answer. |
| Verify | Provide evidence that validates the result. |
| Write down | Obtain the answer(s), usually by extracting information. Little or no calculation is required. Working does not need to be shown. |

# Notation list

There are various systems of notation in use, and the IB has chosen to adopt a system of notation based on the recommendations of the International Organization for Standardization (ISO). This notation is used in the examination papers for this course without explanation. If forms of notation other than those listed in this guide are used on a particular examination paper, they are defined within the question in which they appear.

Because students are required to recognize, though not necessarily use, IB notation in examinations, it is recommended that teachers introduce students to this notation at the earliest opportunity. Students are **not** allowed access to information about this notation in the examinations.

Students must always use correct mathematical notation, not calculator notation.

## SL and HL

| | |
|---|---|
| $\mathbb{N}$ | the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$ |
| $\mathbb{Z}$ | the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$ |
| $\mathbb{Z}^+$ | the set of positive integers, $\{1, 2, 3, ...\}$ |
| $\mathbb{Q}$ | the set of rational numbers |
| $\mathbb{Q}^+$ | the set of positive rational numbers, $\{x \mid x \in \mathbb{Q}, x > 0\}$ |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}^+$ | the set of positive real numbers, $\{x \mid x \in \mathbb{R}, x > 0\}$ |
| $\{x_1, x_2, \ldots\}$ | the set with elements $x_1, x_2, ...$ |
| $n(A)$ | the number of elements in the finite set $A$ |
| $\{x \mid \quad\}$ | the set of all $x$ such that |
| $\in$ | is an element of |
| $\notin$ | is not an element of |
| $\varnothing$ | the empty (null) set |
| $U$ | the universal set |
| $\cup$ | union |
| $\cap$ | intersection |
| $A'$ | the complement of the set $A$ |
| $a^{1/2}, \sqrt{a}$ | $a$ to the power $\frac{1}{2}$, square root of $a$ (if $a \geq 0$ then $\sqrt{a} \geq 0$) |
| $a^{1/n}, \sqrt[n]{a}$ | $a$ to the power of $\frac{1}{n}$, $n^{\text{th}}$ root of $a$ (if $a \geq 0$ then $\sqrt[n]{a} \geq 0$) |
| $a^{-n} = \frac{1}{a^n}$ | $a$ to the power of $-n$, reciprocal of $a^n$ |

| | |
|---|---|
| $\lvert x \rvert$ | the modulus or absolute value of $x$, that is $\begin{cases} x & \text{for} \quad x \geq 0, \quad x \in \mathbb{R} \\ -x & \text{for} \quad x < 0, \quad x \in \mathbb{R} \end{cases}$ |
| $\approx$ | is approximately equal to |
| $>$ | is greater than |
| $\geq$ | is greater than or equal to |
| $<$ | is less than |
| $\leq$ | is less than or equal to |
| $\ngtr$ | is not greater than |
| $\nless$ | is not less than |
| $\Rightarrow$ | implies |
| $u_n$ | the $n^{\text{th}}$ term of a sequence or series |
| $d$ | the common difference of an arithmetic sequence |
| $r$ | the common ratio of a geometric sequence |
| $S_n$ | the sum of the first $n$ terms of a sequence, $u_1 + u_2 + \ldots + u_n$ |
| $\sum_{i=1}^{n} u_i$ | $u_1 + u_2 + \ldots + u_n$ |
| $f(x)$ | the image of $x$ under the function $f$ |
| $f^{-1}$ | the inverse function of the function $f$ |
| $\frac{\mathrm{d}y}{\mathrm{d}x}$ | the derivative of $y$ with respect to $x$ |
| $f'(x)$ | the derivative of $f(x)$ with respect to $x$ |
| $\int y \, \mathrm{d}\, x$ | the indefinite integral of $y$ with respect to $x$ |
| $\int_a^b y \, \mathrm{d}\, x$ | the definite integral of $y$ with respect to $x$ between the limits $x = a$ and $x = b$ |
| $\mathrm{e}^x$ | the exponential function of $x$ |
| $\log_a x$ | the logarithm to the base $a$ of $x$ |
| $\ln x$ | the natural logarithm of $x$, $\log_{\mathrm{e}} x$ |
| sin, cos, tan | the circular functions |
| $\mathrm{A}(x, y)$ | the point A in the plane with Cartesian coordinates $x$ and $y$ |
| $[\mathrm{AB}]$ | the line segment with end points A and B |
| $\mathrm{AB}$ | the length of $[\mathrm{AB}]$ |
| $(\mathrm{AB})$ | the line containing points A and B |
| $\hat{A}$ | the angle at A |
| $\mathrm{C\hat{A}B}$ | the angle between $[\mathrm{CA}]$ and $[\mathrm{AB}]$ |
| $\triangle \mathrm{ABC}$ | the triangle whose vertices are A, B and C |
| $\mathrm{P}(A)$ | probability of event $A$ |

| | |
|---|---|
| $\mathrm{P}(A')$ | probability of the event "not $A$" |
| $\mathrm{P}(A \mid B)$ | probability of the event $A$ given $B$ |
| $x_1, \quad x_2, \ldots$ | observations |
| $f_1, \quad f_2, \ldots$ | frequencies with which the observations $x_1, \ x_2, \ \ldots$ occur |
| $\mathrm{E}(X)$ | the expected value of the random variable $X$ |
| $\mu$ | population mean |
| $\sigma^2$ | population variance |
| $\sigma$ | population standard deviation |
| $\bar{x}$ | the sample mean of a set $\{x_1, x_2, \ldots, x_n\}$ of $n$ observations |
| $\mathrm{P}(X = x)$ | the probability that the random variable $X$ takes the value $x$ |
| $\mathrm{B}(n, p)$ | binomial distribution with parameters $n$ and $p$ |
| $\mathrm{N}(\mu, \sigma^2)$ | normal distribution with mean $\mu$ and variance $\sigma^2$ |
| $X \sim \mathrm{B}(n, p)$ | the random variable $X$ has a binomial distribution with parameters $n$ and $p$ |
| $X \sim \mathrm{N}(\mu, \sigma^2)$ | the random variable $X$ has a normal distribution with mean $\mu$ and variance $\sigma^2$ |
| $r$ | Pearson's product-moment correlation coefficient |
| $r_s$ | Spearman's rank correlation coefficient |
| $\nu$ | number of degrees of freedom |
| $\chi^2$ | chi-squared distribution |
| $\chi^2_{calc}$ | the chi-squared test statistic |
| $\mathrm{H}_0$ | the null hypothesis |
| $\mathrm{H}_1$ | the alternative hypothesis |

# HL only

| | |
|---|---|
| $\mathbb{C}$ | the set of complex numbers, $\{a + b\mathrm{i} \mid a, b \in \mathbb{R}\}$ |
| $\mathrm{i}$ | $\sqrt{-1}$ where $\mathrm{i}^2 = -1$ |
| $z$ | a complex number |
| $z^*$ | the complex conjugate of $z$ |
| $\lvert z \rvert$ | the modulus of $z$ |
| $\arg z$ | the argument of $z$ |
| $\mathrm{Re}\, z$ | the real part of $z$ |
| $\mathrm{Im}\, z$ | the imaginary part of $z$ |
| $\mathrm{cis}\theta$ | $\cos\theta + \mathrm{i}\sin\theta$ |
| $\mathrm{e}^{\mathrm{i}\theta}$ | Euler/exponential form of a complex number |
| $\Leftarrow$ | is implied by |

| | |
|---|---|
| $\Leftrightarrow$ | implies and is implied by |
| $[a, b]$ | the closed interval $a \leq x \leq b$ |
| $]a, b[$ | the open interval $a < x < b$ |
| $S_\infty$ | the sum to infinity of a sequence, $u_1 + u_2 + ...$ |
| $n!$ | $n(n-1)(n-2)...3 \times 2 \times 1$ |
| $\Delta$ | The discriminant of a quadratic equation, $\Delta = b^2 - 4ac$ |
| $f: \quad A \to B$ | $f$ is a function under which each element of set $A$ has an image in set $B$ |
| $f \circ g$ | the composite function of $f$ and $g$ |
| $\lim\limits_{x \to a} f(x)$ | the limit of $f(x)$ as $x$ tends to $a$ |
| $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ | the second derivative of $y$ with respect to $x$ |
| $f''(x)$ | the second derivative of $f(x)$ with respect to $x$ |
| $\dot{x}$ | the first derivative of $x$ with respect to time ($t$) |
| $\ddot{x}$ | the second derivative of $x$ with respect to time ($t$) |
| $\left.\begin{array}{ll} \arcsin, & \sin^{-1} \\ \arccos, & \cos^{-1} \\ \arctan, & \tan^{-1} \end{array}\right\}$ | the inverse circular functions |
| $\boldsymbol{v}$ | the vector $\boldsymbol{v}$ |
| $\overrightarrow{\mathrm{AB}}$ | the vector represented in magnitude and direction by the directed line segment from A to B |
| $\boldsymbol{a}$ | the position vector $\overrightarrow{\mathrm{OA}}$ |
| $\boldsymbol{i}, \quad \boldsymbol{j}, \quad \boldsymbol{k}$ | unit vectors in the directions of the Cartesian coordinate axes |
| $\lvert\boldsymbol{a}\rvert$ | the magnitude of $\boldsymbol{a}$ |
| $\lvert\overrightarrow{\mathrm{AB}}\rvert$ | the magnitude of $\overrightarrow{\mathrm{AB}}$ |
| $\boldsymbol{v} \cdot \boldsymbol{w}$ | the scalar product of $\boldsymbol{v}$ and $\boldsymbol{w}$ |
| $\boldsymbol{v} \times \boldsymbol{w}$ | the vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$ |
| $\boldsymbol{A}$ | the matrix $\boldsymbol{A}$ |
| $\boldsymbol{A}^{-1}$ | the inverse of the non-singular matrix $\boldsymbol{A}$ |
| $\det\boldsymbol{A}$ | the determinant of the square matrix $\boldsymbol{A}$ |
| $\boldsymbol{I}$ | the identity matrix |
| $\boldsymbol{0}$ | the zero matrix |
| $\boldsymbol{s}_0$ | an initial state matrix |
| $\boldsymbol{T}$ | a transition matrix |
| $\boldsymbol{A}_G$ | the adjacency matrix of a graph $G$ |
| $\boldsymbol{P}$ | a matrix of eigenvectors |
| $\boldsymbol{D}$ | a diagonal matrix of eigenvalues |
| $\mathrm{Var}(X)$ | the variance of the random variable $X$ |

| | |
|---|---|
| $s_n^2$ | sample variance |
| $s_n$ | standard deviation of the sample |
| $s_{n-1}^2$ | unbiased estimate of the population variance |
| $\mathrm{Po}(m)$ | Poisson distribution with mean $m$ |
| $X \sim \mathrm{Po}(m)$ | the random variable $X$ has a Poisson distribution with mean $m$ |
| $\rho$ | Pearson's product-moment correlation coefficient for the population |
| $SS_{res}$ | the sum of square residuals |
| $R^2$ | the coefficient of determination |
| $k_n$ | a complete graph with $n$ vertices |

# Updates to the publication

This section outlines the updates made to this publication over the past two years. The changes are ordered from the most recent to the oldest updates. Minor spelling and typographical corrections are not listed.

## Changes for November 2024

### Assessment > "Assessment outline—HL"

Correction of error in the previous version.

For Paper 3, the duration of "60 minutes" has been replaced with "75 minutes".

### Assessment > External assessment

"External assessment details—HL"

Correction of error in the previous version.

For Paper 3, the duration of "1 hour" has been replaced with "1 hour and 15 minutes".