PLAN AND PREPARE

Main Ideas

In this chapter students will perform translations with vectors, algebra and matrices. They will reflect figures in a given line, rotate figures about a point, identify line and rotational symmetry, and perform dilations using drawing tools and matrices.

Prerequisite Skills

Skills Readiness, available online, provides review and practice for the Skills and Algebra Check portion of the Prerequisite Skills quiz.

How student answers the exercises	What to assign from *Skills Readiness*
Any of Exs. 4–5 answered incorrectly	**Skill 40** Use reflections
Any of Exs. 6–7 answered incorrectly	**Skill 35** Use similar figures
All exercises answered correctly	Chapter Enrichment

Additional skills review and practice is available in the Skills Review Handbook and the @HomeTutor.

9 Properties of Transformations

COMMON CORE

Lesson	
9.1	CC.9-12.G.CO.5
9.2	CC.9-12.N.VM.8
9.3	CC.9-12.G.CO.5
9.4	CC.9-12.G.CO.5
9.5	CC.9-12.G.CO.5
9.6	CC.9-12.G.CO.3
9.7	CC.9-12.G.SRT.1

Before

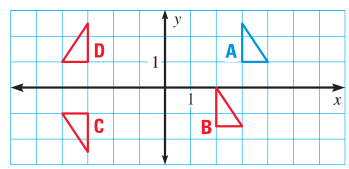
Previously, you learned the following skills, which you'll use in this chapter: translating, reflecting, and rotating polygons, and using similar polygons.

Prerequisite Skills

VOCABULARY CHECK

Match the transformation of Triangle A with its graph.

1. Translation of Triangle A **Triangle B**
2. Reflection of Triangle A **Triangle D**
3. Rotation of Triangle A **Triangle C**



SKILLS AND ALGEBRA CHECK

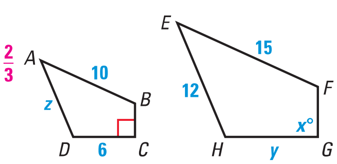
The vertices of *JKLM* are *J*(−1, 6), *K*(2, 5), *L*(2, 2), and *M*(−1, 1). Graph its image after the transformation described.

4. Reflect in the x-axis.
5. Reflect in the y-axis.

4, 5. See margin.

In the diagram, $ABCD \sim EFGH$.

6. Find the scale factor of $ABCD$ to $EFGH$. $\frac{2}{3}$
7. Find the values of x, y, and z. **90, 9, 8**



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MATHEMATICAL PRACTICES The Common Core Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. Opportunities to develop these practices are integrated throughout this program. Some examples are provided below.

1. Make sense of problems and persevere in solving them. Pages 589, 598, 626.
2. Reason abstractly and quantitatively. Pages 567, 570, 583, 623.
3. Construct viable arguments and critique the reasoning of others. Pages 568, 577, 586, 595, 599.
4. Model with mathematics. Pages 575, 578, 583, 623.
5. Use appropriate tools strategically. Pages 580, 590, 599, 617, 625.
6. Attend to precision. Pages 568, 580, 585, 603, 621.
7. Look for and make use of structure. Pages 582, 591, 601, 602.
8. Look for and express regularity in repeated reasoning. Pages 564, 566, 573, 591, 595.

COMMON CORE **Standards for Mathematical Content–**High School

Vector and Matrix Quantities		Lesson
CC.9-12.N.VM.8(+)	Add, subtract, and multiply matrices of appropriate dimensions.	9-2
Congruence		
CC.9-12.G.CO.3	Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.	9-6
CC.9-12.G.CO.5	Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.	9-1, 9-3, 9-4, 9-5

1 PLAN AND PREPARE

Warm-Up Exercises

Also available online

1. What is a translation? **a transformation that moves every point of a figure the same distance in the same direction**
2. Translate $A(3, 5)$ 4 units right and 2 units down. What are the coordinates of the image? **(7, 3)**
3. Find the length of $\overline{BC}$ with endpoints $B(-3, 5)$ and $C(1, 2)$. **5**
4. If you translate the points $M(4, 7)$ and $N(-1, 5)$ by using $(x, y) \to (x - 3, y + 6)$, what is the distance from M' to N'? $\sqrt{29}$

Notetaking Guide

Available online

Promotes interactive learning and notetaking skills.

Pacing

Basic: 2 days
Average: 2 days
Advanced: 2 days
Block: 1 block
- See *Teaching Guide/Lesson Plan*.

2 FOCUS AND MOTIVATE

Essential Question

Big Idea 3

How do you translate a figure using a vector? **Tell students they will learn how to answer this question by studying vectors.**

9.1 Translate Figures and Use Vectors

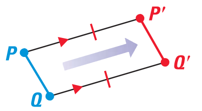
Before	You used a coordinate rule to translate a figure.
Now	You will use a vector to translate a figure.
Why?	So you can find a distance covered on snowshoes, as in Exs. 35–37.

Key Vocabulary
- image
- preimage
- isometry
- vector
  initial point, terminal point, horizontal component, vertical component
- component form
- translation

A *transformation* moves or changes a figure in some way to produce a new figure called an **image**. Another name for the original figure is the **preimage**.

Recall that a *translation* moves every point of a figure the same distance in the same direction. More specifically, a translation maps, or moves, the points P and Q of a plane figure to the points P' (read "P *prime*") and Q', so that one of the following statements is true:

- $PP' = QQ'$ and $\overline{PP'} \parallel \overline{QQ'}$, or
- $PP' = QQ'$ and $\overline{PP'}$ and $\overline{QQ'}$ are collinear.



EXAMPLE 1 Translate a figure in the coordinate plane

Graph quadrilateral *ABCD* with vertices $A(-1, 2)$, $B(-1, 5)$, $C(4, 6)$, and $D(4, 2)$. Find the image of each vertex after the translation $(x, y) \to (x + 3, y - 1)$. Then graph the image using prime notation.

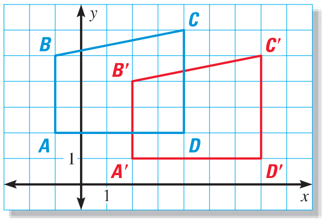
Solution

USE NOTATION
You can use *prime notation* to name an image. For example, if the preimage is $\triangle ABC$, then its image is $\triangle A'B'C'$, read as *"triangle A prime, B prime, C prime."*

First, draw *ABCD*. Find the translation of each vertex by adding 3 to its x-coordinate and subtracting 1 from its y-coordinate. Then graph the image.

$(x, y) \to (x + 3, y - 1)$

$A(-1, 2) \to A'(2, 1)$
$B(-1, 5) \to B'(2, 4)$
$C(4, 6) \to C'(7, 5)$
$D(4, 2) \to D'(7, 1)$



COMMON CORE

CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

✓ GUIDED PRACTICE for Example 1

1. Draw $\triangle RST$ with vertices $R(2, 2)$, $S(5, 2)$, and $T(3, 5)$. Find the image of each vertex after the translation $(x, y) \to (x + 1, y + 2)$. Graph the image using prime notation. **See margin for art; $R'(3, 4)$, $S'(6, 4)$, $T'(4, 7)$.**
2. The image of $(x, y) \to (x + 4, y - 7)$ is $\overline{P'Q'}$ with endpoints $P'(-3, 4)$ and $Q'(2, 1)$. Find the coordinates of the endpoints of the preimage. **$P(-7, 11)$, $Q(-2, 8)$**

COMMON CORE **Standards for Mathematical Content** *High School*

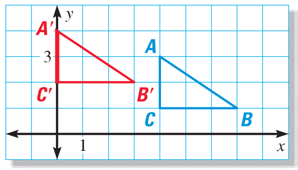
CC.9-12.G.CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

ISOMETRY An **isometry** is a transformation that preserves length and angle measure. Isometry is another word for congruence transformation.

EXAMPLE 2 Write a translation rule and verify congruence

READ DIAGRAMS
In this book, the preimage is always shown in blue, and the image is always shown in red.

Write a rule for the translation of $\triangle ABC$ to $\triangle A'B'C'$. Then verify that the transformation is an isometry.



Solution

To go from A to A', move 4 units left and 1 unit up. So, a rule for the translation is $(x, y) \to (x - 4, y + 1)$.

Use the SAS Congruence Postulate. Notice that $CB = C'B' = 3$, and $AC = A'C' = 2$. The slopes of $\overline{CB}$ and $\overline{C'B'}$ are 0, and the slopes of $\overline{CA}$ and $\overline{C'A'}$ are undefined, so the sides are perpendicular. Therefore, $\angle C$ and $\angle C'$ are congruent right angles. So, $\triangle ABC \cong \triangle A'B'C'$. The translation is an isometry.

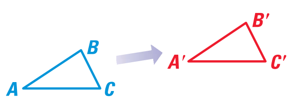
✓ GUIDED PRACTICE for Example 2

3. In Example 2, write a rule to translate $\triangle A'B'C'$ back to $\triangle ABC$.
$(x, y) \to (x + 4, y - 1)$

THEOREM — *For Your Notebook*

THEOREM 9.1 Translation Theorem

A translation is an isometry.



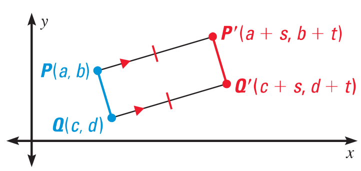
$\triangle ABC \cong \triangle A'B'C'$

PROOF Translation Theorem

A translation is an isometry.

GIVEN ▶ $P(a, b)$ and $Q(c, d)$ are two points on a figure translated by $(x, y) \to (x + s, y + t)$.

PROVE ▶ $PQ = P'Q'$



The translation maps $P(a, b)$ to $P'(a + s, b + t)$ and $Q(c, d)$ to $Q'(c + s, d + t)$.

Use the Distance Formula to find PQ and $P'Q'$. $PQ = \sqrt{(c - a)^2 + (d - b)^2}$.

$$P'Q' = \sqrt{[(c + s) - (a + s)]^2 + [(d + t) - (b + t)]^2}$$
$$= \sqrt{(c + s - a - s)^2 + (d + t - b - t)^2}$$
$$= \sqrt{(c - a)^2 + (d - b)^2}$$

Therefore, $PQ = P'Q'$ by the Transitive Property of Equality.

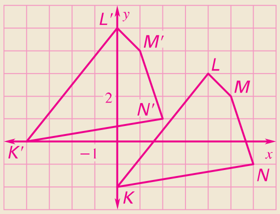
Motivating the Lesson

Ask students to imagine that an airplane is traveling at a constant velocity when it encounters a strong wind. Tell students that in this lesson they will study vectors and that vectors can be used to describe the effect of the wind on the path of the airplane.

3 TEACH

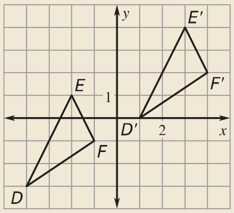
Extra Example 1

Graph quadrilateral $KLMN$ with vertices $K(0, -2)$, $L(4, 3)$, $M(5, 2)$, and $N(6, -1)$. Find the image of each vertex after the translation $(x, y) \to (x - 4, y + 2)$. Then graph the image using prime notation.



Extra Example 2

Write a rule for the translation of $\triangle DEF$ to $\triangle D'E'F'$. Then verify that the transformation is an isometry.



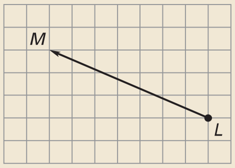
$(x, y) \to (x + 5, y + 3)$; $DE = D'E' = 2\sqrt{5}$, $EF = E'F' = \sqrt{5}$, $DF = D'F' = \sqrt{13}$, so the triangles are congruent by the SSS congruence Postulate.

1. See Additional Answers.

Extra Example 3

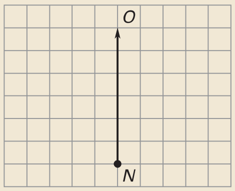
Name the vector and write its component form.

a.



$\overrightarrow{LM}$, ⟨−7, 3⟩

b.



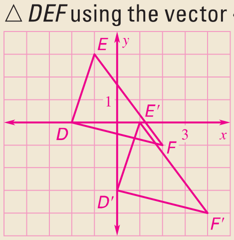
$\overrightarrow{NO}$, ⟨0, 6⟩

MATHEMATICAL PRACTICES ***Key Question Example 3***

- Is $\overrightarrow{BC}$ the same as $\overrightarrow{CB}$? Explain. **No, $\overrightarrow{BC}$ starts at *B* and ends at *C*, while $\overrightarrow{CB}$ starts at *C* and ends at *B*. They have the same length but opposite directions.**

Extra Example 4

The vertices of △*DEF* are *D*(−2, 0), *E*(−1, 3), and *F*(2, −1). Translate △*DEF* using the vector ⟨2, −3⟩.



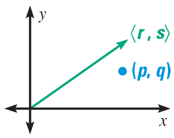
Avoiding Common Errors

Students may forget to use the brackets for the components of a vector. Instead, they will write the components as an ordered pair in parentheses. Show them that there are infinitely many arrows that model the vector ⟨5, 3⟩ but only one point for the ordered pair (5, 3). Stress the correct vector notation throughout the chapter.

VECTORS Another way to describe a translation is by using a vector. A **vector** is a quantity that has both direction and *magnitude,* or size. A vector is represented in the coordinate plane by an arrow drawn from one point to another.

USE NOTATION

Use brackets to write the component form of the vector ⟨*r*, *s*⟩. Use parentheses to write the coordinates of the point (*p*, *q*).



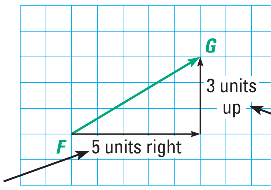
KEY CONCEPT *For Your Notebook*

Vectors

The diagram shows a vector named $\overrightarrow{FG}$, read as "vector *FG*."

The **initial point**, or starting point, of the vector is *F*.

The **terminal point**, or ending point, of the vector is *G*.



vertical component

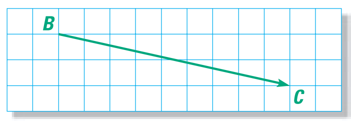
horizontal component

The **component form** of a vector combines the horizontal and vertical components. So, the component form of $\overrightarrow{FG}$ is ⟨5, 3⟩.

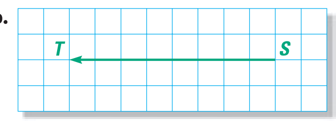
EXAMPLE 3 Identify vector components

Name the vector and write its component form.

a.



b.



Solution

a. The vector is $\overrightarrow{BC}$. From initial point *B* to terminal point *C*, you move 9 units right and 2 units down. So, the component form is ⟨9, −2⟩.

b. The vector is $\overrightarrow{ST}$. From initial point *S* to terminal point *T*, you move 8 units left and 0 units vertically. The component form is ⟨−8, 0⟩.

EXAMPLE 4 Use a vector to translate a figure

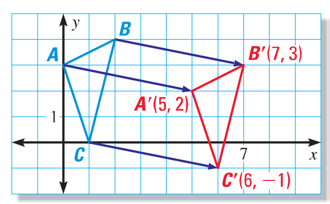
The vertices of △*ABC* are *A*(0, 3), *B*(2, 4), and *C*(1, 0). Translate △*ABC* using the vector ⟨5, −1⟩.

USE VECTORS

Notice that the vector can have different initial points. The vector describes only the direction and magnitude of the translation.

Solution

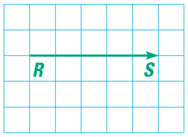
First, graph △*ABC*. Use ⟨5, −1⟩ to move each vertex 5 units to the right and 1 unit down. Label the image vertices. Draw △*A′B′C′*. Notice that the vectors drawn from preimage to image vertices are parallel.

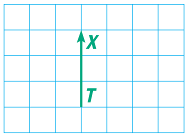


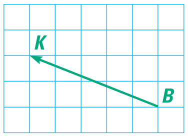
Guided Practice for Examples 3 and 4

4. $\overrightarrow{RS}$, ⟨5, 0⟩
5. $\overrightarrow{TX}$, ⟨0, 3⟩
6. $\overrightarrow{BK}$, ⟨−5, 2⟩

Name the vector and write its component form.

4. 

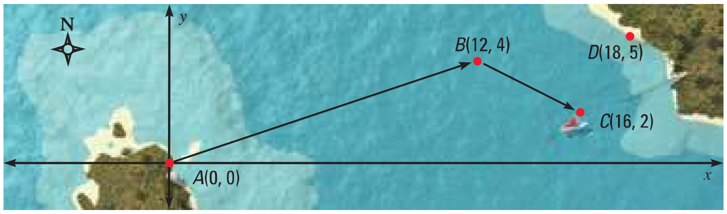
5. 

6. 

7. The vertices of $\triangle LMN$ are $L(2, 2)$, $M(5, 3)$, and $N(9, 1)$. Translate $\triangle LMN$ using the vector ⟨−2, 6⟩. **$L'(0, 8)$, $M'(3, 9)$, $N'(7, 7)$**

EXAMPLE 5 Solve a multi-step problem

NAVIGATION A boat heads out from point A on one island toward point D on another. The boat encounters a storm at B, 12 miles east and 4 miles north of its starting point. The storm pushes the boat off course to point C, as shown.



a. Write the component form of $\overrightarrow{AB}$.

b. Write the component form of $\overrightarrow{BC}$.

c. Write the component form of the vector that describes the straight line path from the boat's current position C to its intended destination D.

Solution

a. The component form of the vector from $A(0, 0)$ to $B(12, 4)$ is

$$\overrightarrow{AB} = \langle 12 - 0, 4 - 0\rangle = \langle 12, 4\rangle.$$

b. The component form of the vector from $B(12, 4)$ to $C(16, 2)$ is

$$\overrightarrow{BC} = \langle 16 - 12, 2 - 4\rangle = \langle 4, -2\rangle.$$

c. The boat is currently at point C and needs to travel to D. The component form of the vector from $C(16, 2)$ to $D(18, 5)$ is

$$\overrightarrow{CD} = \langle 18 - 16, 5 - 2\rangle = \langle 2, 3\rangle.$$

Guided Practice for Example 5

8. **WHAT IF?** In Example 5, suppose there is no storm. Write the component form of the vector that describes the straight path from the boat's starting point A to its final destination D. **⟨18, 5⟩**

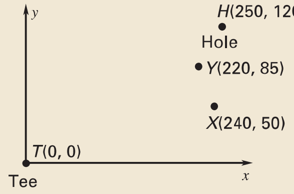
Extra Example 5

Ingrid hit a golf ball from the tee toward the hole. After she hit the ball, it got as far as point Y when a freak gust of wind blew it to point X.

a. Write the component form of $\overrightarrow{TY}$.
⟨220, 85⟩

b [illegible] $\overrightarrow{YX}$.
⟨20, −35⟩

b [illegible] X where the ball landed to the hole H.



⟨10, 70⟩

MATHEMATICAL PRACTICES Key Question Example 5

- Find the component form of $\overrightarrow{AC}$.
⟨16, 2⟩

Closing the Lesson

Have students summarize the major points of the lesson and answer the Essential Question: How do you translate a figure using a vector?

- **A translation is an isometry.**
- **A vector is a quantity with magnitude and direction.**

You add the vector components to the preimage coordinates to get the image coordinates.

Differentiated Instruction

Kinesthetic Learners In a wide, open area, choose a middle point to be the origin of a coordinate grid. Mark the x-axis and y-axis with masking tape or string. Have students begin at the origin. For **Guided Practice Exercises 4–6**, have them follow the path of the vector on the grid, each step representing one unit on the grid. Have them count the number of units they move up or down and left or right from the origin to determine the component form of each vector.

See also the *Differentiated Instruction Resources* for more strategies.

4 PRACTICE AND APPLY

Assignment Guide

Answers for all exercises available online

Basic:
Day 1:
Exs. 1–14, 28, 29, 33, 34
Day 2:
Exs. 15–27, 35–40

Average:
Day 1:
Exs. 1–6, 8–10, 12, 14, 28–31, 33, 34
Day 2:
Exs. 16, 17, 19–27, 36–43

Advanced:
Day 1:
Exs. 1–6, 8–10, 12, 14, 28–34*
Day 2:
Exs. 16, 17, 20–27, 37–46*

Block:
Exs. 1–6, 8–10, 12, 14, 16, 17, 19–31, 33, 34, 36–43

Differentiated Instruction

See *Differentiated Instruction Resources* for suggestions on addressing the needs of a diverse classroom.

Homework Check

For a quick check of student understanding of key concepts, go over the following exercises:

Basic: 4, 12, 16, 33, 36
Average: 6, 13, 20, 34, 36
Advanced: 10, 14, 22, 34, 37

Extra Practice

- Student Edition
- Chapter Resource Book: Practice levels A, B, C

Practice Worksheet

An easily-readable reduced practice page can be found at the beginning of this chapter.

9.1 EXERCISES

HOMEWORK KEY
○ = See **WORKED-OUT SOLUTIONS** Exs. 7, 11, and 35
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 14, and 42

SKILL PRACTICE

A

1. **VOCABULARY** Copy and complete: A _?_ is a quantity that has both _?_ and magnitude. **vector, direction**

2. ★ **WRITING** *Describe* the difference between a vector and a ray. **See margin.**

EXAMPLE 1 for Exs. 3–10

IMAGE AND PREIMAGE Use the translation $(x, y) \to (x - 8, y + 4)$.

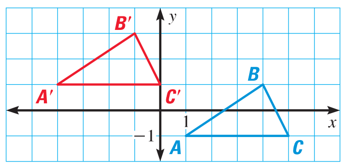
3. What is the image of $A(2, 6)$? **$A'(-6, 10)$**
4. What is the image of $B(-1, 5)$? **$B'(-9, 9)$**
5. What is the preimage of $C'(-3, -10)$? **$C(5, -14)$**
6. What is the preimage of $D'(4, -3)$? **$D(12, -7)$**

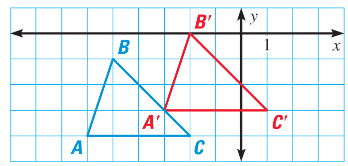
GRAPHING AN IMAGE The vertices of $\triangle PQR$ are $P(-2, 3)$, $Q(1, 2)$, and $R(3, -1)$. Graph the image of the triangle using prime notation. **7–10. See margin.**

7. $(x, y) \to (x + 4, y + 6)$
8. $(x, y) \to (x + 9, y - 2)$
9. $(x, y) \to (x - 2, y - 5)$
10. $(x, y) \to (x - 1, y + 3)$

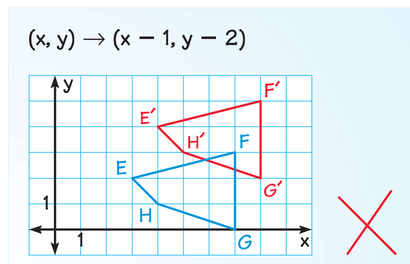
EXAMPLE 2 for Exs. 11–14

WRITING A RULE $\triangle A'B'C'$ is the image of $\triangle ABC$ after a translation. Write a rule for the translation. Then *verify* that the translation is an isometry. **11, 12. See margin.**

11.


12.


13. **ERROR ANALYSIS** *Describe* and correct the error in graphing the translation of quadrilateral *EFGH*.
The image should be 1 unit to the left instead of right and 2 units down instead of up; see margin for art.

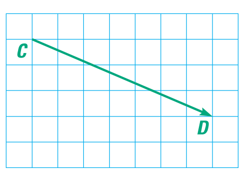


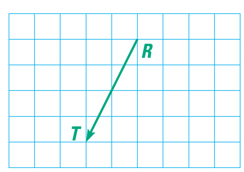
14. ★ **MULTIPLE CHOICE** Translate $Q(0, -8)$ using $(x, y) \to (x - 3, y + 2)$. **C**

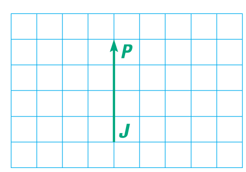
Ⓐ $Q'(-2, 5)$ Ⓑ $Q'(3, -10)$ Ⓒ $Q'(-3, -6)$ Ⓓ $Q'(2, -11)$

EXAMPLE 3 for Exs. 15–23

IDENTIFYING VECTORS Name the vector and write its component form.

15.


16.


17.


15. $\overrightarrow{CD}$, $\langle 7, -3 \rangle$
16. $\overrightarrow{RT}$, $\langle -2, -4 \rangle$
17. $\overrightarrow{JP}$, $\langle 0, 4 \rangle$

2. A vector is a quantity that has both direction and magnitude. A ray, $\overrightarrow{AB}$, consists of an initial point A and all points on $\overleftrightarrow{AB}$ that are on the same side of A as point B.

7–10. See Additional Answers.

11. $(x, y) \to (x - 5, y + 2)$; $AB = A'B' = \sqrt{13}$, $AC = A'C' = 4$, and $BC = B'C' = \sqrt{5}$. $\triangle ABC \cong \triangle A'B'C'$ using the SSS Congruence Postulate.

12. $(x, y) \to (x + 3, y + 1)$; $AB = A'B' = \sqrt{10}$, $AC = A'C' = 4$, and $BC = B'C' = 3\sqrt{2}$. $\triangle ABC \cong \triangle A'B'C'$ using the SSS Congruence Postulate.

13. See Additional Answers.

VECTORS **Use the point $P(-3, 6)$. Find the component form of the vector that describes the translation to P'.**

18. $P'(0, 1)$ $\langle 3, -5\rangle$ 19. $P'(-4, 8)$ $\langle -1, 2\rangle$ 20. $P'(-2, 0)$ $\langle 1, -6\rangle$ 21. $P'(-3, -5)$ $\langle 0, -11\rangle$

TRANSLATIONS **Think of each translation as a vector. *Describe* the vertical component of the vector. *Explain*.**

22.

23.

22. The vertical component is 0 since the bottles are translated left or right but not up or down.

23. The vertical component is the distance from the ground up to the plane entrance.

EXAMPLE 4 for Exs. 24–27

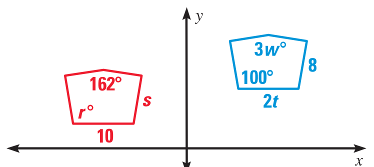
TRANSLATING A TRIANGLE **The vertices of $\triangle DEF$ are $D(2, 5)$, $E(6, 3)$, and $F(4, 0)$. Translate $\triangle DEF$ using the given vector. Graph $\triangle DEF$ and its image.** 24–27. See margin.

24. $\langle 6, 0\rangle$ 25. $\langle 5, -1\rangle$ 26. $\langle -3, -7\rangle$ 27. $\langle -2, -4\rangle$

B

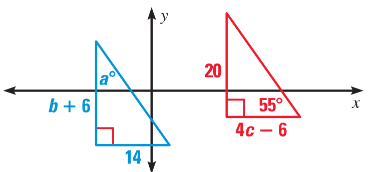
xy **ALGEBRA** **Find the value of each variable in the translation.**

28.



$r = 100$, $s = 8$, $t = 5$, $w = 54$

29.



$a = 35$, $b = 14$, $c = 5$

30. *xy* **ALGEBRA** Translation A maps (x, y) to $(x + n, y + m)$. Translation B maps (x, y) to $(x + s, y + t)$.

a. Translate a point using Translation A, then Translation B. Write a rule for the final image of the point. $(x, y) \to (x + n, y + m) \to (x + n + s, y + m + t)$

b. Translate a point using Translation B, then Translation A. Write a rule for the final image of the point. $(x, y) \to (x + s, y + t) \to (x + s + n, y + t + m)$

c. *Compare* the rules you wrote in parts (a) and (b). Does it matter which translation you do first? *Explain*. They are the same; no; $s + n = n + s$, $m + t = t + m$.

31. **MULTI-STEP PROBLEM** The vertices of a rectangle are $Q(2, -3)$, $R(2, 4)$, $S(5, 4)$, and $T(5, -3)$.

a. Translate $QRST$ 3 units left and 2 units down. Find the areas of $QRST$ and $Q'R'S'T'$. $Q'(-1, -5)$, $R'(-1, 2)$, $S'(2, 2)$, $T'(2, -5)$; 21, 21

b. *Compare* the areas. Make a conjecture about the areas of a preimage and its image after a translation. The areas are the same; the area of an image and its preimage under a translation are the same.

C

32. **CHALLENGE** The vertices of $\triangle ABC$ are $A(2, 2)$, $B(4, 2)$, and $C(3, 4)$. a, b. See margin for art.

a. Graph the image of $\triangle ABC$ after the transformation $(x, y) \to (x + y, y)$. Is the transformation an isometry? *Explain*. Are the areas of $\triangle ABC$ and $\triangle A'B'C'$ the same? No; $\triangle ABC$ is not congruent to $\triangle A'B'C'$; yes.

b. Graph a new triangle, $\triangle DEF$, and its image after the transformation given in part (a). Are the areas of $\triangle DEF$ and $\triangle D'E'F'$ the same? yes

Avoiding Common Errors

Exercises 5–6 Students may do these just as they did Exercises 3 and 4. Explain that in Exercises 5 and 6, the given point is the image, not the preimage. Discuss how they can work back from the image to find the preimage.

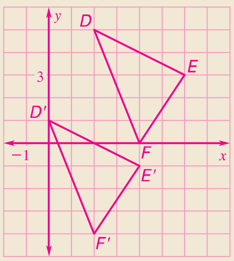
Teaching Strategy

Exercises 18–21 Have students graph P and P'. Draw $\overrightarrow{PP'}$. Explain that P is the initial point and P' the terminal point, so the vector components are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point.

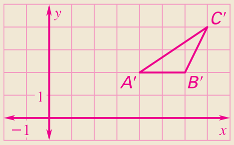
Mathematical Reasoning

Exercise 30 Have students replace n, m, s, and t with whole numbers and complete parts (a) and (b) of this exercise. Then have them generalize back to the variable expressions.

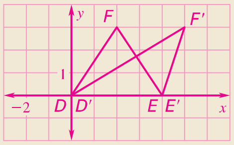
27. $D'(0, 1)$, $E'(4, -1)$, $F'(2, -4)$;



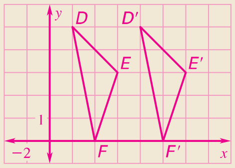
32a.



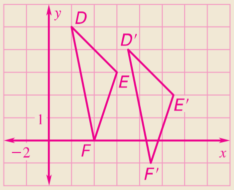
32b. *Sample:*



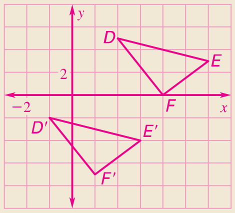
24. $D'(8, 5)$, $E'(12, 3)$, $F'(10, 0)$;



25. $D'(7, 4)$, $E'(11, 2)$, $F'(9, -1)$;



26. $D'(-1, -2)$, $E'(3, -4)$, $F'(1, -7)$;



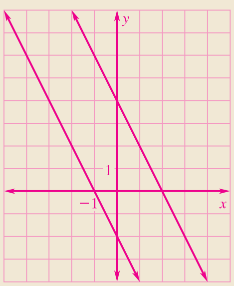
Reading Strategy

Exercise 43 Ask students to examine the diagram carefully. Discuss how the diagram provides visual clues that help explain what is meant by a *grid-indexed microscope slide*.

Teaching Strategy

Exercise 44 It may help students if you discuss how to relate each equation and graph to the graph of $y = x^2$.

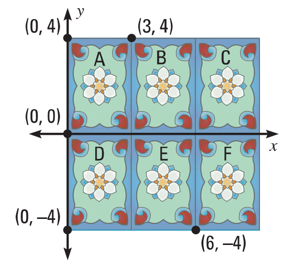
42a.



PROBLEM SOLVING

EXAMPLE 2 **A** for Exs. 33–34

HOME DESIGN **Designers can use computers to make patterns in fabrics or floors. On the computer, a copy of the design in Rectangle A is used to cover an entire floor. The translation $(x, y) \rightarrow (x + 3, y)$ maps Rectangle A to Rectangle B.**



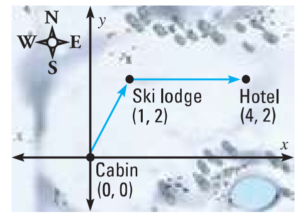
33. Use coordinate notation to describe the translations that map Rectangle A to Rectangles C, D, E, and F.

33. $(x, y) \rightarrow (x + 6, y)$, $(x, y) \rightarrow (x, y - 4)$, $(x, y) \rightarrow (x + 3, y - 4)$, $(x, y) \rightarrow (x + 6, y - 4)$

34. Write a rule to translate Rectangle F back to Rectangle A. $(x, y) \rightarrow (x - 6, y + 4)$

EXAMPLE 5 for Exs. 35–37

SNOWSHOEING **You are snowshoeing in the mountains. The distances in the diagram are in miles. Write the component form of the vector.**

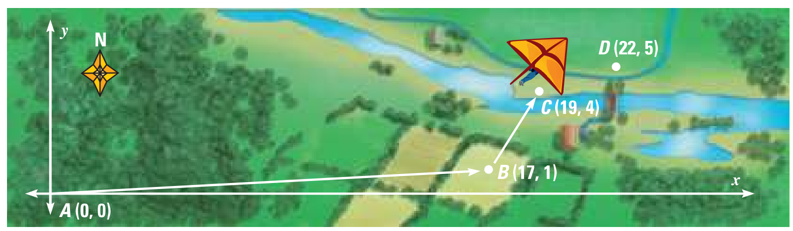


35. From the cabin to the ski lodge ⟨1, 2⟩

36. From the ski lodge to the hotel ⟨3, 0⟩

37. From the hotel back to your cabin ⟨−4, −2⟩

HANG GLIDING **A hang glider travels from point *A* to point *D*. At point *B*, the hang glider changes direction, as shown in the diagram. The distances in the diagram are in kilometers.**



B

38. Write the component form for $\overrightarrow{AB}$ and $\overrightarrow{BC}$. ⟨17, 1⟩, ⟨2, 3⟩

39. Write the component form of the vector that describes the path from the hang glider's current position *C* to its intended destination *D*. ⟨3, 1⟩

40. What is the total distance the hang glider travels? **about 23.8 km**

41. Suppose the hang glider went straight from *A* to *D*. Write the component form of the vector that describes this path. What is this distance? ⟨22, 5⟩; about 22.6 km

42. ★ **EXTENDED RESPONSE** Use the equation $2x + y = 4$.

a. Graph the line and its image after the translation ⟨−5, 4⟩. What is an equation of the image of the line? **See margin for art; $2x + y = -2$.**

b. *Compare* the line and its image. What are the slopes? the *y*-intercepts? the *x*-intercepts? −2, −2; (0, 4), (0, −2); (2, 0), (−1, 0)

c. Write an equation of the image of $2x + y = 4$ after the translation ⟨2, −6⟩ *without* using a graph. *Explain* your reasoning.
$2x + y = 2$. ***Sample answer:*** Translate the *y*-intercept, (0, 4) and *x*-intercept, (2, 0), using the translation ⟨2, −6⟩. Use these points to find the equation of the image.

◯ = See **WORKED-OUT SOLUTIONS** in Student Resources

★ = **STANDARDIZED TEST PRACTICE**

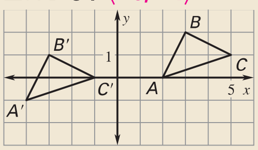
570

1 PLAN AND PREPARE

Warm-Up Exercises

Also available online

1. Write the component form of the vector that translates $\triangle ABC$ to $\triangle A'B'C'$. $\langle -6, -1 \rangle$



2. If $P(-8, 4)$ is translated by $(x, y) \to (x + 6, y + 1)$, what is the image of P? $(-2, 5)$

Notetaking Guide

Available online

Promotes interactive learning and notetaking skills.

Pacing

Basic: 2 days

Average: 2 days

Advanced: 2 days

Block: 1 block

- See *Teaching Guide/Lesson Plan*.

2 FOCUS AND MOTIVATE

Essential Question

Big Idea 3

How do you use matrix operations to translate a figure? **Tell students they will learn how to answer this question by studying matrices that represent figures and transformations.**

9.2 Use Properties of Matrices

Before You performed translations using vectors.

Now You will perform translations using matrix operations.

Why? So you can calculate the total cost of art supplies, as in Ex. 36.

Key Vocabulary
- **matrix**
- **element**
- **dimensions**

COMMON CORE

CC.9-12.N.VM.8(+) Add, subtract, and multiply matrices of appropriate dimensions.

READ VOCABULARY

An element of a matrix may also be called an *entry*.

A **matrix** is a rectangular arrangement of numbers in rows and columns. (The plural of matrix is *matrices*.) Each number in a matrix is called an **element**.

column

$$\text{row} \quad \begin{bmatrix} 5 & 4 & 4 & 9 \\ -3 & 5 & 2 & 6 \\ 3 & -7 & 8 & 7 \end{bmatrix}$$

← **The element in the second row and third column is 2.**

The **dimensions** of a matrix are the numbers of rows and columns. The matrix above has three rows and four columns, so the dimensions of the matrix are 3×4 (read "3 by 4").

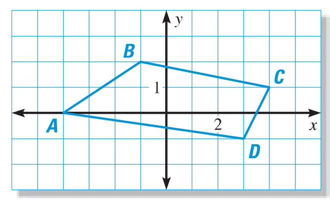
You can represent a figure in the coordinate plane using a matrix with two rows. The first row has the x-coordinate(s) of the vertices. The second row has the corresponding y-coordinate(s). Each column represents a vertex, so the number of columns depends on the number of vertices of the figure.

EXAMPLE 1 Represent figures using matrices

Write a matrix to represent the point or polygon.

a. Point A

b. Quadrilateral $ABCD$



Solution

a. Point matrix for A

$$\begin{bmatrix} -4 \\ 0 \end{bmatrix} \begin{matrix} \leftarrow x\text{-coordinate} \\ \leftarrow y\text{-coordinate} \end{matrix}$$

b. Polygon matrix for $ABCD$

$$\begin{matrix} & A & B & C & D \\ & \end{matrix}$$
$$\begin{bmatrix} -4 & -1 & 4 & 3 \\ 0 & 2 & 1 & -1 \end{bmatrix} \begin{matrix} \leftarrow x\text{-coordinates} \\ \leftarrow y\text{-coordinates} \end{matrix}$$

AVOID ERRORS

The columns in a polygon matrix follow the consecutive order of the vertices of the polygon.

✓ GUIDED PRACTICE for Example 1

1. Write a matrix to represent $\triangle ABC$ with vertices $A(3, 5)$, $B(6, 7)$ and $C(7, 3)$.

   1. $\begin{matrix} A & B & C \end{matrix}$ $\begin{bmatrix} 3 & 6 & 7 \\ 5 & 7 & 3 \end{bmatrix}$

2. How many rows and columns are in a matrix for a hexagon? **2 rows, 6 columns**

COMMON CORE **Standards for Mathematical Content** *High School*

CC.9-12.N.VM.8(+) Add, subtract, and multiply matrices of appropriate dimensions.

ADDING AND SUBTRACTING To add or subtract matrices, you add or subtract corresponding elements. The matrices must have the same dimensions.

EXAMPLE 2 Add and subtract matrices

a. $\begin{bmatrix} 5 & -3 \\ 6 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 5+1 & -3+2 \\ 6+3 & -6+(-4) \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 9 & -10 \end{bmatrix}$

b. $\begin{bmatrix} 6 & 8 & 5 \\ 4 & 9 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -7 & 0 \\ 4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 6-1 & 8-(-7) & 5-0 \\ 4-4 & 9-(-2) & -1-3 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 5 \\ 0 & 11 & -4 \end{bmatrix}$

TRANSLATIONS You can use matrix addition to represent a translation in the coordinate plane. The image matrix for a translation is the sum of the translation matrix and the matrix that represents the preimage.

EXAMPLE 3 Represent a translation using matrices

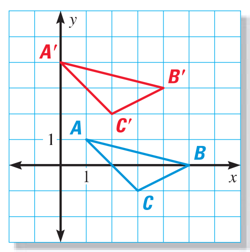
The matrix $\begin{bmatrix} 1 & 5 & 3 \\ 1 & 0 & -1 \end{bmatrix}$ represents $\triangle ABC$. Find the image matrix that represents the translation of $\triangle ABC$ 1 unit left and 3 units up. Then graph $\triangle ABC$ and its image.

Solution

The translation matrix is $\begin{bmatrix} -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix}$.

Add this to the polygon matrix for the preimage to find the image matrix.

$$\underset{\text{Translation matrix}}{\begin{bmatrix} -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix}} + \underset{\text{Polygon matrix}}{\overset{A \quad B \quad C}{\begin{bmatrix} 1 & 5 & 3 \\ 1 & 0 & -1 \end{bmatrix}}} = \underset{\text{Image matrix}}{\overset{A' \quad B' \quad C'}{\begin{bmatrix} 0 & 4 & 2 \\ 4 & 3 & 2 \end{bmatrix}}}$$



AVOID ERRORS
In order to add two matrices, they must have the same dimensions, so the translation matrix here must have three columns like the polygon matrix.

✓ GUIDED PRACTICE for Examples 2 and 3

In Exercises 3 and 4, add or subtract.

3. $\begin{bmatrix} -3 & 7 \end{bmatrix} + \begin{bmatrix} 2 & -5 \end{bmatrix}$ $\begin{bmatrix} -1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -4 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix}$ $\begin{bmatrix} -1 & -7 \\ -4 & -13 \end{bmatrix}$

5. The matrix $\begin{bmatrix} 1 & 2 & 6 & 7 \\ 2 & -1 & 1 & 3 \end{bmatrix}$ represents quadrilateral *JKLM*. Write the translation matrix and the image matrix that represents the translation of *JKLM* 4 units right and 2 units down. Then graph *JKLM* and its image. **See margin.**

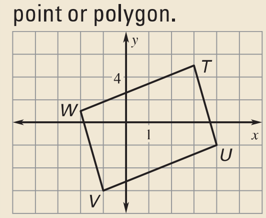
Motivating the Lesson

Remind students that they have used ordered pairs of numbers to represent individual points. Tell them that in this lesson they will learn to use arrays of numbers called matrices to represent points, polygons, and translations.

3 TEACH

Extra Example 1

Write a matrix that represents the point or polygon.



a. Point *T* $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

b. Quadrilateral *TUVW*

$\overset{T \quad U \quad V \quad W}{\begin{bmatrix} 3 & 4 & -1 & -2 \\ 5 & -2 & -6 & 1 \end{bmatrix}}$

Extra Example 2

Add or subtract.

a. $\begin{bmatrix} 7 & 2 \\ -5 & 9 \end{bmatrix} + \begin{bmatrix} -8 & 1 \\ 4 & 0 \end{bmatrix}$ $\begin{bmatrix} -1 & 3 \\ -1 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 3 & 5 \\ 7 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 12 & -2 & 1 \\ 6 & 3 & -4 \end{bmatrix}$

$\begin{bmatrix} -10 & 5 & 4 \\ 1 & -4 & 12 \end{bmatrix}$

5. See Additional Answers.

Differentiated Instruction

Auditory Learners Adding and subtracting matrices requires adding or subtracting several corresponding elements. Encourage students to say the numbers they are adding or subtracting aloud as they add or subtract the matrices. Describing the process aloud will help ensure they have added or subtracted the correct corresponding elements.

See also the *Differentiated Instruction Resources* for more strategies.

MULTIPLYING MATRICES The product of two matrices A and B is defined only when the number of columns in A is equal to the number of rows in B. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix.

USE NOTATION
Recall that the dimensions of a matrix are always written as rows × columns.

$$\begin{array}{ccccc} A & \cdot & B & = & AB \\ (m \text{ by } n) & \cdot & (n \text{ by } p) & = & (m \text{ by } p) \end{array}$$

equal (the two n's) **dimensions of AB**

You will use matrix multiplication in later lessons to represent transformations.

EXAMPLE 4 Multiply matrices

Multiply $\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix}$.

Solution

The matrices are both 2×2, so their product is defined. Use the following steps to find the elements of the product matrix.

STEP 1 **Multiply** the numbers in the first row of the first matrix by the numbers in the first column of the second matrix. Put the result in the first row, first column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & ? \\ ? & ? \end{bmatrix}$$

STEP 2 **Multiply** the numbers in the first row of the first matrix by the numbers in the second column of the second matrix. Put the result in the first row, second column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ ? & ? \end{bmatrix}$$

STEP 3 **Multiply** the numbers in the second row of the first matrix by the numbers in the first column of the second matrix. Put the result in the second row, first column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & ? \end{bmatrix}$$

STEP 4 **Multiply** the numbers in the second row of the first matrix by the numbers in the second column of the second matrix. Put the result in the second row, second column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & 4(-3) + 5(8) \end{bmatrix}$$

STEP 5 **Simplify** the product matrix.

$$\begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & 4(-3) + 5(8) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 28 \end{bmatrix}$$

Animated Geometry at my.hrw.com

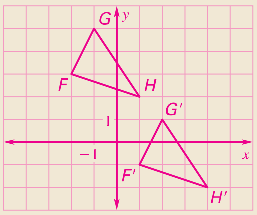
Avoiding Common Errors

If students write matrices for polygons in two columns rather than two rows, point out the error and have them correct their work. Tell them that the reason for using two rows will become clear when they use matrices to perform other types of transformations.

Extra Example 3

The matrix $\begin{bmatrix} -2 & -1 & 1 \\ 3 & 5 & 2 \end{bmatrix}$ represents $\triangle FGH$. Find the image matrix that represents the translation of $\triangle FGH$ 3 units right and 4 units down. Then graph $\triangle FGH$ and its image.

$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$



Extra Example 4

Multiply $\begin{bmatrix} 3 & -5 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$.

$\begin{bmatrix} 6 & -33 \\ 2 & 23 \end{bmatrix}$

Animated Geometry
my.hrw.com

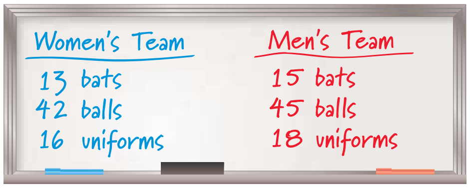
An **Animated Geometry** activity is available online for **Example 4.** This activity is also part of **Power Presentations.**

Study Strategy

Students may find it helpful to remember the following description of how to find the matrix product $A \cdot B$. To find the first row of $A \cdot B$, multiply the first row of A by each column of B. To find the second row of $A \cdot B$, multiply the second row of A by each column of B.

EXAMPLE 5 Solve a real-world problem

SOFTBALL Two softball teams submit equipment lists for the season. A bat costs $20, a ball costs $5, and a uniform costs $40. Use matrix multiplication to find the total cost of equipment for each team.



Solution

First, write the equipment lists and the costs per item in matrix form. You will use matrix multiplication, so you need to set up the matrices so that the number of columns of the equipment matrix matches the number of rows of the cost per item matrix.

EQUIPMENT · **COST** = **TOTAL COST**

$$\begin{array}{r} \\ \text{Women} \\ \text{Men} \end{array}\begin{array}{c} \text{Bats}\quad\text{Balls}\quad\text{Uniforms} \\ \begin{bmatrix} 13 & 42 & 16 \\ 15 & 45 & 18 \end{bmatrix} \end{array} \cdot \begin{array}{r} \text{Bats} \\ \text{Balls} \\ \text{Uniforms} \end{array}\overset{\text{Dollars}}{\begin{bmatrix} 20 \\ 5 \\ 40 \end{bmatrix}} = \begin{array}{r} \text{Women} \\ \text{Men} \end{array}\overset{\text{Dollars}}{\begin{bmatrix} ? \\ ? \end{bmatrix}}$$

You can find the total cost of equipment for each team by multiplying the equipment matrix by the cost per item matrix. The equipment matrix is 2 × 3 and the cost per item matrix is 3 × 1, so their product is a 2 × 1 matrix.

$$\begin{bmatrix} 13 & 42 & 16 \\ 15 & 45 & 18 \end{bmatrix}\begin{bmatrix} 20 \\ 5 \\ 40 \end{bmatrix} = \begin{bmatrix} 13(20) + 42(5) + 16(40) \\ 15(20) + 45(5) + 18(40) \end{bmatrix} = \begin{bmatrix} 1110 \\ 1245 \end{bmatrix}$$

▶ The total cost of equipment for the women's team is $1110, and the total cost for the men's team is $1245.

✓ GUIDED PRACTICE for Examples 4 and 5

Use the matrices below. Is the product defined? *Explain.*

$A = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 6.7 & 0 \\ -9.3 & 5.2 \end{bmatrix}$

6. AB

7. BA **Yes; the number of columns in *B* is equal to the number of rows in *A*.**

8. AC **No; the number of columns in *A* is not equal to the number of rows in *C*.**

Multiply.

9. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -4 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 5 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \end{bmatrix}$ **$\begin{bmatrix} -17 \end{bmatrix}$**

11. $\begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 5 & 1 \end{bmatrix}$

12. WHAT IF? In Example 5, find the total cost for each team if a bat costs $25, a ball costs $4, and a uniform costs $35. **women: $1053, men: $1185**

ANOTHER WAY
You could solve this problem arithmetically, multiplying the number of bats by the price of bats, and so on, then adding the costs for each team.

6. Yes; the number of columns in *A* is equal to the number of rows in *B*.

9. $\begin{bmatrix} 3 & 8 \\ 4 & -7 \end{bmatrix}$

11. $\begin{bmatrix} 15 & -19 \\ -3 & -5 \end{bmatrix}$

Extra Example 5

Jenny and Arthur are going to the store to buy tomatoes, peppers, and cucumbers. If a tomato costs $.89, a pepper $.59, and a cucumber $.45, use matrix multiplication to find the total amount each person spent.

Jenny	Arthur
3 tomatoes	7 tomatoes
2 peppers	4 peppers
4 cucumbers	2 cucumbers

Jenny: $5.65; Arthur: $9.49

MATHEMATICAL PRACTICES *Key Question Example 5*

- Could you multiply the matrices in reverse order? Explain. **No; there is one element in each row of the cost matrix, but there are two elements in each column of the equipment matrix.**

Closing the Lesson

Have students summarize the major points of the lesson and answer the Essential Question: How do you use matrix operations to translate a figure?

- **A matrix is a rectangular array of numbers in rows and columns.**
- **To add or subtract matrices, first check that the dimensions are the same. Then add or subtract corresponding elements.**
- **To multiply matrices the number of columns of the first matrix must be equal to the number of rows of the second matrix.**

To translate a figure, add the matrix for the point or polygon to the translation matrix.

4 PRACTICE AND APPLY

Assignment Guide

Answers for all exercises available online

Basic:
Day 1: SRH p. SR2 Exs. 1–8
Exs. 1–17
Day 2:
Exs. 18–25, 31–34

Average:
Day 1:
Exs. 1–6, 8–12, 14–17, 27, 28
Day 2:
Exs. 18–26, 29, 31–36

Advanced:
Day 1:
Exs. 1, 2, 3–6, 8–12, 14–17, 27–30*
Day 2:
Exs. 18–26, 31–37*

Block:
Exs. 1–6, 8–12, 14–29, 31–36

Differentiated Instruction

See *Differentiated Instruction Resources* for suggestions on addressing the needs of a diverse classroom.

Homework Check

For a quick check of student understanding of key concepts, go over the following exercises:

Basic: 4, 8, 14, 18, 31

Average: 5, 10, 15, 20, 31

Advanced: 6, 12, 16, 22, 31

Extra Practice

- Student Edition
- Chapter Resource Book: Practice levels A, B, C

Practice Worksheet

An easily-readable reduced practice page can be found at the beginning of this chapter.

9.2 EXERCISES

HOMEWORK KEY

○ = See **WORKED-OUT SOLUTIONS** Exs. 13, 19, and 31

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 17, 24, 25, and 35

SKILL PRACTICE

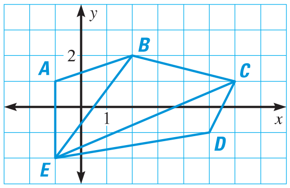
A

1. **VOCABULARY** Copy and complete: To find the sum of two matrices, add corresponding __?__. **elements**

2. ★ **WRITING** How can you determine whether two matrices can be added? How can you determine whether two matrices can be multiplied? **See margin.**

EXAMPLE 1 for Exs. 3–6

USING A DIAGRAM **Use the diagram to write a matrix to represent the given polygon.** **3–6. See margin.**



3. $\triangle EBC$
4. $\triangle ECD$
5. Quadrilateral $BCDE$
6. Pentagon $ABCDE$

8. $\begin{bmatrix} -10 & 2 \\ 1 & 4 \end{bmatrix}$

EXAMPLE 2 for Exs. 7–12

MATRIX OPERATIONS **Add or subtract.**

7. $\begin{bmatrix} 3 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 2 \end{bmatrix}$ $\begin{bmatrix} 12 & 7 \end{bmatrix}$

8. $\begin{bmatrix} -12 & 5 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 8 \end{bmatrix}$

9. $\begin{bmatrix} 9 & 8 \\ -2 & 3 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ 2 & -3 \\ -5 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 16 & 9 \\ 0 & 0 \\ -5 & -3 \end{bmatrix}$

10. $\begin{bmatrix} 4.6 & 8.1 \end{bmatrix} - \begin{bmatrix} 3.8 & -2.1 \end{bmatrix}$ $\begin{bmatrix} 0.8 & 10.2 \end{bmatrix}$

11. $\begin{bmatrix} -5 & 6 \\ -8 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 10 \\ 4 & -7 \end{bmatrix}$

12. $\begin{bmatrix} 1.2 & 6 \\ 5.3 & 1.1 \end{bmatrix} - \begin{bmatrix} 2.5 & -3.3 \\ 7 & 4 \end{bmatrix}$

EXAMPLE 3 for Exs. 13–17

TRANSLATIONS **Find the image matrix that represents the translation of the polygon. Then graph the polygon and its image.** **13–16. See margin.**

11. $\begin{bmatrix} -13 & -4 \\ -12 & 16 \end{bmatrix}$

12. $\begin{bmatrix} -1.3 & 9.3 \\ -1.7 & -2.9 \end{bmatrix}$

13. $\begin{matrix} A & B & C \end{matrix}$ $\begin{bmatrix} -2 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix}$; 4 units up

14. $\begin{matrix} F & G & H & J \end{matrix}$ $\begin{bmatrix} 2 & 5 & 8 & 5 \\ 2 & 3 & 1 & -1 \end{bmatrix}$; 2 units left and 3 units down

15. $\begin{matrix} L & M & N & P \end{matrix}$ $\begin{bmatrix} 2 & 0 & 2 & 3 \\ -1 & 3 & 3 & -1 \end{bmatrix}$; 4 units right and 2 units up

16. $\begin{matrix} Q & R & S \end{matrix}$ $\begin{bmatrix} -5 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}$; 3 units right and 1 unit down

17. ★ **MULTIPLE CHOICE** The matrix that represents quadrilateral $ABCD$ is $\begin{bmatrix} 3 & 8 & 9 & 7 \\ 3 & 7 & 3 & 1 \end{bmatrix}$. Which matrix represents the image of the quadrilateral after translating it 3 units right and 5 units up? **A**

Ⓐ $\begin{bmatrix} 6 & 11 & 12 & 10 \\ 8 & 12 & 8 & 6 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 0 & 5 & 6 & 4 \\ 8 & 12 & 8 & 6 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 6 & 11 & 12 & 10 \\ -2 & 2 & -2 & -4 \end{bmatrix}$ Ⓓ $\begin{bmatrix} 0 & 6 & 6 & 4 \\ -2 & 3 & -2 & -4 \end{bmatrix}$

2. If they have the same dimensions they can be added; if the number of columns in the first matrix matches the number of rows in the second matrix they can be multiplied.

3. $\begin{matrix} E & B & C \end{matrix}$ $\begin{bmatrix} -1 & 2 & 6 \\ -2 & 2 & 1 \end{bmatrix}$ **4.** $\begin{matrix} E & C & D \end{matrix}$ $\begin{bmatrix} -1 & 6 & 5 \\ -2 & 1 & -1 \end{bmatrix}$ **5.** $\begin{matrix} B & C & D & E \end{matrix}$ $\begin{bmatrix} 2 & 6 & 5 & -1 \\ 2 & 1 & -1 & -2 \end{bmatrix}$ **6.** $\begin{matrix} A & B & C & D & E \end{matrix}$ $\begin{bmatrix} -1 & 2 & 6 & 5 & -1 \\ 1 & 2 & 1 & -1 & -2 \end{bmatrix}$

13–16. See Additional Answers.

EXAMPLE 4 **B**
for Exs. 18–26

MATRIX OPERATIONS **Multiply.**

18. $\begin{bmatrix} 5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **[26]**

19. $\begin{bmatrix} 1.2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ -1.5 \end{bmatrix}$ **[−6.9]**

20. $\begin{bmatrix} 6 & 7 \\ -5 & 8 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 9 & -3 \end{bmatrix}$

20. $\begin{bmatrix} 75 & -15 \\ 62 & -29 \end{bmatrix}$

21. $\begin{bmatrix} 0.4 & 6 \\ -6 & 2.3 \end{bmatrix}\begin{bmatrix} 5 & 8 \\ -1 & 2 \end{bmatrix}$ **See margin.**

22. $\begin{bmatrix} 4 & 8 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ **[23]**

23. $\begin{bmatrix} 9 & 1 & 2 \\ 8 & -1 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 38 \\ 36 \end{bmatrix}$

24. ★ **MULTIPLE CHOICE** Which product is not defined? **C**

Ⓐ $\begin{bmatrix} 1 & 7 \\ 3 & 12 \end{bmatrix}\begin{bmatrix} 6 \\ 15 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 3 & 20 \end{bmatrix}\begin{bmatrix} 9 \\ 30 \end{bmatrix}$ Ⓒ $\begin{bmatrix} 15 \\ -3 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 4 & 0 \end{bmatrix}$ Ⓓ $\begin{bmatrix} 30 \\ -7 \end{bmatrix}\begin{bmatrix} 5 & 5 \end{bmatrix}$

25. ★ **OPEN-ENDED MATH** Write two matrices that have a defined product. Then find the product.

25. *Sample answer:* $\begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$; $\begin{bmatrix} 0 & -1 \\ 8 & 4 \end{bmatrix}$

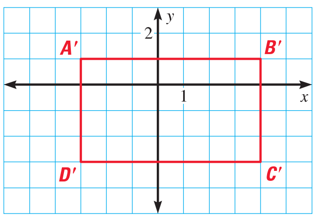
26. **ERROR ANALYSIS** *Describe* and correct the error in the computation.

$$\begin{bmatrix} 9 & -2 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} 9(-6) & -2(12) \\ 4(3) & 10(-6) \end{bmatrix}$$ ✗

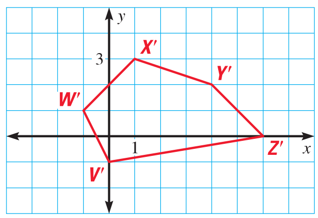
Corresponding elements were multiplied rather than rows and columns; $\begin{bmatrix} 9(-6) + -2(3) & 9(12) + -2(-6) \\ 4(-6) + 10(3) & 4(12) + 10(-6) \end{bmatrix}$

TRANSLATIONS **Use the described translation and the graph of the image to find the matrix that represents the preimage.** **27, 28. See margin.**

27. 4 units right and 2 units down



28. 6 units left and 5 units up



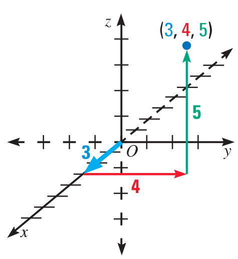
29. **MATRIX EQUATION** Use the description of a translation of a triangle to find the value of each variable. *Explain* your reasoning. What are the coordinates of the vertices of the image triangle? **See margin.**

$$\begin{bmatrix} 12 & 12 & w \\ -7 & v & -7 \end{bmatrix} + \begin{bmatrix} 9 & a & b \\ 6 & -2 & c \end{bmatrix} = \begin{bmatrix} m & 20 & -8 \\ n & -9 & 13 \end{bmatrix}$$

C **30.** **CHALLENGE** A point in space has three coordinates (x, y, z), as shown at the right. From the origin, a point can be forward or back on the x-axis, left or right on the y-axis, and up or down on the z-axis.

a. You translate a point three units forward, four units right, and five units up. Write a translation matrix for the point.

b. You translate a figure that has five vertices. Write a translation matrix to move the figure five units back, ten units left, and six units down. **See margin.**



30a. $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

Graphing Calculator

Exercises 7–12, 18–23 Teach students how to enter matrices on their graphing calculators and then how to add, subtract, and multiply matrices using the calculator.

Avoiding Common Errors

Exercises 18–23 Students sometimes think that matrix multiplication is an application of the Distributive Property of multiplication over addition. Help them by reviewing Example 4 carefully. Point out that none of the sums in the matrix equation in Step 4 could be the result of applying the Distributive Property.

21. $\begin{bmatrix} -4 & 15.2 \\ -32.3 & -43.4 \end{bmatrix}$

27. $\begin{matrix} A & B & C & D \end{matrix}$ $\begin{bmatrix} -7 & 0 & 0 & -7 \\ 3 & 3 & -1 & -1 \end{bmatrix}$

28. $\begin{matrix} V & W & X & Y & Z \end{matrix}$ $\begin{bmatrix} 6 & 5 & 7 & 10 & 12 \\ -6 & -4 & -2 & -3 & -5 \end{bmatrix}$

29. $a = 8$, $b = -20$, $c = 20$, $m = 21$, $n = -1$, $v = -7$, $w = 12$; **the numbers across each row of the transition matrix must be equal and the sum of the corresponding elements on the left equals the corresponding elements on the right; (21, −1), (20, −9), (−8, 13).**

30b. $\begin{bmatrix} -5 & -5 & -5 & -5 & -5 \\ -10 & -10 & -10 & -10 & -10 \\ -6 & -6 & -6 & -6 & -6 \end{bmatrix}$

Vocabulary

Exercises 33–35 Review the Commutative, Associative, and Distributive Properties for whole numbers before having the students complete these exercises. Be sure they know what the properties mean and understand how to check whether the properties extend to matrices.

33b. $\begin{bmatrix} -3 & 15 \\ -14 & 30 \end{bmatrix}, \begin{bmatrix} 25 & -7 \\ 10 & 2 \end{bmatrix},$

$AB \neq BA$

34b. $\begin{bmatrix} -81 & 3 \\ -178 & -26 \end{bmatrix}, \begin{bmatrix} -81 & 3 \\ -178 & -26 \end{bmatrix},$

$A(BC) = (AB)C$

35. $\begin{bmatrix} 2 & 36 \\ 16 & 68 \end{bmatrix}, \begin{bmatrix} 2 & 36 \\ 16 & 68 \end{bmatrix},$

the Distributive Property holds for matrices.

PROBLEM SOLVING

EXAMPLE 5 **A** for Ex. 31

31. **COMPUTERS** Two computer labs submit equipment lists. A mouse costs \$10, a package of CDs costs \$32, and a keyboard costs \$15. Use matrix multiplication to find the total cost of equipment for each lab.

Lab 1	Lab 2
25 Mice	15 Mice
10 CDs	20 CDs
18 Keyboards	12 Keyboards

31. Lab 1: \$840, Lab 2: \$970

32. **SWIMMING** Two swim teams submit equipment lists. The women's team needs 30 caps and 26 goggles. The men's team needs 15 caps and 25 goggles. A cap costs \$10 and goggles cost \$15.

a. Use matrix addition to find the total number of caps and the total number of goggles for each team. **32a. 45 caps, 51 goggles**

b. Use matrix multiplication to find the total equipment cost for each team. **Women: \$690; Men: \$525**

c. Find the total cost for both teams. **\$1215**

B **MATRIX PROPERTIES** **In Exercises 33–35, use matrices *A*, *B*, and *C*.**

$$A = \begin{bmatrix} 5 & 1 \\ 10 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 4 \\ -5 & 1 \end{bmatrix}$$

33. **MULTI-STEP PROBLEM** Use the 2×2 matrices above to explore the Commutative Property of Multiplication.

a. What does it mean that multiplication is *commutative*? $AB = BA$

b. Find and *compare* AB and BA. **See margin.**

c. Based on part (b), make a conjecture about whether matrix multiplication is commutative. **Matrix multiplication is not commutative.**

34. **MULTI-STEP PROBLEM** Use the 2×2 matrices above to explore the Associative Property of Multiplication.

a. What does it mean that multiplication is *associative*? $A(BC) = (AB)C$

b. Find and *compare* $A(BC)$ and $(AB)C$. **See margin.**

c. Based on part (b), make a conjecture about whether matrix multiplication is associative. **Matrix multiplication is associative.**

35. ★ **SHORT RESPONSE** Find and *compare* $A(B + C)$ and $AB + AC$. Make a conjecture about matrices and the Distributive Property. **See margin.**

36. **ART** Two art classes are buying supplies. A brush is \$4 and a paint set is \$10. Each class has only \$225 to spend. Use matrix multiplication to find the maximum number of brushes Class A can buy and the maximum number of paint sets Class B can buy. *Explain.* **26 brushes, 15 paint sets; solve $4x + 120 \leq 225$ and $72 + 10y \leq 225$.**

Class A	Class B
x brushes	18 brushes
12 paint sets	y paint sets

○ = See **WORKED-OUT SOLUTIONS** in Student Resources

★ = **STANDARDIZED TEST PRACTICE**

37a. 2002: Iowa about 1928 million, Illinois about 1471 million, Nebraska about 942 million, Minnesota about 1049 million; 2003: Iowa about 1881 million, Illinois about 1810 million, Nebraska about 1123 million, Minnesota about 971 million.

C **37. CHALLENGE** The total United States production of corn was 8,967 million bushels in 2002, and 10,114 million bushels in 2003. The table shows the percents of the total grown by four states.

a. Use matrix multiplication to find the number of bushels (in millions) harvested in each state each year.

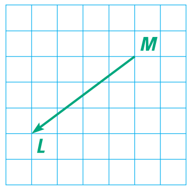
b. How many bushels (in millions) were harvested in these two years in Iowa? **about 3809 million bushels**

c. The price for a bushel of corn in Nebraska was \$2.32 in 2002, and \$2.45 in 2003. Use matrix multiplication to find the total value of corn harvested in Nebraska in these two years. **about \$4935 million**

	2002	2003
Iowa	21.5%	18.6%
Illinois	16.4%	17.9%
Nebraska	10.5%	11.1%
Minnesota	11.7%	9.6%

QUIZ

1. In the diagram shown, name the vector and write its component form. $\overrightarrow{ML}$, $\langle -4, -3 \rangle$



Use the translation $(x, y) \to (x + 3, y - 2)$.

2. What is the image of $(-1, 5)$? **(2, 3)**

3. What is the image of $(6, 3)$? **(9, 1)**

4. What is the preimage of $(-4, -1)$? **(−7, 1)**

Add, subtract, or multiply.

5. $\begin{bmatrix} 5 & -3 \\ 8 & -2 \end{bmatrix} + \begin{bmatrix} -9 & 6 \\ 4 & -7 \end{bmatrix}$ **6.** $\begin{bmatrix} -6 & 1 \\ 3 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 15 \\ -7 & 8 \end{bmatrix}$ **7.** $\begin{bmatrix} 7 & -6 & 2 \\ 8 & 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -9 & 0 \\ 3 & -7 \end{bmatrix}$

5. $\begin{bmatrix} -4 & 3 \\ 12 & -9 \end{bmatrix}$

6. $\begin{bmatrix} -10 & -14 \\ 10 & 4 \end{bmatrix}$

7. $\begin{bmatrix} 95 & 0 \\ 28 & -19 \end{bmatrix}$

See **EXTRA PRACTICE** in Student Resources **ONLINE QUIZ** at my.hrw.com

5 ASSESS AND RETEACH

Daily Homework Quiz

Also available online

Add or subtract

1. $\begin{bmatrix} 2.8 & -9.2 \end{bmatrix} + \begin{bmatrix} 3.5 & 6.1 \end{bmatrix}$

$\begin{bmatrix} 6.3 & -3.1 \end{bmatrix}$

2. $\begin{bmatrix} 8 & -2 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 1 & -7 \end{bmatrix}$

$\begin{bmatrix} 4 & -5 \\ -4 & 12 \end{bmatrix}$

3. Triangle *ABC* has vertices $A(2, -1)$, $B(1, 3)$, and $C(-2, -2)$. Write a matrix equation that shows how to find the image matrix that represents the translation of $\triangle ABC$ 4 units left and 5 units up.

$\begin{bmatrix} -4 & -4 & -4 \\ 5 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & -2 \end{bmatrix} =$

$\begin{bmatrix} -2 & -3 & -6 \\ 4 & 8 & 3 \end{bmatrix}$

4. Multiply $\begin{bmatrix} 2 & -5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 3 & -2 \end{bmatrix}$.

$\begin{bmatrix} -3 & 8 \\ 6 & -1 \end{bmatrix}$

Online Quiz

Available at **my.hrw.com**

Diagnosis/Remediation

- Practice A, B, C in Chapter Resource Book
- Study Guide in Chapter Resource Book
- Practice Workbook
- @HomeTutor

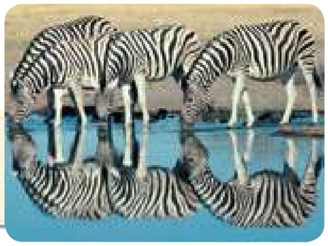
Challenge

Additional challenge is available in the Chapter Resource Book.

Quiz

An easily-readable reduced copy of the quiz from the Assessment Book can be found at the beginning of this chapter.

9.3 Perform Reflections



Before	You reflected a figure in the *x*- or *y*-axis.
Now	You will reflect a figure in any given line.
Why?	So you can identify reflections, as in Exs. 31–33.

Key Vocabulary
- **line of reflection**
- **reflection**

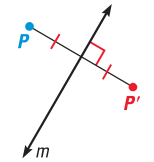
COMMON CORE

CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

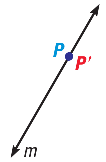
A *reflection* is a transformation that uses a line like a mirror to reflect an image. The mirror line is called the **line of reflection**.

A reflection in a line m maps every point P in the plane to a point P', so that for each point one of the following properties is true:

- If P is not on m, then m is the perpendicular bisector of $\overline{PP'}$, or
- If P is on m, then $P = P'$.



Point *P* not on *m*



Point *P* on *m*

EXAMPLE 1 Graph reflections in horizontal and vertical lines

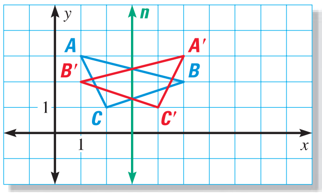
The vertices of $\triangle ABC$ are $A(1, 3)$, $B(5, 2)$, and $C(2, 1)$. Graph the reflection of $\triangle ABC$ described.

a. In the line n: $x = 3$

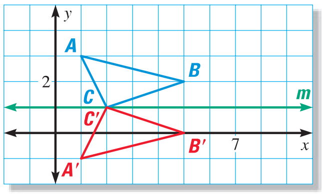
b. In the line m: $y = 1$

Solution

a. Point A is 2 units left of n, so its reflection A' is 2 units right of n at (5, 3). Also, B' is 2 units left of n at (1, 2), and C' is 1 unit right of n at (4, 1).



b. Point A is 2 units above m, so A' is 2 units below m at (1, −1). Also, B' is 1 unit below m at (5, 0). Because point C is on line m, you know that $C = C'$.



✓ **GUIDED PRACTICE** for Example 1

Graph a reflection of $\triangle ABC$ from Example 1 in the given line.

1–3. See Additional Answers beginning.

1. $y = 4$ **2.** $x = -3$ **3.** $y = 2$

1 PLAN AND PREPARE

Warm-Up Exercises

Also available online

1. The *y*-axis is the perpendicular bisector of $\overline{AB}$. If point A is at (−3, 5), what is the location of point B? **(3, 5)**

2. What is the slope of the segment with endpoints at (−7, 4) and (4, −7)? **−1**

3. Multiply $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & -2 & 1 \\ 3 & 4 & 7 \end{bmatrix}$.

$\begin{bmatrix} -6 & 2 & -1 \\ 3 & 4 & 7 \end{bmatrix}$

Notetaking Guide

Available online

Promotes interactive learning and notetaking skills.

Pacing

Basic: 2 days
Average: 2 days
Advanced: 2 days
Block: 1 block
- See *Teaching Guide/Lesson Plan*.

2 FOCUS AND MOTIVATE

Essential Question

Big Idea 1

How do you reflect a figure in the line $y = x$? **Tell students they will learn how to answer this question by studying coordinate rules and matrices for reflections.**

COMMON CORE

Standards for Mathematical Content *High School*

CC.9-12.G.CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

CC.9-12.G.CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Motivating the Lesson

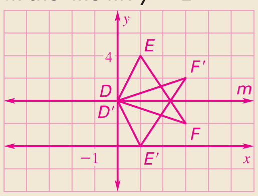
Have students imagine that they are playing miniature golf and want to hit the ball off a wall so it will land in the hole. Tell students that in this lesson they will learn about geometric transformations that can help solve this problem.

3 TEACH

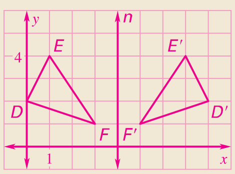
Extra Example 1

The vertices of $\triangle DEF$ are $D(0, 2)$, $E(1, 4)$, and $F(3, 1)$. Graph the reflection of $\triangle DEF$ described.

a. In the line m: $y = 2$

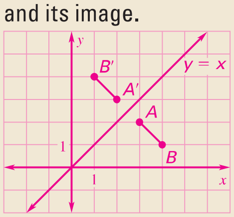


b. In the line n: $x = 4$



Extra Example 2

The endpoints of $\overline{AB}$ are $A(3, 2)$ and $B(4, 1)$. Reflect the segment in the line $y = x$. Graph the segment and its image.



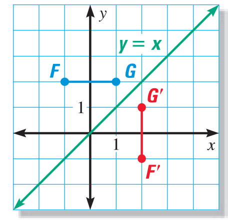
EXAMPLE 2 Graph a reflection in $y = x$

The endpoints of $\overline{FG}$ are $F(-1, 2)$ and $G(1, 2)$. Reflect the segment in the line $y = x$. Graph the segment and its image.

Solution

REVIEW SLOPE
The product of the slopes of perpendicular lines is -1.

The slope of $y = x$ is 1. The segment from F to its image, $\overline{FF'}$, is perpendicular to the line of reflection $y = x$, so the slope of $\overline{FF'}$ will be -1 (because $1(-1) = -1$). From F, move 1.5 units right and 1.5 units down to $y = x$. From that point, move 1.5 units right and 1.5 units down to locate $F'(2, -1)$.



The slope of $\overline{GG'}$ will also be -1. From G, move 0.5 units right and 0.5 units down to $y = x$. Then move 0.5 units right and 0.5 units down to locate $G'(2, 1)$.

COORDINATE RULES You can use coordinate rules to find the images of points reflected in four special lines.

KEY CONCEPT *For Your Notebook*

Coordinate Rules for Reflections

- If (a, b) is reflected in the x-axis, its image is the point $(a, -b)$.
- If (a, b) is reflected in the y-axis, its image is the point $(-a, b)$.
- If (a, b) is reflected in the line $y = x$, its image is the point (b, a).
- If (a, b) is reflected in the line $y = -x$, its image is the point $(-b, -a)$.

EXAMPLE 3 Graph a reflection in $y = -x$

Reflect $\overline{FG}$ from Example 2 in the line $y = -x$. Graph $\overline{FG}$ and its image.

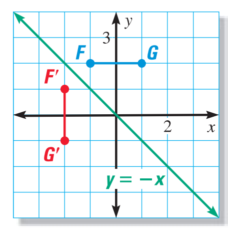
Solution

Use the coordinate rule for reflecting in $y = -x$.

$$(a, b) \rightarrow (-b, -a)$$

$F(-1, 2) \rightarrow F'(-2, 1)$

$G(1, 2) \rightarrow G'(-2, -1)$



Animated Geometry at my.hrw.com

GUIDED PRACTICE for Examples 2 and 3

4. Graph $\triangle ABC$ with vertices $A(1, 3)$, $B(4, 4)$, and $C(3, 1)$. Reflect $\triangle ABC$ in the lines $y = -x$ and $y = x$. Graph each image. **See margin.**

5. In Example 3, *verify* that $\overline{FF'}$ is perpendicular to $y = -x$.

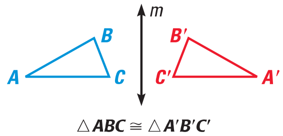
5. Slope of $y = -x$ is -1. The slope of $\overline{FF'}$ is 1. The product of their slopes is -1 making them perpendicular.

REFLECTION THEOREM Recall that the image of a translation is congruent to the original figure. The same is true for a reflection.

THEOREM *For Your Notebook*

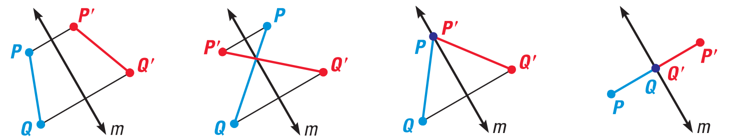
THEOREM 9.2 Reflection Theorem

A reflection is an isometry.



WRITE PROOFS
Some theorems, such as the Reflection Theorem, have more than one case. To prove this type of theorem, each case must be proven.

PROVING THE THEOREM To prove the Reflection Theorem, you need to show that a reflection preserves the length of a segment. Consider a segment $\overline{PQ}$ that is reflected in a line m to produce $\overline{P'Q'}$. There are four cases to prove:



Case 1 P and Q are on the same side of m.

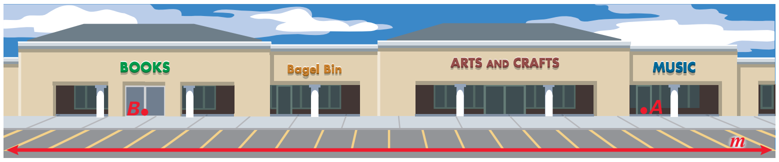
Case 2 P and Q are on opposite sides of m.

Case 3 P lies on m, and $\overline{PQ}$ is not ⊥ to m.

Case 4 Q lies on m, and $\overline{PQ} \perp m$.

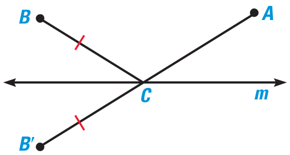
EXAMPLE 4 Find a minimum distance

PARKING You are going to buy books. Your friend is going to buy CDs. Where should you park to minimize the distance you both will walk?



Solution

Reflect B in line m to obtain B'. Then draw $\overline{AB'}$. Label the intersection of $\overline{AB'}$ and m as C. Because AB' is the shortest distance between A and B' and $BC = B'C$, park at point C to minimize the combined distance, $AC + BC$, you both have to walk.

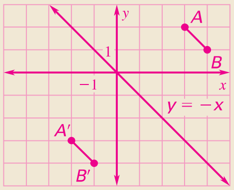


✓ **GUIDED PRACTICE** for Example 4

6. Look back at Example 4. Answer the question by using a reflection of point A instead of point B. **Reflect A in line m obtaining A'. Then draw $\overline{A'B}$. Label the intersection of m and $\overline{A'B}$ as C. Because $A'B$ is the shortest distance between A' and B and $AC = A'C$, park at point C to minimize the combined distance, $AC + BC$, you both have to walk.**

Extra Example 3

Reflect $\overline{AB}$ from Extra Example 2 in the line $y = -x$. Graph $\overline{AB}$ and its image.



Animated Geometry my.hrw.com

An **Animated Geometry** activity is available online for **Example 3.** This activity is also part of **Power Presentations.**

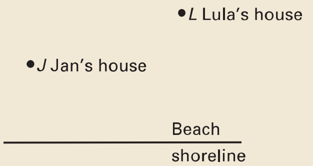
MATHEMATICAL PRACTICES

Key Questions Example 2 and 3

- If a point and its image are in the same quadrant after a reflection in the line $y = x$, in which quadrant is the point located? **I or III**
- If a point and its image are in the same quadrant after a reflection in the line $y = -x$, in which quadrant is the point located? **II or IV**

Extra Example 4

Jan and Lula are going to meet on the beach shoreline. Where will they meet to minimize the distance they both must walk?

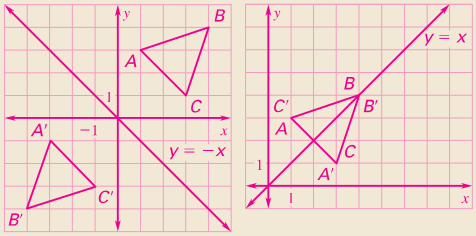


at the point where the beach shoreline intersects $\overline{LJ'}$, where J' is the image of J in the shoreline

Study Strategy

Have the students create a way to help them remember the Coordinate Rules for Reflections.

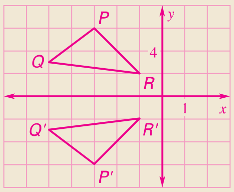
4.



Extra Example 5

The vertices of $\triangle PQR$ are $P(-3, 6)$, $Q(-5, 3)$, and $R(-1, 2)$. Find the reflection of $\triangle PQR$ in the x-axis. Graph $\triangle PQR$ and its image.

$$\begin{matrix} P' & Q' & R' \end{matrix}$$
$$\begin{bmatrix} -3 & -5 & -1 \\ -6 & -3 & -2 \end{bmatrix}$$



Key Question Example 5

- How do the coordinates of the image compare to the coordinates of the preimage? **The x-coordinates are opposites, the y-coordinates are the same.**

Closing the Lesson

Have students summarize the major points of the lesson and answer the Essential Question: How do you reflect a figure in the line $y = x$?

- **Coordinate Rules for Reflections can be used to find the images of points reflected in the x-axis, y-axis, $y = x$, and $y = -x$.**
- **Matrices can be used to find the images of points reflected in the x-axis and y-axis.**

To reflect a figure in the line $y = x$, switch the x- and y-coordinates of each point of the figure.

REFLECTION MATRIX You can find the image of a polygon reflected in the x-axis or y-axis using matrix multiplication. Write the reflection matrix to the *left* of the polygon matrix, then multiply.

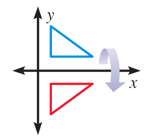
Notice that because matrix multiplication is not commutative, the order of the matrices in your product is important. The reflection matrix must be first followed by the polygon matrix.

KEY CONCEPT *For Your Notebook*

Reflection Matrices

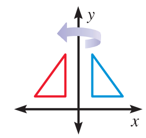
Reflection in the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Reflection in the y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



EXAMPLE 5 Use matrix multiplication to reflect a polygon

The vertices of $\triangle DEF$ are $D(1, 2)$, $E(3, 3)$, and $F(4, 0)$. Find the reflection of $\triangle DEF$ in the y-axis using matrix multiplication. Graph $\triangle DEF$ and its image.

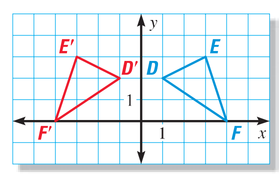
Solution

STEP 1 **Multiply** the polygon matrix by the matrix for a reflection in the y-axis.

$$\underset{\text{Reflection matrix}}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{\text{Polygon matrix}}{\overset{\begin{matrix} D & E & F \end{matrix}}{\begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix}}} = \begin{bmatrix} -1(1) + 0(2) & -1(3) + 0(3) & -1(4) + 0(0) \\ 0(1) + 1(2) & 0(3) + 1(3) & 0(4) + 1(0) \end{bmatrix}$$

$$= \underset{\text{Image matrix}}{\overset{\begin{matrix} D' & E' & F' \end{matrix}}{\begin{bmatrix} -1 & -3 & -4 \\ 2 & 3 & 0 \end{bmatrix}}}$$

STEP 2 **Graph** $\triangle DEF$ and $\triangle D'E'F'$.



GUIDED PRACTICE for Example 5

The vertices of $\triangle LMN$ are $L(-3, 3)$, $M(1, 2)$, and $N(-2, 1)$. Find the described reflection using matrix multiplication.

7. Reflect $\triangle LMN$ in the x-axis. **$L'(-3, -3)$, $M'(1, -2)$, $N'(-2, -1)$**

8. Reflect $\triangle LMN$ in the y-axis. **$L'(3, 3)$, $M'(-1, 2)$, $N'(2, 1)$**

Differentiated Instruction

Visual Learners To help students visualize how matrix multiplication can be used to describe a reflection, have them sketch $\triangle LMN$ and its image for **Guided Practice Exercises 7–8**. For students who need additional help with understanding reflections, have them trace the preimage of $\triangle LMN$, cut it out, and manually reflect it in the x-axis and y-axis.

See also the *Differentiated Instruction Resources* for more strategies.

9.3 EXERCISES

HOMEWORK KEY
◯ = See **WORKED-OUT SOLUTIONS** Exs. 5, 13, and 33
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 12, 25, and 40

SKILL PRACTICE

A

1. VOCABULARY What is a *line of reflection*?
a line which acts like a mirror to reflect an image across the line

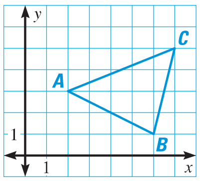
2. ★ WRITING *Explain* how to find the distance from a point to its image if you know the distance from the point to the line of reflection. See margin.

2. Multiply it by 2 because the distance from a point to the line of reflection is the same as the distance from the point's image to the line of reflection.

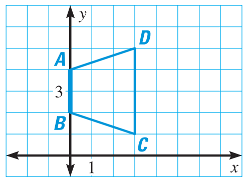
EXAMPLE 1 for Exs. 3–8

REFLECTIONS **Graph the reflection of the polygon in the given line.** 3–11. See margin.

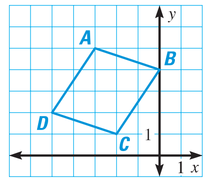
3. x-axis



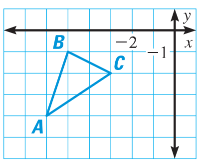
4. y-axis



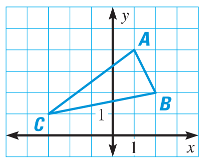
5. $y = 2$



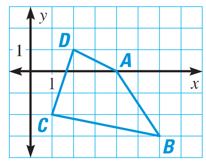
6. $x = -1$



7. y-axis

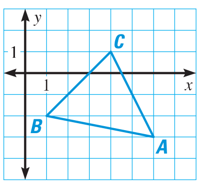


8. $y = -3$

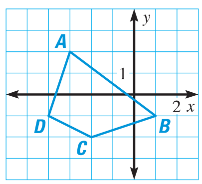


EXAMPLES 2 and 3 for Exs. 9–12

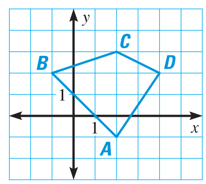
9. $y = x$



10. $y = -x$

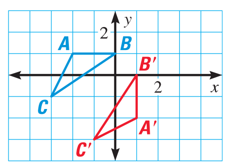


11. $y = x$



12. ★ MULTIPLE CHOICE What is the line of reflection for △*ABC* and its image? D

Ⓐ $y = 0$ (the x-axis) Ⓑ $y = -x$
Ⓒ $x = 1$ Ⓓ $y = x$



EXAMPLE 5 for Exs. 13–17

USING MATRIX MULTIPLICATION **Use matrix multiplication to find the image. Graph the polygon and its image.** 13, 14. See margin.

13. Reflect $\begin{bmatrix} \overset{A}{-2} & \overset{B}{3} & \overset{C}{4} \\ 5 & -3 & 6 \end{bmatrix}$ in the x-axis.

14. Reflect $\begin{bmatrix} \overset{P}{2} & \overset{Q}{6} & \overset{R}{5} & \overset{S}{2} \\ -2 & -3 & -8 & -5 \end{bmatrix}$ in the y-axis.

4 PRACTICE AND APPLY

Assignment Guide

Answers for all exercises available online

Basic:
Day 1:
Exs. 1–12
Day 2:
Exs. 13–21, 31–37

Average:
Day 1:
Exs. 1, 2, 6–12, 26–28
Day 2:
Exs. 15–25, 32–40

Advanced:
Day 1:
Exs. 1, 2, 7–12, 26–30*
Day 2:
Exs. 16–25, 34–41*

Block:
Exs. 1, 2, 6–12, 15–28, 32–40

Differentiated Instruction

See *Differentiated Instruction Resources* for suggestions on addressing the needs of a diverse classroom.

Homework Check

For a quick check of student understanding of key concepts, go over the following exercises:

Basic: 4, 9, 14, 31, 34
Average: 6, 10, 16, 32, 34
Advanced: 8, 11, 17, 34, 35

Extra Practice

- Student Edition
- Chapter Resource Book: Practice levels A, B, C

Practice Worksheet

An easily-readable reduced practice page can be found at the beginning of this chapter.

3–11, 13, 14. See Additional Answers.

Teaching Strategy

Exercises 3–11 Students will do better with these exercises if they use a graph to find the image of each vertex in the given line. This will help give meaning to the coordinate rules stated. Urge students to use their graphs to confirm the coordinate rules.

Avoiding Common Errors

Exercises 19–21 Students may find the midpoint between the points rather than finding a point on the x-axis. Or they may find the point on the x-axis where the x-coordinate is the average of the x-coordinates of the given points. Have them graph the points, reflect one point over the x-axis, and see where the line joining the image to the other point intersects the x-axis.

Study Strategy

Exercise 26 Relate this construction to the construction of a line perpendicular to a given line through a point not on the line. When you reflect A over m you are constructing a point A' such that $\overline{AA'}$ is perpendicular to m.

15. $\begin{array}{ccc} A & B & C \end{array}$ $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 2 & -2 \end{bmatrix}$; $\begin{array}{ccc} A' & B' & C' \end{array}$ $\begin{bmatrix} -1 & -4 & -3 \\ 2 & 2 & -2 \end{bmatrix}$

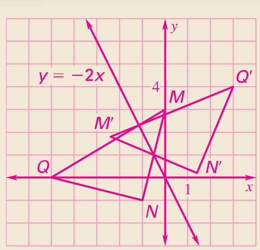
16. $\begin{array}{cccc} A & B & C & D \end{array}$ $\begin{bmatrix} -2 & 4 & 3 & 0 \\ 1 & 1 & -2 & -1 \end{bmatrix}$; $\begin{array}{cccc} A' & B' & C' & D' \end{array}$ $\begin{bmatrix} -2 & 4 & 3 & 0 \\ -1 & -1 & 2 & 1 \end{bmatrix}$

17. $\begin{array}{ccc} A & B & C \end{array}$ $\begin{bmatrix} -4 & 3 & 2 \\ -2 & 1 & -3 \end{bmatrix}$; $\begin{array}{ccc} A' & B' & C' \end{array}$ $\begin{bmatrix} 4 & -3 & -2 \\ -2 & 1 & -3 \end{bmatrix}$

18. The reflection matrix should be $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, not $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -5 & 4 & -2 \\ 4 & 8 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 2 \\ 4 & 8 & -1 \end{bmatrix}$.

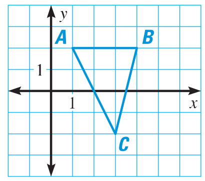
22–24, 26. See Additional Answers.

29.

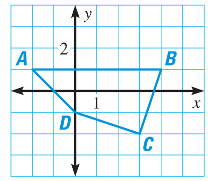


B **FINDING IMAGE MATRICES** **Write a matrix for the polygon. Then find the image matrix that represents the polygon after a reflection in the given line.** 15–17. See margin.

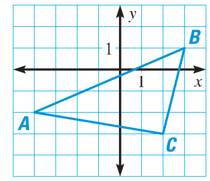
15. y-axis



16. x-axis



17. y-axis



18. **ERROR ANALYSIS** *Describe* and correct the error in finding the image matrix of $\triangle PQR$ reflected in the y-axis. See margin.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -5 & 4 & -2 \\ 4 & 8 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -2 \\ -4 & -8 & -1 \end{bmatrix} \times$$

MINIMUM DISTANCE **Find point C on the x-axis so $AC + BC$ is a minimum.**

19. $A(1, 4)$, $B(6, 1)$ $C(5, 0)$ **20.** $A(4, -3)$, $B(12, -5)$ $C(7, 0)$ **21.** $A(-8, 4)$, $B(-1, 3)$ $C(-4, 0)$

TWO REFLECTIONS **The vertices of $\triangle FGH$ are $F(3, 2)$, $G(1, 5)$, and $H(-1, 2)$. Reflect $\triangle FGH$ in the first line. Then reflect $\triangle F'G'H'$ in the second line. Graph $\triangle F'G'H'$ and $\triangle F''G''H''$.** 22–24. See margin.

22. In $y = 2$, then in $y = -1$ **23.** In $y = -1$, then in $x = 2$ **24.** In $y = x$, then in $x = -3$

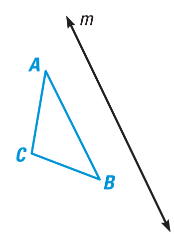
25. ★ **SHORT RESPONSE** Use your graphs from Exercises 22–24. What do you notice about the order of vertices in the preimages and images? The order is reversed.

26. **CONSTRUCTION** Use these steps to construct a reflection of $\triangle ABC$ in line m using a straightedge and a compass. See margin.

STEP 1 Draw $\triangle ABC$ and line m.

STEP 2 Use one compass setting to find two points that are equidistant from A on line m. Use the same compass setting to find a point on the other side of m that is the same distance from line m. Label that point A'.

STEP 3 Repeat Step 2 to find points B' and C'. Draw $\triangle A'B'C'$.

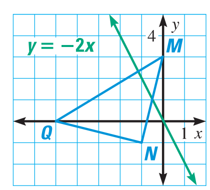


27. **ALGEBRA** The line $y = 3x + 2$ is reflected in the line $y = -1$. What is the equation of the image? $y = -3x - 4$

28. **ALGEBRA** Reflect the graph of the quadratic equation $y = 2x^2 - 5$ in the x-axis. What is the equation of the image? $y = -2x^2 + 5$

C **29.** **REFLECTING A TRIANGLE** Reflect $\triangle MNQ$ in the line $y = -2x$. See margin.

30. **CHALLENGE** Point $B'(1, 4)$ is the image of $B(3, 2)$ after a reflection in line c. Write an equation of line c. $y = x + 1$



○ = See WORKED-OUT SOLUTIONS in Student Resources ★ = STANDARDIZED TEST PRACTICE

Differentiated Instruction

Advanced Extend **Exercise 27** by having students generalize. If the line $y = mx + b$ is reflected in the line $y = k$, what is the equation of the image? Also generalize on **Exercise 28** finding the equation of $y = ax^2 + c$ reflected in the x-axis or the y-axis and finding the equation of the image.

See also the *Differentiated Instruction Resources* for more strategies.

PROBLEM SOLVING

A **REFLECTIONS** **Identify the case of the Reflection Theorem represented.**

31. 

Case 4

32. 

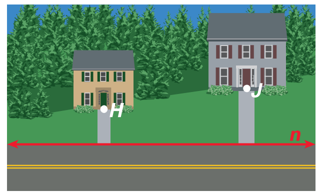
Case 3

33. 

Case 1

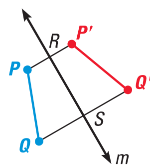
EXAMPLE 4 for Ex. 34

34. DELIVERING PIZZA You park at some point K on line n. You deliver a pizza to house H, go back to your car, and deliver a pizza to house J. Assuming that you can cut across both lawns, how can you determine the parking location K that minimizes the total walking distance?



34. *Sample answer:* Reflect point H across line n and label it H'. Draw $\overline{JH'}$. Label the point where line n intersects at P. Park the car at P.

35. PROVING THEOREM 9.2 Prove Case 1 of the Reflection Theorem.



Case 1 The segment does not intersect the line of reflection.

GIVEN ▶ A reflection in m maps P to P' and Q to Q'.

PROVE ▶ $PQ = P'Q'$

Plan for Proof

a, b. See margin.

a. Draw $\overline{PP'}$, $\overline{QQ'}$, $\overline{RQ}$, and $\overline{RQ'}$. Prove that $\triangle RSQ \cong \triangle RSQ'$.

b. Use the properties of congruent triangles and perpendicular bisectors to prove that $PQ = P'Q'$.

B **PROVING THEOREM 9.2** **In Exercises 36–38, write a proof for the given case of the Reflection Theorem. (Refer to the diagrams in the theorem box.)** 36–38. See margin.

36. Case 2 The segment intersects the line of reflection.

GIVEN ▶ A reflection in m maps P to P' and Q to Q'. Also, $\overline{PQ}$ intersects m at point R.

PROVE ▶ $PQ = P'Q'$

37. Case 3 One endpoint is on the line of reflection, and the segment is not perpendicular to the line of reflection.

GIVEN ▶ A reflection in m maps P to P' and Q to Q'. Also, P lies on line m, and $\overline{PQ}$ is not perpendicular to m.

PROVE ▶ $PQ = P'Q'$

38. Case 4 One endpoint is on the line of reflection, and the segment is perpendicular to the line of reflection.

GIVEN ▶ A reflection in m maps P to P' and Q to Q'. Also, Q lies on line m, and $\overline{PQ}$ is perpendicular to line m.

PROVE ▶ $PQ = P'Q'$

Teaching Strategy

Exercise 30 Elicit that the line of reflection c is the perpendicular bisector of the segment $\overline{BB'}$. Ask students how they could find the coordinates of a point on line c. Then ask how they could find the slope of line c.

Mathematical Reasoning

Exercises 35–38 Ask students if they can think of yet another special case of the Reflection Theorem. Ask them to explain why this case does not merit a complete proof. **the case in which $\overline{PQ}$ is contained in line m; $\overline{P'Q'}$ and $\overline{PQ}$ coincide, so it is trivially true that $\overline{P'Q'} \cong \overline{PQ}$.**

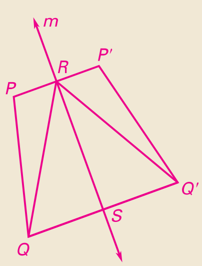
Internet Reference

Exercises 40–41 For more information about the Law of Reflection, visit acept.la.asu.edu/PiN/rdg/reflection/reflection.shtml

35b. Using corresponding parts of congruent triangles are congruent, $\overline{RQ} \cong \overline{RQ'}$. Using the definition of a line of reflection, $\overline{PR} \cong \overline{P'R}$. Since $\overline{PP'}$ and $\overline{QQ'}$ are both perpendicular to m, they are parallel. Using the Alternate Interior Angles Theorem, $\angle SQ'R \cong \angle P'RQ'$ and $\angle SQR \cong \angle PRQ$. Using corresponding parts of congruent triangles are congruent, $\angle SQ'R \cong \angle SQR$. Using the Transitive Property of Angle Congruence, $\angle P'RQ' \cong \angle PRQ$. $\triangle PRQ \cong \triangle P'RQ'$ using the SAS Congruence Postulate. Using corresponding parts of congruent triangles are congruent, $\overline{PQ} \cong \overline{P'Q'}$ which implies $PQ = P'Q'$.

36–38. See Additional Answers.

35a.



Given: A reflection in m maps P to P' and Q to Q'. Using the definition of a line of reflection, $\overline{QS} \cong \overline{Q'S}$ and $\angle QSR \cong \angle Q'SR$. Using the Reflexive Property of Segment Congruence, $\overline{RS} \cong \overline{RS}$. Using the SAS Congruence Postulate, $\triangle RSQ \cong \triangle RSQ'$.

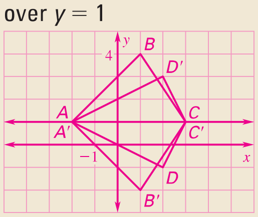
5 ASSESS AND RETEACH

Daily Homework Quiz

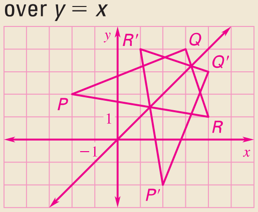
Also available online

Graph the polygon and its reflection in the given line.

1. Quadrilateral with vertices $A(-2, 1)$, $B(1, 4)$, $C(3, 1)$, $D(2, -1)$ over $y = 1$



2. Triangle $P(-2, 2)$, $Q(3, 4)$, $R(4, 1)$ over $y = x$



3. Use matrix multiplication to find the image matrix when $\triangle ABC$ is reflected in the y-axis.

$$\begin{array}{ccc} A & B & C \end{array} \quad \begin{array}{ccc} A' & B' & C' \end{array}$$
$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

4. Find point C on the x-axis so $AC + BC$ is a minimum for $A(2, 5)$ and $B(7, 3)$. $\left(\frac{41}{8}, 0\right)$

Online Quiz

Available at **my.hrw.com**

Diagnosis/Remediation

- Practice A, B, C in Chapter Resource Book
- Study Guide in Chapter Resource Book
- Practice Workbook
- @HomeTutor

Challenge

Additional challenge is available in the Chapter Resource Book.

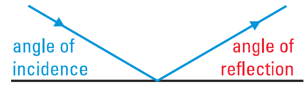
39. REFLECTING POINTS Use $C(1, 3)$.

a. Point A has coordinates $(-1, 1)$. Find point B on $\overrightarrow{AC}$ so $AC = CB$. **$B(3, 5)$**

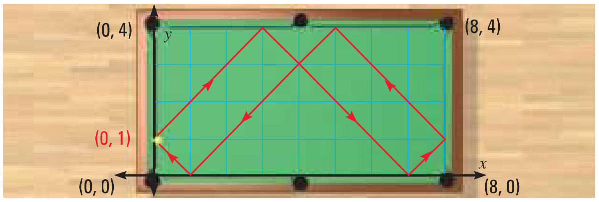
b. The endpoints of $\overline{FG}$ are $F(2, 0)$ and $G(3, 2)$. Find point H on $\overrightarrow{FC}$ so $FC = CH$. Find point J on $\overrightarrow{GC}$ so $GC = CJ$. **$H(0, 6)$; $J(-1, 4)$**

c. Explain why parts (a) and (b) can be called *reflection in a point.* **In each case point C bisects each line segment.**

PHYSICS The Law of Reflection states that the angle of incidence is congruent to the angle of reflection. Use this information in Exercises 40 and 41.



40. ★ SHORT RESPONSE Suppose a billiard table has a coordinate grid on it. If a ball starts at the point (0, 1) and rolls at a 45° angle, it will eventually return to its starting point. Would this happen if the ball started from other points on the y-axis between (0, 0) and (0, 4)? *Explain.*

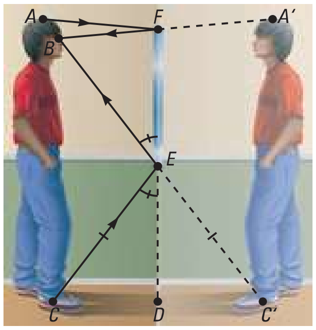


41. CHALLENGE Use the diagram to prove that you can see your full self in a mirror that is only half of your height. Assume that you and the mirror are both perpendicular to the floor.

a. Think of a light ray starting at your foot and reflected in a mirror. Where does it have to hit the mirror in order to reflect to your eye?

b. Think of a light ray starting at the top of your head and reflected in a mirror. Where does it have to hit the mirror in order to reflect to your eye?

c. Show that the distance between the points you found in parts (a) and (b) is half your height.



40. Yes. ***Sample answer:*** **Starting at (0, 3) the ball would follow the following path: (1, 4), (5, 0), (8, 3), (7, 4), (3, 0) and end up at (0, 3).**

41a. at a point that is directly across from the midpoint of the distance between your eye and your foot

41b. at a point that is directly across from the midpoint of the distance between your eye and the top of your head

41c. If you are $\overline{AC}$, with B on $\overline{AC}$ and $\overline{AC} \parallel \overline{FD}$. Let F' be the point on $\overline{AC}$ so that $\overline{FF'} \perp \overline{AC}$ and $\overline{FF'} \perp \overline{FD}$ and let E' be the point on $\overline{AC}$ so that $\overline{EE'} \perp \overline{AC}$ and $\overline{EE'} \perp \overline{FD}$. F' is the midpoint of $\overline{AB}$ and E' is the midpoint of $\overline{BC}$. Then $EF = E'F' = \frac{1}{2}AB + \frac{1}{2}BC$, because $AC = AB + BC$, $EF = \frac{1}{2}AC$.

C

See **EXTRA PRACTICE** in Student Resources

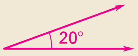
ONLINE QUIZ at my.hrw.com

1 PLAN AND PREPARE

Warm-Up Exercises

Also available online

1. Use a protractor to draw an angle with measure 20°.



2. Given the points $A(3, 5)$, $B(-5, 3)$, and the origin $O(0, 0)$, find OA, OB, and $m\angle BOA$. **$\sqrt{34}$, $\sqrt{34}$, 90°**

3. Multiply $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$.

$\begin{bmatrix} 5 & -7 \\ -2 & -3 \end{bmatrix}$

Notetaking Guide

Available online

Promotes interactive learning and notetaking skills.

Pacing

Basic: 2 days

Average: 2 days

Advanced: 2 days

Block: 1 block

- See *Teaching Guide/Lesson Plan*.

2 FOCUS AND MOTIVATE

Essential Question

Big Idea 1

How do you rotate a figure 90°, 180°, or 270° about the origin?
Tell students they will learn how to answer this question by using rules and matrices.

9.4 Perform Rotations

Before You rotated figures about the origin.

Now You will rotate figures about a point.

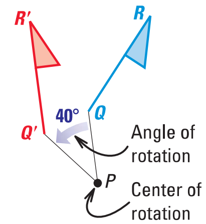
Why? So you can classify transformations, as in Exs. 3–5.

Key Vocabulary
- **center of rotation**
- **angle of rotation**
- **rotation**

Recall that a *rotation* is a transformation in which a figure is turned about a fixed point called the **center of rotation**. Rays drawn from the center of rotation to a point and its image form the **angle of rotation**.

A rotation about a point P through an angle of $x°$ maps every point Q in the plane to a point Q' so that one of the following properties is true:

- If Q is not the center of rotation P, then $QP = Q'P$ and $m\angle QPQ' = x°$, or
- If Q is the center of rotation P, then the image of Q is Q.



A 40° counterclockwise rotation is shown at the right. Rotations can be *clockwise* or *counterclockwise*. In this chapter, all rotations are counterclockwise.

DIRECTION OF ROTATION

clockwise

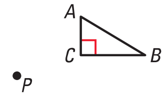
counterclockwise

COMMON CORE

CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

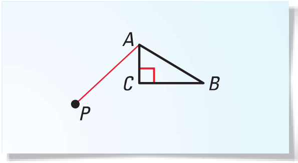
EXAMPLE 1 Draw a rotation

Draw a 120° rotation of △*ABC* about *P*.

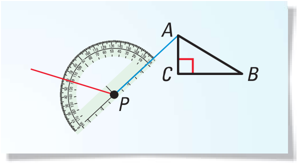


Solution

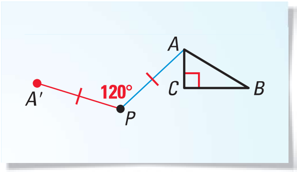
STEP 1 **Draw** a segment from A to P.



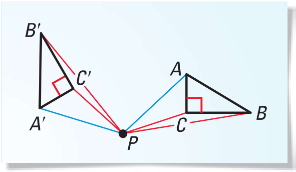
STEP 2 **Draw** a ray to form a 120° angle with $\overline{PA}$.



STEP 3 **Draw** A' so that $PA' = PA$.



STEP 4 **Repeat** Steps 1–3 for each vertex. Draw $\triangle A'B'C'$.



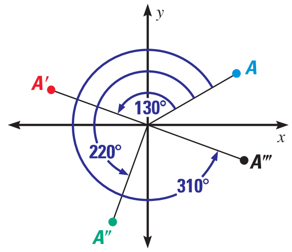
COMMON CORE **Standards for Mathematical Content** *High School*

CC.9-12.G.CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

USE ROTATIONS
You can rotate a figure more than 360°. However, the effect is the same as rotating the figure by the angle minus 360°.

ROTATIONS ABOUT THE ORIGIN You can rotate a figure more than 180°. The diagram shows rotations of point A 130°, 220°, and 310° about the origin. A rotation of 360° returns a figure to its original coordinates.

There are coordinate rules that can be used to find the coordinates of a point after rotations of 90°, 180°, or 270° about the origin.

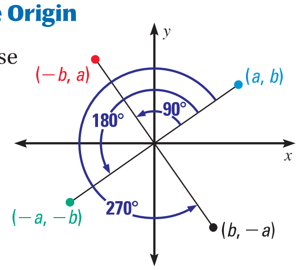


KEY CONCEPT *For Your Notebook*

Coordinate Rules for Rotations about the Origin

When a point (a, b) is rotated counterclockwise about the origin, the following are true:

1. For a rotation of 90°, $(a, b) \to (-b, a)$.
2. For a rotation of 180°, $(a, b) \to (-a, -b)$.
3. For a rotation of 270°, $(a, b) \to (b, -a)$.



EXAMPLE 2 Rotate a figure using the coordinate rules

Graph quadrilateral $RSTU$ with vertices $R(3, 1)$, $S(5, 1)$, $T(5, -3)$, and $U(2, -1)$. Then rotate the quadrilateral 270° about the origin.

ANOTHER WAY
For an alternative method for solving the problem in Example 2, see the **Problem Solving Workshop**.

Solution

Graph $RSTU$. Use the coordinate rule for a 270° rotation to find the images of the vertices.

$(a, b) \to (b, -a)$

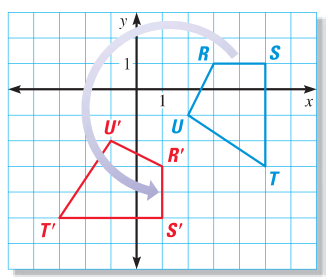
$R(3, 1) \to R'(1, -3)$

$S(5, 1) \to S'(1, -5)$

$T(5, -3) \to T'(-3, -5)$

$U(2, -1) \to U'(-1, -2)$

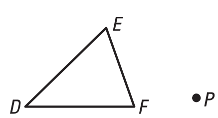
Graph the image $R'S'T'U'$.



Animated Geometry at my.hrw.com

✓ GUIDED PRACTICE for Examples 1 and 2

1. Trace $\triangle DEF$ and P. Then draw a 50° rotation of $\triangle DEF$ about P. **See margin.**



2. Graph $\triangle JKL$ with vertices $J(3, 0)$, $K(4, 3)$, and $L(6, 0)$. Rotate the triangle 90° about the origin. **See margin.**

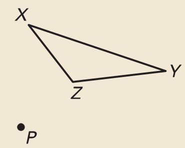
Motivating the Lesson

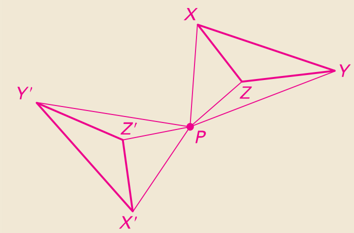
Have students think of devices in everyday life that have parts that turn around a fixed point. Tell them that in this lesson they will learn how to describe the motion of the turning parts of such devices.

3 TEACH

Extra Example 1

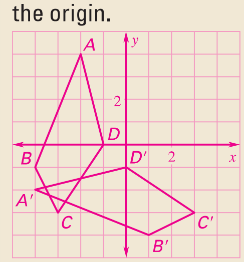
Draw a 150° rotation of $\triangle XYZ$ about P.





Extra Example 2

Graph quadrilateral $ABCD$ with vertices $A(-2, 4)$, $B(-4, -1)$, $C(-3, -3)$, and $D(-1, 0)$. Then rotate the quadrilateral 90° about the origin.



1–2. See Additional Answers.

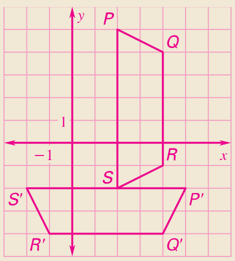
Animated Geometry
my.hrw.com

An **Animated Geometry** activity is available online for **Example 2.** This activity is also part of **Power Presentations.**

Extra Example 3

Trapezoid *PQRS* has vertices *P*(2, 5), *Q*(4, 4), *R*(4, −1), and *S*(2, −2). Find the image matrix for a 270° rotation about the origin. Graph *PQRS* and its image.

$$\begin{array}{cccc} P' & Q' & R' & S' \end{array}$$
$$\begin{bmatrix} 5 & 4 & -1 & -2 \\ -2 & -4 & -4 & -2 \end{bmatrix}$$



Teaching Strategy

After class discussion of Example 3, you may wish to show students how they can easily find the rotation matrices without memorizing them. This method also works for reflection matrices. In the first column of the transformation matrix, show the coordinates for the image of the *x*-axis unit point (1, 0). In the second column of the transformation matrix, show the coordinates for the image of the *y*-axis unit point (0, 1). The coordinates of the image points can easily be found from a rough sketch. Illustrate the method with the rotation matrices. Encourage students to verify that the method also works for the reflection matrices in previous lesson.

Vocabulary

Have students write their own explanations of the terms rotation matrix, reflection matrix, polygon matrix, and image matrix.

USING MATRICES You can find certain images of a polygon rotated about the origin using matrix multiplication. Write the rotation matrix to the left of the polygon matrix, then multiply.

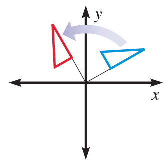
READ VOCABULARY
Notice that a 360° rotation returns the figure to its original position. Multiplying by the matrix that represents this rotation gives you the polygon matrix you started with, which is why it is also called the *identity matrix*.

KEY CONCEPT — *For Your Notebook*

Rotation Matrices (Counterclockwise)

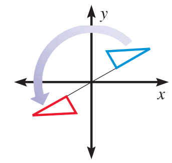
90° rotation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



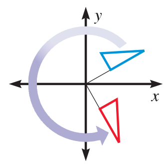
180° rotation

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



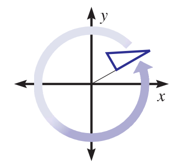
270° rotation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



360° rotation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



EXAMPLE 3 Use matrices to rotate a figure

Trapezoid *EFGH* has vertices *E*(−3, 2), *F*(−3, 4), *G*(1, 4), and *H*(2, 2). Find the image matrix for a 180° rotation of *EFGH* about the origin. Graph *EFGH* and its image.

Solution

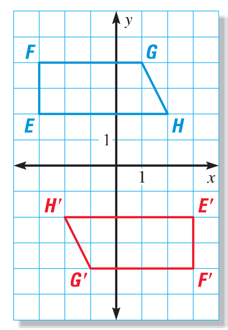
STEP 1 **Write** the polygon matrix:

$$\begin{array}{cccc} E & F & G & H \end{array}$$
$$\begin{bmatrix} -3 & -3 & 1 & 2 \\ 2 & 4 & 4 & 2 \end{bmatrix}$$

STEP 2 **Multiply** by the matrix for a 180° rotation.

$$\underset{\text{Rotation matrix}}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}} \underset{\text{Polygon matrix}}{\overset{\begin{array}{cccc} E & F & G & H \end{array}}{\begin{bmatrix} -3 & -3 & 1 & 2 \\ 2 & 4 & 4 & 2 \end{bmatrix}}} = \underset{\text{Image matrix}}{\overset{\begin{array}{cccc} E' & F' & G' & H' \end{array}}{\begin{bmatrix} 3 & 3 & -1 & -2 \\ -2 & -4 & -4 & -2 \end{bmatrix}}}$$

STEP 3 **Graph** the preimage *EFGH*.
Graph the image *E′F′G′H′*.



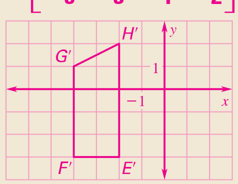
AVOID ERRORS
Because matrix multiplication is not commutative, you should always write the rotation matrix first, then the polygon matrix.

✓ **GUIDED PRACTICE** for Example 3

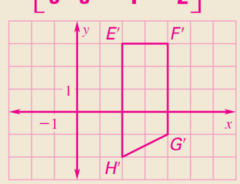
Use the quadrilateral *EFGH* in Example 3. Find the image matrix after the rotation about the origin. Graph the image. **3–5. See margin.**

3. 90° **4.** 270° **5.** 360°

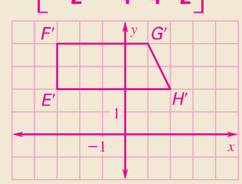
3. $\begin{bmatrix} -2 & -4 & -4 & -2 \\ -3 & -3 & 1 & 2 \end{bmatrix}$ (columns E', F', G', H');



4. $\begin{bmatrix} 2 & 4 & 4 & 2 \\ 3 & 3 & -1 & -2 \end{bmatrix}$ (columns E', F', G', H');



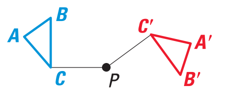
5. $\begin{bmatrix} -3 & -3 & 1 & 2 \\ 2 & 4 & 4 & 2 \end{bmatrix}$ (columns E', F', G', H');



THEOREM — *For Your Notebook*

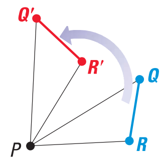
THEOREM 9.3 Rotation Theorem

A rotation is an isometry.

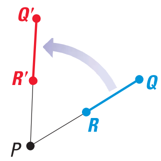


$\triangle ABC \cong \triangle A'B'C'$

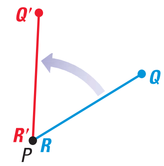
CASES OF THEOREM 9.3 To prove the Rotation Theorem, you need to show that a rotation preserves the length of a segment. Consider a segment $\overline{QR}$ rotated about point P to produce $\overline{Q'R'}$. There are three cases to prove:



Case 1 R, Q, and P are noncollinear.



Case 2 R, Q, and P are collinear.



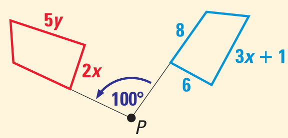
Case 3 P and R are the same point.

★ EXAMPLE 4 Standardized Test Practice

The quadrilateral is rotated about *P*. What is the value of *y*?

Ⓐ $\frac{8}{5}$ Ⓑ 2

Ⓒ 3 Ⓓ 10



Solution

By Theorem 9.3, the rotation is an isometry, so corresponding side lengths are equal. Then $2x = 6$, so $x = 3$. Now set up an equation to solve for y.

$5y = 3x + 1$ **Corresponding lengths in an isometry are equal.**

$5y = 3(3) + 1$ **Substitute 3 for *x*.**

$y = 2$ **Solve for *y*.**

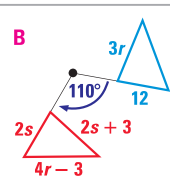
▶ The correct answer is B. Ⓐ Ⓑ Ⓒ Ⓓ

✓ GUIDED PRACTICE for Example 4

6. Find the value of r in the rotation of the triangle. **B**

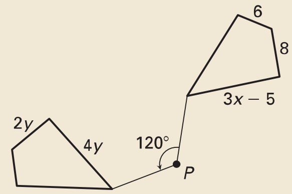
Ⓐ 3 Ⓑ 5

Ⓒ 6 Ⓓ 15



Extra Example 4

The quadrilateral is rotated about P. What is the value of x? **D**



Ⓐ 2 Ⓑ 4

Ⓒ $\frac{17}{3}$ Ⓓ 7

Closing the Lesson

Have students summarize the major points of the lesson and answer the Essential Question: How do you rotate a figure 90°, 180°, or 270° about the origin?

- **A figure can be rotated 90° about the origin using the rule $(a, b) \to (-b, a)$ or the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.**
- **A figure can be rotated 180° about the origin using the rule $(a, b) \to (-a, -b)$ or the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.**
- **A figure can be rotated 270° about the origin using the rule $(a, b) \to (b, -a)$ or the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.**
- **A rotation is an isometry.**

You rotate a figure 90°, 180°, or 270° about the origin by multiplying the polygon matrix by the rotation matrix or by using the coordinate rules for rotations about the origin.

Differentiated Instruction

Below Level Have students use their graphing calculators to enter and save the matrices for reflections in the x-axis, y-axis, $y = x$, and $y = -x$ and the matrices for rotations of 90°, 180°, and 270° about the origin. Teach them how to enter the coefficient matrix and multiply it by the matrix for the specific transformation they wish to perform.

See also the *Differentiated Instruction Resources* for more strategies.

4 PRACTICE AND APPLY

Assignment Guide

Answers for all exercises available online

Basic:
Day 1:
Exs. 1–14
Day 2:
Exs. 15–23, 29–35

Average:
Day 1:
Exs. 1–8, 10, 11, 13, 14, 23, 24
Day 2:
Exs. 16–22, 29–37

Advanced:
Day 1:
Exs. 1, 2, 6–8, 10, 11, 13, 14, 25–28*
Day 2:
Exs. 15–17, 20–24, 31–40*

Block:
Exs. 1–8, 10, 11, 13, 14, 16–24, 29–37

Differentiated Instruction

See *Differentiated Instruction Resources* for suggestions on addressing the needs of a diverse classroom.

Homework Check

For a quick check of student understanding of key concepts, go over the following exercises:

Basic: 4, 12, 16, 20, 30

Average: 8, 14, 18, 21, 31

Advanced: 10, 14, 17, 21, 32

Extra Practice

- Student Edition
- Chapter Resource Book: Practice levels A, B, C

Practice Worksheet

An easily-readable reduced practice page can be found at the beginning of this chapter.

9.4 EXERCISES

HOMEWORK KEY

○ = See **WORKED-OUT SOLUTIONS** Exs. 13, 15, and 29

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 20, 21, 24, and 37

SKILL PRACTICE

A

1. **VOCABULARY** What is a *center of rotation*?
a point which a figure is turned about during a rotation transformation

2. ★ **WRITING** *Compare* the coordinate rules and the rotation matrices for a rotation of 90°. **In the coordinate rotation the *x* and the *y* values are switched with the new *x* value being the opposite of the old *y* value. The rotation matrix for 90° has the same result.**

EXAMPLE 1 for Exs. 3–11

IDENTIFYING TRANSFORMATIONS Identify the type of transformation, *translation*, *reflection*, or *rotation*, in the photo. *Explain* your reasoning.

3. 

4. 

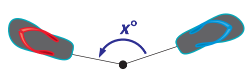
5. 

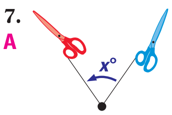
3. Reflection; the horses are reflected across the edge of the stream which acts like a line of symmetry.

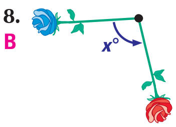
4. Rotation; as the steering wheel turns everything rotates around the center point.

5. Translation; the train moves horizontally from right to left.

ANGLE OF ROTATION Match the diagram with the angle of rotation.

6. **C** 

7. **A** 

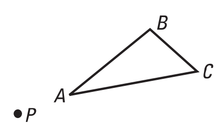
8. **B** 

A. 70° **B.** 100° **C.** 150°

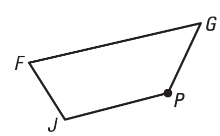
Animated Geometry at my.hrw.com

ROTATING A FIGURE Trace the polygon and point *P* on paper. Then draw a rotation of the polygon the given number of degrees about *P*. **9–11. See margin.**

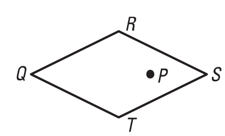
9. 30°



10. 150°



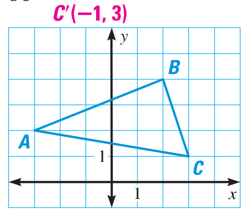
11. 130°



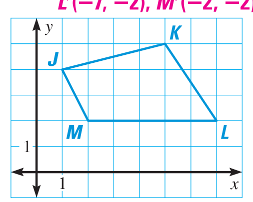
EXAMPLE 2 for Exs. 12–14

USING COORDINATE RULES Rotate the figure the given number of degrees about the origin. List the coordinates of the vertices of the image. **12–14. See margin for art.**

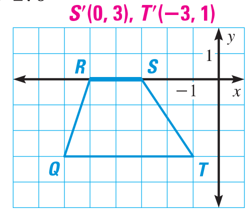
12. 90° **$A'(-2, -3)$, $B'(-4, 2)$, $C'(-1, 3)$**

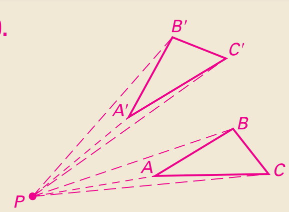


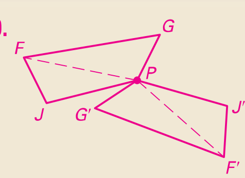
13. 180° **$J'(-1, -4)$, $K'(-5, -5)$, $L'(-7, -2)$, $M'(-2, -2)$**

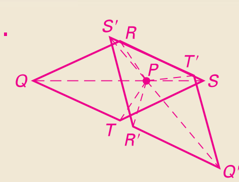


14. 270° **$Q'(-3, 6)$, $R'(0, 5)$, $S'(0, 3)$, $T'(-3, 1)$**



9. 

10. 

11. 

12–14. See Additional Answers.

EXAMPLE 3
for Exs. 15–19

USING MATRICES **Find the image matrix that represents the rotation of the polygon about the origin. Then graph the polygon and its image.** 15–17. See margin.

15. $\begin{matrix} A & B & C \end{matrix}$ $\begin{bmatrix} 1 & 5 & 4 \\ 4 & 6 & 3 \end{bmatrix}$; 90°

16. $\begin{matrix} J & K & L \end{matrix}$ $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -3 \end{bmatrix}$; 180°

17. $\begin{matrix} P & Q & R & S \end{matrix}$ $\begin{bmatrix} -4 & 2 & 2 & -4 \\ -4 & -2 & -5 & -7 \end{bmatrix}$; 270°

ERROR ANALYSIS **The endpoints of $\overline{AB}$ are $A(-1, 1)$ and $B(2, 3)$. *Describe* and correct the error in setting up the matrix multiplication for a 270° rotation about the origin.** 18, 19. See margin.

18.

270° rotation of $\overline{AB}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$ ✗

19.

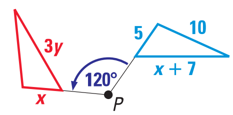
270° rotation of $\overline{AB}$

$\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ✗

EXAMPLE 4
for Exs. 20–21

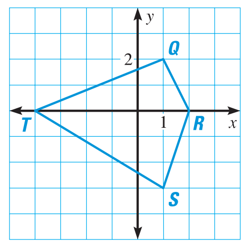
20. ★ **MULTIPLE CHOICE** What is the value of y in the rotation of the triangle about P? A

Ⓐ 4 Ⓑ 5 Ⓒ $\frac{17}{3}$ Ⓓ 10



21. ★ **MULTIPLE CHOICE** Suppose quadrilateral $QRST$ is rotated 180° about the origin. In which quadrant is Q'? C

Ⓐ I Ⓑ II Ⓒ III Ⓓ IV



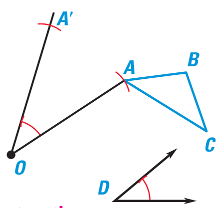
B **22.** **FINDING A PATTERN** The vertices of $\triangle ABC$ are $A(2, 0)$, $B(3, 4)$, and $C(5, 2)$. Make a table to show the vertices of each image after a 90°, 180°, 270°, 360°, 450°, 540°, 630°, and 720° rotation. What would be the coordinates of A' after a rotation of 1890°? *Explain.*
See margin for table; (0, 2); divide 1890 by 360 and use the remainder as your rotation.

23. **CONSTRUCTION** Use these steps to construct a rotation of $\triangle ABC$ by angle D around a point O using a straightedge and a compass.

STEP 1 Draw $\triangle ABC$, $\angle D$, and O, the center of rotation.

STEP 2 Draw $\overline{OA}$. Use the construction for copying an angle to copy $\angle D$ at O, as shown. Then use distance OA and center O to find A'.

STEP 3 Repeat Step 2 to find points B' and C'. Draw $\triangle A'B'C'$.



Check constructions.

24. ★ **SHORT RESPONSE** Rotate the triangle in Exercise 12 90° about the origin. Show that corresponding sides of the preimage and image are perpendicular. *Explain.* See margin.

C **25.** **VISUAL REASONING** A point in space has three coordinates (x, y, z). What is the image of point (3, 2, 0) rotated 180° about the origin in the xz-plane? (*See* Exercise 30 in the lesson *Use Properties of Matrices*.) (−3, 2, 0)

CHALLENGE **Rotate the line the given number of degrees (a) about the x-intercept and (b) about the y-intercept. Write the equation of each image.**

26. $y = 2x - 3$; 90°
a. $y = -\frac{1}{2}x + \frac{3}{4}$ b. $y = -\frac{1}{2}x - 3$

27. $y = -x + 8$; 180°
a. $y = -x + 8$ b. $y = -x + 8$

28. $y = \frac{1}{2}x + 5$; 270°
a. $y = -2x - 20$ b. $y = -2x + 5$

Avoiding Common Errors

Exercises 9–11 Some students may rotate the figures clockwise. Remind them of our convention to use counterclockwise rotations except when there is an explicit instruction to do otherwise.

Teaching Strategy

Exercises 12–17 Use these as examples and have students show that the result is the same regardless of the method used to obtain the image. Have them use a drawing, the coordinate rules, and matrix multiplication.

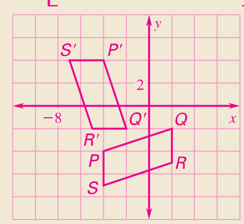
Avoiding Common Errors

Exercises 15–17 Students will run into trouble if they write the matrices in the wrong order when they set up the matrix multiplication expression. Be sure students write the rotation matrix to the left of the coordinate matrix in order to multiply.

Mathematical Reasoning

Exercise 25 Have students make a model of a 3-dimensional coordinate system to show the point and its image after the rotation.

17. $\begin{matrix} P' & Q' & R' & S' \end{matrix}$ $\begin{bmatrix} -4 & -2 & -5 & -7 \\ 4 & -2 & -2 & 4 \end{bmatrix}$;



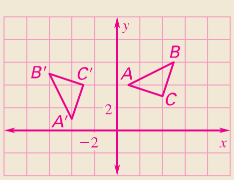
18. The wrong rotation matrix is being used; $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$.

19. The rotation matrix should be first; $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$.

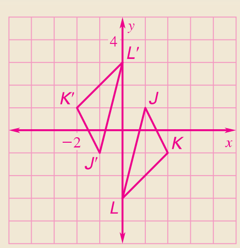
22. See Additional Answers.

24. The slope of $\overline{AB}$ is $\frac{2}{5}$ and the slope of $\overline{A'B'}$ is $-\frac{5}{2}$; the slope of $\overline{BC}$ is −3 and the slope of $\overline{B'C'}$ is $\frac{1}{3}$; the slope of $\overline{AC}$ is $-\frac{1}{6}$ and the slope of $\overline{A'C'}$ is 6. They are perpendicular since the product of the corresponding slopes is −1.

15. $\begin{matrix} A' & B' & C' \end{matrix}$ $\begin{bmatrix} -4 & -6 & -3 \\ 1 & 5 & 4 \end{bmatrix}$;



16. $\begin{matrix} J' & K' & L' \end{matrix}$ $\begin{bmatrix} -1 & -2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$;



Teaching Strategy

Exercise 36 Have students locate two points on the line, write a coordinate matrix for those two points, and multiply by the appropriate rotation matrix. To graph the image line, they need only graph the two image points and draw a line through them. Have students write an equation for their new line. Compare the slopes and y-intercepts of the original line and the new line.

30. 180°; A' and A are on opposite sides of a line segment containing the point of rotation.

31. 120°; the line segment joining A' to the center of rotation is rotated $\frac{1}{3}$ of a circle from the line segment joining A to the center of rotation.

34. Given: A rotation about P maps Q to Q' and R to R'. P, Q, and R are collinear. Using the definition of rotation about a point P, $PR = PR'$ and $PQ = PQ'$. Using the Segment Addition Postulate, $PQ = PR + RQ$ and $PQ' = PR' + R'Q'$. Using the Transitive Property of Equality, $PR + RQ = PR' + R'Q'$. Using the Subtraction Property of Equality, $RQ = R'Q'$.

35. Given: A rotation about P maps Q to Q' and R to R'. P and R are the same point. Using the definition of rotation about a point P, $PQ = PQ'$ and P, R, and R' are the same point. Substituting R for P on the left and R' for P on the right side, you get $RQ = R'Q'$.

36a. Rotating 90° produces perpendicular lines. Rotating 180° produces parallel lines. Rotating 270° produces perpendicular lines. Rotating 360° produces the same line.

PROBLEM SOLVING

A **ANGLE OF ROTATION** **Use the photo to find the angle of rotation that maps A onto A'. *Explain* your reasoning.** 30, 31. See margin.

29. 270°; the line segment joining A' to the center of rotation is perpendicular to the line segment joining A to the center of rotation.

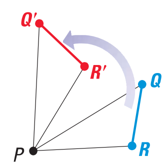
30.

31.

32. REVOLVING DOOR You enter a revolving door and rotate the door 180°. What does this mean in the context of the situation? Now, suppose you enter a revolving door and rotate the door 360°. What does this mean in the context of the situation? *Explain.*

If you started inside the building you will end up outside, and if you started outside the building you will end up inside; you will end up back where you started.

33. PROVING THEOREM 9.3 Copy and complete the proof of Case 1.



Case 1 The segment is noncollinear with the center of rotation.

GIVEN ▶ A rotation about P maps Q to Q' and R to R'.

PROVE ▶ $QR = Q'R'$

STATEMENTS	REASONS
1. $PQ = PQ'$, $PR = PR'$, $m\angle QPQ' = m\angle RPR'$	1. Definition of _?_ a rotation about a point
2. $m\angle QPQ' = m\angle QPR' + m\angle R'PQ'$; $m\angle RPR' = m\angle RPQ + m\angle QPR'$	2. _?_ Angle Addition Postulate
3. $m\angle QPR' + m\angle R'PQ' = m\angle RPQ + m\angle QPR'$	3. _?_ Property of Equality Transitive
4. $m\angle QPR = m\angle Q'PR'$	4. _?_ Property of Equality Subtraction
5. _?_ ≅ _?_ $\triangle RPQ \cong \triangle R'PQ'$	5. SAS Congruence Postulate
6. $\overline{QR} \cong \overline{Q'R'}$	6. _?_ Corr. Parts of ≅ △ are ≅
7. $QR = Q'R'$	7. _?_ definition of segment congruence

B **PROVING THEOREM 9.3** **Write a proof for Case 2 and Case 3. (Refer to the diagrams below the theorem box.)** 34, 35. See margin.

34. Case 2 The segment is collinear with the center of rotation.

GIVEN ▶ A rotation about P maps Q to Q' and R to R'. P, Q, and R are collinear.

PROVE ▶ $QR = Q'R'$

35. Case 3 The center of rotation is one endpoint of the segment.

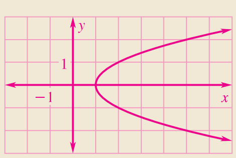
GIVEN ▶ A rotation about P maps Q to Q' and R to R'. P and R are the same point.

PROVE ▶ $QR = Q'R'$

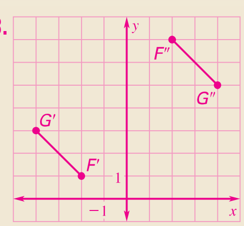
○ = See WORKED-OUT SOLUTIONS in Student Resources

★ = STANDARDIZED TEST PRACTICE

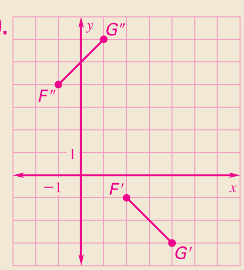
37a.



38.

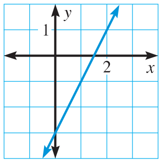


39.



36b. Yes; choose two points on the line and rotate the two points and look at the slope of each image.

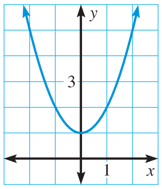
36. MULTI-STEP PROBLEM Use the graph of $y = 2x - 3$.



a. Rotate the line 90°, 180°, 270°, and 360° about the origin. *Describe* the relationship between the equation of the preimage and each image. **See margin.**

b. Do you think that the relationships you described in part (a) are true for *any* line? *Explain* your reasoning.

37. ★ EXTENDED RESPONSE Use the graph of the quadratic equation $y = x^2 + 1$ at the right.



a. Rotate the *parabola* by replacing y with x and x with y in the original equation, then graph this new equation. **See margin.**

b. What is the angle of rotation? **270°**

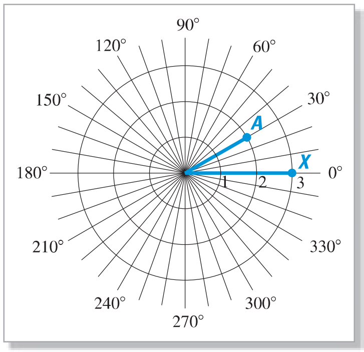
c. Are the image and the preimage both functions? *Explain.* **No; the image does not pass the vertical line test.**

C **TWO ROTATIONS** **The endpoints of $\overline{FG}$ are $F(1, 2)$ and $G(3, 4)$. Graph $\overline{F'G'}$ and $\overline{F''G''}$ after the given rotations.** **38, 39. See margin.**

38. Rotation: 90° about the origin
Rotation: 180° about (0, 4)

39. Rotation: 270° about the origin
Rotation: 90° about (−2, 0)

40. CHALLENGE A polar coordinate system locates a point in a plane by its distance from the origin O and by the measure of an angle with its vertex at the origin. For example, the point $A(2, 30°)$ at the right is 2 units from the origin and $m\angle XOA = 30°$. What are the polar coordinates of the image of point A after a 90° rotation? 180° rotation? 270° rotation? *Explain.* **(2, 120°); (2, 210°); (2, 300°); the distance from the origin to A never changes. Add 90°, 180°, and 270° to 30° to find each angle measure.**

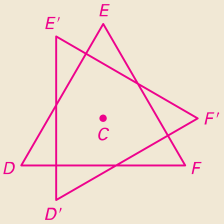


5 ASSESS AND RETEACH

Daily Homework Quiz

Also available online

1. Construct an equilateral triangle *DEF*. Then draw the image of △*DEF* for a 30° rotation about the center *C* of the triangle.

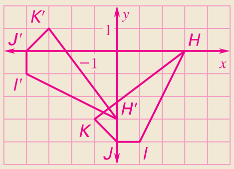


For Exercises 2 and 3, use the quadrilateral *HIJK* with vertices $H(3, 0)$, $I(1, -4)$, $J(0, -4)$, and $K(-1, -3)$.

2. Use the coordinate rules to name the vertices of the image of *HIJK* for a rotation of 180° about the origin. **$H'(-3, 0)$, $I'(-1, 4)$, $J'(0, 4)$, $K'(1, 3)$**

3. Use matrix multiplication to find the image matrix that represents a 270° rotation of *HIJK* about the origin. Then graph the polygon and its image.

$$\begin{matrix} H' & I' & J' & K' \end{matrix}$$
$$\begin{bmatrix} 0 & -4 & -4 & -3 \\ -3 & -1 & 0 & 1 \end{bmatrix}$$



Online Quiz

Available at **my.hrw.com**

Diagnosis/Remediation

- Practice A, B, C in Chapter Resource Book
- Study Guide in Chapter Resource Book
- Practice Workbook
- @HomeTutor

Challenge

Additional challenge is available in the Chapter Resource Book.

MATHEMATICAL PRACTICES Make sense of problems and persevere in solving them.

Alternative Strategy

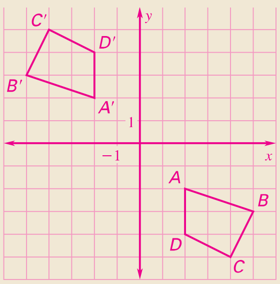
Example 2 in this lesson can be solved by using tracing paper. This method allows the student to visualize how the figure is rotating about the point.

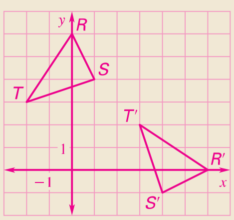
Avoiding Common Errors

Students need to be sure to keep the intersection of the axes on the tracing paper in the same place as the origin on the graph in Practice Exercises 1–3.

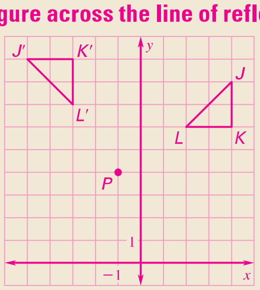
Mathematical Reasoning

Multiple Representations Extend Practice Exercise 5 to have the students generalize what happens to the x-coordinate and y-coordinate of a point when it is rotated 90° about the origin. **The x-coordinate becomes the opposite of the original y-coordinate and the y-coordinate becomes the original x-coordinate.**

1.


2.


4. Trace the figure then reflect the figure across the line of reflection.

6.


PROBLEM SOLVING WORKSHOP LESSON 9.4

Using ALTERNATIVE METHODS

Another Way to Solve Example 2

MATHEMATICAL PRACTICES Make sense of problems and persevere in solving them.

MULTIPLE REPRESENTATIONS In Example 2, you saw how to use a coordinate rule to rotate a figure. You can also *use tracing paper* and move a copy of the figure around the coordinate plane.

PROBLEM

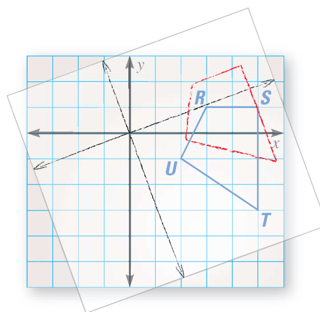
Graph quadrilateral $RSTU$ with vertices $R(3, 1)$, $S(5, 1)$, $T(5, -3)$, and $U(2, -1)$. Then rotate the quadrilateral 270° about the origin.

METHOD

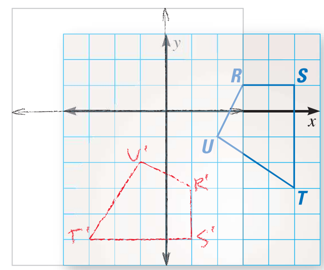
Using Tracing Paper You can use tracing paper to rotate a figure.

STEP 1 **Graph** the original figure in the coordinate plane.

STEP 2 **Trace** the quadrilateral and the axes on tracing paper.



STEP 3 **Rotate** the tracing paper 270°. Then transfer the resulting image onto the graph paper.



PRACTICE

1. **GRAPH** Graph quadrilateral $ABCD$ with vertices $A(2, -2)$, $B(5, -3)$, $C(4, -5)$, and $D(2, -4)$. Then rotate the quadrilateral 180° about the origin using tracing paper. **See margin.**
2. **GRAPH** Graph $\triangle RST$ with vertices $R(0, 6)$, $S(1, 4)$, and $T(-2, 3)$. Then rotate the triangle 270° about the origin using tracing paper. **See margin.**
3. **SHORT RESPONSE** *Explain* why rotating a figure 90° clockwise is the same as rotating the figure 270° counterclockwise. **Since they are rotating in opposite directions they will each place you at 90° below your reference line.**
4. **SHORT RESPONSE** *Explain* how you could use tracing paper to do a reflection. **See margin.**
5. **REASONING** If you rotate the point (3, 4) 90° about the origin, what happens to the x-coordinate? What happens to the y-coordinate? **The x-coordinate is now −4; the y-coordinate is now 3.**
6. **GRAPH** Graph $\triangle JKL$ with vertices $J(4, 8)$, $K(4, 6)$, and $L(2, 6)$. Then rotate the triangle 90° about the point $(-1, 4)$ using tracing paper. **See margin.**

1 PLAN AND PREPARE

Warm-Up Exercises

Also available online

Find the image of (2, 3) under each transformation.

1. translation $(x, y) \rightarrow (x - 6, y - 2)$ **(−4, 1)**
2. reflection in the y-axis **(−2, 3)**
3. reflection in the line $y = 6$ **(2, 9)**
4. rotation 90° about the origin **(−3, 2)**

Notetaking Guide

Available online

Promotes interactive learning and notetaking skills.

Pacing

Basic: 2 days
Average: 2 days
Advanced: 2 days
Block: 1 block
• See *Teaching Guide/Lesson Plan*.

2 FOCUS AND MOTIVATE

Essential Question

Big Idea 1

What is a glide reflection? **Tell students they will learn how to answer this question by studying compositions of two or more transformations.**

9.5 Apply Compositions of Transformations

Before You performed rotations, reflections, or translations.
Now You will perform combinations of two or more transformations.
Why? So you can describe the transformations that represent a rowing crew, as in Ex. 30.

Key Vocabulary
- **glide reflection**
- **composition of transformations**

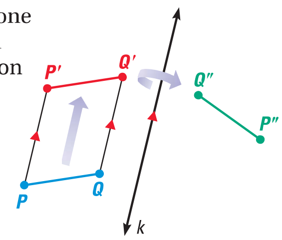
COMMON CORE

CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

A translation followed by a reflection can be performed one after the other to produce a *glide reflection*. A translation can be called a glide. A **glide reflection** is a transformation in which every point P is mapped to a point P'' by the following steps.

STEP 1 **First,** a translation maps P to P'.

STEP 2 **Then,** a reflection in a line k parallel to the direction of the translation maps P' to P''.



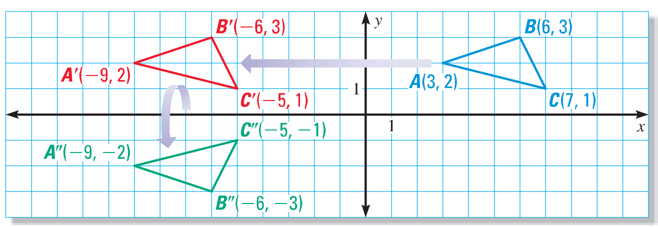
EXAMPLE 1 Find the image of a glide reflection

The vertices of $\triangle ABC$ are $A(3, 2)$, $B(6, 3)$, and $C(7, 1)$. Find the image of $\triangle ABC$ after the glide reflection.

Translation: $(x, y) \rightarrow (x - 12, y)$
Reflection: in the x-axis

Solution

Begin by graphing $\triangle ABC$. Then graph $\triangle A'B'C'$ after a translation 12 units left. Finally, graph $\triangle A''B''C''$ after a reflection in the x-axis.



AVOID ERRORS
The line of reflection must be parallel to the direction of the translation to be a glide reflection.

✓ GUIDED PRACTICE for Example 1

1. Suppose $\triangle ABC$ in Example 1 is translated 4 units down, then reflected in the y-axis. What are the coordinates of the vertices of the image? **A(−3, −2), B(−6, −1), C(−7, −3)**
2. In Example 1, *describe* a glide reflection from $\triangle A''B''C''$ to $\triangle ABC$. **$(x, y) \rightarrow (x + 12, y)$ followed by a reflection in x.**

COMMON CORE **Standards for Mathematical Content** *High School*

CC.9-12.G.CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

COMPOSITIONS When two or more transformations are combined to form a single transformation, the result is a **composition of transformations**. A glide reflection is an example of a composition of transformations.

In this lesson, a composition of transformations uses isometries, so the final image is congruent to the preimage. This suggests the Composition Theorem.

THEOREM *For Your Notebook*

THEOREM 9.4 Composition Theorem

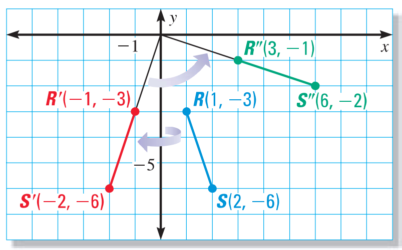
The composition of two (or more) isometries is an isometry.

EXAMPLE 2 Find the image of a composition

The endpoints of $\overline{RS}$ are $R(1, -3)$ and $S(2, -6)$. Graph the image of $\overline{RS}$ after the composition.

Reflection: in the y-axis
Rotation: 90° about the origin

Solution



STEP 1 **Graph** $\overline{RS}$.

STEP 2 **Reflect** $\overline{RS}$ in the y-axis. $\overline{R'S'}$ has endpoints $R'(-1, -3)$ and $S'(-2, -6)$.

STEP 3 **Rotate** $\overline{R'S'}$ 90° about the origin. $\overline{R''S''}$ has endpoints $R''(3, -1)$ and $S''(6, -2)$.

AVOID ERRORS
Unless you are told otherwise, do the transformations in the order given.

TWO REFLECTIONS Compositions of two reflections result in either a translation or a rotation, as described in the following two theorems.

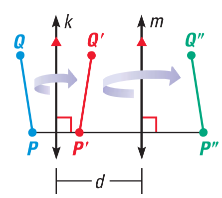
THEOREM *For Your Notebook*

THEOREM 9.5 Reflections in Parallel Lines Theorem

If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is the same as a translation.

If P'' is the image of P, then:

1. $\overline{PP''}$ is perpendicular to k and m, and
2. $PP'' = 2d$, where d is the distance between k and m.



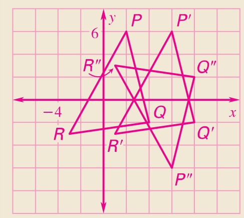
Motivating the Lesson

Tell students that patterns in tiled floors are often created by combining reflections, translations, and rotations. Tell students that in this lesson they will learn about the results of performing more than one transformation.

3 TEACH

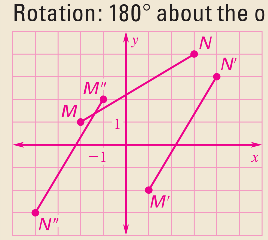
Extra Example 1

The vertices of $\triangle PQR$ are $P(2, 6)$, $Q(4, -2)$, and $R(-3, -3)$. Find the image of $\triangle PQR$ after the glide reflection.
Translation: $(x, y) \rightarrow (x + 4, y)$;
Reflection: in the x-axis



Extra Example 2

The endpoints of $\overline{MN}$ are $M(-2, 1)$ and $N(3, 4)$. Graph the image of $\overline{MN}$ after the composition.
Reflection: in the line (x, y)
Rotation: 180° about the origin

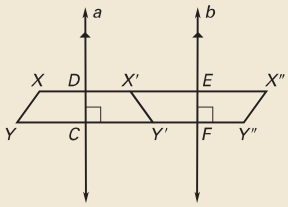


MATHEMATICAL PRACTICES ***Key Question Example 2***

- Would you get the same result if you performed the rotation first? Explain. **No; the final image would have endpoints at (−3, 1) and (−6, 2).**

Extra Example 3

In the diagram, a reflection in line a maps $\overline{XY}$ to $\overline{X'Y'}$. A reflection in line b maps $\overline{X'Y'}$ to $\overline{X''Y''}$. Also, $XD = 6$ and $EX'' = 9$.



a. Name any segments congruent to each segment: $\overline{XY}$, $\overline{XD}$, and $\overline{YC}$. $\overline{X'Y'}$, $\overline{X''Y''}$; $\overline{DX'}$; $\overline{Y'C}$

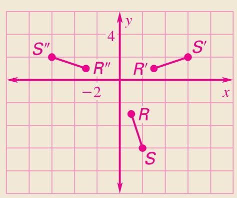
b. Does $CF = DE$? Explain. **Yes, *CDEF* is a rectangle.**

c. What is the length of $\overline{YY''}$? **30**

MATHEMATICAL PRACTICES **Key Question Example 3**

- What type of figure is $HGG''H''$? **parallelogram**

3.



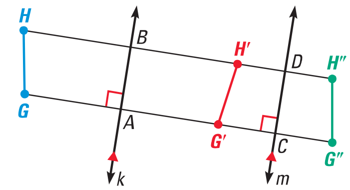
EXAMPLE 3 Use Theorem 9.5

In the diagram, a reflection in line k maps $\overline{GH}$ to $\overline{G'H'}$. A reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$. Also, $HB = 9$ and $DH'' = 4$.

a. Name any segments congruent to each segment: $\overline{HG}$, $\overline{HB}$, and $\overline{GA}$.

b. Does $AC = BD$? Explain.

c. What is the length of $\overline{GG''}$?



Solution

a. $\overline{HG} \cong \overline{H'G'}$, and $\overline{HG} \cong \overline{H''G''}$. $\overline{HB} \cong \overline{H'B}$. $\overline{GA} \cong \overline{G'A}$.

b. Yes, $AC = BD$ because $\overline{GG''}$ and $\overline{HH''}$ are perpendicular to both k and m, so $\overline{BD}$ and $\overline{AC}$ are opposite sides of a rectangle.

c. By the properties of reflections, $H'B = 9$ and $H'D = 4$. Theorem 9.5 implies that $GG'' = HH'' = 2 \cdot BD$, so the length of $\overline{GG''}$ is $2(9 + 4)$, or 26 units.

✓ GUIDED PRACTICE for Examples 2 and 3

3. See margin for art; yes; the resulting segment $R''S''$ is not the same.

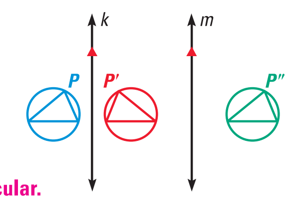
3. Graph $\overline{RS}$ from Example 2. Do the rotation first, followed by the reflection. Does the order of the transformations matter? *Explain*.

4. In Example 3, part (c), *explain* how you know that $GG'' = HH''$. **They are opposite sides of a parallelogram.**

Use the figure below for Exercises 5 and 6. The distance between line k and line m is 1.6 centimeters.

5. The preimage is reflected in line k, then in line m. *Describe* a single transformation that maps the blue figure to the green figure. **translation**

6. What is the distance between P and P''? If you draw $\overline{PP'}$, what is its relationship with line k? *Explain*. **3.2 cm; they are perpendicular.**

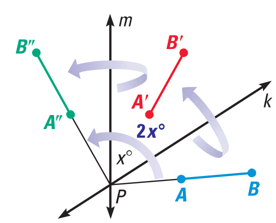


THEOREM *For Your Notebook*

THEOREM 9.6 Reflections in Intersecting Lines Theorem

If lines k and m intersect at point P, then a reflection in k followed by a reflection in m is the same as a rotation about point P.

The angle of rotation is $2x°$, where $x°$ is the measure of the acute or right angle formed by k and m.



$m\angle BPB'' = 2x°$

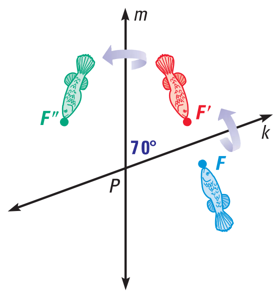
Differentiated Instruction

Inclusion Some students may have difficulty grasping how two reflections can result in a translation. Assist students with **Guided Practice Exercise 5** by having them trace the preimage P. Ask them to cut out the preimage and place it over P on the page. Then have them manipulate P to create P''. Help them to describe in words how P is transformed to create P''.

See also the *Differentiated Instruction Resources* for more strategies.

EXAMPLE 4 Use Theorem 9.6

In the diagram, the figure is reflected in line k. The image is then reflected in line m. Describe a single transformation that maps F to F''.



Solution

The measure of the acute angle formed between lines k and m is 70°. So, by Theorem 9.6, a single transformation that maps F to F'' is a 140° rotation about point P.

You can check that this is correct by tracing lines k and m and point F, then rotating the point 140°.

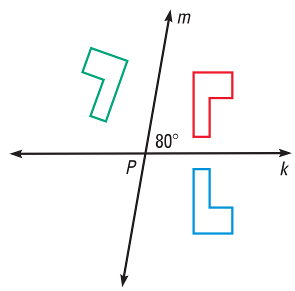
Animated Geometry at my.hrw.com

✓ GUIDED PRACTICE for Example 4

7. a rotation of 160° about point P

7. In the diagram at the right, the preimage is reflected in line k, then in line m. Describe a single transformation that maps the blue figure onto the green figure.

8. A rotation of 76° maps C to C'. To map C to C' using two reflections, what is the angle formed by the intersecting lines of reflection? **38°**



9.5 EXERCISES

HOMEWORK KEY
○ = See **WORKED-OUT SOLUTIONS** Exs. 7, 17, and 27
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 25, 29, and 34

SKILL PRACTICE

A

1. **VOCABULARY** Copy and complete: In a glide reflection, the direction of the translation must be _?_ to the line of reflection. **parallel**

2. ★ **WRITING** *Explain* why a glide reflection is an isometry.
It preserves length and angle measure.

EXAMPLE 1 for Exs. 3–6

GLIDE REFLECTION **The endpoints of $\overline{CD}$ are $C(2, -5)$ and $D(4, 0)$. Graph the image of $\overline{CD}$ after the glide reflection.** **3–6. See margin.**

3. **Translation:** $(x, y) \to (x, y - 1)$
Reflection: in the y-axis

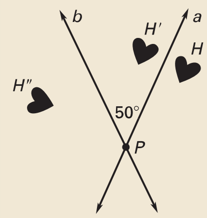
4. **Translation:** $(x, y) \to (x - 3, y)$
Reflection: in $y = -1$

5. **Translation:** $(x, y) \to (x, y + 4)$
Reflection: in $x = 3$

6. **Translation:** $(x, y) \to (x + 2, y + 2)$
Reflection: in $y = x$

Extra Example 4

In the diagram, the figure is reflected in line a. The image is then reflected in line b. Describe a single transformation that maps H to H''. **100° rotation about P**



Animated Geometry my.hrw.com

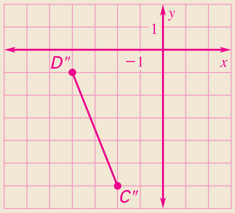
An **Animated Geometry** activity is available online for **Example 4.** This activity is also part of **Power Presentations.**

Closing the Lesson

Have students summarize the major points of the lesson and answer the Essential Question: What is a glide reflection?

- **The composition of two or more isometries is an isometry.**
- **Successive reflections in two parallel lines give a translation.**
- **Successive reflections in two intersecting lines give a rotation.**

A glide reflection is a translation followed by a reflection over a line parallel to the direction of the translation.

3.


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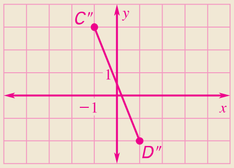

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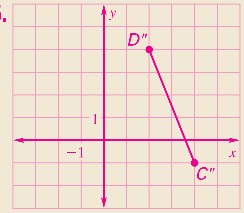
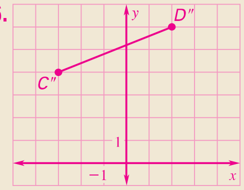


6.


Assignment Guide

Answers for all exercises available online

Basic:
Day 1:
Exs. 1–14, 27–30
Day 2:
Exs. 15–23, 31–35

Average:
Day 1:
Exs. 1, 2, 4–6, 8–14, 23, 24, 27–30
Day 2:
Exs. 15–22, 25, 31–39

Advanced:
Day 1:
Exs. 1, 2, 4–6, 9–14, 23, 24, 26–30*
Day 2:
Exs. 15–22, 25, 31–41*

Block:
Exs. 1, 2, 4–6, 8–25, 27–39

Differentiated Instruction

See *Differentiated Instruction Resources* for suggestions on addressing the needs of a diverse classroom.

Homework Check

For a quick check of student understanding of key concepts, go over the following exercises:

Basic: 4, 8, 16, 20, 27
Average: 5, 12, 18, 21, 28
Advanced: 6, 14, 19, 21, 30

Extra Practice

- Student Edition
- Chapter Resource Book: Practice levels A, B, C

Practice Worksheet

An easily-readable reduced practice page can be found at the beginning of this chapter.

EXAMPLE 2 for Exs. 7–14

GRAPHING COMPOSITIONS The vertices of $\triangle PQR$ are $P(2, 4)$, $Q(6, 0)$, and $R(7, 2)$. Graph the image of $\triangle PQR$ after a composition of the transformations in the order they are listed. 7–10. See margin.

7. **Translation:** $(x, y) \to (x, y - 5)$
   Reflection: in the y-axis

8. **Translation:** $(x, y) \to (x - 3, y + 2)$
   Rotation: $90°$ about the origin

9. **Translation:** $(x, y) \to (x + 12, y + 4)$
   Translation: $(x, y) \to (x - 5, y - 9)$

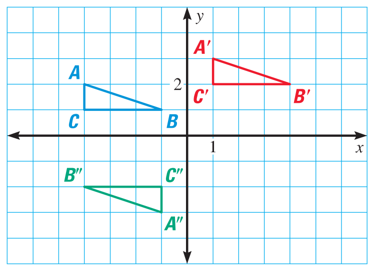
10. **Reflection:** in the x-axis
    Rotation: $90°$ about the origin

REVERSING ORDERS Graph $\overline{F''G''}$ after a composition of the transformations in the order they are listed. Then perform the transformations in reverse order. Does the order affect the final image $\overline{F''G''}$? 11, 12. See margin for art.

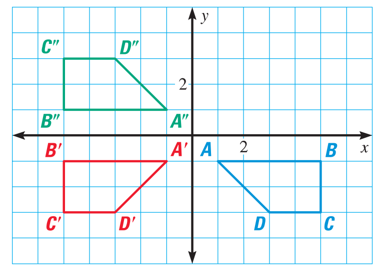
11. $F(-5, 2)$, $G(-2, 4)$
    Translation: $(x, y) \to (x + 3, y - 8)$
    Reflection: in the x-axis yes

12. $F(-1, -8)$, $G(-6, -3)$
    Reflection: in the line $y = 2$
    Rotation: $90°$ about the origin yes

DESCRIBING COMPOSITIONS *Describe* the composition of transformations.

13. 

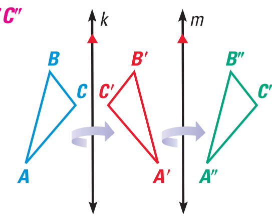
13. $(x, y) \to (x + 5, y + 1)$ followed by a rotation of 180° about the origin.

14. 

14. a reflection in the y-axis followed by a reflection in the x-axis

EXAMPLE 3 for Exs. 15–19

USING THEOREM 9.5 In the diagram, $k \parallel m$, $\triangle ABC$ is reflected in line k, and $\triangle A'B'C'$ is reflected in line m.



15. A translation maps $\triangle ABC$ onto which triangle? $\triangle A''B''C''$

16. Which lines are perpendicular to $\overleftrightarrow{AA''}$? line k and line m

17. Name two segments parallel to $\overleftrightarrow{BB''}$. Sample answer: $\overline{AA'}$, $\overline{AA''}$

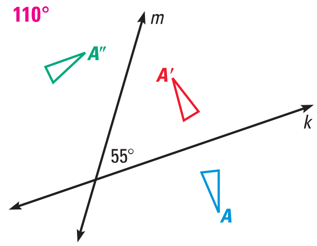
18. If the distance between k and m is 2.6 inches, what is the length of $\overline{CC''}$? 5.2 in.

19. Is the distance from B' to m the same as the distance from B'' to m? *Explain.*
    yes; definition of reflection of a point over a line

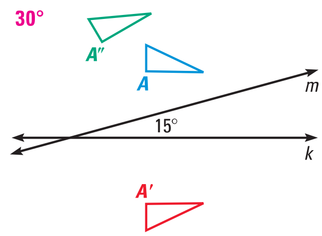
EXAMPLE 4 for Exs. 20–21

USING THEOREM 9.6 Find the angle of rotation that maps A onto A''.

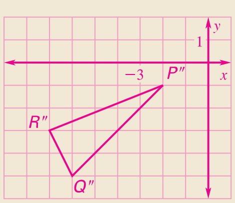
20. 110°

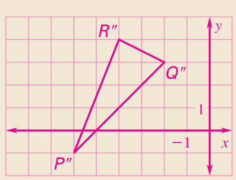


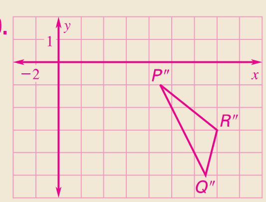
21. 30°

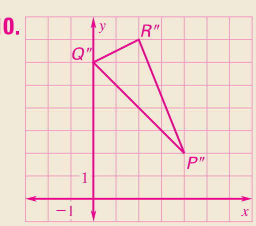


○ = See WORKED-OUT SOLUTIONS in Student Resources
★ = STANDARDIZED TEST PRACTICE

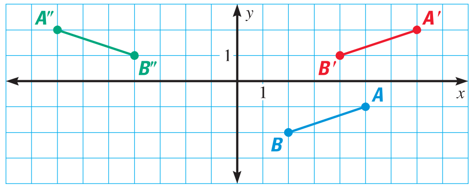
7. 

8. 

9. 

10. 

B **22. ERROR ANALYSIS** A student described the translation of $\overline{AB}$ to $\overline{A'B'}$ followed by the reflection of $\overline{A'B'}$ to $\overline{A''B''}$ in the y-axis as a glide reflection. *Describe* and correct the student's error.



22. The line of reflection is not parallel to the direction of the translation; this is not a glide reflection.

USING MATRICES **The vertices of $\triangle PQR$ are $P(1, 4)$, $Q(3, -2)$, and $R(7, 1)$. Use matrix operations to find the image matrix that represents the composition of the given transformations. Then graph $\triangle PQR$ and its image.** 23, 24. See margin.

23. Translation: $(x, y) \rightarrow (x, y + 5)$
Reflection: in the y-axis

24. Reflection: in the x-axis
Translation: $(x, y) \rightarrow (x - 9, y - 4)$

25. ★ OPEN-ENDED MATH Sketch a polygon. Apply three transformations of your choice on the polygon. What can you say about the congruence of the preimage and final image after multiple transformations? *Explain.*

25. Check students work. Since the three transformations are isometries, the preimage and the final image are congruent because an isometry preserves length and angle measure.

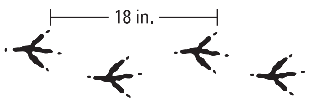
C **26. CHALLENGE** The vertices of $\triangle JKL$ are $J(1, -3)$, $K(2, 2)$, and $L(3, 0)$. Find the image of the triangle after a 180° rotation about the point $(-2, 2)$, followed by a reflection in the line $y = -x$. $J''(-7, 5)$, $K''(-2, 6)$, $L''(-4, 7)$

PROBLEM SOLVING

EXAMPLE 1 **A** for Exs. 27–30

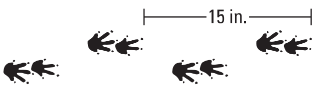
ANIMAL TRACKS **The left and right prints in the set of animal tracks can be related by a glide reflection. Copy the tracks and *describe* a translation and reflection that combine to create the glide reflection.**

27. bald eagle (2 legs)



27. *Sample answer:* $(x, y) \rightarrow (x + 9, y)$, reflected over a horizontal line that separates the left and right prints

28. armadillo (4 legs)



Sample answer: $(x, y) \rightarrow (x + 7.5, y)$, reflected over a horizontal line that separates the left and right prints

29. ★ MULTIPLE CHOICE Which is *not* a glide reflection? C

Ⓐ The teeth of a closed zipper

Ⓑ The tracks of a walking duck

Ⓒ The keys on a computer keyboard

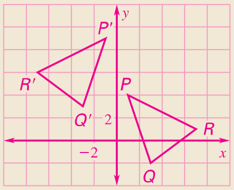
Ⓓ The red squares on two adjacent rows of a checkerboard

30. ROWING *Describe* the transformations that are combined to represent an eight-person rowing shell. glide reflection

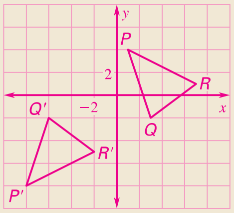
Avoiding Common Errors

Exercises 23–24 Some students may attempt to use matrix multiplication to perform the translations. Remind them that translations are performed by using matrix addition.

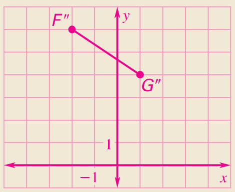
23. $\begin{bmatrix} -1 & -3 & -7 \\ 9 & 3 & 6 \end{bmatrix}$; (columns P', Q', R')



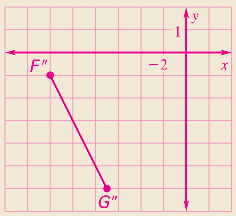
24. $\begin{bmatrix} -8 & -6 & -2 \\ -8 & -2 & -5 \end{bmatrix}$; (columns P', Q', R')



11.



12.



Teaching Strategy

Exercises 35–38 Divide the students into groups and assign one problem to each group. Have them present their proofs to the class.

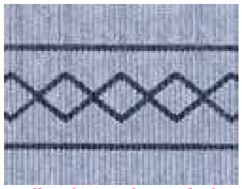
36. A reflection followed by a rotation, a reflection followed by a translation, a rotation followed by a translation, a rotation followed by a reflection, a translation followed by a rotation, or a translation followed by a reflection. *Sample answer:* Given: a reflection in m mapping P to P' and Q to Q' followed by a rotation about R mapping P' to P'' and Q' to Q''. Using the Reflection Theorem, $PQ = P'Q'$. Using the Rotation Theorem, $P'Q' = P''Q''$. Using the Transitive Property of Equality, $PQ = P''Q''$.

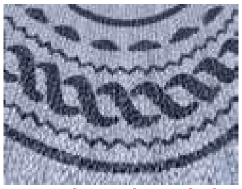
37a. Given: A reflection in ℓ maps $\overline{JK}$ to $\overline{J'K'}$, a reflection in m maps $\overline{J'K'}$ to $\overline{J''K''}$, $\ell \parallel m$, and the distance between ℓ and m is d. Using the definition of reflection, ℓ is the perpendicular bisector of $\overline{KK'}$ and m is the perpendicular bisector of $\overline{K'K''}$. Using the Segment Addition Postulate, $KK' + K'K'' = KK''$. It follows that $\overline{KK''}$ is perpendicular to ℓ and m.

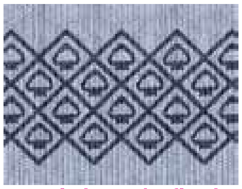
37b. Using the definition of reflection, the distance from K to ℓ is the same as the distance from ℓ to K' and the distance from K' to m is the same as the distance from m to K''. Since the distance from ℓ to K' plus the distance from K' to m is d, it follows that $K'K'' = 2d$.

38a–b. See Additional Answers.

B **SWEATER PATTERNS** In Exercises 31–33, *describe* the transformations that are combined to make each sweater pattern.

31. 
reflection and translation

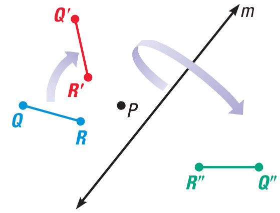
32. 
rotation and translation

33. 
translation and reflection

34. ★ **SHORT RESPONSE** Use the Reflections in Parallel Lines Theorem to *explain* how you can make a glide reflection using three reflections. How are the lines of reflection related?

34. Reflect the object across two parallel lines, and then reflect it across a third line perpendicular to the first two lines.

35. PROVING THEOREM 9.4 Write a plan for proof for one case of the Composition Theorem.



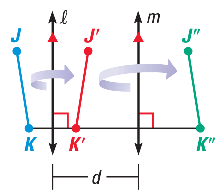
GIVEN ▶ A rotation about P maps Q to Q' and R to R'. A reflection in m maps Q' to Q'' and R' to R''.

PROVE ▶ $QR = Q''R''$

Use the Rotation Theorem followed by the Reflection Theorem.

36. PROVING THEOREM 9.4 A composition of a rotation and a reflection, as in Exercise 35, is one case of the Composition Theorem. List all possible cases, and prove the theorem for another pair of compositions. **See margin.**

37. PROVING THEOREM 9.5 Prove the Reflection in Parallel Lines Theorem. **See margin.**

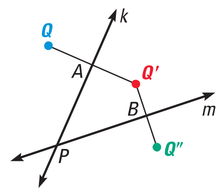


GIVEN ▶ A reflection in line ℓ maps $\overline{JK}$ to $\overline{J'K'}$, a reflection in line m maps $\overline{J'K'}$ to $\overline{J''K''}$, and $\ell \parallel m$.

PROVE ▶ **a.** $\overleftrightarrow{KK''}$ is perpendicular to ℓ and m.

b. $KK'' = 2d$, where d is the distance between ℓ and m.

38. PROVING THEOREM 9.6 Prove the Reflection in Intersecting Lines Theorem. **See margin.**



GIVEN ▶ Lines k and m intersect at point P. Q is any point not on k or m.

PROVE ▶ **a.** If you reflect point Q in k, and then reflect its image Q' in m, Q'' is the image of Q after a rotation about point P.

b. $m\angle QPQ'' = 2(m\angle APB)$

Plan for Proof First show $k \perp \overline{QQ'}$ and $\overline{QA} \cong \overline{Q'A}$. Then show $\triangle QAP \cong \triangle Q'AP$. In the same way, show $\triangle Q'BP \cong \triangle Q''BP$. Use congruent triangles and substitution to show that $\overline{QP} \cong \overline{Q''P}$. That proves part (a) by the definition of a rotation. Then use congruent triangles to prove part (b).

39. VISUAL REASONING You are riding a bicycle along a flat street.

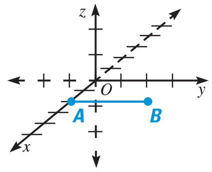
a. What two transformations does the wheel's motion use? **translation and a rotation**

b. *Explain* why this is not a composition of transformations.

39b. One transformation is not followed by the second. They are done simultaneously.

★ = STANDARDIZED TEST PRACTICE

C **40. MULTI-STEP PROBLEM** A point in space has three coordinates (x, y, z). From the origin, a point can be forward or back on the x-axis, left or right on the y-axis, and up or down on the z-axis. The endpoints of segment $\overline{AB}$ in space are $A(2, 0, 0)$ and $B(2, 3, 0)$, as shown at the right.



a. Rotate $\overline{AB}$ 90° about the x-axis with center of rotation A. What are the coordinates of $\overline{A'B'}$? **$A'(2, 0, 0)$, $B'(2, 0, 3)$**

b. Translate $\overline{A'B'}$ using the vector $\langle 4, 0, -1\rangle$. What are the coordinates of $\overline{A''B''}$? **$A''(6, 0, -1)$, $B''(6, 0, 2)$**

41. CHALLENGE *Justify* the following conjecture or provide a counterexample.

Conjecture When performing a composition of two transformations of the *same type*, order does not matter.

41. *Sample answer:* The conjecture is not always true. Consider a reflection of a point (a, b) in the x-axis followed by a reflection in the line $y = x$.

QUIZ

The vertices of $\triangle ABC$ are $A(7, 1)$, $B(3, 5)$, and $C(10, 7)$. Graph the reflection in the line. **1–3. See margin.**

1. y-axis
2. $x = -4$
3. $y = -x$

Find the coordinates of the image of $P(2, -3)$ after the rotation about the origin.

4. 180° rotation **$(-2, 3)$**
5. 90° rotation **$(3, 2)$**
6. 270° rotation **$(-3, -2)$**

The vertices of $\triangle PQR$ are $P(-8, 8)$, $Q(-5, 0)$, and $R(-1, 3)$. Graph the image of $\triangle PQR$ after a composition of the transformations in the order they are listed. **7–10. See margin.**

7. **Translation:** $(x, y) \rightarrow (x + 6, y)$
   Reflection: in the y-axis
8. **Reflection:** in the line $y = -2$
   Rotation: 90° about the origin
9. **Translation:** $(x, y) \rightarrow (x - 5, y)$
   Translation: $(x, y) \rightarrow (x + 2, y + 7)$
10. **Rotation:** 180° about the origin
    Translation: $(x, y) \rightarrow (x + 4, y - 3)$

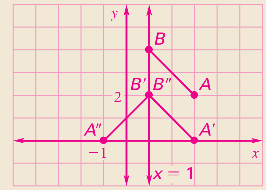
See **EXTRA PRACTICE** in Student Resources **ONLINE QUIZ** at my.hrw.com

5 ASSESS AND RETEACH

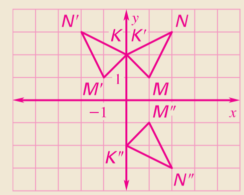
Daily Homework Quiz

Also available online

1. The endpoints of $\overline{AB}$ are $A(3, 2)$ and $B(1, 4)$. Graph the image of $\overline{AB}$ after the glide reflection.
   Translation: $(x, y) \rightarrow (x, y - 2)$
   Reflection: in the line $x = 1$



2. The vertices of $\triangle MNK$ are $M(1, 1)$, $N(2, 3)$, and $K(0, 2)$. Graph the image of $\triangle MNK$ after the composition of the reflection followed by the rotation.
   Reflection: in the y-axis
   Rotation: 180° about the origin



Online Quiz

Available at **my.hrw.com**

Diagnosis/Remediation

- Practice A, B, C in Chapter Resource Book
- Study Guide in Chapter Resource Book
- Practice Workbook
- @HomeTutor

Challenge

Additional challenge is available in the Chapter Resource Book.

Quiz

An easily-readable reduced copy of the quiz from the Assessment Book can be found at the beginning of this chapter.

Quiz 1–3, 7–10. See Additional Answers.

EXAMPLE 2 Draw a tessellation using one shape

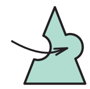
Change a triangle to make a tessellation.

Solution

STEP 1

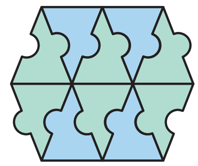
Cut a piece from the triangle.

STEP 2



Slide the piece to another side.

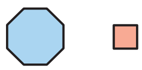
STEP 3



Translate and reflect the figure to make a tessellation.

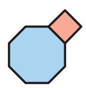
EXAMPLE 3 Draw a tessellation using two shapes

Draw a tessellation using the given floor tiles.



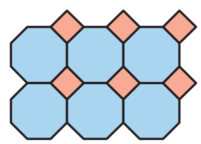
Solution

STEP 1



Combine one octagon and one square by connecting sides of the same length.

STEP 2



Translate the pair of polygons to make a tessellation

Animated Geometry at my.hrw.com

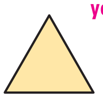
READ VOCABULARY
Notice that in the tessellation in Example 3, the same combination of regular polygons meet at each vertex. This type of tessellation is called *semi-regular.*

PRACTICE

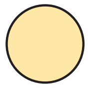
EXAMPLE 1 for Exs. 1–4

REGULAR TESSELLATIONS **Does the shape tessellate? If so, tell whether the tessellation is regular.**

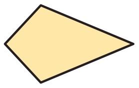
1. Equilateral triangle **yes; regular**



2. Circle **no**

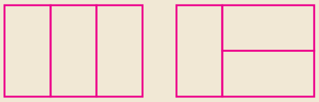


3. Kite **yes; not regular**



4. ★ **OPEN-ENDED MATH** Draw a rectangle. Use the rectangle to make two different tessellations. **See margin.**

4. Sample:



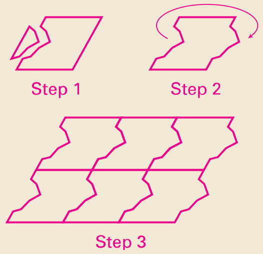
Key Question Example 1

MATHEMATICAL PRACTICES

- Why can a regular polygon with more than six sides not tessellate a plane? **The measure of each angle will not be a factor of 360°.**

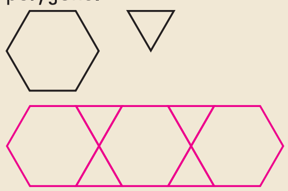
Extra Example 2

Change a quadrilateral to make a tessellation. **Cut a piece from the quadrilateral, slide it to another side, and translate the figure.**



Extra Example 3

Draw a tessellation using the given polygons.



Teaching Strategy

Have the students bring in real world examples of tessellations. For example, floor tile patterns, brick patterns, patio paver stone patterns, and so on. Have students discuss these tessellations in class.

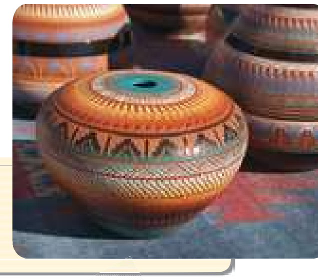
Closing the Lesson

Have students summarize the major points of the lesson and answer the Essential Question: What regular polygons can tessellate a plane?

- **A tessellation is a collection of figures that cover a plane.**
- **A regular tessellation is a tessellation of congruent regular polygons.**

An equilateral triangle, square, and regular hexagon can tessellate a plane.

9.6 Identify Symmetry



Before You reflected or rotated figures.
Now You will identify line and rotational symmetries of a figure.
Why? So you can identify the symmetry in a bowl, as in Ex. 11.

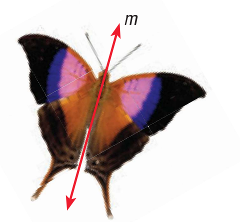
Key Vocabulary
- line symmetry
- line of symmetry
- rotational symmetry
- center of symmetry

COMMON CORE

CC.9-12.G.CO.3 Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

REVIEW REFLECTION
Notice that the lines of symmetry are also lines of reflection.

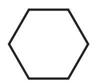
A figure in the plane has **line symmetry** if the figure can be mapped onto itself by a reflection in a line. This line of reflection is a **line of symmetry**, such as line *m* at the right. A figure can have more than one line of symmetry.

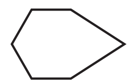


EXAMPLE 1 Identify lines of symmetry

How many lines of symmetry does the hexagon have?

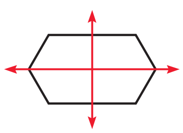
a. 

b. 

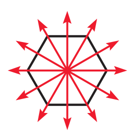
c. 

Solution

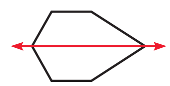
a. Two lines of symmetry



b. Six lines of symmetry



c. One line of symmetry



Animated Geometry at my.hrw.com

✓ GUIDED PRACTICE for Example 1

How many lines of symmetry does the object appear to have?

1.  8

2.  5

3.  1

4. Draw a hexagon with no lines of symmetry. **See margin.**

1 PLAN AND PREPARE

Warm-Up Exercises

Also available online

Find the coordinates of the image of $A(-3, 8)$ for each transformation.

1. Reflection in the line $y = x$ **(8, −3)**

2. Rotation of 90° about the origin **(−8, −3)**

3. Translation 4 units down **(−3, 4)**

Notetaking Guide

Available online

Promotes interactive learning and notetaking skills.

Pacing

Basic: 1 day
Average: 1 day
Advanced: 1 day
Block: 0.5 block with next lesson
- See *Teaching Guide/Lesson Plan*.

2 FOCUS AND MOTIVATE

Essential Question

Big Idea 2

When does a figure have line symmetry? **Tell students they will learn how to answer this question by studying how to draw symmetry lines.**

4. See Additional Answers.

COMMON CORE **Standards for Mathematical Content** *High School*

CC.9-12.G.CO.3 Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

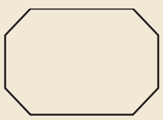
Motivating the Lesson

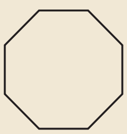
Tell students that artists use symmetry to create designs. Tell them that in this lesson they will study line symmetry and rotational symmetry of various figures.

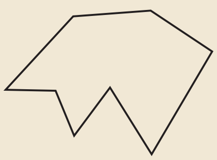
3 TEACH

Extra Example 1

How many lines of symmetry does each octagon have?

a.  **2**

b.  **8**

c.  **0**

MATHEMATICAL PRACTICES Key Question Example 1

- How many lines of symmetry will a regular polygon with n sides have? Describe them. **It will have n symmetry lines. If n is even, then half of them will be perpendicular bisectors of a pair of sides and the others will pass through a pair of vertices. If n is odd, then the lines of symmetry will be the perpendicular bisectors of the sides.**

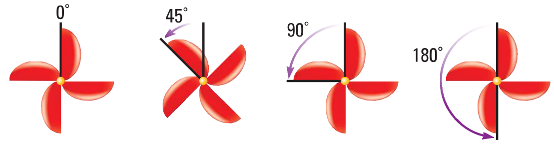
Animated Geometry
my.hrw.com

An **Animated Geometry** activity is available online for **Example 1.** This activity is also part of **Power Presentations.**

ROTATIONAL SYMMETRY A figure in a plane has **rotational symmetry** if the figure can be mapped onto itself by a rotation of 180° or less about the center of the figure. This point is the **center of symmetry**. Note that the rotation can be either clockwise or counterclockwise.

> **REVIEW ROTATION**
> For a figure with rotational symmetry, the *angle of rotation* is the smallest angle that maps the figure onto itself.

For example, the figure below has rotational symmetry, because a rotation of either 90° or 180° maps the figure onto itself (although a rotation of 45° does not).

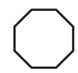


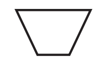
The figure above also has *point symmetry*, which is 180° rotational symmetry.

EXAMPLE 2 Identify rotational symmetry

Does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

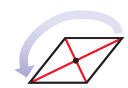
a. Parallelogram 

b. Regular octagon 

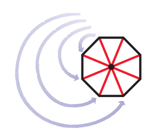
c. Trapezoid 

Solution

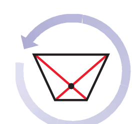
a. The parallelogram has rotational symmetry. The center is the intersection of the diagonals. A 180° rotation about the center maps the parallelogram onto itself.



b. The regular octagon has rotational symmetry. The center is the intersection of the diagonals. Rotations of 45°, 90°, 135°, or 180° about the center all map the octagon onto itself.



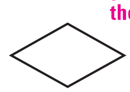
c. The trapezoid does not have rotational symmetry because no rotation of 180° or less maps the trapezoid onto itself.



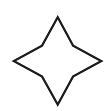
✓ GUIDED PRACTICE for Example 2

Does the figure have rotational symmetry? If so, *describe* any rotations that map the figure onto itself.

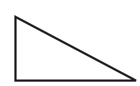
5. Rhombus **yes; 180° about the center**



6. Octagon **yes; 90° or 180° about the center**

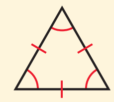


7. Right triangle **no**



★ EXAMPLE 3 Standardized Test Practice

Identify the line symmetry and rotational symmetry of the equilateral triangle at the right.



Ⓐ 3 lines of symmetry, 60° rotational symmetry

Ⓑ 3 lines of symmetry, 120° rotational symmetry

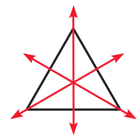
Ⓒ 1 line of symmetry, 180° rotational symmetry

Ⓓ 1 line of symmetry, no rotational symmetry

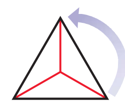
Solution

ELIMINATE CHOICES
An equilateral triangle can be mapped onto itself by reflecting over any of three different lines. So, you can eliminate choices C and D.

The triangle has line symmetry. Three lines of symmetry can be drawn for the figure.



For a figure with s lines of symmetry, the smallest rotation that maps the figure onto itself has the measure $\frac{360°}{s}$. So, the equilateral triangle has $\frac{360°}{3}$, or 120° rotational symmetry.



▶ The correct answer is B. Ⓐ Ⓑ Ⓒ Ⓓ

✓ GUIDED PRACTICE for Example 3

8. *Describe* the lines of symmetry and rotational symmetry of a non-equilateral isosceles triangle.
the altitude, no rotational symmetry

9.6 EXERCISES

HOMEWORK KEY
○ = See **WORKED-OUT SOLUTIONS** Exs. 7, 13, and 31
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 13, 14, 21, and 23

SKILL PRACTICE

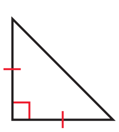
A

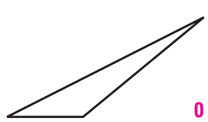
1. **VOCABULARY** What is a *center of symmetry*?
If a figure has rotational symmetry it is the point about which the figure is rotated.

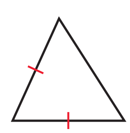
2. ★ **WRITING** Draw a figure that has one line of symmetry and does not have rotational symmetry. Can a figure have two lines of symmetry and no rotational symmetry? **See margin for art; no.**

EXAMPLE 1 for Exs. 3–5

LINE SYMMETRY **How many lines of symmetry does the triangle have?**

3.  **1**

4.  **0**

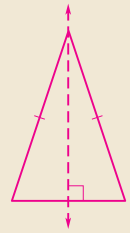
5.  **1**

Differentiated Instruction

Kinesthetic Learners Have students work with a partner to find an object in the classroom that has both line symmetry and rotational symmetry. Have them determine how many lines of symmetry the object has and what the angle of rotation is. Invite pairs of students to describe their findings to the class.

See also the *Differentiated Instruction Resources* for more strategies.

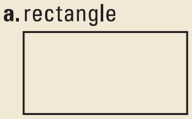
2. *Sample:*



Extra Example 2

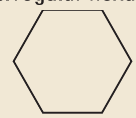
Does the figure have rotational symmetry? If so, describe the rotations that map the figure onto itself.

a. rectangle



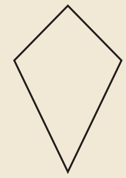
yes, 180° about the intersection of the diagonals

b. regular hexagon



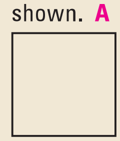
yes, 60°, 120°, 180° about the center

c. kite **no**



Extra Example 3

Identify the line symmetry and rotational symmetry of the square shown. **A**



Ⓐ 4 lines of symmetry, 90° rotational symmetry

Ⓑ 4 lines of symmetry, 45° rotational symmetry

Ⓒ 2 lines of symmetry, 90° rotational symmetry

Ⓓ 2 lines of symmetry, no rotational symmetry

Closing the Lesson

Have students summarize the major points of the lesson and answer the Essential Question: When does a figure have line symmetry?

- **A line of symmetry for a figure is a line in which you can reflect the figure to map it onto itself.**
- **A figure has rotational symmetry if it can be mapped onto itself by a rotation of 180° or less about the center of the figure.**

A figure has line symmetry when it can be mapped onto itself by a reflection in a line.

4 PRACTICE AND APPLY

Assignment Guide

Answers for all exercises available online

Basic:
Day 1:
Exs. 1–8, 10–18, 27–32

Average:
Day 1:
Exs. 1, 2, 4, 5, 7–16, 19–23, 28–35

Advanced:
Day 1:
Exs. 1, 2, 4, 5, 8–14, 19–26*, 29–36*

Block:
Exs. 1, 2, 4, 5, 7–16, 19–23, 28–35 (with next lesson)

Differentiated Instruction

See *Differentiated Instruction Resources* for suggestions on addressing the needs of a diverse classroom.

Homework Check

For a quick check of student understanding of key concepts, go over the following exercises:

Basic: 3, 6, 10, 17, 28

Average: 4, 8, 11, 20, 29

Advanced: 5, 9, 12, 20, 30

Extra Practice

- Student Edition
- Chapter Resource Book: Practice levels A, B, C

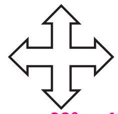
Practice Worksheet

An easily-readable reduced practice page can be found at the beginning of this chapter.

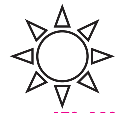
11. Line symmetry, rotational symmetry; there are four lines of symmetry, two passing through the outer opposite pairs of leaves and two passing through the inner opposite pairs of leaves; 90° or 180° about the center.

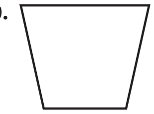
EXAMPLE 2
for Exs. 6–9

ROTATIONAL SYMMETRY **Does the figure have rotational symmetry? If so, *describe* any rotations that map the figure onto itself.**

6. 
yes; 90° or 180° about the center

7. 
yes; 72° or 144° about the center

8. 
yes; 45°, 90°, 135°, or 180° about the center

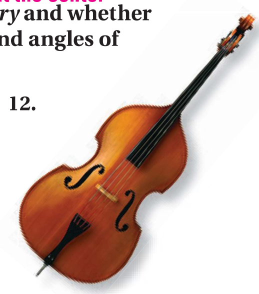
9. 
no

EXAMPLE 3
for Exs. 10–16

SYMMETRY **Determine whether the figure has *line symmetry* and whether it has *rotational symmetry*. Identify all lines of symmetry and angles of rotation that map the figure onto itself.**

10. 

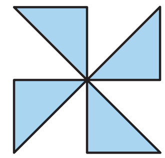
11. 
See margin.

12. 

10. Line symmetry, rotational symmetry; the 5 lines of symmetry run through the center of each seed; 72° or 144° about the center.

12. Line symmetry, no rotational symmetry; the line of symmetry runs through the violin between the 2 center strings.

13. ★ **MULTIPLE CHOICE** Identify the line symmetry and rotational symmetry of the figure at the right. **C**



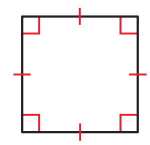
Ⓐ 1 line of symmetry, no rotational symmetry

Ⓑ 1 line of symmetry, 180° rotational symmetry

Ⓒ No lines of symmetry, 90° rotational symmetry

Ⓓ No lines of symmetry, no rotational symmetry

14. ★ **MULTIPLE CHOICE** Which statement best describes the rotational symmetry of a square? **D**



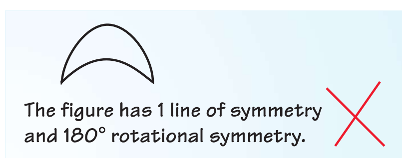
Ⓐ The square has no rotational symmetry.

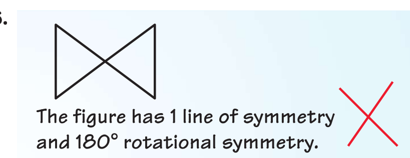
Ⓑ The square has 90° rotational symmetry.

Ⓒ The square has point symmetry.

Ⓓ Both B and C are correct.

ERROR ANALYSIS ***Describe*** **and correct the error made in describing the symmetry of the figure.**

15. 

16. 

15. There is no rotational symmetry; the figure has 1 line of symmetry but no rotational symmetry.

16. There are 2 lines of symmetry; the figure has 2 lines of symmetry and 180° rotational symmetry.

B

DRAWING FIGURES **In Exercises 17–20, use the description to draw a figure. If not possible, write *not possible*.**

17. A quadrilateral with no line of symmetry **See margin.**

18. An octagon with exactly two lines of symmetry **not possible**

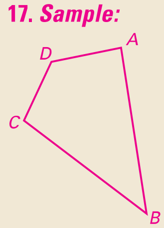
19. A hexagon with no point symmetry **See margin.**

20. A trapezoid with rotational symmetry **not possible**

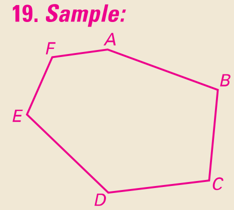
◯ = See **WORKED-OUT SOLUTIONS** in Student Resources

★ = **STANDARDIZED TEST PRACTICE**

17. *Sample:*

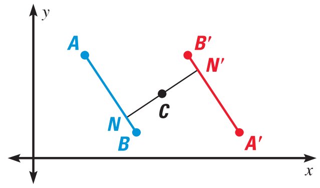


19. *Sample:*



21. ★ **OPEN-ENDED MATH** Draw a polygon with 180° rotational symmetry and with exactly two lines of symmetry. **See margin.**

22. **POINT SYMMETRY** In the graph, $\overline{AB}$ is reflected in the point C to produce the image $\overline{A'B'}$. To make a reflection in a point C for each point N on the preimage, locate N' so that $N'C = NC$ and N' is on $\overleftrightarrow{NC}$. *Explain* what kind of rotation would produce the same image. What kind of symmetry does quadrilateral $AB'A'B$ have?
a rotation of 180° about *C*; rotational symmetry of 180°



23. No; what's on the left and right of the first line would have to be the same as what's on the left and right of the second line which is not possible.

23. ★ **SHORT RESPONSE** A figure has more than one line of symmetry. Can two of the lines of symmetry be parallel? *Explain*.

C

24. **REASONING** How many lines of symmetry does a circle have? How many angles of rotational symmetry does a circle have? *Explain*. **See margin.**

25. **VISUAL REASONING** How many planes of symmetry does a cube have? **9 planes**

26. **CHALLENGE** What can you say about the rotational symmetry of a regular polygon with n sides? *Explain*.
The regular polygon would have rotational symmetry about the center of the *n*-gon and the smallest angle of rotation would be $\frac{360°}{n}$.

PROBLEM SOLVING

EXAMPLES A **1 and 2** for Exs. 27–30

WORDS Identify the line symmetry and rotational symmetry (if any) of each word.

27. MOW   28. RADAR   29. OHIO   30. pod

No line symmetry, it has rotational symmetry of 180° about the center of o.

27. No line symmetry, rotational symmetry of 180° about the center of the letter O.

28. No line symmetry, no rotational symmetry

29. It has a line of symmetry passing horizontally through the center of each O, no rotational symmetry.

KALEIDOSCOPES In Exercises 31–33, use the following information about kaleidoscopes.

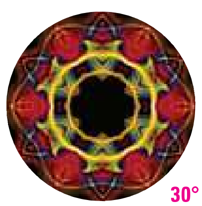
Inside a kaleidoscope, two mirrors are placed next to each other to form a V, as shown at the right. The angle between the mirrors determines the number of lines of symmetry in the image. Use the formula $n(m\angle 1) = 180°$ to find the measure of $\angle 1$ between the mirrors or the number n of lines of symmetry in the image.

B

Calculate the angle at which the mirrors must be placed for the image of a kaleidoscope to make the design shown.

31.

22.5°

32.

30°

33.

15°

Avoiding Common Errors

Exercise 9 Students may think this figure can be rotated 180° onto itself. Have them copy it onto tracing paper and rotate it so they can see the image does not match the original figure.

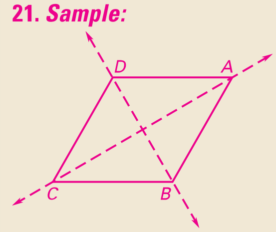
Mathematical Reasoning

Exercise 25 Challenge students to devise models to illustrate the planes of symmetry of a cube.

Internet Reference

Exercise 35 More information about the Castillo de San Marcos can be found at www.nps.gov/casa

21. *Sample:*



24. Infinitely many; infinitely many; any line passing through the center of the circle is a line of symmetry and any rotation about the center is rotational symmetry. There are an infinite number in both cases.

9.7 Identify and Perform Dilations

Before You used a coordinate rule to draw a dilation.

Now You will use drawing tools and matrices to draw dilations.

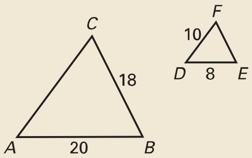
Why? So you can determine the scale factor of a photo, as in Ex. 37.

1 PLAN AND PREPARE

Warm-Up Exercises

Also available online

If $\triangle ABC \sim \triangle DEF$, find each value.



1. EF $\frac{36}{5}$ **2.** AC 25

3. scale factor $\frac{5}{2}$

Notetaking Guide

Available online

Promotes interactive learning and notetaking skills.

Pacing

Basic: 1 day

Average: 1 day

Advanced: 1 day

Block: 0.5 block with previous lesson

• See *Teaching Guide/Lesson Plan.*

2 FOCUS AND MOTIVATE

Essential Question

Big Idea 1

How do you use matrices to draw a dilation? **Tell students they will learn how to answer this question by studying how to use matrices to find the coordinates of the image of a dilation.**

Key Vocabulary
- **scalar multiplication**
- **dilation**
- **reduction**
- **enlargement**

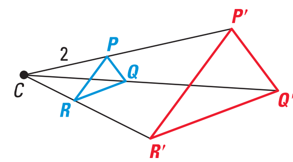
COMMON CORE

CC.9-12.G.SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor; a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged. b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

A dilation is a transformation in which the original figure and its image are similar.

A dilation with center C and scale factor k maps every point P in a figure to a point P' so that one of the following statements is true:

- If P is not the center point C, then the image point P' lies on $\overrightarrow{CP}$. The scale factor k is a positive number such that $k = \frac{CP'}{CP}$ and $k \neq 1$, or



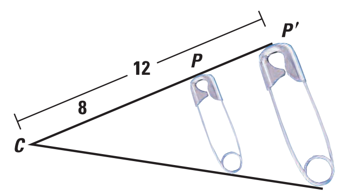
- If P is the center point C, then $P = P'$.

The dilation is a *reduction* if $0 < k < 1$ and it is an *enlargement* if $k > 1$.

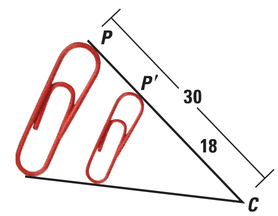
EXAMPLE 1 Identify dilations

Find the scale factor of the dilation. Then tell whether the dilation is a *reduction* or an *enlargement*.

a.



b.



Solution

a. Because $\frac{CP'}{CP} = \frac{12}{8}$, the scale factor is $k = \frac{3}{2}$. The image P' is an enlargement.

b. Because $\frac{CP'}{CP} = \frac{18}{30}$, the scale factor is $k = \frac{3}{5}$. The image P' is a reduction.

Animated Geometry at my.hrw.com

COMMON CORE **Standards for Mathematical Content** High School

CC.9-12.G.CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

CC.9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

CC.9-12.G.SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor: a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged. b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

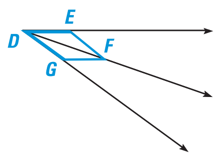
CC.9-12.G.SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

EXAMPLE 2 Draw a dilation

Draw and label ▱*DEFG*. Then construct a dilation of ▱*DEFG* with point *D* as the center of dilation and a scale factor of 2.

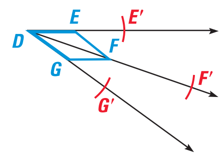
Solution

STEP 1



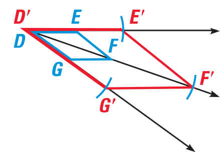
Draw *DEFG*. Draw rays from *D* through vertices *E*, *F*, and *G*.

STEP 2



Open the compass to the length of $\overline{DE}$. Locate *E′* on $\overrightarrow{DE}$ so $DE' = 2(DE)$. Locate *F′* and *G′* the same way.

STEP 3



Add a second label *D′* to point *D*. Draw the sides of *D′E′F′G′*.

✓ GUIDED PRACTICE for Examples 1 and 2

1. In a dilation, $CP' = 3$ and $CP = 12$. Tell whether the dilation is a *reduction* or an *enlargement* and find its scale factor. **reduction, $\frac{1}{4}$**

2. Draw and label △*RST*. Then construct a dilation of △*RST* with *R* as the center of dilation and a scale factor of 3. **See margin.**

MATRICES **Scalar multiplication** is the process of multiplying each element of a matrix by a real number or *scalar*.

EXAMPLE 3 Scalar multiplication

Simplify the product: $4\begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & -3 \end{bmatrix}$.

Solution

$$4\begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 4(3) & 4(0) & 4(1) \\ 4(2) & 4(-1) & 4(-3) \end{bmatrix}$$ **Multiply each element in the matrix by 4.**

$$= \begin{bmatrix} 12 & 0 & 4 \\ 8 & -4 & -12 \end{bmatrix}$$ **Simplify.**

✓ GUIDED PRACTICE for Example 3

Simplify the product.

3. $5\begin{bmatrix} 2 & 1 & -10 \\ 3 & -4 & 7 \end{bmatrix}$ $\begin{bmatrix} 10 & 5 & -50 \\ 15 & -20 & 35 \end{bmatrix}$

4. $-2\begin{bmatrix} -4 & 1 & 0 \\ 9 & -5 & -7 \end{bmatrix}$ $\begin{bmatrix} 8 & -2 & 0 \\ -18 & 10 & 14 \end{bmatrix}$

Motivating the Lesson

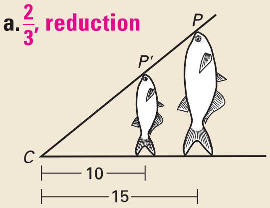
Tell students that enlarging a photograph involves creating a dilation with a scale factor greater than 1. Tell them that in this lesson they will learn two methods for creating dilations of geometric figures.

3 TEACH

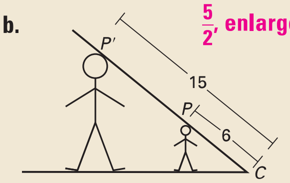
Extra Example 1

Find the scale factor of the dilation. Then tell whether the dilation is a reduction or an enlargement.

a. $\frac{2}{3}$, **reduction**



b. $\frac{5}{2}$, **enlargement**

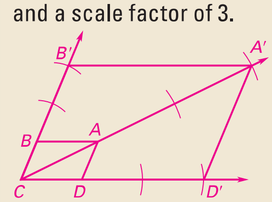


Animated Geometry
my.hrw.com

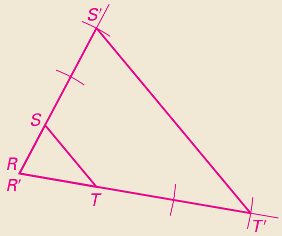
An **Animated Geometry** activity is available online for **Example 1.** This activity is also part of **Power Presentations.**

Extra Example 2

Draw and label parallelogram *ABCD*. Then construct a dilation of *ABCD* with point *C* as the center and a scale factor of 3.



2. Sample:



Extra Example 3

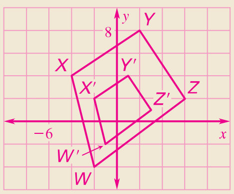
Simplify the product:

$7\begin{bmatrix} 2 & -1 & 5 \\ 3 & 4 & -2 \end{bmatrix}$. $\begin{bmatrix} 14 & -7 & 35 \\ 21 & 28 & -14 \end{bmatrix}$

Extra Example 4

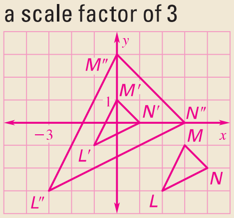
The vertices of *XYZW* are *X*(−4, 4), *Y*(2, 8), *Z*(6, 2), and *W*(−2, −4). Use scalar multiplication to find the image of *XYZW* after a dilation with its center at the origin and a scale factor of $\frac{1}{2}$. Graph *XYZW* and its image.

$\begin{matrix} X' & Y' & Z' & W' \end{matrix}$
$\begin{bmatrix} -2 & 1 & 3 & -1 \\ 2 & 4 & 1 & -2 \end{bmatrix}$



Extra Example 5

The vertices of △*LMN* are *L*(2, −3), *M*(3, −1), and *N*(4, −2). Find the image of △*LMN* after the given composition.
Translation: (*x*, *y*) → (*x* − 3, *y* + 2)
Dilation: centered at the origin with a scale factor of 3



Closing the Lesson

Have students summarize the major points of the lesson and answer the Essential Question: How do you use matrices to draw a dilation?

- **A dilation results in an image similar to the preimage. Lines through corresponding points are concurrent.**
- **A dilation can be found by scalar multiplication.**

Multiply each element of the matrix for the preimage by the scale factor. Then use the resulting matrix to draw the image.

DILATIONS USING MATRICES You can use scalar multiplication to represent a dilation centered at the origin in the coordinate plane. To find the image matrix for a dilation centered at the origin, use the scale factor as the scalar.

EXAMPLE 4 Use scalar multiplication in a dilation

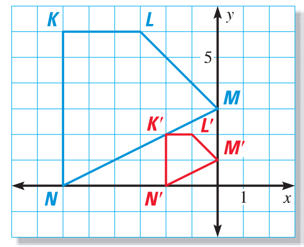
The vertices of quadrilateral *KLMN* are *K*(−6, 6), *L*(−3, 6), *M*(0, 3), and *N*(−6, 0). Use scalar multiplication to find the image of *KLMN* after a dilation with its center at the origin and a scale factor of $\frac{1}{3}$. Graph *KLMN* and its image.

Solution

$$\frac{1}{3}\begin{bmatrix} -6 & -3 & 0 & -6 \\ 6 & 6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & -2 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

Columns of the polygon matrix: *K*, *L*, *M*, *N*; columns of the image matrix: *K*′, *L*′, *M*′, *N*′.

Scale factor ($\frac{1}{3}$) — Polygon matrix — Image matrix



EXAMPLE 5 Find the image of a composition

The vertices of △*ABC* are *A*(−4, 1), *B*(−2, 2), and *C*(−2, 1). Find the image of △*ABC* after the given composition.

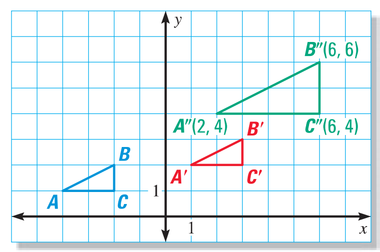
Translation: (*x*, *y*) → (*x* + 5, *y* + 1)
Dilation: centered at the origin with a scale factor of 2

Solution

STEP 1 **Graph** the preimage △*ABC* on the coordinate plane.

STEP 2 **Translate** △*ABC* 5 units to the right and 1 unit up. Label it △*A*′*B*′*C*′.

STEP 3 **Dilate** △*A*′*B*′*C*′ using the origin as the center and a scale factor of 2 to find △*A*″*B*″*C*″.



✓ GUIDED PRACTICE for Examples 4 and 5

5. The vertices of △*RST* are *R*(1, 2), *S*(2, 1), and *T*(2, 2). Use scalar multiplication to find the vertices of △*R*′*S*′*T*′ after a dilation with its center at the origin and a scale factor of 2. ***R*′(2, 4), *S*′(4, 2), *T*′(4, 4)**

6. A segment has the endpoints *C*(−1, 1) and *D*(1, 1). Find the image of $\overline{CD}$ after a 90° rotation about the origin followed by a dilation with its center at the origin and a scale factor of 2. ***C*′(−2, −2), *D*′(−2, 2)**

9.7 EXERCISES

HOMEWORK KEY
○ = See **WORKED-OUT SOLUTIONS** Exs. 7, 19, and 35
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 24, 25, 27, 29, and 38

SKILL PRACTICE

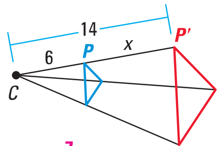
A

1. **VOCABULARY** What is a *scalar*? **a real number**

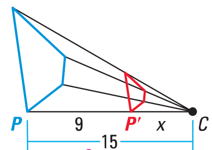
2. ★ **WRITING** If you know the scale factor, *explain* how to determine if an image is larger or smaller than the preimage. **If the scale factor is greater than 1, the image is larger. If the scale factor is between 0 and 1, the image is smaller.**

EXAMPLE 1 Exs. 3–6

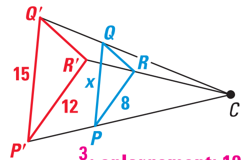
IDENTIFYING DILATIONS **Find the scale factor. Tell whether the dilation is a *reduction* or an *enlargement*. Find the value of *x*.**

3. 

$\frac{7}{3}$; enlargement; 8

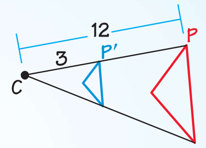
4. 

$\frac{2}{5}$; reduction; 6

5. 

$\frac{3}{2}$; enlargement; 10

6. **ERROR ANALYSIS** *Describe* and correct the error in finding the scale factor k of the dilation. **The ratio should be $\frac{CP'}{CP}$; $k = \frac{CP'}{CP} = \frac{3}{12} = \frac{1}{4}$.**

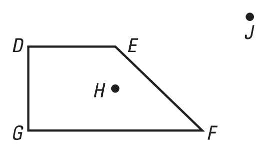


$k = \frac{CP}{CP'}$

$k = \frac{12}{3} = 4$ ✗

EXAMPLE 2 for Exs. 7–14

CONSTRUCTION **Copy the diagram. Then draw the given dilation.** **7–14. See margin.**



7. Center H; $k = 2$
8. Center H; $k = 3$
9. Center J; $k = 2$
10. Center F; $k = 2$
11. Center J; $k = \frac{1}{2}$
12. Center F; $k = \frac{3}{2}$
13. Center D; $k = \frac{3}{2}$
14. Center G; $k = \frac{1}{2}$

EXAMPLE 3 for Exs. 15–17

SCALAR MULTIPLICATION **Simplify the product.** **15–17. See margin.**

15. $4\begin{bmatrix} 3 & 7 & 4 \\ 0 & 9 & -1 \end{bmatrix}$

16. $-5\begin{bmatrix} -2 & -5 & 7 & 3 \\ 1 & 4 & 0 & -1 \end{bmatrix}$

17. $9\begin{bmatrix} 0 & 3 & 2 \\ -1 & 7 & 0 \end{bmatrix}$

EXAMPLE 4 for Exs. 18–20

DILATIONS WITH MATRICES **Find the image matrix that represents a dilation of the polygon centered at the origin with the given scale factor. Then graph the polygon and its image.** **18–20. See margin.**

18. $\begin{matrix} D & E & F \end{matrix}$ $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix}$; $k = 2$

19. $\begin{matrix} G & H & J \end{matrix}$ $\begin{bmatrix} -2 & 0 & 6 \\ -4 & 2 & -2 \end{bmatrix}$; $k = \frac{1}{2}$

20. $\begin{matrix} J & L & M & N \end{matrix}$ $\begin{bmatrix} -6 & -3 & 3 & 3 \\ 0 & 3 & 0 & -3 \end{bmatrix}$; $k = \frac{2}{3}$

4 PRACTICE AND APPLY

Assignment Guide

Answers for all exercises available online

Basic:
Day 1:
Exs. 1–10, 15–25, 33–38

Average:
Day 1:
Exs. 1, 2, 4–6, 10–12, 16, 17, 19–30, 34–41

Advanced:
Day 1:
Exs. 1, 2, 4, 5, 13–16, 20–24 even, 25–32*, 35–42*

Block:
Exs. 1, 2, 4–6, 10–12, 16, 17, 19–30, 34–41 (with previous lesson)

Differentiated Instruction

See *Differentiated Instruction Resources* for suggestions on addressing the needs of a diverse classroom.

Homework Check

For a quick check of student understanding of key concepts, go over the following exercises:

Basic: 8, 15, 18, 21, 33
Average: 10, 16, 20, 22, 34
Advanced: 14, 16, 20, 22, 35

Extra Practice

- Student Edition
- Chapter Resource Book: Practice levels A, B, C

Practice Worksheet

An easily-readable reduced practice page can be found at the beginning of this chapter.

7–14. See Additional Answers.

15. $\begin{bmatrix} 12 & 28 & 16 \\ 0 & 36 & -4 \end{bmatrix}$

16. $\begin{bmatrix} 10 & 25 & -35 & -15 \\ -5 & -20 & 0 & 5 \end{bmatrix}$

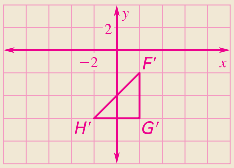
17. $\begin{bmatrix} 0 & 27 & 18 \\ -9 & 63 & 0 \end{bmatrix}$

18–20. See Additional Answers.

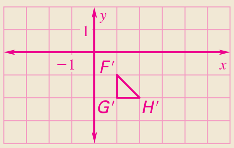
Mathematical Reasoning

Exercise 29 Students may have difficulty with this exercise. Since *x* is multiplied in choice A and *y* is multiplied in choice B, they may not realize that both coordinates must be multiplied by the same number to be a dilation. Have them experiment by graphing a point and its image under the transformations in choices A and B to see whether the line through the image and preimage passes through the origin.

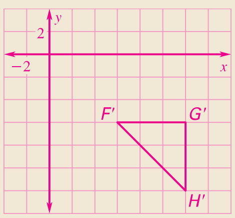
21.



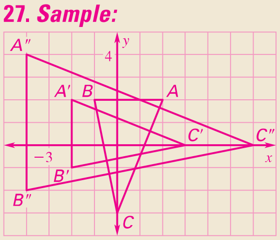
22.



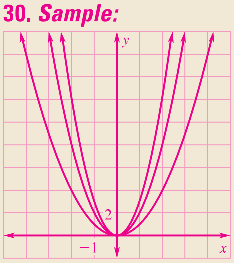
23.



27. Sample:



30. Sample:



EXAMPLE 5 B
for Exs. 21–23

COMPOSING TRANSFORMATIONS **The vertices of $\triangle FGH$ are $F(-2, -2)$, $G(-2, -4)$, and $H(-4, -4)$. Graph the image of the triangle after a composition of the transformations in the order they are listed.** **21–23. See margin.**

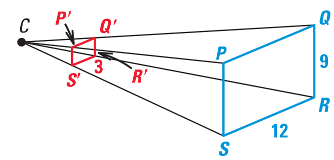
21. **Translation:** $(x, y) \rightarrow (x + 3, y + 1)$
Dilation: centered at the origin with a scale factor of 2

22. **Dilation:** centered at the origin with a scale factor of $\frac{1}{2}$
Reflection: in the y-axis

23. **Rotation:** 90° about the origin
Dilation: centered at the origin with a scale factor of 3

24. ★ **WRITING** Is a composition of transformations that includes a dilation ever an isometry? *Explain.* **No; dilation does not preserve length.**

25. ★ **MULTIPLE CHOICE** In the diagram, the center of the dilation of ▱*PQRS* is point *C*. The length of a side of ▱*P′Q′R′S′* is what percent of the length of the corresponding side of ▱*PQRS*? **A**



Ⓐ 25% Ⓑ 33% Ⓒ 300% Ⓓ 400%

26. **REASONING** The distance from the center of dilation to the image of a point is shorter than the distance from the center of dilation to the preimage. Is the dilation a *reduction* or an *enlargement*? *Explain.*
Reduction; the ratio of corresponding image to preimage lengths is between 0 and 1.

27. ★ **SHORT RESPONSE** Graph a triangle in the coordinate plane. Rotate the triangle, then dilate it. Then do the same dilation first, followed by the rotation. In this composition of transformations, does it matter in which order the triangle is dilated and rotated? *Explain* your answer.
See margin for art; no; the result is the same.

28. **REASONING** A dilation maps $A(5, 1)$ to $A'(2, 1)$ and $B(7, 4)$ to $B'(6, 7)$.

a. Find the scale factor of the dilation. **2**

b. Find the center of the dilation. **(8, 1)**

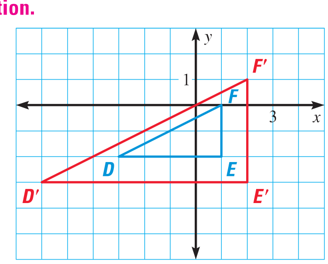
29. ★ **MULTIPLE CHOICE** Which transformation of (x, y) is a dilation? **C**

Ⓐ $(3x, y)$ Ⓑ $(-x, 3y)$ Ⓒ $(3x, 3y)$ Ⓓ $(x + 3, y + 3)$

30. *xy* **ALGEBRA** Graph parabolas of the form $y = ax^2$ using three different values of a. Describe the effect of changing the value of a. Is this a dilation? *Explain.* **See margin for art; as *a* increases the parabola becomes steeper; no; there is no center of dilation.**

31. Sample answer: **Dilation with its center at the origin and a scale factor of 2 followed by a translation $(x, y) \rightarrow (x, y + 1)$** C

31. **REASONING** *Verify* that the two figures in the graph are similar by describing a composition of transformations, involving a dilation then a translation, that maps $\triangle DEF$ to $\triangle D'E'F'$.



32. **CHALLENGE** $\triangle ABC$ has vertices $A(4, 2)$, $B(4, 6)$, and $C(7, 2)$. Find the vertices that represent a dilation of $\triangle ABC$ centered at $(4, 0)$ with a scale factor of 2. **$A'(4, 4)$, $B'(4, 12)$, $C'(10, 4)$**

○ = See **WORKED-OUT SOLUTIONS** in Student Resources

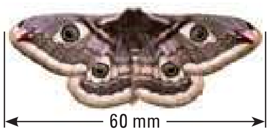
★ = **STANDARDIZED TEST PRACTICE**

PROBLEM SOLVING

EXAMPLE 1 A
for Exs. 33–35

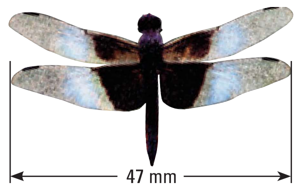
SCIENCE **You are using magnifying glasses. Use the length of the insect and the magnification level to determine the length of the image seen through the magnifying glass.**

33. Emperor moth **300 mm**
magnification 5x



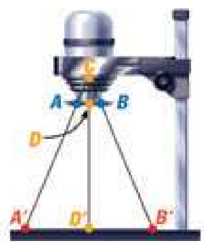
34. Ladybug **45 mm**
magnification 10x

(35.) Dragonfly **940 mm**
magnification 20x



36. MURALS A painter sketches plans for a mural. The plans are 2 feet by 4 feet. The actual mural will be 25 feet by 50 feet. What is the scale factor? Is this a dilation? *Explain.* **$\frac{25}{2}$; yes; the center point is (0, 0) with scale factor $\frac{25}{2}$.**

B **37. PHOTOGRAPHY** By adjusting the distance between the negative and the enlarged print in a photographic enlarger, you can make prints of different sizes. In the diagram shown, you want the enlarged print to be 9 inches wide ($A'B'$). The negative is 1.5 inches wide (AB), and the distance between the light source and the negative is 1.75 inches (CD).

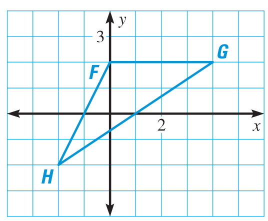


a. What is the scale factor of the enlargement? **$\frac{6}{1}$**

b. What is the distance between the negative and the enlarged print? **8.75 in.**

38. ★ **OPEN-ENDED MATH** Graph a polygon in a coordinate plane. Draw a figure that is similar but not congruent to the polygon. What is the scale factor of the dilation you drew? What is the center of the dilation? **See margin.**

39. MULTI-STEP PROBLEM Use the figure at the right.



a. Write a polygon matrix for the figure. Multiply the matrix by the scalar −2. **a–c. See margin.**

b. Graph the polygon represented by the new matrix.

c. Repeat parts (a) and (b) using the scalar $-\frac{1}{2}$.

d. Make a conjecture about the effect of multiplying a polygon matrix by a negative scale factor.
A reflection in both the *x*-axis and *y*-axis occurs as well as dilation.

40. AREA You have an 8 inch by 10 inch photo.

a. What is the area of the photo? **80 in.^{2}**

b. You photocopy the photo at 50%. What are the dimensions of the image? What is the area of the image? **4 in. by 5 in.; 20 in.^{2}**

c. How many images of this size would you need to cover the original photo? **4 images**

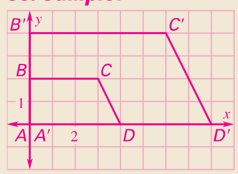
Mathematical Reasoning

Exercise 39 Have students rotate the given figure by 180° and then dilate the image with a scale factor of 2. How does the final image compare with the image obtained in parts (a) and (b)? **The images are the same.**

Internet Reference

Exercise 33 More information about the Emperor moth can be found at www.arkive.org/emperor-moth/saturnia-pavonia/info.html

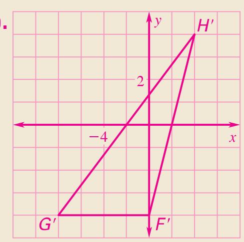
38. Sample:



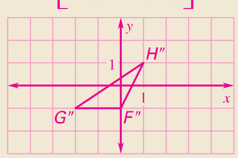
2; (0, 0)

39a. $\begin{matrix} F & G & H \end{matrix}$ $\begin{bmatrix} 0 & 4 & -2 \\ 2 & 2 & -2 \end{bmatrix}$; $\begin{matrix} F' & G' & H' \end{matrix}$ $\begin{bmatrix} 0 & -8 & 4 \\ -4 & -4 & 4 \end{bmatrix}$

39b.



39c. $\begin{matrix} F'' & G'' & H'' \end{matrix}$ $\begin{bmatrix} 0 & -2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$;

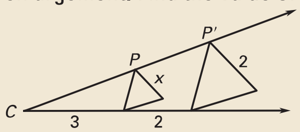


5 ASSESS AND RETEACH

Daily Homework Quiz

Also available online

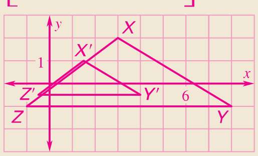
1. Find the scale factor. Tell whether the dilation is a reduction or an enlargement. Find the value of x.



$\frac{5}{3}$; enlargement; 1.2

2. Find the image matrix that represents a dilation of $\triangle XYZ$ with vertices $X(3, 2)$, $Y(8, -1)$, and $Z(-1, -1)$ centered at the origin with scale factor $\frac{1}{2}$. Then graph the polygon and its image.

$$\begin{array}{ccc} X' & Y' & Z' \end{array}$$
$$\begin{bmatrix} 1.5 & 4 & -0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix}$$



Online Quiz

Available at **my.hrw.com**

Diagnosis/Remediation

- Practice A, B, C in Chapter Resource Book
- Study Guide in Chapter Resource Book
- Practice Workbook
- @HomeTutor

Challenge

Additional challenge is available in the Chapter Resource Book.

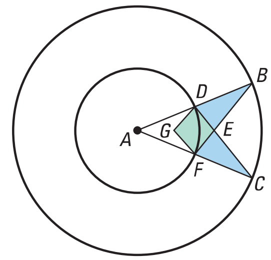
Quiz

An easily-readable reduced copy of the quiz from the Assessment Book can be found at the beginning of this chapter.

42. See Additional Answers.

41. REASONING You put a reduction of a page on the original page. *Explain* why there is a point that is in the same place on both pages.
It's the center point of the dilation.

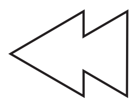
C **42. CHALLENGE** Draw two concentric circles with center A. Draw $\overline{AB}$ and $\overline{AC}$ to the larger circle to form a 45° angle. Label points D and F, where $\overline{AB}$ and $\overline{AC}$ intersect the smaller circle. Locate point E at the intersection of $\overline{BF}$ and $\overline{CD}$. Choose a point G and draw quadrilateral $DEFG$. Use A as the center of the dilation and a scale factor of $\frac{1}{2}$. Dilate $DEFG$, $\triangle DBE$, and $\triangle CEF$ two times. Sketch each image on the circles. *Describe* the result.
See margin for art; kaleidoscope image.



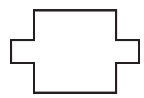
QUIZ

Determine whether the figure has *line symmetry* and/or *rotational symmetry*. Identify the number of lines of symmetry and/or the rotations that map the figure onto itself.

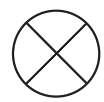
1.



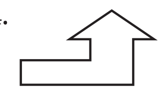
2.



3.



4.



1. line symmetry, no rotational symmetry; one

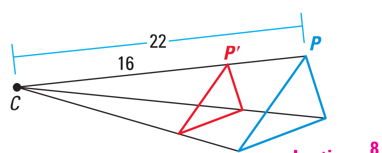
2. line symmetry, rotational symmetry; two, 180° about the center

3. line symmetry, rotational symmetry; four, 90° or 180° about the center

4. no line symmetry, no rotational symmetry

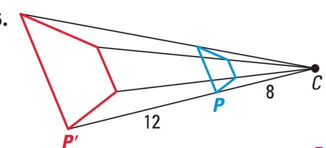
Tell whether the dilation is a *reduction* or an *enlargement* and find its scale factor.

5.



reduction, $\frac{8}{11}$

6.



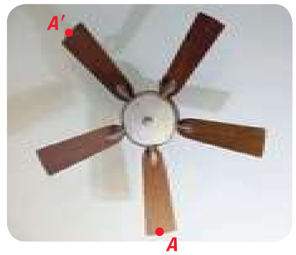
enlargement, $\frac{5}{2}$

7. The vertices of $\triangle RST$ are $R(3, 1)$, $S(0, 4)$, and $T(-2, 2)$. Use scalar multiplication to find the image of the triangle after a dilation centered at the origin with scale factor $4\frac{1}{2}$. **$R'\left(13\frac{1}{2}, 4\frac{1}{2}\right)$, $S'(0, 18)$, $T'(-9, 9)$**

MIXED REVIEW *of Problem Solving*

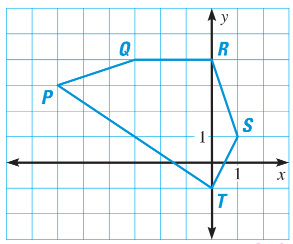
MATHEMATICAL PRACTICES — Make sense of problems and persevere in solving them.

1. GRIDDED ANSWER What is the angle of rotation, in degrees, that maps A to A' in the photo of the ceiling fan below? **216**



2. SHORT RESPONSE The vertices of $\triangle DEF$ are $D(-3, 2)$, $E(2, 3)$, and $F(3, -1)$. Graph $\triangle DEF$. Rotate $\triangle DEF$ 90° about the origin. Compare the slopes of corresponding sides of the preimage and image. What do you notice? **See margin.**

3. MULTI–STEP PROBLEM Use pentagon $PQRST$ shown below.



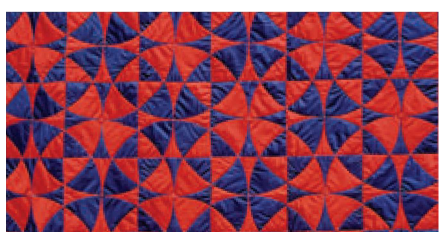
a–c. See margin.

a. Write the polygon matrix for $PQRST$.

b. Find the image matrix for a 270° rotation about the origin.

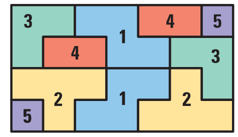
c. Graph the image.

4. SHORT RESPONSE *Describe* the transformations that can be found in the quilt pattern below.



translation, rotation and reflection

5. MULTI-STEP PROBLEM The diagram shows the pieces of a puzzle.



a. Which pieces are translated? **4, 5**

b. Which pieces are reflected? **1**

c. Which pieces are glide reflected? **2, 3**

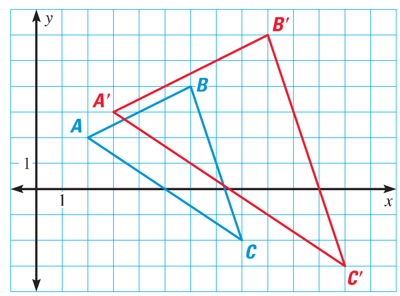
6. OPEN-ENDED Draw a figure that has the given type(s) of symmetry. **a–c. See margin.**

a. Line symmetry only

b. Rotational symmetry only

c. Both line symmetry and rotational symmetry

7. EXTENDED RESPONSE In the graph below, $\triangle A'B'C'$ is a dilation of $\triangle ABC$.



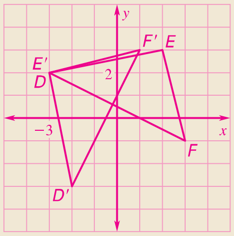
a. Is the dilation a *reduction* or an *enlargement*? **enlargement**

b. What is the scale factor? *Explain* your steps. **b–d. See margin.**

c. What is the polygon matrix? What is the image matrix?

d. When you perform a composition of a dilation and a translation on a figure, does order matter? *Justify* your answer using the translation $(x, y) \to (x + 3, y - 1)$ and the dilation of $\triangle ABC$.

2.

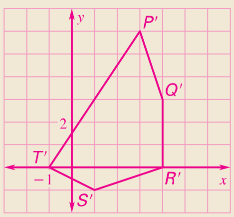


They are negative reciprocals of each other.

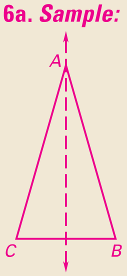
3a. $\begin{matrix} P & Q & R & S & T \end{matrix}$ $\begin{bmatrix} -6 & -3 & 0 & 1 & 0 \\ 3 & 4 & 4 & 1 & -1 \end{bmatrix}$

3b. $\begin{matrix} P' & Q' & R' & S' & T' \end{matrix}$ $\begin{bmatrix} 3 & 4 & 4 & 1 & -1 \\ 6 & 3 & 0 & -1 & 0 \end{bmatrix}$

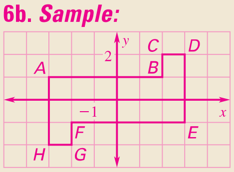
3c.



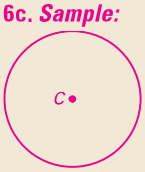
6a. *Sample:*



6b. *Sample:*



6c. *Sample:*



7b. $\frac{3}{2}$; find $\frac{A'B'}{AB} = \frac{3\sqrt{5}}{2\sqrt{5}} = \frac{3}{2}$

7c. $\begin{matrix} A & B & C \end{matrix}$ $\begin{bmatrix} 2 & 6 & 8 \\ 2 & 4 & -2 \end{bmatrix}$; $\begin{matrix} A' & B' & C' \end{matrix}$ $\begin{bmatrix} 3 & 9 & 12 \\ 3 & 6 & -3 \end{bmatrix}$

7d. yes; $\begin{bmatrix} 2 & 6 & 8 \\ 2 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 5 & 9 & 11 \\ 1 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 7.5 & 13.5 & 16.5 \\ 1.5 & 4.5 & -4.5 \end{bmatrix}$; $\begin{bmatrix} 2 & 6 & 8 \\ 2 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 3 & 9 & 12 \\ 3 & 6 & -3 \end{bmatrix} \to \begin{bmatrix} 6 & 12 & 15 \\ 2 & 5 & -4 \end{bmatrix}$

9 CHAPTER SUMMARY

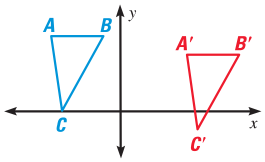
BIG IDEAS *For Your Notebook*

Big Idea 1

Performing Congruence and Similarity Transformations

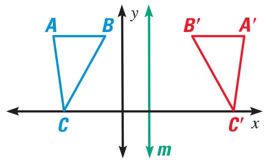
Translation

Translate a figure right or left, up or down.



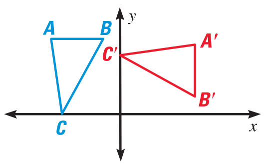
Reflection

Reflect a figure in a line.



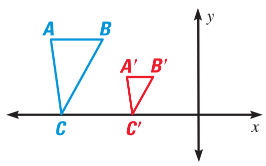
Rotation

Rotate a figure about a point.



Dilation

Dilate a figure to change the size but not the shape.

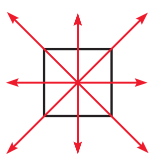


You can combine congruence and similarity transformations to make a composition of transformations, such as a glide reflection.

Big Idea 2

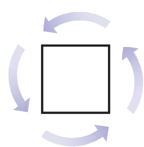
Making Real-World Connections to Symmetry and Tessellations

Line symmetry



4 lines of symmetry

Rotational symmetry



90° rotational symmetry

Big Idea 3

Applying Matrices and Vectors in Geometry

You can use matrices to represent points and polygons in the coordinate plane. Then you can use matrix addition to represent translations, matrix multiplication to represent reflections and rotations, and scalar multiplication to represent dilations. You can also use vectors to represent translations.

Additional Resources

The following resources are available to help review the materials in this chapter.

Chapter Resource Book

- Chapter Review Games and Activities
- Cumulative Practice

Student Resources in Spanish

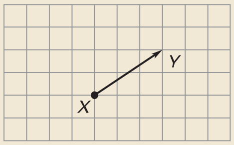
@HomeTutor

Vocabulary Practice

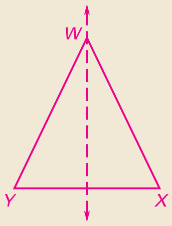
Vocabulary practice is available at **my.hrw.com**

Extra Example 1

Name the vector and write its component form. $\overrightarrow{XY}$, $\langle 3, 2\rangle$



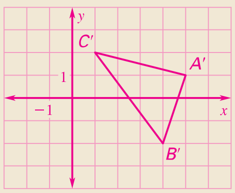
2. ***Sample:***



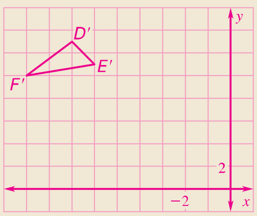
3. Count the number of rows, *n*, and the number of columns, *m*. The dimensions are $n \times m$.

Sample answer: $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & 7 \end{bmatrix}$ **is 2×3.**

6.



7.



9 CHAPTER REVIEW

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my.hrw.com
- Multi-Language Glossary
- Vocabulary practice

REVIEW KEY VOCABULARY

For a list of postulates and theorems, see p. PT2.

- image
- preimage
- isometry
- vector
  initial point, terminal point, horizontal component, vertical component
- component form
- matrix
- element
- dimensions
- line of reflection
- center of rotation
- angle of rotation
- glide reflection
- composition of transformations
- line symmetry
- line of symmetry
- rotational symmetry
- center of symmetry
- scalar multiplication

VOCABULARY EXERCISES

1. Copy and complete: A(n) __?__ is a transformation that preserves lengths. **isometry**
2. Draw a figure with exactly one line of symmetry. **See margin.**
3. **WRITING** *Explain* how to identify the dimensions of a matrix. Include an example with your explanation. **See margin.**

Match the point with the appropriate name on the vector.

4. *T* **B**
5. *H* **A**

A. Initial point
B. Terminal point

REVIEW EXAMPLES AND EXERCISES

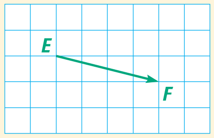
Use the review examples and exercises below to check your understanding of the concepts you have learned in each lesson of this chapter.

9.1 Translate Figures and Use Vectors

EXAMPLE

Name the vector and write its component form.

The vector is $\overrightarrow{EF}$. From initial point *E* to terminal point *F*, you move 4 units right and 1 unit down. So, the component form is $\langle 4, 1\rangle$.



EXERCISES

EXAMPLES 1 and 4 for Exs. 6–7

6. The vertices of $\triangle ABC$ are $A(2, 3)$, $B(1, 0)$, and $C(-2, 4)$. Graph the image of $\triangle ABC$ after the translation $(x, y) \rightarrow (x + 3, y - 2)$. **See margin.**
7. The vertices of $\triangle DEF$ are $D(-6, 7)$, $E(-5, 5)$, and $F(-8, 4)$. Graph the image of $\triangle DEF$ after the translation using the vector $\langle -1, 6\rangle$. **See margin.**

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Chapter Review Practice

9.2 Use Properties of Matrices

EXAMPLE

Add $\begin{bmatrix} -9 & 12 \\ 5 & -4 \end{bmatrix} + \begin{bmatrix} 20 & 18 \\ 11 & 25 \end{bmatrix}$.

These two matrices have the same dimensions, so you can perform the addition. To add matrices, you add corresponding elements.

$$\begin{bmatrix} -9 & 12 \\ 5 & -4 \end{bmatrix} + \begin{bmatrix} 20 & 18 \\ 11 & 25 \end{bmatrix} = \begin{bmatrix} -9+20 & 12+18 \\ 5+11 & -4+25 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 16 & 21 \end{bmatrix}$$

EXERCISES

EXAMPLE 3 for Exs. 8–9

Find the image matrix that represents the translation of the polygon. Then graph the polygon and its image. **8, 9. See margin.**

8. $\begin{bmatrix} A & B & C \\ 2 & 8 & 1 \\ 4 & 3 & 2 \end{bmatrix}$; 5 units up and 3 units left

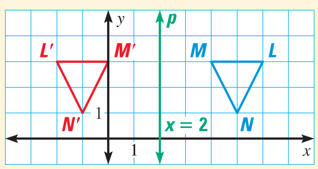
9. $\begin{bmatrix} D & E & F & G \\ -2 & 3 & 4 & -1 \\ 3 & 6 & 4 & -1 \end{bmatrix}$; 2 units down

9.3 Perform Reflections

EXAMPLE

The vertices of $\triangle MLN$ are $M(4, 3)$, $L(6, 3)$, and $N(5, 1)$. Graph the reflection of $\triangle MLN$ in the line p with equation $x = 2$.

Point M is 2 units to the right of p, so its reflection M' is 2 units to the left of p at (0, 3). Similarly, L' is 4 units to the left of p at (−2, 3) and N' is 3 units to the left of p at (−1, 1).

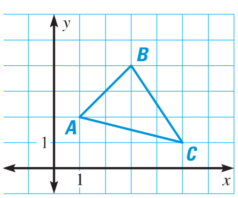


EXERCISES

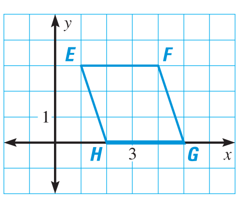
EXAMPLES 1 and 2 for Exs. 10–12

Graph the reflection of the polygon in the given line. **10–12. See margin.**

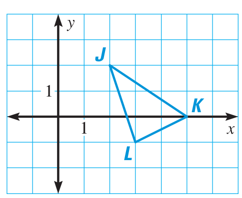
10. $x = 4$



11. $y = 3$



12. $y = x$



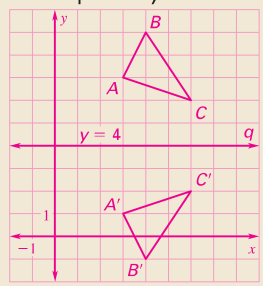
Extra Example 2

Subtract $\begin{bmatrix} 6 & 7 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} 16 & -9 \\ -11 & 3 \end{bmatrix}$.

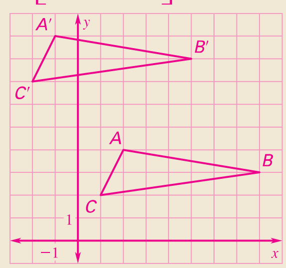
$\begin{bmatrix} -10 & 16 \\ 9 & 2 \end{bmatrix}$

Extra Example 3

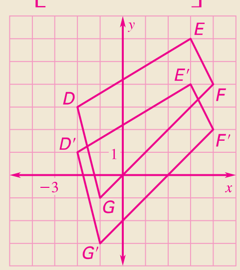
The vertices of $\triangle ABC$ are $A(3, 7)$, $B(4, 9)$, and $C(6, 6)$. Graph the reflection of $\triangle ABC$ in the line q with equation $y = 4$.



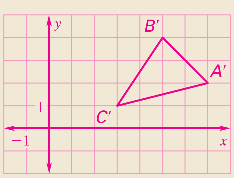
8. $\begin{bmatrix} A' & B' & C' \\ -1 & 5 & -2 \\ 9 & 8 & 7 \end{bmatrix}$;



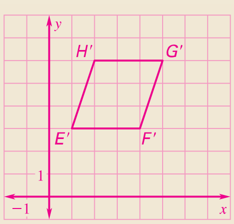
9. $\begin{bmatrix} D' & E' & F' & G' \\ -2 & 3 & 4 & -1 \\ 1 & 4 & 2 & -3 \end{bmatrix}$;



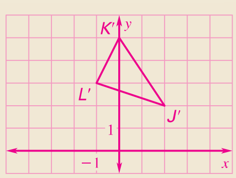
10.



11.

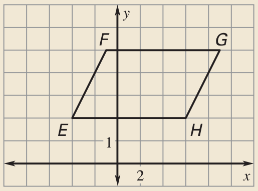


12.



Extra Example 4

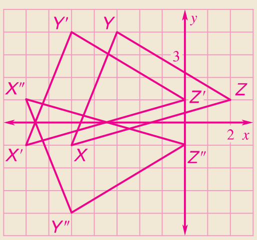
Find the image matrix that represents the 90° rotation of *EFGH* about the origin.



$$\begin{matrix} E' & F' & G' & H' \end{matrix}$$
$$\begin{bmatrix} -2 & -5 & -5 & -2 \\ -4 & -1 & 9 & 6 \end{bmatrix}$$

Extra Example 5

The vertices of $\triangle XYZ$ are $X(-5, -1)$, $Y(-3, 4)$, and $Z(2, 1)$. Graph the image of $\triangle XYZ$ after the glide reflection.
Translation: $(x, y) \to (x - 2, y)$
Reflection: in the *x*-axis

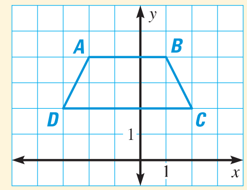


9 CHAPTER REVIEW

9.4 Perform Rotations

EXAMPLE

Find the image matrix that represents the 90° rotation of *ABCD* about the origin.



The polygon matrix for *ABCD* is $\begin{bmatrix} -2 & 1 & 2 & -3 \\ 4 & 4 & 2 & 2 \end{bmatrix}$.

Multiply by the matrix for a 90° rotation.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} -2 & 1 & 2 & -3 \\ 4 & 4 & 2 & 2 \end{bmatrix}} = \overset{\begin{matrix} A' & B' & C' & D' \end{matrix}}{\begin{bmatrix} -4 & -4 & -2 & -2 \\ -2 & 1 & 2 & -3 \end{bmatrix}}$$

EXERCISES

EXAMPLE 3 for Exs. 13–14

Find the image matrix that represents the given rotation of the polygon about the origin. Then graph the polygon and its image. 13, 14. See margin.

13. $\overset{\begin{matrix} Q & R & S \end{matrix}}{\begin{bmatrix} 3 & 4 & 1 \\ 0 & 5 & -2 \end{bmatrix}}$; 180°

14. $\overset{\begin{matrix} L & M & N & P \end{matrix}}{\begin{bmatrix} -1 & 3 & 5 & -2 \\ 6 & 5 & 0 & -3 \end{bmatrix}}$; 270°

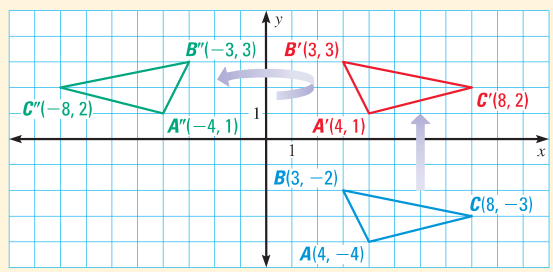
9.5 Apply Compositions of Transformations

EXAMPLE

The vertices of $\triangle ABC$ are $A(4, -4)$, $B(3, -2)$, and $C(8, -3)$. Graph the image of $\triangle ABC$ after the glide reflection.

Translation: $(x, y) \to (x, y + 5)$
Reflection: in the *y*-axis

Begin by graphing $\triangle ABC$. Then graph the image $\triangle A'B'C'$ after a translation of 5 units up. Finally, graph the image $\triangle A''B''C''$ after a reflection in the *y*-axis.



EXERCISES

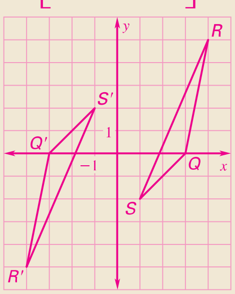
EXAMPLE 1 for Exs. 15–16

Graph the image of $H(-4, 5)$ after the glide reflection. 15, 16. See margin.

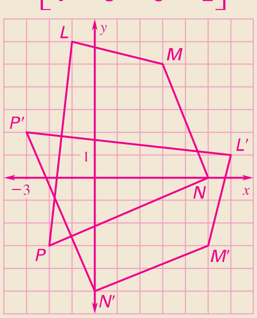
15. Translation: $(x, y) \to (x + 6, y - 2)$
Reflection: in $x = 3$

16. Translation: $(x, y) \to (x - 4, y - 5)$
Reflection: in $y = x$

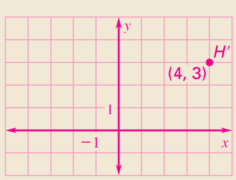
13. $\overset{\begin{matrix} Q' & R' & S' \end{matrix}}{\begin{bmatrix} -3 & -4 & -1 \\ 0 & -5 & 2 \end{bmatrix}}$;



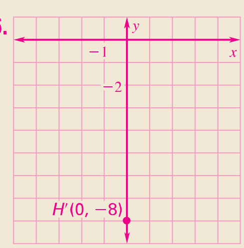
14. $\overset{\begin{matrix} L' & M' & N' & P' \end{matrix}}{\begin{bmatrix} 6 & 5 & 0 & -3 \\ 1 & -3 & -5 & 2 \end{bmatrix}}$;



15.



16.



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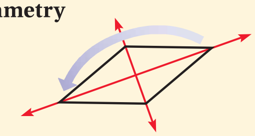
Chapter Review Practice

9.6 Identify Symmetry

EXAMPLE

Determine whether the rhombus has *line symmetry* and/or *rotational symmetry*. Identify the number of lines of symmetry and/or the rotations that map the figure onto itself.

The rhombus has two lines of symmetry. It also has rotational symmetry, because a 180° rotation maps the rhombus onto itself.



EXERCISES

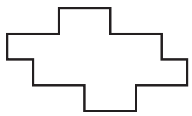
EXAMPLES 1 and 2 for Exs. 17–19

Determine whether the figure has *line symmetry* and/or *rotational symmetry*. Identify the number of lines of symmetry and/or the rotations that map the figure onto itself.

17.

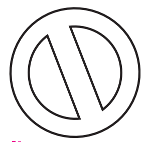
line symmetry, no rotational symmetry; one

18.



no line symmetry, rotational symmetry; 180° about the center

19.



line symmetry, rotational symmetry; two, 180° about the center

9.7 Identify and Perform Dilations

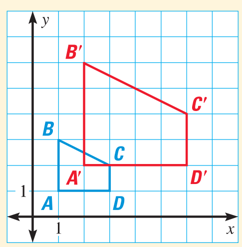
EXAMPLE

Quadrilateral *ABCD* has vertices *A*(1, 1), *B*(1, 3), *C*(3, 2), and *D*(3, 1). Use scalar multiplication to find the image of *ABCD* after a dilation with its center at the origin and a scale factor of 2. Graph *ABCD* and its image.

To find the image matrix, multiply each element of the polygon matrix by the scale factor.

$$\begin{matrix} & A & B & C & D \\ 2 & \Big[1 & 1 & 3 & 3 \\ & 1 & 3 & 2 & 1 \Big] \end{matrix} = \begin{matrix} A' & B' & C' & D' \\ \Big[2 & 2 & 6 & 6 \\ 2 & 6 & 4 & 2 \Big] \end{matrix}$$

Scale factor　Polygon matrix　Image matrix



EXERCISES

EXAMPLE 4 for Exs. 20–21

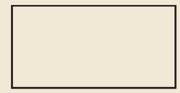
Find the image matrix that represents a dilation of the polygon centered at the origin with the given scale factor. Then graph the polygon and its image. 20, 21. See margin.

20. $\begin{bmatrix} Q & R & S \\ 2 & 4 & 8 \\ 2 & 4 & 2 \end{bmatrix}$; $k = \frac{1}{4}$

21. $\begin{bmatrix} L & M & N \\ -1 & 1 & 2 \\ -2 & 3 & 4 \end{bmatrix}$; $k = 3$

Extra Example 6

Determine whether the rectangle has line symmetry and/or rotational symmetry. Identify the number of lines of symmetry and/or the rotations that map the figure onto itself.

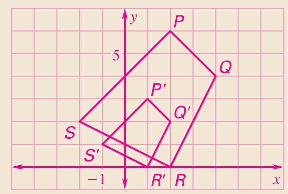


2 lines of symmetry, 180° rotational symmetry

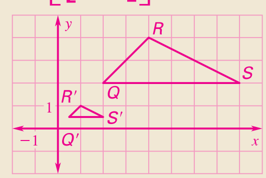
Extra Example 7

Quadrilateral *PQRS* has vertices *P*(2, 6), *Q*(4, 4), *R*(2, 0), and *S*(−2, 2). Use scalar multiplication to find the image of *PQRS* after a dilation with its center at the origin and a scale factor of $\frac{1}{2}$. Graph *PQRS* and its image.

$\begin{bmatrix} P' & Q' & R' & S' \\ 1 & 2 & 1 & -1 \\ 3 & 2 & 0 & 1 \end{bmatrix}$



20. $\begin{bmatrix} Q' & R' & S' \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$;



21. $\begin{bmatrix} L' & M' & N' \\ -3 & 3 & 6 \\ -6 & 9 & 12 \end{bmatrix}$;