C H A P T E R 4

Probability and Counting Rules

Objectives

After completing this chapter, you should be able to

1. Determine sample spaces and find the probability of an event, using classical probability or empirical probability.
2. Find the probability of compound events, using the addition rules.
3. Find the probability of compound events, using the multiplication rules.
4. Find the conditional probability of an event.
5. Find the total number of outcomes in a sequence of events, using the fundamental counting rule.
6. Find the number of ways that *r* objects can be selected from *n* objects, using the permutation rule.
7. Find the number of ways that *r* objects can be selected from *n* objects without regard to order, using the combination rule.
8. Find the probability of an event, using the counting rules.

Outline

Statistics Today

Would You Bet Your Life?

Humans not only bet money when they gamble, but also bet their lives by engaging in unhealthy activities such as smoking, drinking, using drugs, and exceeding the speed limit when driving. Many people don't care about the risks involved in these activities since they do not understand the concepts of probability. On the other hand, people may fear activities that involve little risk to health or life because these activities have been sensationalized by the press and media.

In his book *Probabilities in Everyday Life* (Ivy Books, p. 191), John D. McGervey states

> *When people have been asked to estimate the frequency of death from various causes, the most overestimated categories are those involving pregnancy, tornadoes, floods, fire, and homicide. The most underestimated categories include deaths from diseases such as diabetes, strokes, tuberculosis, asthma, and stomach cancer (although cancer in general is overestimated).*

The question then is, Would you feel safer if you flew across the United States on a commercial airline or if you drove? How much greater is the risk of one way to travel over the other? See Statistics Today—Revisited at the end of the chapter for the answer.

In this chapter, you will learn about probability—its meaning, how it is computed, and how to evaluate it in terms of the likelihood of an event actually happening.

4–1 Introduction

A cynical person once said, "The only two sure things are death and taxes." This philosophy no doubt arose because so much in people's lives is affected by chance. From the time a person awakes until he or she goes to bed, that person makes decisions regarding the possible events that are governed at least in part by chance. For example, should I carry an umbrella to work today? Will my car battery last until spring? Should I accept that new job?

Probability as a general concept can be defined as the chance of an event occurring. Many people are familiar with probability from observing or playing games of chance, such as card games, slot machines, or lotteries. In addition to being used in games of chance, probability theory is used in the fields of insurance, investments, and weather forecasting and in various other areas. Finally, as stated in Chapter 1, probability is the basis of inferential statistics. For example, predictions are based on probability, and hypotheses are tested by using probability.

The basic concepts of probability are explained in this chapter. These concepts include *probability experiments, sample spaces,* the *addition* and *multiplication rules,* and the *probabilities of complementary events.* Also in this chapter, you will learn the rule for counting, the differences between permutations and combinations, and how to figure out how many different combinations for specific situations exist. Finally, Section 4–6 explains how the counting rules and the probability rules can be used together to solve a wide variety of problems.

4–2 Sample Spaces and Probability

The theory of probability grew out of the study of various games of chance using coins, dice, and cards. Since these devices lend themselves well to the application of concepts of probability, they will be used in this chapter as examples. This section begins by explaining some basic concepts of probability. Then the types of probability and probability rules are discussed.

Basic Concepts

Processes such as flipping a coin, rolling a die, or drawing a card from a deck are called *probability experiments.*

Objective 1

Determine sample spaces and find the probability of an event, using classical probability or empirical probability.

A **probability experiment** is a chance process that leads to well-defined results called outcomes.

An **outcome** is the result of a single trial of a probability experiment.

A trial means flipping a coin once, rolling one die once, or the like. When a coin is tossed, there are two possible outcomes: head or tail. (*Note:* We exclude the possibility of a coin landing on its edge.) In the roll of a single die, there are six possible outcomes: 1, 2, 3, 4, 5, or 6. In any experiment, the set of all possible outcomes is called the *sample space.*

A **sample space** is the set of all possible outcomes of a probability experiment.

Some sample spaces for various probability experiments are shown here.

Experiment	Sample space
Toss one coin	Head, tail
Roll a die	1, 2, 3, 4, 5, 6
Answer a true/false question	True, false
Toss two coins	Head-head, tail-tail, head-tail, tail-head

It is important to realize that when two coins are tossed, there are *four* possible outcomes, as shown in the fourth experiment above. Both coins could fall heads up. Both coins could fall tails up. Coin 1 could fall heads up and coin 2 tails up. Or coin 1 could fall tails up and coin 2 heads up. Heads and tails will be abbreviated as H and T throughout this chapter.

Example 4–1

Find the sample space for rolling two dice.

Solution

Since each die can land in six different ways, and two dice are rolled, the sample space can be presented by a rectangular array, as shown in Figure 4–1. The sample space is the list of pairs of numbers in the chart.

Figure 4–1
Sample Space for Rolling Two Dice (Example 4–1)

	Die 2					
Die 1	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

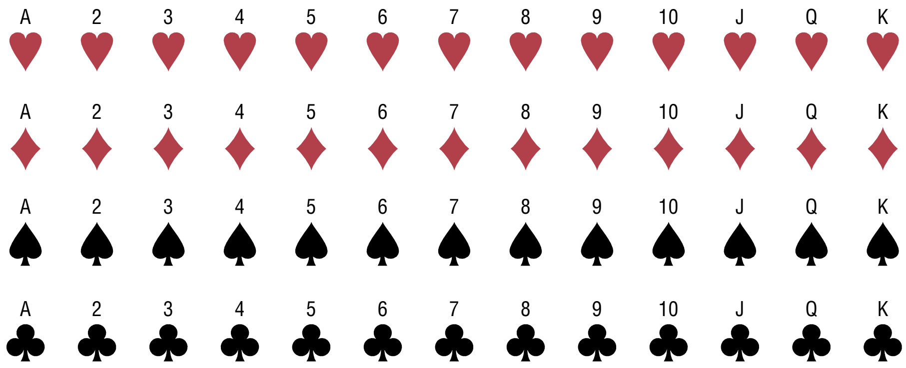
Example 4–2

Find the sample space for drawing one card from an ordinary deck of cards.

Solution

Since there are 4 suits (hearts, clubs, diamonds, and spades) and 13 cards for each suit (ace through king), there are 52 outcomes in the sample space. See Figure 4–2.

Figure 4–2
Sample Space for Drawing a Card (Example 4–2)



Example 4–3

Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.

Solution

There are two genders, male and female, and each child could be either gender. Hence, there are eight possibilities, as shown here.

BBB BBG BGB GBB GGG GGB GBG BGG

In Examples 4–1 through 4–3, the sample spaces were found by observation and reasoning; however, another way to find all possible outcomes of a probability experiment is to use a *tree diagram.*

A **tree diagram** is a device consisting of line segments emanating from a starting point and also from the outcome point. It is used to determine all possible outcomes of a probability experiment.

Example 4–4

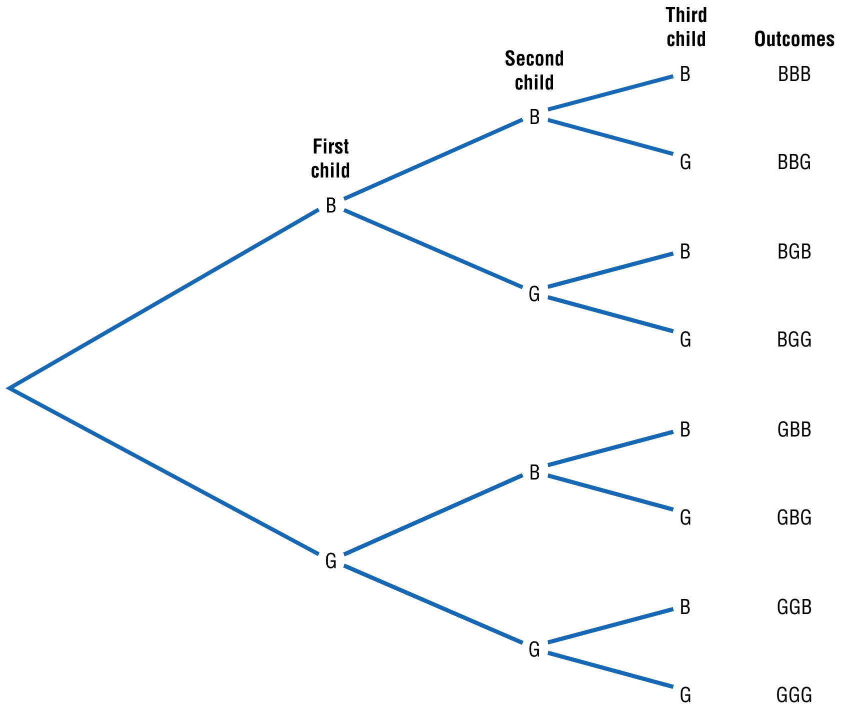
Use a tree diagram to find the sample space for the gender of three children in a family, as in Example 4–3.

Solution

Since there are two possibilities (boy or girl) for the first child, draw two branches from a starting point and label one B and the other G. Then if the first child is a boy, there are two possibilities for the second child (boy or girl), so draw two branches from B and label one B and the other G. Do the same if the first child is a girl. Follow the same procedure for the third child. The completed tree diagram is shown in Figure 4–3. To find the outcomes for the sample space, trace through all the possible branches, beginning at the starting point for each one.

Figure 4–3

Tree Diagram for Example 4–4



Historical Note

The famous Italian astronomer Galileo (1564–1642) found that a sum of 10 occurs more often than any other sum when three dice are tossed. Previously, it was thought that a sum of 9 occurred more often than any other sum.

Historical Note

A mathematician named Jerome Cardan (1501–1576) used his talents in mathematics and probability theory to make his living as a gambler. He is thought to be the first person to formulate the definition of classical probability.

An outcome was defined previously as the result of a single trial of a probability experiment. In many problems, one must find the probability of two or more outcomes. For this reason, it is necessary to distinguish between an outcome and an event.

An **event** consists of a set of outcomes of a probability experiment.

An event can be one outcome or more than one outcome. For example, if a die is rolled and a 6 shows, this result is called an *outcome,* since it is a result of a single trial. An event with one outcome is called a **simple event.** The event of getting an odd number when a die is rolled is called a **compound event,** since it consists of three outcomes or three simple events. In general, a compound event consists of two or more outcomes or simple events.

There are three basic interpretations of probability:

1. Classical probability
2. Empirical or relative frequency probability
3. Subjective probability

Historical Note

During the mid-1600s, a professional gambler named Chevalier de Méré made a considerable amount of money on a gambling game. He would bet unsuspecting patrons that in four rolls of a die, he could get at least one 6. He was so successful at the game that some people refused to play. He decided that a new game was necessary to continue his winnings. By reasoning, he figured he could roll at least one double 6 in 24 rolls of two dice, but his reasoning was incorrect and he lost systematically. Unable to figure out why, he contacted a mathematician named Blaise Pascal (1623–1662) to find out why.

Pascal became interested and began studying probability theory. He corresponded with a French government official, Pierre de Fermat (1601–1665), whose hobby was mathematics. Together the two formulated the beginnings of probability theory.

Classical Probability

Classical probability uses sample spaces to determine the numerical probability that an event will happen. One does not actually have to perform the experiment to determine that probability. Classical probability is so named because it was the first type of probability studied formally by mathematicians in the 17th and 18th centuries.

Classical probability assumes that all outcomes in the sample space are equally likely to occur. For example, when a single die is rolled, each outcome has the same probability of occurring. Since there are six outcomes, each outcome has a probability of $\frac{1}{6}$. When a card is selected from an ordinary deck of 52 cards, one assumes that the deck has been shuffled, and each card has the same probability of being selected. In this case, it is $\frac{1}{52}$.

Equally likely events are events that have the same probability of occurring.

Formula for Classical Probability

The probability of any event E is

$$\frac{\text{Number of outcomes in } E}{\text{Total number of outcomes in the sample space}}$$

This probability is denoted by

$$P(E) = \frac{n(E)}{n(S)}$$

This probability is called *classical probability,* and it uses the sample space S.

Probabilities can be expressed as fractions, decimals, or—where appropriate—percentages. If one asks, "What is the probability of getting a head when a coin is tossed?" typical responses can be any of the following three.

"One-half."

"Point five."

"Fifty percent."[1]

These answers are all equivalent. In most cases, the answers to examples and exercises given in this chapter are expressed as fractions or decimals, but percentages are used where appropriate.

[1]Strictly speaking, a percent is not a probability. However, in everyday language, probabilities are often expressed as percents (i.e., there is a 60% chance of rain tomorrow). For this reason, some probabilities will be expressed as percents throughout this book.

Rounding Rule for Probabilities Probabilities should be expressed as reduced fractions or rounded to two or three decimal places. When the probability of an event is an extremely small decimal, it is permissible to round the decimal to the first nonzero digit after the point. For example, 0.0000587 would be 0.00006. When obtaining probabilities from one of the tables in Appendix C, use the number of decimal places given in the table. If decimals are converted to percentages to express probabilities, move the decimal point two places to the right and add a percent sign.

Example 4–5

For a card drawn from an ordinary deck, find the probability of getting a queen.

Solution

Since there are 52 cards in a deck and there are 4 queens, $P(\text{queen}) = \frac{4}{52} = \frac{1}{13}$.

Example 4–6

If a family has three children, find the probability that all the children are girls.

Solution

The sample space for the gender of children for a family that has three children is BBB, BBG, BGB, GBB, GGG, GGB, GBG, and BGG (see Examples 4–3 and 4–4). Since there is one way in eight possibilities for all three children to be girls,

$$P(\text{GGG}) = \frac{1}{8}$$

Historical Note

Ancient Greeks and Romans made crude dice from animal bones, various stones, minerals, and ivory. When the dice were tested mathematically, some were found to be quite accurate.

In probability theory, it is important to understand the meaning of the words *and* and *or.* For example, if you were asked to find the probability of getting a queen *and* a heart when you are drawing a single card from a deck, you would be looking for the queen of hearts. Here the word *and* means "at the same time." The word *or* has two meanings. For example, if you were asked to find the probability of selecting a queen *or* a heart when one card is selected from a deck, you would be looking for one of the 4 queens or one of the 13 hearts. In this case, the queen of hearts would be included in both cases and counted twice. So there would be $4 + 13 - 1 = 16$ possibilities.

On the other hand, if you were asked to find the probability of getting a queen *or* a king, you would be looking for one of the 4 queens or one of the 4 kings. In this case, there would be $4 + 4 = 8$ possibilities. In the first case, both events can occur at the same time; we say that this is an example of the *inclusive or.* In the second case, both events cannot occur at the same time, and we say that this is an example of the *exclusive or.*

Example 4–7

A card is drawn from an ordinary deck. Find these probabilities.

a. Of getting a jack
b. Of getting the 6 of clubs (i.e., a 6 and a club)
c. Of getting a 3 or a diamond
d. Of getting a 3 or a 6

Solution

a. Refer to the sample space in Figure 4–2. There are 4 jacks so there are 4 outcomes in event E and 52 possible outcomes in the sample space. Hence,

$$P(\text{jack}) = \frac{4}{52} = \frac{1}{13}$$

b. Since there is only one 6 of clubs in event E, the probability of getting a 6 of clubs is

$P(\text{6 of clubs}) = \frac{1}{52}$

c. There are four 3s and 13 diamonds, but the 3 of diamonds is counted twice in this listing. Hence, there are 16 possibilities of drawing a 3 or a diamond, so

$P(\text{3 or diamond}) = \frac{16}{52} = \frac{4}{13}$

This is an example of the inclusive or.

d. Since there are four 3s and four 6s,

$P(\text{3 or 6}) = \frac{8}{52} = \frac{2}{13}$

This is an example of the exclusive or.

There are four basic probability rules. These rules are helpful in solving probability problems, in understanding the nature of probability, and in deciding if your answers to the problems are correct.

Historical Note

Paintings in tombs excavated in Egypt show that the Egyptians played games of chance. One game called *Hounds and Jackals* played in 1800 B.C. is similar to the present-day game of *Snakes and Ladders.*

Probability Rule 1

The probability of any event E is a number (either a fraction or decimal) between and including 0 and 1. This is denoted by $0 \leq P(E) \leq 1$.

Rule 1 states that probabilities cannot be negative or greater than 1.

Probability Rule 2

If an event E cannot occur (i.e., the event contains no members in the sample space), its probability is 0.

Example 4–8

When a single die is rolled, find the probability of getting a 9.

Solution

Since the sample space is 1, 2, 3, 4, 5, and 6, it is impossible to get a 9. Hence, the probability is $P(9) = \frac{0}{6} = 0$.

Probability Rule 3

If an event E is certain, then the probability of E is 1.

In other words, if $P(E) = 1$, then the event E is certain to occur. This rule is illustrated in Example 4–9.

Example 4–9

When a single die is rolled, what is the probability of getting a number less than 7?

Solution

Since all outcomes—1, 2, 3, 4, 5, and 6—are less than 7, the probability is

$$P(\text{number less than 7}) = \frac{6}{6} = 1$$

The event of getting a number less than 7 is certain.

In other words, probability values range from 0 to 1. When the probability of an event is close to 0, its occurrence is highly unlikely. When the probability of an event is near 0.5, there is about a 50-50 chance that the event will occur; and when the probability of an event is close to 1, the event is highly likely to occur.

Probability Rule 4

The sum of the probabilities of all the outcomes in the sample space is 1.

For example, in the roll of a fair die, each outcome in the sample space has a probability of $\frac{1}{6}$. Hence, the sum of the probabilities of the outcomes is as shown.

Outcome	1		2		3		4		5		6
Probability	$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$
Sum	$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6} = \frac{6}{6} = 1$

Complementary Events

Another important concept in probability theory is that of *complementary events.* When a die is rolled, for instance, the sample space consists of the outcomes 1, 2, 3, 4, 5, and 6. The event E of getting odd numbers consists of the outcomes 1, 3, and 5. The event of not getting an odd number is called the *complement* of event E, and it consists of the outcomes 2, 4, and 6.

The **complement of an event** E is the set of outcomes in the sample space that are not included in the outcomes of event E. The complement of E is denoted by $\bar{E}$ (read "E bar").

Example 4–10 further illustrates the concept of complementary events.

Example 4–10

Find the complement of each event.

a. Rolling a die and getting a 4

b. Selecting a letter of the alphabet and getting a vowel

c. Selecting a month and getting a month that begins with a J

d. Selecting a day of the week and getting a weekday

Solution

a. Getting a 1, 2, 3, 5, or 6

b. Getting a consonant (assume *y* is a consonant)

c. Getting February, March, April, May, August, September, October, November, or December

d. Getting Saturday or Sunday

The outcomes of an event and the outcomes of the complement make up the entire sample space. For example, if two coins are tossed, the sample space is HH, HT, TH, and TT. The complement of "getting all heads" is not "getting all tails," since the event "all heads" is HH, and the complement of HH is HT, TH, and TT. Hence, the complement of the event "all heads" is the event "getting at least one tail."

Since the event and its complement make up the entire sample space, it follows that the sum of the probability of the event and the probability of its complement will equal 1. That is, $P(E) + P(\overline{E}) = 1$. In Example 4–10, let E = all heads, or HH, and let $\overline{E}$ = at least one tail, or HT, TH, TT. Then $P(E) = \frac{1}{4}$ and $P(\overline{E}) = \frac{3}{4}$; hence, $P(E) + P(\overline{E}) = \frac{1}{4} + \frac{3}{4} = 1$.

The rule for complementary events can be stated algebraically in three ways.

Rule for Complementary Events

$$P(\overline{E}) = 1 - P(E) \quad \text{or} \quad P(E) = 1 - P(\overline{E}) \quad \text{or} \quad P(E) + P(\overline{E}) = 1$$

Stated in words, the rule is: *If the probability of an event or the probability of its complement is known, then the other can be found by subtracting the probability from 1.* This rule is important in probability theory because at times the best solution to a problem is to find the probability of the complement of an event and then subtract from 1 to get the probability of the event itself.

Example 4–11

If the probability that a person lives in an industrialized country of the world is $\frac{1}{5}$, find the probability that a person does not live in an industrialized country.

Source: Harper's Index.

Solution

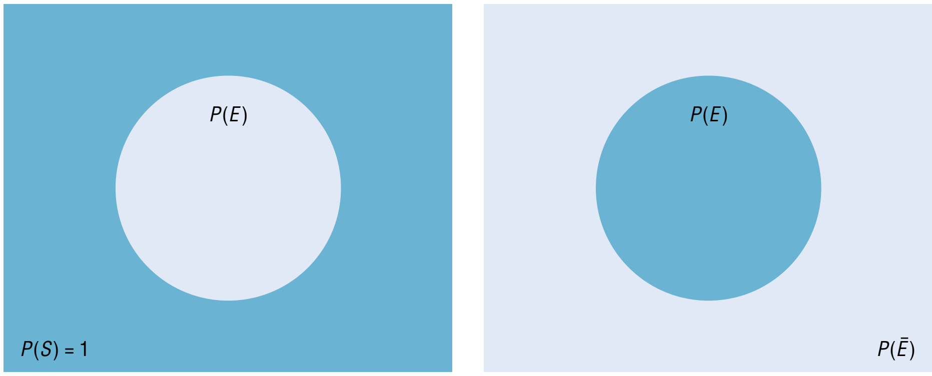
$$\begin{aligned} P(\text{not living in an industrialized country}) &= 1 - P(\text{living in an industrialized country}) \\ &= 1 - \tfrac{1}{5} = \tfrac{4}{5} \end{aligned}$$

Probabilities can be represented pictorially by **Venn diagrams.** Figure 4–4(a) shows the probability of a simple event *E.* The area inside the circle represents the probability of event *E,* that is, $P(E)$. The area inside the rectangle represents the probability of all the events in the sample space $P(S)$.

The Venn diagram that represents the probability of the complement of an event $P(\overline{E})$ is shown in Figure 4–4(b). In this case, $P(\overline{E}) = 1 - P(E)$, which is the area inside the rectangle but outside the circle representing $P(E)$. Recall that $P(S) = 1$ and $P(E) = 1 - P(\overline{E})$. The reasoning is that $P(E)$ is represented by the area of the circle and $P(\overline{E})$ is the probability of the events that are outside the circle.

Figure 4–4

Venn Diagram for the Probability and Complement



(a) Simple probability

(b) $P(\bar{E}) = 1 - P(E)$

Empirical Probability

The difference between classical and **empirical probability** is that classical probability assumes that certain outcomes are equally likely (such as the outcomes when a die is rolled), while empirical probability relies on actual experience to determine the likelihood of outcomes. In empirical probability, one might actually roll a given die 6000 times, observe the various frequencies, and use these frequencies to determine the probability of an outcome. Suppose, for example, that a researcher asked 25 people if they liked the taste of a new soft drink. The responses were classified as "yes," "no," or "undecided." The results were categorized in a frequency distribution, as shown.

Response	Frequency
Yes	15
No	8
Undecided	2
Total	25

Probabilities now can be compared for various categories. For example, the probability of selecting a person who liked the taste is $\frac{15}{25}$, or $\frac{3}{5}$, since 15 out of 25 people in the survey answered yes.

Formula for Empirical Probability

Given a frequency distribution, the probability of an event being in a given class is

$$P(E) = \frac{\text{frequency for the class}}{\text{total frequencies in the distribution}} = \frac{f}{n}$$

This probability is called *empirical probability* and is based on observation.

Example 4–12

In the soft drink survey just described, find the probability that a person responded no.

Solution

$$P(E) = \frac{f}{n} = \frac{8}{25}$$

Note: This is the same relative frequency explained in Chapter 2.

Example 4–13

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a. A person has type O blood.
b. A person has type A or type B blood.
c. A person has neither type A nor type O blood.
d. A person does not have type AB blood.

Source: The American Red Cross.

Solution

Type		Frequency
A		22
B		5
AB		2
O		21
	Total	50

a. $P(\text{O}) = \frac{f}{n} = \frac{21}{50}$

b. $P(\text{A or B}) = \frac{22}{50} + \frac{5}{50} = \frac{27}{50}$

(Add the frequencies of the two classes.)

c. $P(\text{neither A nor O}) = \frac{5}{50} + \frac{2}{50} = \frac{7}{50}$

(Neither A nor O means that a person has either type B or type AB blood.)

d. $P(\text{not AB}) = 1 - P(\text{AB}) = 1 - \frac{2}{50} = \frac{48}{50} = \frac{24}{25}$

(Find the probability of not AB by subtracting the probability of type AB from 1.)

Example 4–14

Hospital records indicated that maternity patients stayed in the hospital for the number of days shown in the distribution.

Number of days stayed	Frequency
3	15
4	32
5	56
6	19
7	5
	127

Find these probabilities.

a. A patient stayed exactly 5 days.
b. A patient stayed less than 6 days.
c. A patient stayed at most 4 days.
d. A patient stayed at least 5 days.

Solution

a. $P(5) = \dfrac{56}{127}$

b. $P(\text{less than 6 days}) = \dfrac{15}{127} + \dfrac{32}{127} + \dfrac{56}{127} = \dfrac{103}{127}$

(Less than 6 days means 3, 4, or 5 days.)

c. $P(\text{at most 4 days}) = \dfrac{15}{127} + \dfrac{32}{127} = \dfrac{47}{127}$

(At most 4 days means 3 or 4 days.)

d. $P(\text{at least 5 days}) = \dfrac{56}{127} + \dfrac{19}{127} + \dfrac{5}{127} = \dfrac{80}{127}$

(At least 5 days means 5, 6, or 7 days.)

Empirical probabilities can also be found by using a relative frequency distribution, as shown in Section 2–3.

For example, the relative frequency distribution of the soft drink data shown before is

Response	Frequency	Relative frequency
Yes	15	0.60
No	8	0.32
Undecided	2	0.08
	25	1.00

Hence, the probability that a person responded no is 0.32, which is equal to $\frac{8}{25}$.

Law of Large Numbers

When a coin is tossed one time, it is common knowledge that the probability of getting a head is $\frac{1}{2}$. But what happens when the coin is tossed 50 times? Will it come up heads 25 times? Not all the time. One should expect about 25 heads if the coin is fair. But due to chance variation, 25 heads will not occur most of the time.

If the empirical probability of getting a head is computed by using a small number of trials, it is usually not exactly $\frac{1}{2}$. However, as the number of trials increases, the empirical probability of getting a head will approach the theoretical probability of $\frac{1}{2}$, if in fact the coin is fair (i.e., balanced). This phenomenon is an example of the **law of large numbers.**

One should be careful to not think that the number of heads and number of tails tend to "even out." As the number of trials increases, the proportion of heads to the total number of trials will approach $\frac{1}{2}$. This law holds for any type of gambling game—tossing dice, playing roulette, and so on.

It should be pointed out that the probabilities that the proportions steadily approach may or may not agree with those theorized in the classical model. If not, it can have important implications, such as "the die is not fair." Pit bosses in Las Vegas watch for empirical trends that do not agree with classical theories, and they will sometimes take a set of dice out of play if observed frequencies are too far out of line with classical expected frequencies.

Subjective Probability

The third type of probability is called *subjective probability.* **Subjective probability** uses a probability value based on an educated guess or estimate, employing opinions and inexact information.

In subjective probability, a person or group makes an educated guess at the chance that an event will occur. This guess is based on the person's experience and evaluation of a solution. For example, a sportswriter may say that there is a 70% probability that the Pirates will win the pennant next year. A physician might say that, on the basis of her diagnosis, there is a 30% chance the patient will need an operation. A seismologist might say there is an 80% probability that an earthquake will occur in a certain area. These are only a few examples of how subjective probability is used in everyday life.

All three types of probability (classical, empirical, and subjective) are used to solve a variety of problems in business, engineering, and other fields.

Probability and Risk Taking

An area in which people fail to understand probability is risk taking. Actually, people fear situations or events that have a relatively small probability of happening rather than those events that have a greater likelihood of occurring. For example, many people think that the crime rate is increasing every year. However, in his book entitled *How Risk Affects Your Everyday Life,* author James Walsh states: "Despite widespread concern about the number of crimes committed in the United States, FBI and Justice Department statistics show that the national crime rate has remained fairly level for 20 years. It even dropped slightly in the early 1990s."

He further states, "Today most media coverage of risk to health and well-being focuses on shock and outrage." Shock and outrage make good stories and can scare us about the wrong dangers. For example, the author states that if a person is 20% overweight, the loss of life expectancy is 900 days (about 3 years), but loss of life expectancy from exposure to radiation emitted by nuclear power plants is 0.02 day. As you can see, being overweight is much more of a threat than being exposed to radioactive emission.

Many people gamble daily with their lives, for example, by using tobacco, drinking and driving, and riding motorcycles. When people are asked to estimate the probabilities or frequencies of death from various causes, they tend to overestimate causes such as accidents, fires, and floods and to underestimate the probabilities of death from diseases (other than cancer), strokes, etc. For example, most people think that their chances of dying of a heart attack are 1 in 20, when in fact they are almost 1 in 3; the chances of dying by pesticide poisoning are 1 in 200,000 (*True Odds* by James Walsh). The reason people think this way is that the news media sensationalize deaths resulting from catastrophic events and rarely mention deaths from disease.

When you are dealing with life-threatening catastrophes such as hurricanes, floods, automobile accidents, or smoking, it is important to get the facts. That is, get the actual numbers from accredited statistical agencies or reliable statistical studies, and then compute the probabilities and make decisions based on your knowledge of probability and statistics.

In summary, then, when you make a decision or plan a course of action based on probability, make sure that you understand the true probability of the event occurring. Also, find out how the information was obtained (i.e., from a reliable source). Weigh the cost of the action and decide if it is worth it. Finally, look for other alternatives or courses of action with less risk involved.

Applying the Concepts 4–2

Tossing a Coin

Assume you are at a carnival and decide to play one of the games. You spot a table where a person is flipping a coin, and since you have an understanding of basic probability, you believe that the odds of winning are in your favor. When you get to the table, you find out that all you have to do is guess which side of the coin will be facing up after it is tossed. You are assured that the coin is fair, meaning that each of the two sides has an equally likely chance of occurring. You think back about what you learned in your statistics class about probability before you decide what to bet on. Answer the following questions about the coin-tossing game.

1. What is the sample space?
2. What are the possible outcomes?
3. What does the classical approach to probability say about computing probabilities for this type of problem?

You decide to bet on heads, believing that it has a 50% chance of coming up. A friend of yours, who had been playing the game for awhile before you got there, tells you that heads has come up the last 9 times in a row. You remember the law of large numbers.

4. What is the law of large numbers, and does it change your thoughts about what will occur on the next toss?
5. What does the empirical approach to probability say about this problem, and could you use it to solve this problem?
6. Can subjective probabilities be used to help solve this problem? Explain.
7. Assume you could win $1 million if you could guess what the results of the next toss will be. What would you bet on? Why?

See page 234 for the answers.

Exercises 4–2

1. What is a probability experiment?

2. Define *sample space*.

3. What is the difference between an outcome and an event?

4. What are equally likely events?

5. What is the range of the values of the probability of an event?

6. When an event is certain to occur, what is its probability?

7. If an event cannot happen, what value is assigned to its probability?

8. What is the sum of the probabilities of all the outcomes in a sample space?

9. If the probability that it will rain tomorrow is 0.20, what is the probability that it won't rain tomorrow? Would you recommend taking an umbrella?

10. A probability experiment is conducted. Which of these cannot be considered a probability of an outcome?

a. $\frac{1}{3}$	*d.* -0.59	*g.* 1
b. $-\frac{1}{5}$	*e.* 0	*h.* 33%
c. 0.80	*f.* 1.45	*i.* 112%

11. Classify each statement as an example of classical probability, empirical probability, or subjective probability.

a. The probability that a person will watch the 6 o'clock evening news is 0.15.

b. The probability of winning at a Chuck-a-Luck game is $\frac{5}{36}$.

c. The probability that a bus will be in an accident on a specific run is about 6%.

d. The probability of getting a royal flush when five cards are selected at random is $\frac{1}{649,740}$.

e. The probability that a student will get a C or better in a statistics course is about 70%.

f. The probability that a new fast-food restaurant will be a success in Chicago is 35%.

g. The probability that interest rates will rise in the next 6 months is 0.50.

12. **(ans)** If a die is rolled one time, find these probabilities.

a. Of getting a 4
b. Of getting an even number
c. Of getting a number greater than 4
d. Of getting a number less than 7
e. Of getting a number greater than 0
f. Of getting a number greater than 3 or an odd number
g. Of getting a number greater than 3 and an odd number

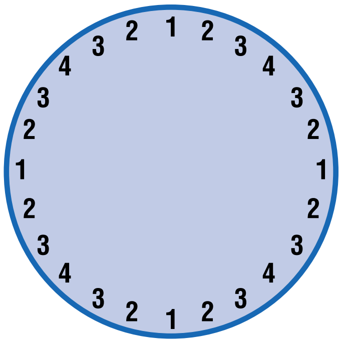
13. If two dice are rolled one time, find the probability of getting these results.

a. A sum of 6
b. Doubles
c. A sum of 7 or 11
d. A sum greater than 9
e. A sum less than or equal to 4

14. **(ans)** If one card is drawn from a deck, find the probability of getting these results.

a. An ace
b. A diamond
c. An ace of diamonds
d. A 4 or a 6
e. A 4 or a club
f. A 6 or a spade
g. A heart or a club
h. A red queen
i. A red card or a 7
j. A black card and a 10

15. A shopping mall has set up a promotion as follows. With any mall purchase, the customer gets to spin the wheel shown here. If the number 1 comes up, the customer wins $10. If the number 2 comes up, the customer wins $5, and if the number 3 or 4 comes up, the customer wins a discount coupon. Find the following probabilities.



a. The customer wins $10.
b. The customer wins money.
c. The customer wins a coupon.

16. Choose one of the 50 states at random.

a. What is the probability that it begins with M?
b. What is the probability that it doesn't begin with a vowel?

17. In a college class of 250 graduating seniors, 50 have jobs waiting, 10 are going to medical school, 20 are going to law school, and 80 are going to various other kinds of graduate schools. Select one graduate at random.

a. What is the probability that the student is going to graduate school?
b. What is the probability that the student is going to medical school?
c. What is the probability that the student will have to start paying back her deferred student loans after 6 months (i.e., does not continue in school)?

18. Sixty-nine percent of adults favor gun licensing in general. Choose one adult at random. What is the probability that the selected adult doesn't believe in gun licensing?

Source: *Time* magazine.

19. For a recent year, 51% of the families in the United States had no children under the age of 18; 20% had one child; 19% had two children; 7% had three children; and 3% had four or more children. If a family is selected at random, find the probability that the family has

a. Two or three children
b. More than one child
c. Less than three children
d. Based on the answers to parts *a*, *b*, and *c*, which is most likely to occur? Explain why.

Source: U.S. Census Bureau.

20. In a recent year, of the 1,184,000 bachelor's degrees conferred, 233,000 were in the field of business, 125,000 were in the social sciences, and 106,000 were in education. If one degree is selected at random, find the following probabilities.

a. The degree was awarded in education.
b. The degree was not awarded in business.

Source: National Center for Education Statistics.

21. A couple has three children. Find each probability.

a. All boys
b. All girls or all boys
c. Exactly two boys or two girls
d. At least one child of each gender

22. In the game of craps using two dice, a person wins on the first roll if a 7 or an 11 is rolled. Find the probability of winning on the first roll.

23. In a game of craps, a player loses on the roll if a 2, 3, or 12 is tossed on the first roll. Find the probability of losing on the first roll.

24. For a specific year a total of 2541 postal workers were bitten by dogs. The top six cities for crunching canines were as follows.

Houston	49	Chicago	37
Miami	35	Los Angeles	32
Brooklyn	22	Cleveland	20

If one bitten postal worker is selected at random, what is the probability that he was bitten in Houston, Chicago, or Los Angeles? What is the probability that he was bitten in some other city?

Source: *N.Y. Times Almanac*.

25. A roulette wheel has 38 spaces numbered 1 through 36, 0, and 00. Find the probability of getting these results.

a. An odd number (Do not count 0 or 00.)

b. A number greater than 27

c. A number that contains the digit 0

d. Based on the answers to parts *a*, *b*, and *c*, which is most likely to occur? Explain why.

26. Thirty-nine of fifty states are currently under court order to alleviate overcrowding and poor conditions in one or more of their prisons. If a state is selected at random, find the probability that it is currently under such a court order.

Source: *Harper's Index*.

27. A CBS News/*New York Times* poll found that of 764 adults surveyed nationwide, 34% felt that we are spending too much on space exploration, 19% felt that we are spending too little, 35% felt that we are spending the right amount, and the rest said "don't know" or had no answer. If one of the respondents is selected at random, what is the probability that the person felt that we are spending the right amount or too little?

Source: www.pollingreport.com.

28. In a survey, 16 percent of American children said they use flattery to get their parents to buy them things. If a child is selected at random, find the probability that the child said he or she does not use parental flattery.

Source: *Harper's Index*.

29. Roll two dice and multiply the numbers together.

a. Write out the sample space.

b. What is the probability that the product is a multiple of 6?

c. What is the probability that the product is less than 10?

30. The source of federal government revenue for a specific year is

50% from individual income taxes
32% from social insurance payroll taxes
10% from corporate income taxes
3% from excise taxes
5% other

If a revenue source is selected at random, what is the probability that it comes from individual or corporate income taxes?

Source: *N.Y. Times Almanac*.

31. A box contains a $1 bill, a $5 bill, a $10 bill, and a $20 bill. A bill is selected at random, and it is not replaced; then a second bill is selected at random. Draw a tree diagram and determine the sample space.

32. Draw a tree diagram and determine the sample space for tossing four coins.

33. Four balls numbered 1 through 4 are placed in a box. A ball is selected at random, and its number is noted; then it is replaced. A second ball is selected at random, and its number is noted. Draw a tree diagram and determine the sample space.

34. Kimberly decides to have a computer custom-made. She can select one option from each category:

Megabytes	Monitor	Color
128	15 inches	Tan
256	17 inches	Ivory
512		

Draw a tree diagram for all possible types of computers she can select.

35. Betty and Claire play a tennis tournament consisting of three games. Draw a tree diagram for all possible outcomes of the tournament.

36. A coin is tossed; if it falls heads up, it is tossed again. If it falls tails up, a die is rolled. Draw a tree diagram and determine the outcomes.

Extending the Concepts

37. The distribution of ages of CEOs is as follows:

Age	Frequency
21–30	1
31–40	8
41–50	27
51–60	29
61–70	24
71–up	11

Source: Information based on *USA TODAY* Snapshot.

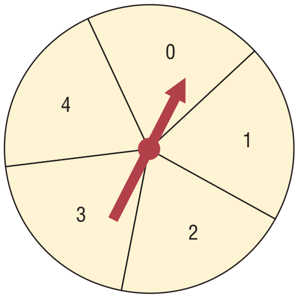
If a CEO is selected at random, find the probability that his or her age is

a. Between 31 and 40
b. Under 31
c. Over 30 and under 51
d. Under 31 or over 60

38. A person flipped a coin 100 times and obtained 73 heads. Can the person conclude that the coin was unbalanced?

39. A medical doctor stated that with a certain treatment, a patient has a 50% chance of recovering without surgery. That is, "Either he will get well or he won't get well." Comment on this statement.

40. The wheel spinner shown here is spun twice. Find the sample space, and then determine the probability of the following events.



a. An odd number on the first spin and an even number on the second spin (*Note:* 0 is considered even.)
b. A sum greater than 4
c. Even numbers on both spins
d. A sum that is odd
e. The same number on both spins

41. Toss three coins 128 times and record the number of heads (0, 1, 2, or 3); then record your results with the theoretical probabilities. Compute the empirical probabilities of each.

42. Toss two coins 100 times and record the number of heads (0, 1, 2). Compute the probabilities of each outcome, and compare these probabilities with the theoretical results.

43. Odds are used in gambling games to make them fair. For example, if a person rolled a die and won every time he or she rolled a 6, then the person would win on average once every 6 times. So that the game is fair, the odds of 5 to 1 are given. This means that if the person bet $1 and won, he or she could win $5. On average, the player would win $5 once in 6 rolls and lose $1 on the other 5 rolls—hence the term *fair game.*

In most gambling games, the odds given are not fair. For example, if the odds of winning are really 20 to 1, the house might offer 15 to 1 in order to make a profit.

Odds can be expressed as a fraction or as a ratio, such as $\frac{5}{1}$, 5:1, or 5 to 1. Odds are computed in favor of the event or against the event. The formulas for odds are

$$\text{Odds in favor} = \frac{P(E)}{1 - P(E)}$$

$$\text{Odds against} = \frac{P(\overline{E})}{1 - P(\overline{E})}$$

In the die example,

$$\text{Odds in favor of a 6} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5} \text{ or } 1{:}5$$

$$\text{Odds against a 6} = \frac{\frac{5}{6}}{\frac{1}{6}} = \frac{5}{1} \text{ or } 5{:}1$$

Find the odds in favor of and against each event.

a. Rolling a die and getting a 2
b. Rolling a die and getting an even number
c. Drawing a card from a deck and getting a spade
d. Drawing a card and getting a red card
e. Drawing a card and getting a queen
f. Tossing two coins and getting two tails
g. Tossing two coins and getting one tail

4–3

The Addition Rules for Probability

Objective 2

Find the probability of compound events, using the addition rules.

Many problems involve finding the probability of two or more events. For example, at a large political gathering, one might wish to know, for a person selected at random, the probability that the person is a female or is a Republican. In this case, there are three possibilities to consider:

1. The person is a female.
2. The person is a Republican.
3. The person is both a female and a Republican.

Consider another example. At the same gathering there are Republicans, Democrats, and Independents. If a person is selected at random, what is the probability that the person is a Democrat or an Independent? In this case, there are only two possibilities:

1. The person is a Democrat.
2. The person is an Independent.

The difference between the two examples is that in the first case, the person selected can be a female and a Republican at the same time. In the second case, the person selected cannot be both a Democrat and an Independent at the same time. In the second case, the two events are said to be *mutually exclusive;* in the first case, they are not mutually exclusive.

Two events are **mutually exclusive events** if they cannot occur at the same time (i.e., they have no outcomes in common).

In another situation, the events of getting a 4 and getting a 6 when a single card is drawn from a deck are mutually exclusive events, since a single card cannot be both a 4 and a 6. On the other hand, the events of getting a 4 and getting a heart on a single draw are not mutually exclusive, since one can select the 4 of hearts when drawing a single card from an ordinary deck.

Historical Note

The first book on probability, *The Book of Chance and Games,* was written by Jerome Cardan (1501–1576). Cardan was an astrologer, philosopher, physician, mathematician, and gambler. This book contained techniques on how to cheat and how to catch others at cheating.

Example 4–15

Determine which events are mutually exclusive and which are not, when a single die is rolled.

a. Getting an odd number and getting an even number

b. Getting a 3 and getting an odd number

c. Getting an odd number and getting a number less than 4

d. Getting a number greater than 4 and getting a number less than 4

Solution

a. The events are mutually exclusive, since the first event can be 1, 3, or 5 and the second event can be 2, 4, or 6.

b. The events are not mutually exclusive, since the first event is a 3 and the second can be 1, 3, or 5. Hence, 3 is contained in both events.

c. The events are not mutually exclusive, since the first event can be 1, 3, or 5 and the second can be 1, 2, or 3. Hence, 1 and 3 are contained in both events.

d. The events are mutually exclusive, since the first event can be 5 or 6 and the second event can be 1, 2, or 3.

Example 4–16

Determine which events are mutually exclusive and which are not, when a single card is drawn from a deck.

a. Getting a 7 and getting a jack
b. Getting a club and getting a king
c. Getting a face card and getting an ace
d. Getting a face card and getting a spade

Solution

Only the events in parts *a* and *c* are mutually exclusive.

The probability of two or more events can be determined by the *addition rules.* The first addition rule is used when the events are mutually exclusive.

Addition Rule 1

When two events A and B are mutually exclusive, the probability that A or B will occur is

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 4–17

A box contains 3 glazed doughnuts, 4 jelly doughnuts, and 5 chocolate doughnuts. If a person selects a doughnut at random, find the probability that it is either a glazed doughnut or a chocolate doughnut.

Solution

Since the box contains 3 glazed doughnuts, 5 chocolate doughnuts, and a total of 12 doughnuts, $P(\text{glazed or chocolate}) = P(\text{glazed}) + P(\text{chocolate}) = \frac{3}{12} + \frac{5}{12} = \frac{8}{12} = \frac{2}{3}$. The events are mutually exclusive.

Example 4–18

At a political rally, there are 20 Republicans, 13 Democrats, and 6 Independents. If a person is selected at random, find the probability that he or she is either a Democrat or an Independent.

Solution

$$P(\text{Democrat or Independent}) = P(\text{Democrat}) + P(\text{Independent})$$
$$= \frac{13}{39} + \frac{6}{39} = \frac{19}{39}$$

Example 4–19

A day of the week is selected at random. Find the probability that it is a weekend day.

Solution

$$P(\text{Saturday or Sunday}) = P(\text{Saturday}) + P(\text{Sunday}) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$$

When two events are not mutually exclusive, we must subtract one of the two probabilities of the outcomes that are common to both events, since they have been counted twice. This technique is illustrated in Example 4–20.

Example 4–20

A single card is drawn from a deck. Find the probability that it is a king or a club.

Solution

Since the king of clubs means a king and a club, it has been counted twice—once as a king and once as a club; therefore, one of the outcomes must be subtracted, as shown.

$$P(\text{king or club}) = P(\text{king}) + P(\text{club}) - P(\text{king of clubs})$$
$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

When events are not mutually exclusive, addition rule 2 can be used to find the probability of the events.

Addition Rule 2

If A and B are *not* mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Note: This rule can also be used when the events are mutually exclusive, since $P(A \text{ and } B)$ will always equal 0. However, it is important to make a distinction between the two situations.

Example 4–21

In a hospital unit there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

Solution

The sample space is shown here.

Staff	Females	Males	Total
Nurses	7	1	8
Physicians	3	2	5
Total	10	3	13

The probability is

$$P(\text{nurse or male}) = P(\text{nurse}) + P(\text{male}) - P(\text{male nurse})$$
$$= \frac{8}{13} + \frac{3}{13} - \frac{1}{13} = \frac{10}{13}$$

Example 4–22

On New Year's Eve, the probability of a person driving while intoxicated is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving accident while intoxicated is 0.06. What is the probability of a person driving while intoxicated or having a driving accident?

Solution

$$\begin{aligned} P(\text{intoxicated or accident}) &= P(\text{intoxicated}) + P(\text{accident}) \\ &\quad - P(\text{intoxicated and accident}) \\ &= 0.32 + 0.09 - 0.06 = 0.35 \end{aligned}$$

In summary, then, when the two events are mutually exclusive, use addition rule 1. When the events are not mutually exclusive, use addition rule 2.

The probability rules can be extended to three or more events. For three mutually exclusive events A, B, and C,

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$$

For three events that are *not* mutually exclusive,

$$\begin{aligned} P(A \text{ or } B \text{ or } C) = &\ P(A) + P(B) + P(C) - P(A \text{ and } B) - P(A \text{ and } C) \\ &- P(B \text{ and } C) + P(A \text{ and } B \text{ and } C) \end{aligned}$$

See Exercises 23, 24, and 25 in this section.

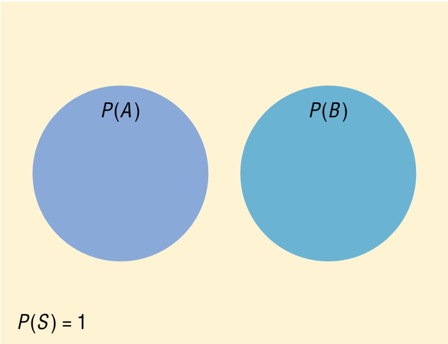
Figure 4–5(a) shows a Venn diagram that represents two mutually exclusive events A and B. In this case, $P(A \text{ or } B) = P(A) + P(B)$, since these events are mutually exclusive and do not overlap. In other words, the probability of occurrence of event A or event B is the sum of the areas of the two circles.

Figure 4–5(b) represents the probability of two events that are *not* mutually exclusive. In this case, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. The area in the intersection or overlapping part of both circles corresponds to $P(A \text{ and } B)$; and when the area of circle A is added to the area of circle B, the overlapping part is counted twice. It must therefore be subtracted once to get the correct area or probability.

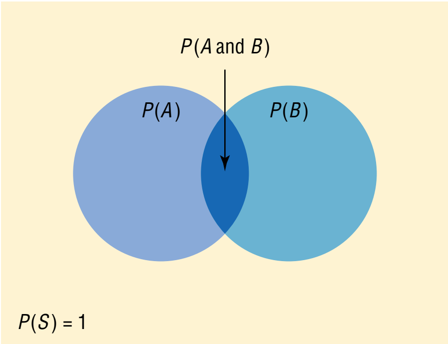
Note: Venn diagrams were developed by mathematician John Venn (1834–1923) and are used in set theory and symbolic logic. They have been adapted to probability theory also. In set theory, the symbol $\cup$ represents the *union* of two sets, and $A \cup B$ corresponds to A or B. The symbol $\cap$ represents the *intersection* of two sets, and $A \cap B$ corresponds to A and B. Venn diagrams show only a general picture of the probability rules and do not portray all situations, such as $P(A) = 0$, accurately.

Figure 4–5

Venn Diagrams for the Addition Rules



(a) Mutually exclusive events
$P(A \text{ or } B) = P(A) + P(B)$



(b) Non-mutually exclusive events
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Applying the Concepts 4–3

Which Pain Reliever Is Best?

Assume that following an injury you received from playing your favorite sport, you obtain and read information on new pain medications. In that information you read of a study that was conducted to test the side effects of two new pain medications. Use the following table to answer the questions and decide which, if any, of the two new pain medications you will use.

	Number of side effects in 12-week clinical trial		
Side effect	**Placebo** $n = 192$	**Drug A** $n = 186$	**Drug B** $n = 188$
Upper respiratory congestion	10	32	19
Sinus headache	11	25	32
Stomach ache	2	46	12
Neurological headache	34	55	72
Cough	22	18	31
Lower respiratory congestion	2	5	1

1. How many subjects were in the study?
2. How long was the study?
3. What were the variables under study?
4. What type of variables are they, and what level of measurement are they on?
5. Are the numbers in the table exact figures?
6. What is the probability that a randomly selected person was receiving a placebo?
7. What is the probability that a person was receiving a placebo or drug A? Are these mutually exclusive events? What is the complement to this event?
8. What is the probability that a randomly selected person was receiving a placebo or experienced a neurological headache?
9. What is the probability that a randomly selected person was not receiving a placebo or experienced a sinus headache?

See page 234 for the answers.

Exercises 4–3

1. Define mutually exclusive events, and give an example of two events that are mutually exclusive and two events that are not mutually exclusive.

2. Determine whether these events are mutually exclusive.

a. Roll a die: Get an even number, and get a number less than 3.

b. Roll a die: Get a prime number (2, 3, 5), and get an odd number.

c. Roll a die: Get a number greater than 3, and get a number less than 3.

d. Select a student in your class: The student has blond hair, and the student has blue eyes.

e. Select a student in your college: The student is a sophomore, and the student is a business major.

f. Select any course: It is a calculus course, and it is an English course.

g. Select a registered voter: The voter is a Republican, and the voter is a Democrat.

3. An automobile dealer decides to select a month for its annual sale. Find the probability that it will be September or October. Assume all months have an equal probability of being selected. Compute the probability of selecting September or October, using days, and compare the answers.

4. At a community swimming pool there are 2 managers, 8 lifeguards, 3 concession stand clerks, and 2 maintenance people. If a person is selected at random, find the probability that the person is either a lifeguard or a manager.

5. At a convention there are 7 mathematics instructors, 5 computer science instructors, 3 statistics instructors, and 4 science instructors. If an instructor is selected, find the probability of getting a science instructor or a math instructor.

6. A media rental store rented the following number of movie titles in each of these categories: 170 horror, 230 drama, 120 mystery, 310 romance, and 150 comedy. If a person selects a movie to rent, find the probability that it is a romance or a comedy. Is this event likely or unlikely to occur? Explain your answer.

7. A recent study of 200 nurses found that of 125 female nurses, 56 had bachelor's degrees; and of 75 male nurses, 34 had bachelor's degrees. If a nurse is selected at random, find the probability that the nurse is

a. A female nurse with a bachelor's degree
b. A male nurse
c. A male nurse with a bachelor's degree
d. Based on your answers to parts *a*, *b*, and *c*, explain which is most likely to occur. Explain why.

8. The probability that a student owns a car is 0.65, and the probability that a student owns a computer is 0.82. If the probability that a student owns both is 0.55, what is the probability that a given student owns neither a car nor a computer?

9. At a particular school with 200 male students, 58 play football, 40 play basketball, and 8 play both. What is the probability that a randomly selected male student plays neither sport?

10. A single card is drawn from a deck. Find the probability of selecting the following.

a. A 4 or a diamond
b. A club or a diamond
c. A jack or a black card

11. In a statistics class there are 18 juniors and 10 seniors; 6 of the seniors are females, and 12 of the juniors are males. If a student is selected at random, find the probability of selecting the following.

a. A junior or a female
b. A senior or a female
c. A junior or a senior

12. At a used-book sale, 100 books are adult books and 160 are children's books. Of the adult books, 70 are nonfiction while 60 of the children's books are nonfiction. If a book is selected at random, find the probability that it is

a. Fiction
b. Not a children's nonfiction book
c. An adult book or a children's nonfiction book

13. The Bargain Auto Mall has these cars in stock.

	SUV	Compact	Mid-sized
Foreign	20	50	20
Domestic	65	100	45

If a car is selected at random, find the probability that it is

a. Domestic
b. Foreign and mid-sized
c. Domestic or an SUV

14. The numbers of endangered species for several groups are listed here.

	Mammals	Birds	Reptiles	Amphibians
United States	63	78	14	10
Foreign	251	175	64	8

If one endangered species is selected at random, find the probability that it is

a. Found in the United States and is a bird
b. Foreign or a mammal
c. Warm-blooded

Source: *N.Y. Times Almanac.*

15. A grocery store employs cashiers, stock clerks, and deli personnel. The distribution of employees according to marital status is shown here.

Marital status	Cashiers	Stock clerks	Deli personnel
Married	8	12	3
Not married	5	15	2

If an employee is selected at random, find these probabilities.

a. The employee is a stock clerk or married.
b. The employee is not married.
c. The employee is a cashier or is not married.

16. In a certain geographic region, newspapers are classified as being published daily morning, daily evening, and weekly. Some have a comics section and others do not. The distribution is shown here.

Have comics section	Morning	Evening	Weekly
Yes	2	3	1
No	3	4	2

If a newspaper is selected at random, find these probabilities.

a. The newspaper is a weekly publication.

b. The newspaper is a daily morning publication or has comics.

c. The newspaper is published weekly or does not have comics.

17. Three cable channels (6, 8, and 10) have quiz shows, comedies, and dramas. The number of each is shown here.

Type of show	Channel 6	Channel 8	Channel 10
Quiz show	5	2	1
Comedy	3	2	8
Drama	4	4	2

If a show is selected at random, find these probabilities.

a. The show is a quiz show, or it is shown on channel 8.

b. The show is a drama or a comedy.

c. The show is shown on channel 10, or it is a drama.

18. A local postal carrier distributes first-class letters, advertisements, and magazines. For a certain day, she distributed the following numbers of each type of item.

Delivered to	First-class letters	Ads	Magazines
Home	325	406	203
Business	732	1021	97

If an item of mail is selected at random, find these probabilities.

a. The item went to a home.

b. The item was an ad, or it went to a business.

c. The item was a first-class letter, or it went to a home.

19. The frequency distribution shown here illustrates the number of medical tests conducted on 30 randomly selected emergency patients.

Number of tests performed	Number of patients
0	12
1	8
2	2
3	3
4 or more	5

If a patient is selected at random, find these probabilities.

a. The patient has had exactly 2 tests done.

b. The patient has had at least 2 tests done.

c. The patient has had at most 3 tests done.

d. The patient has had 3 or fewer tests done.

e. The patient has had 1 or 2 tests done.

20. This distribution represents the length of time a patient spends in a hospital.

Days	Frequency
0–3	2
4–7	15
8–11	8
12–15	6
16+	9

If a patient is selected, find these probabilities.

a. The patient spends 3 days or fewer in the hospital.

b. The patient spends fewer than 8 days in the hospital.

c. The patient spends 16 or more days in the hospital.

d. The patient spends a maximum of 11 days in the hospital.

21. A sales representative who visits customers at home finds she sells 0, 1, 2, 3, or 4 items according to the following frequency distribution.

Items sold	Frequency
0	8
1	10
2	3
3	2
4	1

Find the probability that she sells the following.

a. Exactly 1 item

b. More than 2 items

c. At least 1 item

d. At most 3 items

22. A recent study of 300 patients found that of 100 alcoholic patients, 87 had elevated cholesterol levels, and of 200 nonalcoholic patients, 43 had elevated cholesterol levels. If a patient is selected at random, find the probability that the patient is the following.

a. An alcoholic with elevated cholesterol level

b. A nonalcoholic

c. A nonalcoholic with nonelevated cholesterol level

23. If one card is drawn from an ordinary deck of cards, find the probability of getting the following.

a. A king or a queen or a jack

b. A club or a heart or a spade

c. A king or a queen or a diamond

d. An ace or a diamond or a heart

e. A 9 or a 10 or a spade or a club

24. Two dice are rolled. Find the probability of getting

a. A sum of 5, 6, or 7

b. Doubles or a sum of 6 or 8

c. A sum greater than 8 or less than 3

d. Based on the answers to parts *a*, *b*, and *c*, which is least likely to occur? Explain why.

25. An urn contains 6 red balls, 2 green balls, 1 blue ball, and 1 white ball. If a ball is drawn, find the probability of getting a red or a white ball.

26. Three dice are rolled. Find the probability of getting

a. Triples *b.* A sum of 5

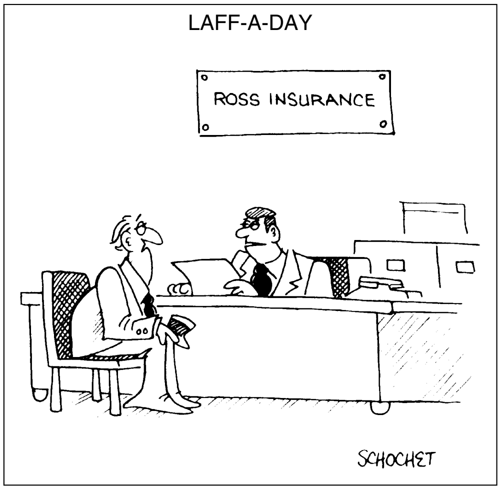
Extending the Concepts

27. The probability that a customer selects a pizza with mushrooms or pepperoni is 0.43, and the probability that the customer selects only mushrooms is 0.32. If the probability that he or she selects only pepperoni is 0.17, find the probability of the customer selecting both items.

28. In building new homes, a contractor finds that the probability of a home buyer selecting a two-car garage is 0.70 and of selecting a one-car garage is 0.20. Find the probability that the buyer will select no garage. The builder does not build houses with three-car or more garages.

29. In Exercise 28, find the probability that the buyer will not want a two-car garage.

30. Suppose that $P(A) = 0.42$, $P(B) = 0.38$, and $P(A \cup B) = 0.70$. Are A and B mutually exclusive? Explain.



"I know you haven't had an accident in thirteen years. We're raising your rates because you're about due one."

Source: Reprinted with special permission of King Features Syndicate.

Technology *Step by Step*

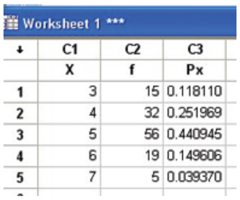
MINITAB Step by Step

Calculate Relative Frequency Probabilities

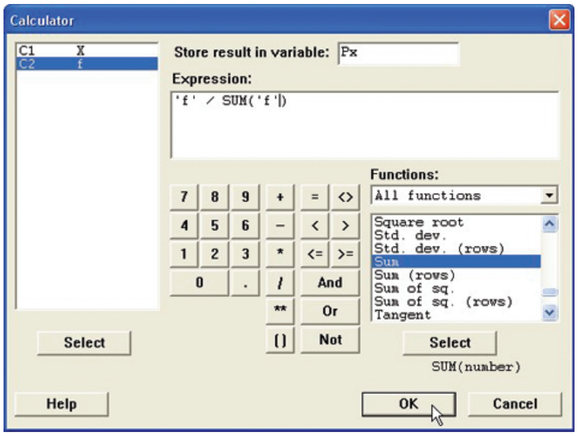
The random variable X represents the number of days patients stayed in the hospital from Example 4–14.

1. In C1 of a worksheet, type in the values of X. Name the column **X.**
2. In C2 enter the frequencies. Name the column **f.**
3. To calculate the relative frequencies and store them in a new column named Px:
   a) Select **Calc>Calculator.**
   b) Type **Px** in the box for Store result in variable:.
   c) Click in the Expression box, then double-click C2 f.
   d) Type or click the division operator.
   e) Scroll down the function list to Sum, then click [Select].
   f) Double-click C2 f to select it.
   g) Click [OK].

The dialog box and completed worksheet are shown.



↓	C1	C2	C3
	X	f	Px
1	3	15	0.118110
2	4	32	0.251969
3	5	56	0.440945
4	6	19	0.149606
5	7	5	0.039370



If the original data, rather than the table, are in a worksheet, use **Stat>Tables>Tally** to make the tables with percents (Section 2–2).

MINITAB can also make a two-way classification table.

Construct a Contingency Table

1. Select **File>Open Worksheet** to open the Databank.mtw file.
2. Select **Stat>Tables>Crosstabulation . . .**
   a) Double-click C4 SMOKING STATUS to select it For rows:.
   b) Select C11 GENDER for the For Columns: Field.
   c) Click on option for Counts and then [OK].

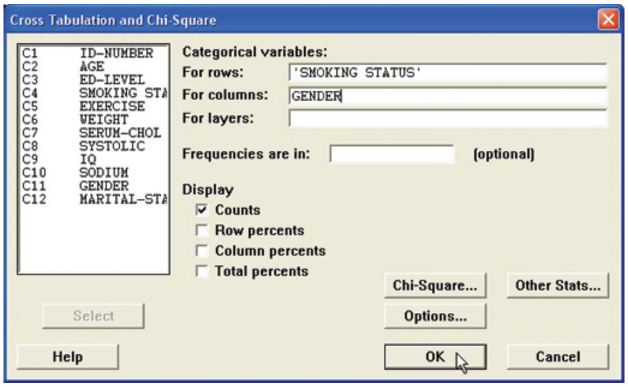
The session window and completed dialog box are shown.

Tabulated statistics: SMOKING STATUS, GENDER

Rows: SMOKING STATUS Columns: GENDER

	F	M	All
0	25	22	47
1	18	19	37
2	7	9	16
All	50	50	100

Cell Contents: Count



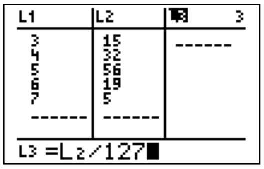
In this sample of 100 there are 25 females who do not smoke compared to 22 men. Sixteen individuals smoke 1 pack or more per day.

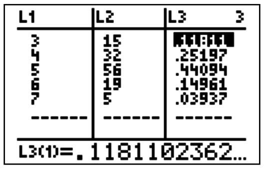
TI-83 Plus or TI-84 Plus
Step by Step

To construct a relative frequency table:

1. Enter the data values in L_1 and the frequencies in L_2.
2. Move the cursor to the top of the L_3 column so that L_3 is highlighted.
3. Type L_2 divided by the sample size, then press **ENTER.**

Use the data from Example 4–14.



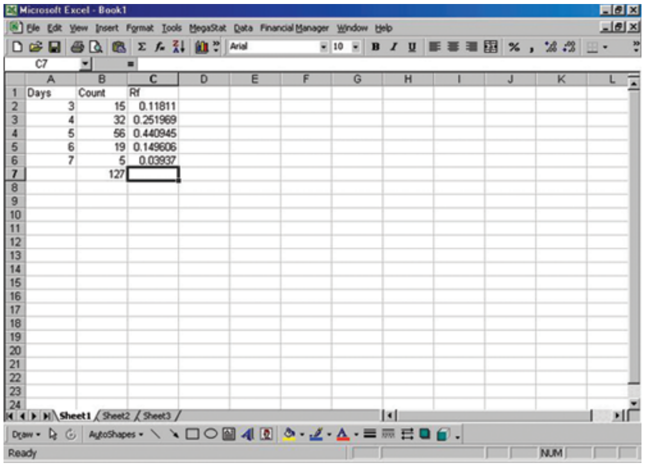


Excel
Step by Step

Constructing a Relative Frequency Distribution

Use the data from Example 4–14.

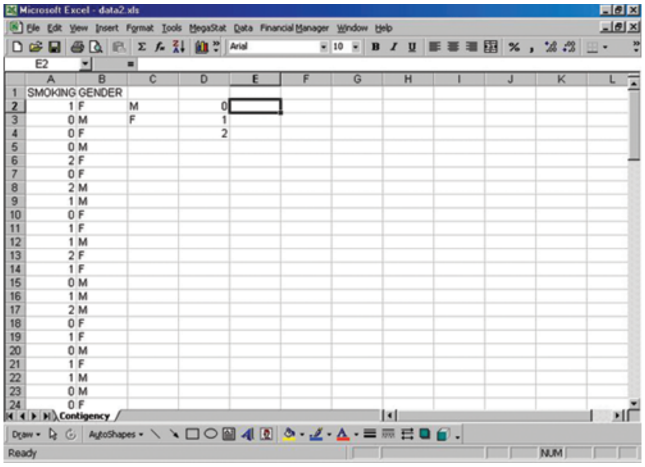
1. In a new worksheet, type the label **DAYS** in cell A1. Beginning in cell A2, type in the data from this variable.
2. In cell B1, type the label for the frequency, **COUNT.** Beginning in cell B2, type in the frequencies.
3. In the cell below the frequencies, compute the sum of the frequencies by selecting the sum icon (Σ) from the toolbar. Highlight the frequencies in column B and **Enter.**
4. In cell C1, type a label for the relative frequencies, **Rf.** Select cell C2 and type **=(B2)/(B7)** and **Enter.** Repeat for each frequency.



Constructing a Contingency Table

Although Excel was not specifically designed to construct a contingency table, you may do so with the use of the MegaStat Add-in available on your CD and Online Learning Center.

1. Select **File>Open** to open the Databank.xls file as directed in Example XL1–1, on page 24.
2. Highlight the column labeled SMOKING STATUS and select **Edit>Copy.**
3. On the toolbar, select **File>New** to open a new worksheet. Paste the contents of the copied column in column A of the new worksheet by selecting cell A1, then **Edit>Paste.**
4. Return to the previous file Databank.xls by selecting **File>DataBank.xls.**
5. Highlight the column labeled GENDER. Copy and paste these data in column B of the worksheet with the SMOKING STATUS data.
6. In an open cell on the new worksheet, type **M** for male and in the cell directly below in the same column, type **F** for female. In an open column, type in the categories for the SMOKING STATUS data **0, 1,** and **2** in separate cells.



7. Select **MegaStat** from the toolbar. Select **Chi-Square/Crosstab>Crosstabulation.**
8. In the first Data range box, type **A1:A101.** In the Specification range box, type in the range of cells containing the labels for the values of the SMOKING STATUS variable.
9. In the second Data range box, type **B1:B101.** In the Specification range box, type in the range of cells containing the labels for the values of the GENDER variable.
10. Remove any checks from the Output options in the Crosstabulation dialog box. Then click [OK].

This table is obtained in a new sheet labeled Output.

Crosstabulation

		GENDER		
		M	F	Total
SMOKING STATUS	0	22	25	47
	1	19	18	37
	2	9	7	16
	Total	50	50	100

4–4 The Multiplication Rules and Conditional Probability

Section 4–3 showed that the addition rules are used to compute probabilities for mutually exclusive and non-mutually exclusive events. This section introduces the multiplication rules.

Objective 3

Find the probability of compound events, using the multiplication rules.

The Multiplication Rules

The *multiplication rules* can be used to find the probability of two or more events that occur in sequence. For example, if a coin is tossed and then a die is rolled, one can find the probability of getting a head on the coin *and* a 4 on the die. These two events are said to be *independent* since the outcome of the first event (tossing a coin) does not affect the probability outcome of the second event (rolling a die).

Two events *A* and *B* are **independent events** if the fact that *A* occurs does not affect the probability of *B* occurring.

Here are other examples of independent events:

Rolling a die and getting a 6, and then rolling a second die and getting a 3.

Drawing a card from a deck and getting a queen, replacing it, and drawing a second card and getting a queen.

To find the probability of two independent events that occur in sequence, one must find the probability of each event occurring separately and then multiply the answers. For example, if a coin is tossed twice, the probability of getting two heads is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. This result can be verified by looking at the sample space HH, HT, TH, TT. Then $P(\text{HH}) = \frac{1}{4}$.

Multiplication Rule 1

When two events are independent, the probability of both occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example 4–23

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution

$$P(\text{head and } 4) = P(\text{head}) \cdot P(4) = \tfrac{1}{2} \cdot \tfrac{1}{6} = \tfrac{1}{12}$$

Note that the sample space for the coin is H, T; and for the die it is 1, 2, 3, 4, 5, 6.

The problem in Example 4–23 can also be solved by using the sample space

H1 H2 H3 H4 H5 H6 T1 T2 T3 T4 T5 T6

The solution is $\frac{1}{12}$, since there is only one way to get the head-4 outcome.

Example 4–24

A card is drawn from a deck and replaced; then a second card is drawn. Find the probability of getting a queen and then an ace.

Solution

The probability of getting a queen is $\frac{4}{52}$, and since the card is replaced, the probability of getting an ace is $\frac{4}{52}$. Hence, the probability of getting a queen and an ace is

$$P(\text{queen and ace}) = P(\text{queen}) \cdot P(\text{ace}) = \frac{4}{52} \cdot \frac{4}{52} = \frac{16}{2704} = \frac{1}{169}$$

Example 4–25

An urn contains 3 red balls, 2 blue balls, and 5 white balls. A ball is selected and its color noted. Then it is replaced. A second ball is selected and its color noted. Find the probability of each of these.

a. Selecting 2 blue balls

b. Selecting 1 blue ball and then 1 white ball

c. Selecting 1 red ball and then 1 blue ball

Solution

a. $P(\text{blue and blue}) = P(\text{blue}) \cdot P(\text{blue}) = \frac{2}{10} \cdot \frac{2}{10} = \frac{4}{100} = \frac{1}{25}$

b. $P(\text{blue and white}) = P(\text{blue}) \cdot P(\text{white}) = \frac{2}{10} \cdot \frac{5}{10} = \frac{10}{100} = \frac{1}{10}$

c. $P(\text{red and blue}) = P(\text{red}) \cdot P(\text{blue}) = \frac{3}{10} \cdot \frac{2}{10} = \frac{6}{100} = \frac{3}{50}$

Multiplication rule 1 can be extended to three or more independent events by using the formula

$$P(A \text{ and } B \text{ and } C \text{ and } \ldots \text{ and } K) = P(A) \cdot P(B) \cdot P(C) \cdots P(K)$$

When a small sample is selected from a large population and the subjects are not replaced, the probability of the event occurring changes so slightly that for the most part, it is considered to remain the same. Examples 4–26 and 4–27 illustrate this concept.

Example 4–26

A Harris poll found that 46% of Americans say they suffer great stress at least once a week. If three people are selected at random, find the probability that all three will say that they suffer great stress at least once a week.

Source: 100% American.

Solution

Let S denote stress. Then

$$\begin{aligned} P(S \text{ and } S \text{ and } S) &= P(S) \cdot P(S) \cdot P(S) \\ &= (0.46)(0.46)(0.46) \approx 0.097 \end{aligned}$$

Example 4–27

Approximately 9% of men have a type of color blindness that prevents them from distinguishing between red and green. If 3 men are selected at random, find the probability that all of them will have this type of red-green color blindness.

Source: USA TODAY.

Solution

Let C denote red-green color blindness. Then

$$\begin{aligned} P(C \text{ and } C \text{ and } C) &= P(C) \cdot P(C) \cdot P(C) \\ &= (0.09)(0.09)(0.09) \\ &= 0.000729 \end{aligned}$$

Hence, the rounded probability is 0.0007.

In Examples 4–23 through 4–27, the events were independent of one another, since the occurrence of the first event in no way affected the outcome of the second event. On the other hand, when the occurrence of the first event changes the probability of the occurrence of the second event, the two events are said to be *dependent.* For example, suppose a card is drawn from a deck and *not* replaced, and then a second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Before an answer to the question can be given, one must realize that the events are dependent. The probability of selecting an ace on the first draw is $\frac{4}{52}$. If that card is *not* replaced, the probability of selecting a king on the second card is $\frac{4}{51}$, since there are 4 kings and 51 cards remaining. The outcome of the first draw has affected the outcome of the second draw.

Dependent events are formally defined now.

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be **dependent events.**

Here are some examples of dependent events:

Drawing a card from a deck, not replacing it, and then drawing a second card.

Selecting a ball from an urn, not replacing it, and then selecting a second ball.

Being a lifeguard and getting a suntan.

Having high grades and getting a scholarship.

Parking in a no-parking zone and getting a parking ticket.

To find probabilities when events are dependent, use the multiplication rule with a modification in notation. For the problem just discussed, the probability of getting an ace on the first draw is $\frac{4}{52}$, and the probability of getting a king on the second draw is $\frac{4}{51}$. By the multiplication rule, the probability of both events occurring is

$$\frac{4}{52} \cdot \frac{4}{51} = \frac{16}{2652} = \frac{4}{663}$$

The event of getting a king on the second draw *given* that an ace was drawn the first time is called a *conditional probability.*

The **conditional probability** of an event B in relationship to an event A is the probability that event B occurs after event A has already occurred. The notation for conditional probability is $P(B|A)$. This notation does not mean that B is divided by A; rather, it means the probability that event B occurs given that event A has already occurred. In the card example, $P(B|A)$ is the probability that the second card is a king given that the first card is an ace, and it is equal to $\frac{4}{51}$ since the first card was *not* replaced.

Multiplication Rule 2

When two events are dependent, the probability of both occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Example 4–28

A person owns a collection of 30 CDs, of which 5 are country music. If 2 CDs are selected at random, find the probability that both are country music.

Solution

Since the events are dependent,

$$P(C_1 \text{ and } C_2) = P(C_1) \cdot P(C_2|C_1) = \frac{5}{30} \cdot \frac{4}{29} = \frac{20}{870} = \frac{2}{87}$$

Example 4–29

The World Wide Insurance Company found that 53% of the residents of a city had homeowner's insurance (H) with the company. Of these clients, 27% also had automobile insurance (A) with the company. If a resident is selected at random, find the probability that the resident has both homeowner's and automobile insurance with the World Wide Insurance Company.

Solution

$$P(\text{H and A}) = P(\text{H}) \cdot P(\text{A}|\text{H}) = (0.53)(0.27) = 0.1431$$

This multiplication rule can be extended to three or more events, as shown in Example 4–30.

Example 4–30

Three cards are drawn from an ordinary deck and not replaced. Find the probability of these.

a. Getting 3 jacks
b. Getting an ace, a king, and a queen in order
c. Getting a club, a spade, and a heart in order
d. Getting 3 clubs

Solution

a. $P(\text{3 jacks}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{24}{132{,}600} = \frac{1}{5525}$

b. $P(\text{ace and king and queen}) = \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = \frac{64}{132{,}600} = \frac{8}{16{,}575}$

c. $P(\text{club and spade and heart}) = \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} = \frac{2197}{132{,}600} = \frac{169}{10{,}200}$

d. $P(\text{3 clubs}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} = \frac{1716}{132{,}600} = \frac{11}{850}$

Tree diagrams can be used as an aid to finding the solution to probability problems when the events are sequential. Example 4–31 illustrates the use of tree diagrams.

Example 4–31

Box 1 contains 2 red balls and 1 blue ball. Box 2 contains 3 blue balls and 1 red ball. A coin is tossed. If it falls heads up, box 1 is selected and a ball is drawn. If it falls tails up, box 2 is selected and a ball is drawn. Find the probability of selecting a red ball.

Solution

With the use of a tree diagram, the sample space can be determined as shown in Figure 4–6. First, assign probabilities to each branch. Next, using the multiplication rule, multiply the probabilities for each branch.

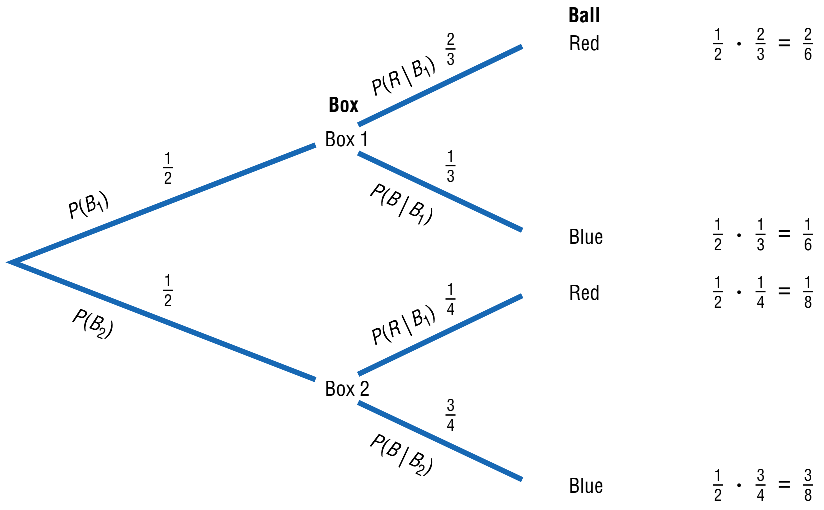
Finally, use the addition rule, since a red ball can be obtained from box 1 or box 2.

$$P(\text{red}) = \tfrac{2}{6} + \tfrac{1}{8} = \tfrac{8}{24} + \tfrac{3}{24} = \tfrac{11}{24}$$

(*Note:* The sum of all final probabilities will always be equal to 1.)

Figure 4–6

Tree Diagram for Example 4–31



Tree diagrams can be used when the events are independent or dependent, and they can also be used for sequences of three or more events.

Conditional Probability

Objective 4

Find the conditional probability of an event.

The conditional probability of an event B in relationship to an event A was defined as the probability that event B occurs after event A has already occurred.

The conditional probability of an event can be found by dividing both sides of the equation for multiplication rule 2 by $P(A)$, as shown:

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

$$\frac{P(A \text{ and } B)}{P(A)} = \frac{\cancel{P(A)} \cdot P(B|A)}{\cancel{P(A)}}$$

$$\frac{P(A \text{ and } B)}{P(A)} = P(B|A)$$

Formula for Conditional Probability

The probability that the second event B occurs given that the first event A has occurred can be found by dividing the probability that both events occurred by the probability that the first event has occurred. The formula is

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Examples 4–32, 4–33, and 4–34 illustrate the use of this rule.

Example 4–32

A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip *and* a white chip is $\frac{15}{56}$, and the probability of selecting a black chip on the first draw is $\frac{3}{8}$, find the probability of selecting the white chip on the second draw, *given* that the first chip selected was a black chip.

Solution

Let

$$B = \text{selecting a black chip} \qquad W = \text{selecting a white chip}$$

Then

$$P(W|B) = \frac{P(B \text{ and } W)}{P(B)} = \frac{\frac{15}{56}}{\frac{3}{8}}$$

$$= \frac{15}{56} \div \frac{3}{8} = \frac{15}{56} \cdot \frac{8}{3} = \frac{\overset{5}{\cancel{15}}}{\underset{7}{\cancel{56}}} \cdot \frac{\overset{1}{\cancel{8}}}{\underset{1}{\cancel{3}}} = \frac{5}{7}$$

Hence, the probability of selecting a white chip on the second draw given that the first chip selected was black is $\frac{5}{7}$.

Example 4–33

The probability that Sam parks in a no-parking zone *and* gets a parking ticket is 0.06, and the probability that Sam cannot find a legal parking space and has to park in the no-parking zone is 0.20. On Tuesday, Sam arrives at school and has to park in a no-parking zone. Find the probability that he will get a parking ticket.

Solution

Let

$$N = \text{parking in a no-parking zone} \qquad T = \text{getting a ticket}$$

Then

$$P(T|N) = \frac{P(N \text{ and } T)}{P(N)} = \frac{0.06}{0.20} = 0.30$$

Hence, Sam has a 0.30 probability of getting a parking ticket, given that he parked in a no-parking zone.

The conditional probability of events occurring can also be computed when the data are given in table form, as shown in Example 4–34.

Example 4–34

A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown.

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

Find these probabilities.

a. The respondent answered yes, given that the respondent was a female.

b. The respondent was a male, given that the respondent answered no.

Solution

Let

M = respondent was a male Y = respondent answered yes
F = respondent was a female N = respondent answered no

a. The problem is to find $P(Y|F)$. The rule states

$$P(Y|F) = \frac{P(F \text{ and } Y)}{P(F)}$$

The probability $P(F$ and $Y)$ is the number of females who responded yes, divided by the total number of respondents:

$$P(F \text{ and } Y) = \frac{8}{100}$$

The probability $P(F)$ is the probability of selecting a female:

$$P(F) = \frac{50}{100}$$

Then

$$P(Y|F) = \frac{P(F \text{ and } Y)}{P(F)} = \frac{8/100}{50/100}$$

$$= \frac{8}{100} \div \frac{50}{100} = \frac{\overset{4}{\cancel{8}}}{\underset{1}{\cancel{100}}} \cdot \frac{\overset{1}{\cancel{100}}}{\underset{25}{\cancel{50}}} = \frac{4}{25}$$

b. The problem is to find $P(M|N)$.

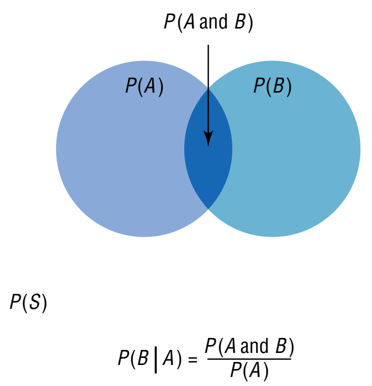
$$P(M|N) = \frac{P(N \text{ and } M)}{P(N)} = \frac{18/100}{60/100}$$

$$= \frac{18}{100} \div \frac{60}{100} = \frac{\overset{3}{\cancel{18}}}{\underset{1}{\cancel{100}}} \cdot \frac{\overset{1}{\cancel{100}}}{\underset{10}{\cancel{60}}} = \frac{3}{10}$$

The Venn diagram for conditional probability is shown in Figure 4–7. In this case,

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Figure 4–7
Venn Diagram for Conditional Probability



which is represented by the area in the intersection or overlapping part of the circles A and B, divided by the area of circle A. The reasoning here is that if one assumes A has occurred, then A becomes the sample space for the next calculation and is the denominator of the probability fraction $\frac{P(A \text{ and } B)}{P(A)}$. The numerator $P(A \text{ and } B)$ represents the probability of the part of B that is contained in A. Hence, $P(A \text{ and } B)$ becomes the numerator of the probability fraction $\frac{P(A \text{ and } B)}{P(A)}$. Imposing a condition reduces the sample space.

Probabilities for "At Least"

The multiplication rules can be used with the complementary event rule (Section 4–2) to simplify solving probability problems involving "at least." Examples 4–35, 4–36, and 4–37 illustrate how this is done.

Example 4–35

A game is played by drawing four cards from an ordinary deck and replacing each card after it is drawn. Find the probability of winning if at least one ace is drawn.

Solution

It is much easier to find the probability that no aces are drawn (i.e., losing) and then subtract that value from 1 than to find the solution directly, because that would involve finding the probability of getting one ace, two aces, three aces, and four aces and then adding the results.

Let E = at least one ace is drawn and $\overline{E}$ = no aces drawn. Then

$$P(\overline{E}) = \frac{48}{52} \cdot \frac{48}{52} \cdot \frac{48}{52} \cdot \frac{48}{52}$$

$$= \frac{12}{13} \cdot \frac{12}{13} \cdot \frac{12}{13} \cdot \frac{12}{13} = \frac{20{,}736}{28{,}561}$$

Hence,

$$P(E) = 1 - P(\overline{E})$$

$$P(\text{winning}) = 1 - P(\text{losing}) = 1 - \frac{20{,}736}{28{,}561} = \frac{7{,}825}{28{,}561} \approx 0.27$$

or a hand with at least one ace will win about 27% of the time.

Example 4–36

A coin is tossed 5 times. Find the probability of getting at least one tail.

Solution

It is easier to find the probability of the complement of the event, which is "all heads," and then subtract the probability from 1 to get the probability of at least one tail.

$$P(E) = 1 - P(\overline{E})$$

$$P(\text{at least 1 tail}) = 1 - P(\text{all heads})$$

$$P(\text{all heads}) = \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{32}$$

Hence,

$$P(\text{at least 1 tail}) = 1 - \tfrac{1}{32} = \tfrac{31}{32}$$

Example 4–37

The Neckware Association of America reported that 3% of ties sold in the United States are bow ties. If 4 customers who purchased a tie are randomly selected, find the probability that at least one purchased a bow tie.

Solution

Let E = at least one bow tie is purchased and $\overline{E}$ = no bow ties are purchased. Then

$$P(E) = 0.03 \qquad \text{and} \qquad P(\overline{E}) = 1 - 0.03 = 0.97$$

P(no bow ties are purchased) = (0.97)(0.97)(0.97)(0.97) ≈ 0.885; hence, P(at least one bow tie is purchased) = 1 − 0.885 = 0.115.

Similar methods can be used for problems involving "at most."

Applying the Concepts 4–4

Guilty or Innocent?

In July 1964, an elderly woman was mugged in Costa Mesa, California. In the vicinity of the crime a tall, bearded man sat waiting in a yellow car. Shortly after the crime was committed, a young, tall woman, wearing her blond hair in a ponytail, was seen running from the scene of the crime and getting into the car, which sped off. The police broadcast a description of the suspected muggers. Soon afterward, a couple fitting the description was arrested and convicted of the crime. Although the evidence in the case was largely circumstantial, the two people arrested were nonetheless convicted of the crime. The prosecutor based his entire case on basic probability theory, showing the unlikeness of another couple being in that area while having all the same characteristics that the elderly woman described. The following probabilities were used.

Characteristic	Assumed probability
Drives yellow car	1 out of 12
Man over 6 feet tall	1 out of 10
Man wearing tennis shoes	1 out of 4
Man with beard	1 out of 11
Woman with blond hair	1 out of 3
Woman with hair in a ponytail	1 out of 13
Woman over 6 feet tall	1 out of 100

1. Compute the probability of another couple being in that area with the same characteristics.
2. Would you use the addition or multiplication rule? Why?
3. Are the characteristics independent or dependent?
4. How are the computations affected by the assumption of independence or dependence?
5. Should any court case be based solely on probabilities?
6. Would you convict the couple who was arrested even if there were no eyewitnesses?
7. Comment on why in today's justice system no person can be convicted solely on the results of probabilities.
8. In actuality, aren't most court cases based on uncalculated probabilities?

See page 235 for the answers.

Exercises 4–4

1. State which events are independent and which are dependent.

a. Tossing a coin and drawing a card from a deck

b. Drawing a ball from an urn, not replacing it, and then drawing a second ball

c. Getting a raise in salary and purchasing a new car

d. Driving on ice and having an accident

e. Having a large shoe size and having a high IQ

f. A father being left-handed and a daughter being left-handed

g. Smoking excessively and having lung cancer

h. Eating an excessive amount of ice cream and smoking an excessive amount of cigarettes

2. If 37% of high school students said that they exercise regularly, find the probability that 5 randomly selected high school students will say that they exercise regularly. Would you consider this event likely or unlikely to occur? Explain your answer.

3. If 84% of all people who do aerobics are women, find the probability that if 2 people who do aerobics are randomly selected, both are women. Would you consider this event likely or unlikely to occur? Explain your answer.

4. The Gallup Poll reported that 52% of Americans used a seat belt the last time they got into a car. If four people are selected at random, find the probability that they all used a seat belt the last time they got into a car.

Source: *100% American.*

5. If 28% of U.S. medical degrees are conferred to women, find the probability that 3 randomly selected medical school graduates are men. Would you consider this event likely or unlikely to occur? Explain your answer.

6. If 25% of U.S. federal prison inmates are not U.S. citizens, find the probability that 2 randomly selected federal prison inmates will not be U.S. citizens.

Source: *Harper's Index.*

7. At a local university 54.3% of incoming first-year students have computers. If 3 students are selected at random, find the following probabilities.

a. None have computers.

b. At least one has a computer.

c. All have computers.

8. If 2 cards are selected from a standard deck of 52 cards without replacement, find these probabilities.

a. Both are spades.

b. Both are the same suit.

c. Both are kings.

9. Of the 216 players on major league soccer rosters, 80.1% are U.S. citizens. If 3 players are selected at random for an exhibition, what is the probability that all are U.S. citizens?

Source: *USA TODAY.*

10. If one-half of Americans believe that the federal government should take "primary responsibility" for eliminating poverty, find the probability that 3 randomly selected Americans will agree that it is the federal government's responsibility to eliminate poverty.

Source: *Harper's Index.*

11. Of fans who own sports league–licensed apparel, 31% have NFL apparel. If 3 of these fans are selected at random, what is the probability that all have NFL apparel?

Source: ESPN Chilton Sports Poll.

12. A flashlight has 6 batteries, 2 of which are defective. If 2 are selected at random without replacement, find the probability that both are defective.

13. Eighty-eight percent of U.S. children are covered by some type of health insurance. If 4 children are selected at random, what is the probability that none are covered?

Source: Federal Interagency Forum on Child and Family Statistics.

14. The U.S. Department of Justice reported that 6% of all U.S. murders are committed without a weapon. If 3 murder cases are selected at random, find the probability that a weapon was not used in any one of them.

Source: *100% American.*

15. In a department store there are 120 customers, 90 of whom will buy at least one item. If 5 customers are selected at random, one by one, find the probability that all will buy at least one item.

16. Three cards are drawn from a deck *without* replacement. Find these probabilities.

a. All are jacks.

b. All are clubs.

c. All are red cards.

17. In a scientific study there are 8 guinea pigs, 5 of which are pregnant. If 3 are selected at random without replacement, find the probability that all are pregnant.

18. In Exercise 17, find the probability that none are pregnant.

19. In a civic organization, there are 38 members; 15 are men and 23 are women. If 3 members are selected to plan the July 4th parade, find the probability that all 3 are women. Would you consider this event likely or unlikely to occur? Explain your answer.

20. In Exercise 19, find the probability that all 3 members are men.

21. A manufacturer makes two models of an item: model I, which accounts for 80% of unit sales, and model II, which accounts for 20% of unit sales. Because of defects, the manufacturer has to replace (or exchange) 10% of its model I and 18% of its model II. If a model is selected at random, find the probability that it will be defective.

22. An automobile manufacturer has three factories, A, B, and C. They produce 50, 30, and 20%, respectively, of a specific model of car. Thirty percent of the cars produced in factory A are white, 40% of those produced in factory B are white, and 25% produced in factory C are white. If an automobile produced by the company is selected at random, find the probability that it is white.

23. An insurance company classifies drivers as low-risk, medium-risk, and high-risk. Of those insured, 60% are low-risk, 30% are medium-risk, and 10% are high-risk. After a study, the company finds that during a 1-year period, 1% of the low-risk drivers had an accident, 5% of the medium-risk drivers had an accident, and 9% of the high-risk drivers had an accident. If a driver is selected at random, find the probability that the driver will have had an accident during the year.

24. In a certain geographic location, 25% of the wage earners have a college degree and 75% do not. Of those who have a college degree, 5% earn more than $100,000 a year. Of those who do not have a college degree, 2% earn more than $100,000 a year. If a wage earner is selected at random, find the probability that she or he earns more than $100,000 a year.

25. Urn 1 contains 5 red balls and 3 black balls. Urn 2 contains 3 red balls and 1 black ball. Urn 3 contains 4 red balls and 2 black balls. If an urn is selected at random and a ball is drawn, find the probability it will be red.

26. For a recent year, 0.99 of the incarcerated population is adults and 0.07 is female. If an incarcerated person is selected at random, find the probability that the person is a female given that she is an adult.

Source: Bureau of Justice.

27. In a certain city, the probability that an automobile will be stolen and found within one week is 0.0009. The probability that an automobile will be stolen is 0.0015. Find the probability that a stolen automobile will be found within one week.

28. A circuit to run a model railroad has 8 switches. Two are defective. If a person selects 2 switches at random and tests them, find the probability that the second one is defective, given that the first one is defective.

29. At the Avonlea Country Club, 73% of the members play bridge and swim, and 82% play bridge. If a member is selected at random, find the probability that the member swims, given that the member plays bridge.

30. At a large university, the probability that a student takes calculus and is on the dean's list is 0.042. The probability that a student is on the dean's list is 0.21. Find the probability that the student is taking calculus, given that he or she is on the dean's list.

31. In Rolling Acres Housing Plan, 42% of the houses have a deck and a garage; 60% have a deck. Find the probability that a home has a garage, given that it has a deck.

32. In a pizza restaurant, 95% of the customers order pizza. If 65% of the customers order pizza and a salad, find the probability that a customer who orders pizza will also order a salad.

33. At an exclusive country club, 68% of the members play bridge and drink champagne, and 83% play bridge. If a member is selected at random, find the probability that the member drinks champagne, given that he or she plays bridge.

34. Eighty students in a school cafeteria were asked if they favored a ban on smoking in the cafeteria. The results of the survey are shown in the table.

Class	Favor	Oppose	No opinion
Freshman	15	27	8
Sophomore	23	5	2

If a student is selected at random, find these probabilities.

a. Given that the student is a freshman, he or she opposes the ban.

b. Given that the student favors the ban, the student is a sophomore.

35. Consider this table concerning utility patents granted for a specific year.

	Corporation	Government	Individual
United States	70,894	921	6129
Foreign	63,182	104	6267

Select one patent at random.

a. What is the probability that it is a foreign patent, given that it was issued to a corporation?

b. What is the probability that it was issued to an individual, given that it was a U.S. patent?

Source: *N.Y. Times Almanac.*

36. The medal distribution from the 2000 Summer Olympic Games is shown in the table.

	Gold	Silver	Bronze
United States	39	25	33
Russia	32	28	28
China	28	16	15
Australia	16	25	17
Others	186	205	235

Choose one medal winner at random.

a. Find the probability that the winner won the gold medal, given that the winner was from the United States.

b. Find the probability that the winner was from the United States, given that she or he won a gold medal.

c. Are the events "medal winner is from United States" and "gold medal was won" independent? Explain.

Source: *N.Y. Times Almanac.*

37. According to the *Statistical Abstract of the United States,* 70.3% of females ages 20 to 24 have never been married. Choose 5 young women in this age category at random. Find the probability that

a. None have ever been married.

b. At least one has been married.

Source: *N.Y. Times Almanac.*

38. The American Automobile Association (AAA) reports that of the fatal car and truck accidents, 54% are caused by car driver error. If 3 accidents are chosen at random, find the probability that

a. All are caused by car driver error.

b. None are caused by car driver error.

c. At least 1 is caused by car driver error.

Source: AAA quoted on CNN.

39. Seventy-six percent of toddlers get their recommended immunizations. Suppose that 6 toddlers are selected at random. What is the probability that at least one has not received the recommended immunizations?

Source: Federal Interagency Forum on Child and Family Statistics.

40. A lot of portable radios contains 15 good radios and 3 defective ones. If 2 are selected and tested, find the probability that at least one will be defective.

41. Fifty-eight percent of American children (ages 3 to 5) are read to every day by someone at home. Suppose 5 children are randomly selected. What is the probability that at least one is read to every day by someone at home?

Source: Federal Interagency Forum on Child and Family Statistics.

42. Of Ph.D. students, 60% have paid assistantships. If 3 students are selected at random, find the probabilities.

a. All have assistantships.

b. None have assistantships.

c. At least one has an assistantship.

Source: U.S. Department of Education/*Chronicle of Higher Education.*

43. If 4 cards are drawn from a deck of 52 and not replaced, find the probability of getting at least one club.

44. At a local clinic there are 8 men, 5 women, and 3 children in the waiting room. If 3 patients are randomly selected, find the probability that there is at least one child among them.

45. It has been found that 6% of all automobiles on the road have defective brakes. If 5 automobiles are stopped and checked by the state police, find the probability that at least one will have defective brakes.

46. A medication is 75% effective against a bacterial infection. Find the probability that if 12 people take the medication, at least one person's infection will not improve.

47. A coin is tossed 5 times; find the probability of getting at least one tail. Would you consider this event likely to happen? Explain your answer.

48. If 3 letters of the alphabet are selected at random, find the probability of getting at least one letter x. Letters can be used more than once. Would you consider this event likely to happen? Explain your answer.

49. A die is rolled 7 times. Find the probability of getting at least one 3. Would you consider this event likely to occur? Explain your answer.

50. At a teachers' conference, there were 4 English teachers, 3 mathematics teachers, and 5 science teachers. If 4 teachers are selected for a committee, find the probability that at least one is a science teacher.

51. If a die is rolled 3 times, find the probability of getting at least one even number.

52. In a large vase, there are 8 roses, 5 daisies, 12 lilies, and 9 orchids. If 4 flowers are selected at random, find the probability that at least one of the flowers is a rose. Would you consider this event likely to occur? Explain your answer.

Extending the Concepts

53. Let A and B be two mutually exclusive events. Are A and B independent events? Explain your answer.

54. The Bargain Auto Mall has the following cars in stock.

	SUV	Compact	Mid-sized
Foreign	20	50	20
Domestic	65	100	45

Are the events "compact" and "domestic" independent? Explain.

55. An admissions director knows that the probability a student will enroll after a campus visit is 0.55, or $P(E) = 0.55$. While students are on campus visits, interviews with professors are arranged. The admissions director computes these conditional probabilities for students enrolling after visiting three professors, DW, LP, and MH.

$P(E|\text{DW}) = 0.95 \qquad P(E|\text{LP}) = 0.55 \qquad P(E|\text{MH}) = 0.15$

Is there something wrong with the numbers? Explain.

56. Event A is the event that a person remembers a certain product commercial. Event B is the event that a person buys the product. If $P(B) = 0.35$, comment on each of these conditional probabilities if you were vice president for sales.

a. $P(B|A) = 0.20$
b. $P(B|A) = 0.35$
c. $P(B|A) = 0.55$

4–5 Counting Rules

Many times one wishes to know the number of all possible outcomes for a sequence of events. To determine this number, three rules can be used: the *fundamental counting rule,* the *permutation rule,* and the *combination rule.* These rules are explained here, and they will be used in Section 4–6 to find probabilities of events.

The first rule is called the **fundamental counting rule.**

The Fundamental Counting Rule

Objective 5

Find the total number of outcomes in a sequence of events, using the fundamental counting rule.

Fundamental Counting Rule

In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3, and so forth, the total number of possibilities of the sequence will be

$$k_1 \cdot k_2 \cdot k_3 \cdots k_n$$

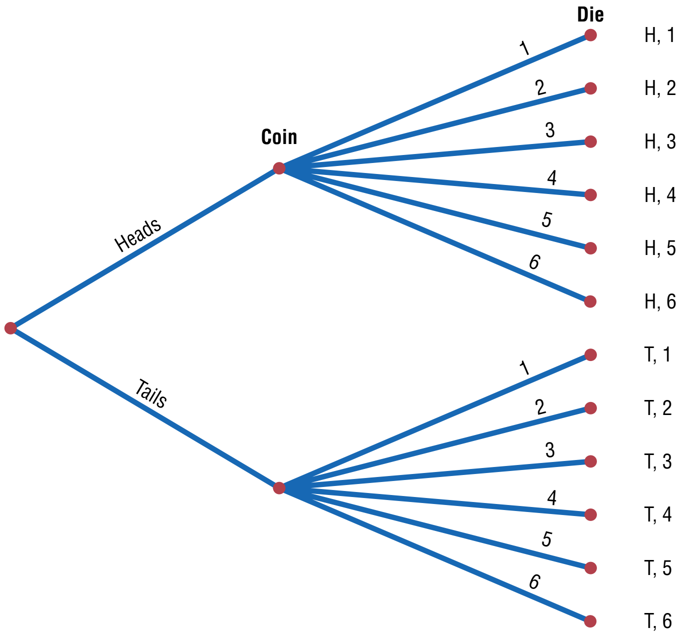
Note: In this case *and* means to multiply.

Examples 4–38 through 4–41 illustrate the fundamental counting rule.

Example 4–38

A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.

Figure 4–8
Complete Tree Diagram for Example 4–38



Interesting Fact

Possible games of chess: 25×10^{115}.

Solution

Since the coin can land either heads up or tails up and since the die can land with any one of six numbers showing face up, there are $2 \cdot 6 = 12$ possibilities. A tree diagram can also be drawn for the sequence of events. See Figure 4–8.

Example 4–39

A paint manufacturer wishes to manufacture several different paints. The categories include

Color	Red, blue, white, black, green, brown, yellow
Type	Latex, oil
Texture	Flat, semigloss, high gloss
Use	Outdoor, indoor

How many different kinds of paint can be made if a person can select one color, one type, one texture, and one use?

Solution

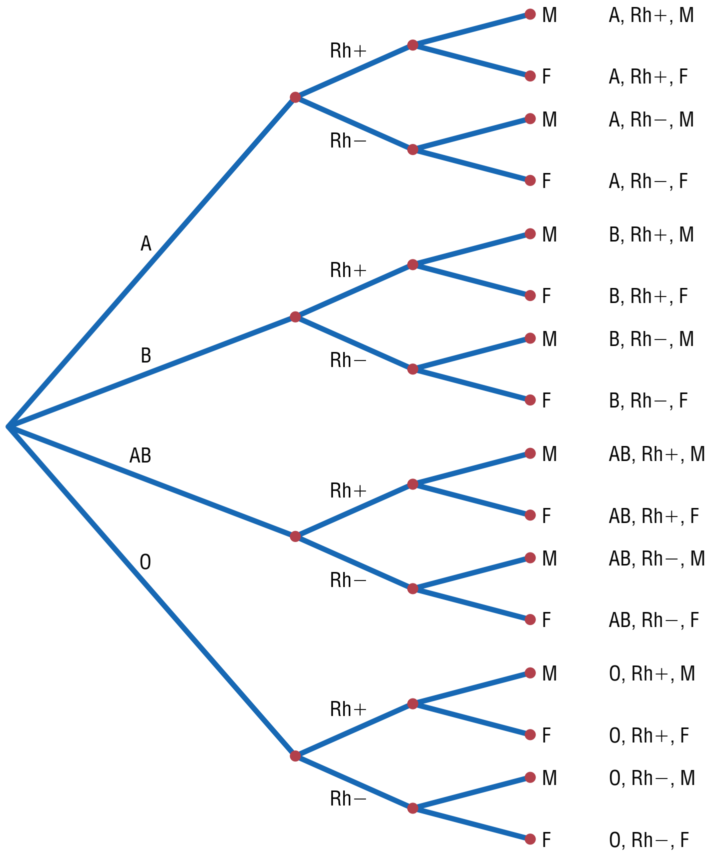
A person can choose one color and one type and one texture and one use. Since there are 7 color choices, 2 type choices, 3 texture choices, and 2 use choices, the total number of possible different paints is

Color		Type		Texture		Use	
7	·	2	·	3	·	2	= 84

Example 4–40

There are four blood types, A, B, AB, and O. Blood can also be Rh+ and Rh−. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labeled?

Figure 4–9
Complete Tree Diagram for Example 4–40



Solution

Since there are 4 possibilities for blood type, 2 possibilities for Rh factor, and 2 possibilities for the gender of the donor, there are $4 \cdot 2 \cdot 2$, or 16, different classification categories, as shown.

Blood type		Rh		Gender	
$\boxed{4}$	$\cdot$	$\boxed{2}$	$\cdot$	$\boxed{2}$	$= 16$

A tree diagram for the events is shown in Figure 4–9.

When determining the number of different possibilities of a sequence of events, one must know whether repetitions are permissible.

Example 4–41

The digits 0, 1, 2, 3, and 4 are to be used in a four-digit ID card. How many different cards are possible if repetitions are permitted?

Solution

Since there are 4 spaces to fill and 5 choices for each space, the solution is

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625$$

Now, what if repetitions are not permitted? For Example 4–41, the first digit can be chosen in 5 ways. But the second digit can be chosen in only 4 ways, since there are only four digits left, etc. Thus, the solution is

$$5 \cdot 4 \cdot 3 \cdot 2 = 120$$

The same situation occurs when one is drawing balls from an urn or cards from a deck. If the ball or card is replaced before the next one is selected, then repetitions are permitted, since the same one can be selected again. But if the selected ball or card is not replaced, then repetitions are not permitted, since the same ball or card cannot be selected the second time.

These examples illustrate the fundamental counting rule. In summary: *If repetitions are permitted, then the numbers stay the same going from left to right. If repetitions are not permitted, then the numbers decrease by 1 for each place left to right.*

Two other rules that can be used to determine the total number of possibilities of a sequence of events are the permutation rule and the combination rule.

Factorial Notation

Historical Note

In 1808 Christian Kramp first used the factorial notation.

These rules use *factorial notation.* The factorial notation uses the exclamation point.

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

To use the formulas in the permutation and combination rules, a special definition of 0! is needed. $0! = 1$.

Factorial Formulas

For any counting n

$$n! = n(n-1)(n-2) \cdots 1$$
$$0! = 1$$

Permutations

A **permutation** is an arrangement of n objects in a specific order.

Examples 4–42 and 4–43 illustrate permutations.

Example 4–42

Suppose a business owner has a choice of five locations in which to establish her business. She decides to rank each location according to certain criteria, such as price of the store and parking facilities. How many different ways can she rank the five locations?

Solution

There are

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

different possible rankings. The reason is that she has 5 choices for the first location, 4 choices for the second location, 3 choices for the third location, etc.

In Example 4–42 all objects were used up. But what happens when not all objects are used up? The answer to this question is given in Example 4–43.

Example 4–43

Suppose the business owner in Example 4–42 wishes to rank only the top three of the five locations. How many different ways can she rank them?

Solution

Using the fundamental counting rule, she can select any one of the five for first choice, then any one of the remaining four locations for her second choice, and finally, any one of the remaining locations for her third choice, as shown.

First choice		Second choice		Third choice	
5	·	4	·	3	= 60

The solutions in Examples 4–42 and 4–43 are permutations.

Objective 6

Find the number of ways that r objects can be selected from n objects, using the permutation rule.

Permutation Rule

The arrangement of n objects in a specific order using r objects at a time is called a *permutation of n objects taking r objects at a time.* It is written as $_nP_r$, and the formula is

$$_nP_r = \frac{n!}{(n-r)!}$$

The notation $_nP_r$ is used for permutations.

$$_6P_4 \text{ means } \frac{6!}{(6-4)!} \quad \text{or} \quad \frac{6!}{2!} = \frac{6\cdot5\cdot4\cdot3\cdot\cancel{2}\cdot\cancel{1}}{\cancel{2}\cdot\cancel{1}} = 360$$

Although Examples 4–42 and 4–43 were solved by the multiplication rule, they can now be solved by the permutation rule.

In Example 4–42, five locations were taken and then arranged in order; hence,

$$_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{1} = 120$$

(Recall that $0! = 1$.)

In Example 4–43, three locations were selected from five locations, so $n = 5$ and $r = 3$; hence

$$_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot4\cdot3\cdot\cancel{2}\cdot\cancel{1}}{\cancel{2}\cdot\cancel{1}} = 60$$

Examples 4–44 and 4–45 illustrate the permutation rule.

Example 4–44

A television news director wishes to use three news stories on an evening show. One story will be the lead story, one will be the second story, and the last will be a closing story. If the director has a total of eight stories to choose from, how many possible ways can the program be set up?

Solution

Since order is important, the solution is

$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336$$

Hence, there would be 336 ways to set up the program.

Example 4–45

How many different ways can a chairperson and an assistant chairperson be selected for a research project if there are seven scientists available?

Solution

$$_7P_2 = \frac{7!}{(7-2)!} = \frac{7!}{5!} = 42$$

Combinations

Suppose a dress designer wishes to select two colors of material to design a new dress, and she has on hand four colors. How many different possibilities can there be in this situation?

Objective 7

Find the number of ways that *r* objects can be selected from *n* objects without regard to order, using the combination rule.

This type of problem differs from previous ones in that the order of selection is not important. That is, if the designer selects yellow and red, this selection is the same as the selection red and yellow. This type of selection is called a *combination.* The difference between a permutation and a combination is that in a combination, the order or arrangement of the objects is not important; by contrast, order *is* important in a permutation. Example 4–46 illustrates this difference.

A selection of distinct objects without regard to order is called a **combination.**

Example 4–46

Given the letters A, B, C, and D, list the permutations and combinations for selecting two letters.

Solution

The permutations are

AB	BA	CA	DA
AC	BC	CB	DB
AD	BD	CD	DC

In permutations, AB is different from BA. But in combinations, AB is the same as BA since the order of the objects does not matter in combinations. Therefore, if duplicates are removed from a list of permutations, what is left is a list of combinations, as shown.

AB	~~BA~~	~~CA~~	~~DA~~
AC	BC	~~CB~~	~~DB~~
AD	BD	CD	~~DC~~

Hence the combinations of A, B, C, and D are AB, AC, AD, BC, BD, and CD. (Alternatively, BA could be listed and AB crossed out, etc.) The combinations have been listed alphabetically for convenience, but this is not a requirement.

Interesting Fact

The total number of hours spent mowing lawns in the United States each year: 2,220,000,000.

Combinations are used when the order or arrangement is not important, as in the selecting process. Suppose a committee of 5 students is to be selected from 25 students. The five selected students represent a combination, since it does not matter who is selected first, second, etc.

Combination Rule

The number of combinations of r objects selected from n objects is denoted by ${}_nC_r$ and is given by the formula

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Example 4–47

How many combinations of 4 objects are there, taken 2 at a time?

Solution

Since this is a combination problem, the answer is

$${}_4C_2 = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{\overset{2}{\cancel{4}} \cdot 3 \cdot \cancel{2!}}{\cancel{2} \cdot 1 \cdot \cancel{2!}} = 6$$

This is the same result shown in Example 4–46.

Notice that the expression for ${}_nC_r$ is

$$\frac{n!}{(n-r)!r!}$$

which is the formula for permutations with $r!$ in the denominator. In other words,

$$_nC_r = \frac{_nP_r}{r!}$$

This $r!$ divides out the duplicates from the number of permutations, as shown in Example 4–46. For each two letters, there are two permutations but only one combination. Hence, dividing the number of permutations by $r!$ eliminates the duplicates. This result can be verified for other values of n and r. *Note:* $_nC_n = 1$.

Example 4–48

A bicycle shop owner has 12 mountain bicycles in the showroom. The owner wishes to select 5 of them to display at a bicycle show. How many different ways can a group of 5 be selected?

Solution

$$_{12}C_5 = \frac{12!}{(12-5)!5!} = \frac{12!}{7!5!} = \frac{12 \cdot 11 \cdot \overset{2}{\cancel{10}} \cdot \overset{3}{\cancel{9}} \cdot \overset{2}{\cancel{8}} \cdot \cancel{7!}}{\cancel{7!} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 792$$

Example 4–49

In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

Solution

Here, one must select 3 women from 7 women, which can be done in $_7C_3$, or 35, ways. Next, 2 men must be selected from 5 men, which can be done in $_5C_2$, or 10, ways. Finally, by the fundamental counting rule, the total number of different ways is $35 \cdot 10 = 350$, since one is choosing both men and women. Using the formula gives

$$_7C_3 \cdot {_5C_2} = \frac{7!}{(7-3)!3!} \cdot \frac{5!}{(5-2)!2!} = 350$$

Table 4–1 summarizes the counting rules.

Table 4–1 Summary of Counting Rules

Rule	Definition	Formula
Fundamental counting rule	The number of ways a sequence of n events can occur if the first event can occur in k_1 ways, the second event can occur in k_2 ways, etc.	$k_1 \cdot k_2 \cdot k_3 \cdots k_n$
Permutation rule	The number of permutations of n objects taking r objects at a time (order is important)	$_nP_r = \frac{n!}{(n-r)!}$
Combination rule	The number of combinations of r objects taken from n objects (order is not important)	$_nC_r = \frac{n!}{(n-r)!r!}$

Applying the Concepts 4–5

Garage Door Openers

Garage door openers originally had a series of four on/off switches so that homeowners could personalize the frequencies that opened their garage doors. If all garage door openers were set at the same frequency, anyone with a garage door opener could open anyone else's garage door.

1. Use a tree diagram to show how many different positions four consecutive on/off switches could be in.

After garage door openers became more popular, another set of four on/off switches was added to the systems.

2. Find a pattern of how many different positions are possible with the addition of each on/off switch.
3. How many different positions are possible with eight consecutive on/off switches?
4. Is it reasonable to assume, if you owned a garage door opener with eight switches, that someone could use his/her garage door opener to open your garage door by trying all the different possible positions?

In 1989 it was reported that the ignition keys for 1988 Dodge Caravans were made from a single blank that had five cuts on it. Each cut was made at one out of five possible levels. In 1988, assume there was 420,000 Dodge Caravans sold in the United States.

5. How many different possible keys can be made from the same key blank?
6. How many different 1988 Dodge Caravans could any one key start?

Look at the ignition key for your car and count the number of cuts on it. Assume that the cuts are made at one of any five possible levels. Most car companies use one key blank for all their makes and models of cars.

7. Conjecture how many cars your car company sold over recent years, and then figure out how many other cars your car key could start. What would you do to decrease the odds of someone being able to open another vehicle with his or her key?

See page 235 for the answers.

Exercises 4–5

1. How many 5-digit zip codes are possible if digits can be repeated? If there cannot be repetitions?

2. How many ways can a baseball manager arrange a batting order of 9 players?

3. How many different ways can 7 different video game cartridges be arranged on a shelf?

4. How many different ways can 6 radio commercials be played during a 1-hour radio program?

5. A store manager wishes to display 8 different brands of shampoo in a row. How many ways can this be done?

6. There are 8 different statistics books, 6 different geometry books, and 3 different trigonometry books. A student must select one book of each type. How many different ways can this be done?

7. At a local cheerleaders' camp, 5 routines must be practiced. A routine may not be repeated. In how many different orders can these 5 routines be presented?

8. The call letters of a radio station must have 4 letters. The first letter must be a K or a W. How many different station call letters can be made if repetitions are not allowed? If repetitions are allowed?

9. How many different 3-digit identification tags can be made if the digits can be used more than once? If the first digit must be a 5 and repetitions are not permitted?

10. How many different ways can 9 trophies be arranged on a shelf?

11. If a baseball manager has 5 pitchers and 2 catchers, how many different possible pitcher-catcher combinations can he field?

12. There are 2 major roads from city *X* to city *Y* and 4 major roads from city *Y* to city *Z*. How many different trips can be made from city *X* to city *Z* passing through city *Y*?

13. Evaluate each of these.

a. 8!	*e.* $_7P_5$	*i.* $_5P_5$
b. 10!	*f.* $_{12}P_4$	*j.* $_6P_2$
c. 0!	*g.* $_5P_3$	
d. 1!	*h.* $_6P_0$	

14. The County Assessment Bureau decides to reassess homes in 8 different areas. How many different ways can this be accomplished?

15. How many different 4-color code stripes can be made on a sports car if each code consists of the colors green, red, blue, and white? All colors are used only once.

16. An inspector must select 3 tests to perform in a certain order on a manufactured part. He has a choice of 7 tests. How many ways can he perform 3 different tests?

17. Anderson Research Co. decides to test-market a product in 6 areas. How many different ways can 3 areas be selected in a certain order for the first test?

18. How many different ways can a city health department inspector visit 5 restaurants in a city with 10 restaurants?

19. How many different 4-letter permutations can be formed from the letters in the word *decagon*?

20. In a board of directors composed of 8 people, how many ways can 1 chief executive officer, 1 director, and 1 treasurer be selected?

21. How many different ID cards can be made if there are 6 digits on a card and no digit can be used more than once?

22. How many different ways can 5 Public Service announcements be run during 1 hour of time?

23. How many different ways can 4 tickets be selected from 50 tickets if each ticket wins a different prize?

24. How many different ways can a researcher select 5 rats from 20 rats and assign each to a different test?

25. How many different signals can be made by using at least 3 distinct flags if there are 5 different flags from which to select?

26. An investigative agency has 7 cases and 5 agents. How many different ways can the cases be assigned if only 1 case is assigned to each agent?

27. **(ans)** Evaluate each expression.

a. $_5C_2$	*d.* $_6C_2$	*g.* $_3C_3$	*j.* $_4C_3$
b. $_8C_3$	*e.* $_6C_4$	*h.* $_9C_7$	
c. $_7C_4$	*f.* $_3C_0$	*i.* $_{12}C_2$	

28. How many ways can 3 cards be selected from a standard deck of 52 cards, disregarding the order of selection?

29. How many ways are there to select 3 bracelets from a box of 10 bracelets, disregarding the order of selection?

30. How many ways can 4 baseball players and 3 basketball players be selected from 12 baseball players and 9 basketball players?

31. How many ways can a committee of 4 people be selected from a group of 10 people?

32. If a person can select 3 presents from 10 presents under a Christmas tree, how many different combinations are there?

33. How many different tests can be made from a test bank of 20 questions if the test consists of 5 questions?

34. The general manager of a fast-food restaurant chain must select 6 restaurants from 11 for a promotional program. How many different possible ways can this selection be done?

35. How many different ways can a theatrical group select 2 musicals and 3 dramas from 11 musicals and 8 dramas to be presented during the year?

36. In a train yard there are 4 tank cars, 12 boxcars, and 7 flatcars. How many ways can a train be made up consisting of 2 tank cars, 5 boxcars, and 3 flatcars? (In this case, order is not important.)

37. There are 7 women and 5 men in a department. How many ways can a committee of 4 people be selected? How many ways can this committee be selected if there must be 2 men and 2 women on the committee? How many ways can this committee be selected if there must be at least 2 women on the committee?

38. Wake Up cereal comes in 2 types, crispy and crunchy. If a researcher has 10 boxes of each, how many ways can she select 3 boxes of each for a quality control test?

39. How many ways can a dinner patron select 3 appetizers and 2 vegetables if there are 6 appetizers and 5 vegetables on the menu?

40. How many ways can a jury of 6 women and 6 men be selected from 10 women and 12 men?

41. How many ways can a foursome of 2 men and 2 women be selected from 10 men and 12 women in a golf club?

42. The state narcotics bureau must form a 5-member investigative team. If it has 25 agents from which to choose, how many different possible teams can be formed?

43. How many different ways can an instructor select 2 textbooks from a possible 17?

44. The Environmental Protection Agency must investigate 9 mills for complaints of air pollution. How many different ways can a representative select 5 of these to investigate this week?

45. How many ways can a person select 7 television commercials from 11 television commercials?

46. How many ways can a person select 8 videotapes from 10 tapes?

47. A buyer decides to stock 8 different posters. How many ways can she select these 8 if there are 20 from which to choose?

48. An advertising manager decides to have an ad campaign in which 8 special calculators will be hidden at various locations in a shopping mall. If she has 17 locations from which to pick, how many different possible combinations can she choose?

Extending the Concepts

49. How many different ways can a person select one or more coins if she has 2 nickels, 1 dime, and 1 half-dollar?

50. In a barnyard there is an assortment of chickens and cows. Counting heads, one gets 15; counting legs, one gets 46. How many of each are there?

51. How many different ways can five people—A, B, C, D, and E—sit in a row at a movie theater if (*a*) A and B must sit together; (*b*) C must sit to the right of, but not necessarily next to, B; (*c*) D and E will not sit next to each other?

52. Using combinations, calculate the number of each poker hand in a deck of cards. (A poker hand consists of 5 cards dealt in any order.)

a. Royal flush
b. Straight flush
c. Four of a kind
d. Full house

Technology *Step by Step*

TI-83 Plus or TI-84 Plus Step by Step

Factorials, Permutations, and Combinations

Factorials *n*!

1. Type the value of *n*.
2. Press **MATH** and move the cursor to PRB, then press **4** for !.
3. Press **ENTER.**

Permutations $_nP_r$

1. Type the value of *n*.
2. Press **MATH** and move the cursor to PRB, then press **2** for $_nP_r$.
3. Type the value of *r*.
4. Press **ENTER.**

Combinations $_nC_r$

1. Type the value of *n*.
2. Press **MATH** and move the cursor to PRB, then press **3** for $_nC_r$.
3. Type the value of *r*.
4. Press **ENTER.**

```
5!
                 120
8 nPr 3
                 336
12 nCr 5
                 792
```

Calculate 5!, $_8P_3$, and $_{12}C_5$ (Examples 4–42, 4–44, and 4–48 from the text).

Excel
Step by Step

Permutations, Combinations, and Factorials

To find a value of a permutation, for example, ${}_5P_3$:

1. Select an open cell in an Excel workbook.
2. Select the paste function (f_x) icon from the toolbar.
3. Select Function category Statistical. Then select the PERMUT function.
4. Type **5** in the Number box.
5. Type **3** in the Number_chosen box and click [OK].

The selected cell will display the answer: 60.

To find a value of a combination, for example, ${}_5C_3$:

1. Select an open cell in an Excel workbook.
2. Select the paste function icon from the toolbar.
3. Select Function category Math & Trig. Then select the COMBIN function.
4. Type **5** in the Number box.
5. Type **3** in the Number_chosen box and click [OK].

The selected cell will display the answer: 10.

To find a factorial of a number, for example, 7!:

1. Select an open cell in an Excel workbook.
2. Select the paste function icon from the toolbar.
3. Select Function category Math & Trig. Then select the FACT function.
4. Type **7** in the Number box and click [OK].

The selected cell will display the answer: 5040.

4–6 Probability and Counting Rules

Objective 8

Find the probability of an event, using the counting rules.

The counting rules can be combined with the probability rules in this chapter to solve many types of probability problems. By using the fundamental counting rule, the permutation rules, and the combination rule, one can compute the probability of outcomes of many experiments, such as getting a full house when 5 cards are dealt or selecting a committee of 3 women and 2 men from a club consisting of 10 women and 10 men.

Example 4–50

Find the probability of getting 4 aces when 5 cards are drawn from an ordinary deck of cards.

Solution

There are ${}_{52}C_5$ ways to draw 5 cards from a deck. There is only one way to get 4 aces (i.e., ${}_4C_4$), but there are 48 possibilities to get the fifth card. Therefore, there are 48 ways to get 4 aces and 1 other card. Hence,

$$P(4\text{ aces}) = \frac{{}_4C_4 \cdot 48}{{}_{52}C_5} = \frac{1 \cdot 48}{2{,}598{,}960} = \frac{48}{2{,}598{,}960} = \frac{1}{54{,}145}$$

Example 4–51

A box contains 24 transistors, 4 of which are defective. If 4 are sold at random, find the following probabilities.

a. Exactly 2 are defective.

b. None is defective.

c. All are defective.

d. At least 1 is defective.

Solution

There are ${}_{24}C_4$ ways to sell 4 transistors, so the denominator in each case will be 10,626.

a. Two defective transistors can be selected as ${}_4C_2$ and 2 nondefective ones as ${}_{20}C_2$. Hence,

$$P(\text{exactly 2 defectives}) = \frac{{}_4C_2 \cdot {}_{20}C_2}{{}_{24}C_4} = \frac{1140}{10{,}626} = \frac{190}{1771}$$

b. The number of ways to choose no defectives is ${}_{20}C_4$. Hence,

$$P(\text{no defectives}) = \frac{{}_{20}C_4}{{}_{24}C_4} = \frac{4845}{10{,}626} = \frac{1615}{3542}$$

c. The number of ways to choose 4 defectives from 4 is ${}_4C_4$, or 1. Hence,

$$P(\text{all defective}) = \frac{1}{{}_{24}C_4} = \frac{1}{10{,}626}$$

d. To find the probability of at least 1 defective transistor, find the probability that there are no defective transistors, and then subtract that probability from 1.

$$P(\text{at least 1 defective}) = 1 - P(\text{no defectives})$$
$$= 1 - \frac{{}_{20}C_4}{{}_{24}C_4} = 1 - \frac{1615}{3542} = \frac{1927}{3542}$$

Example 4–52

A store has 6 *TV Graphic* magazines and 8 *Newstime* magazines on the counter. If two customers purchased a magazine, find the probability that one of each magazine was purchased.

Solution

$$P(1\ \textit{TV Graphic}\text{ and }1\ \textit{Newstime}) = \frac{{}_6C_1 \cdot {}_8C_1}{{}_{14}C_2} = \frac{6 \cdot 8}{91} = \frac{48}{91}$$

Example 4–53

A combination lock consists of the 26 letters of the alphabet. If a 3-letter combination is needed, find the probability that the combination will consist of the letters ABC in that order. The same letter can be used more than once. (*Note:* A combination lock is really a permutation lock.)

Solution

Since repetitions are permitted, there are $26 \cdot 26 \cdot 26 = 17{,}576$ different possible combinations. And since there is only one ABC combination, the probability is $P(\text{ABC}) = 1/26^3 = 1/17{,}576$.

Example 4–54

There are 8 married couples in a tennis club. If 1 man and 1 woman are selected at random to plan the summer tournament, find the probability that they are married to each other.

Solution

Since there are 8 ways to select the man and 8 ways to select the woman, there are $8 \cdot 8$, or 64, ways to select 1 man and 1 woman. Since there are 8 married couples, the solution is $\frac{8}{64} = \frac{1}{8}$.

As indicated at the beginning of this section, the counting rules and the probability rules can be used to solve a large variety of probability problems found in business, gambling, economics, biology, and other fields.

Applying the Concepts 4–6

Counting Rules and Probability

One of the biggest problems for students when doing probability problems is to decide which formula or formulas to use. Another problem is to decide whether two events are independent or dependent. Use the following problem to help develop a better understanding of these concepts.

Assume you are given a 5-question multiple-choice quiz. Each question has 5 possible answers: A, B, C, D, and E.

1. How many events are there?
2. Are the events independent or dependent?
3. If you guess at each question, what is the probability that you get all of them correct?
4. What is the probability that a person would guess answer A for each question?

Assume that you are given a test in which you are to match the correct answers in the right column with the questions in the left column. You can use each answer only once.

5. How many events are there?
6. Are the events independent or dependent?
7. What is the probability of getting them all correct if you are guessing?
8. What is the difference between the two problems?

See page 235 for the answers.

Speaking of Statistics

Probabilities can be used to assess the danger of a situation. For example, the chances of dying from a shark attack are 1 in 100 million while the chances of dying from a heart attack are 1 in 400. Hence, we are at greater risk from dying of heart disease. Using the probabilities in the table, explain why we should be more afraid of smoking or a bad diet than of walking across a street.

How much danger are we really in?

You are 71,500 times more likely to die in a car crash this year than to be killed by anthrax. Indeed, you are 8,300 times more likely to get killed walking across the street, says risk expert Fred Kilbourne, a member of the board of Conference of Consulting Actuaries.[1] He calculates an American's chance of dying this year from any cause—from anthrax to drowning—is 1 in 130.

Based on deaths in the USA, these are each American's risk of dying each year from:

Motor vehicle accidents	1 in 7,000
Being shot by a gun	1 in 10,000
Falling down	1 in 20,000
Poison	1 in 40,000
Walking across the street	1 in 60,000
Drowning	1 in 75,000
House fire	1 in 100,000
Bicycle accident	1 in 500,000
Commercial plane crash	1 in 1 million
Lightning strike	1 in 3 million
Shark attack	1 in 100 million
Roller coaster accident	1 in 300 million
Anthrax	**1 in 500 million**

Kilbourne says the major killers are not accidents but illness and disease.

The risk of death per person per year in the USA for:

Heart disease	1 in 400
Cancer	1 in 600
Stroke	1 in 2,000
Flu and pneumonia	1 in 3,000

1 – Kilbourne cautions that actuarial assessments, the predictive numbers used by the insurance industry, are based on "the assumption that the future will look reasonably like the past. We may not be able to make that assumption on every issue right now."

Source: Copyright 2001, *USA TODAY.* Reprinted with permission.

Exercises 4–6

1. Find the probability of getting 2 face cards (king, queen, or jack) when 2 cards are drawn from a deck without replacement.

2. A parent-teacher committee consisting of 4 people is to be formed from 20 parents and 5 teachers. Find the probability that the committee will consist of these people. (Assume that the selection will be random.)

a. All teachers

b. 2 teachers and 2 parents

c. All parents

d. 1 teacher and 3 parents

3. In a company there are 7 executives: 4 women and 3 men. Three are selected to attend a management seminar. Find these probabilities.

a. All 3 selected will be women.

b. All 3 selected will be men.

c. 2 men and 1 woman will be selected.

d. 1 man and 2 women will be selected.

4. The composition of the Senate of the 107th Congress is

49 Republicans   1 Independent   50 Democrats

A new committee is being formed to study ways to benefit the arts in education. If 3 Senators are selected at random to head the committee, what is the probability that they will all be Republicans? What is the probability that they will all be Democrats? What is the probability that there will be 1 from each party, including the Independent?

Source: *N.Y. Times Almanac.*

5. The signers of the Declaration of Independence came from the Thirteen Colonies as shown.

Massachusetts	5	New York	4
New Hampshire	3	Georgia	3
Virginia	7	North Carolina	3
Maryland	4	South Carolina	4
New Jersey	5	Connecticut	4
Pennsylvania	9	Delaware	3
Rhode Island	2		

Suppose that 4 are chosen at random to be the subject of a documentary. Find the probability that

a. All 4 come from Pennsylvania

b. 2 come from Pennsylvania and 2 from Virginia

Source: *N.Y. Times Almanac.*

6. A package contains 12 resistors, 3 of which are defective. If 4 are selected, find the probability of getting

a. No defective resistors

b. 1 defective resistor

c. 3 defective resistors

7. If 50 tickets are sold and 2 prizes are to be awarded, find the probability that one person will win 2 prizes if that person buys 2 tickets.

8. Find the probability of getting a full house (3 cards of one denomination and 2 of another) when 5 cards are dealt from an ordinary deck.

9. A committee of 4 people is to be formed from 6 doctors and 8 dentists. Find the probability that the committee will consist of

a. All dentists

b. 2 dentists and 2 doctors

c. All doctors

d. 3 doctors and 1 dentist

e. 1 doctor and 3 dentists

10. An insurance sales representative selects 3 policies to review. The group of policies she can select from contains 8 life policies, 5 automobile policies, and 2 homeowner policies. Find the probability of selecting

a. All life policies

b. Both homeowner policies

c. All automobile policies

d. 1 of each policy

e. 2 life policies and 1 automobile policy

11. A drawer contains 11 identical red socks and 8 identical black socks. Suppose that you choose 2 socks at random in the dark

a. What is the probability that you get a pair of red socks?

b. What is the probability that you get a pair of black socks?

c. What is the probability that you get 2 unmatched socks?

d. Where did the other red sock go?

12. Find the probability of selecting 3 science books and 4 math books from 8 science books and 9 math books. The books are selected at random.

13. When 3 dice are rolled, find the probability of getting a sum of 7.

14. Find the probability of randomly selecting 2 mathematics books and 3 physics books from a box containing 4 mathematics books and 8 physics books.

15. Find the probability that if 5 different-sized washers are arranged in a row, they will be arranged in order of size.

16. Using the information in Exercise 52 in Section 4–5, find the probability of each poker hand.

a. Royal flush

b. Straight flush

c. 4 of a kind

4–7 Summary

In this chapter, the basic concepts and rules of probability are explained. The three types of probability are classical, empirical, and subjective. Classical probability uses sample spaces. Empirical probability uses frequency distributions and is based on observation. In subjective probability, the researcher makes an educated guess about the chance of an event occurring.

A probability event consists of one or more outcomes of a probability experiment. Two events are said to be mutually exclusive if they cannot occur at the same time. Events can also be classified as independent or dependent. If events are independent, whether or not the first event occurs does not affect the probability of the next event occurring. If the probability of the second event occurring is changed by the occurrence of the first event, then the events are dependent. The complement of an event is the set of outcomes in the sample space that are not included in the outcomes of the event itself. Complementary events are mutually exclusive.

Probability problems can be solved by using the addition rules, the multiplication rules, and the complementary event rules.

Finally, the fundamental counting rule, the permutation rule, and the combination rule can be used to determine the number of outcomes of events; then these numbers can be used to determine the probabilities of events.

Important Terms

classical probability 176

combination 217

complement of an event 179

compound event 176

conditional probability 202

dependent events 202

empirical probability 181

equally likely events 176

event 175

fundamental counting rule 212

independent events 199

law of large numbers 183

mutually exclusive events 189

outcome 173

permutation 215

probability 173

probability experiment 173

sample space 173

simple event 175

subjective probability 183

tree diagram 175

Venn diagrams 180

Important Formulas

Formula for classical probability:

$$P(E) = \frac{\textbf{number of outcomes in } E}{\textbf{total number of outcomes in sample space}} = \frac{n(E)}{n(S)}$$

Formula for empirical probability:

$$P(E) = \frac{\textbf{frequency for class}}{\textbf{total frequencies in distribution}} = \frac{f}{n}$$

Addition rule 1, for two mutually exclusive events:

$$P(A \text{ or } B) = P(A) + P(B)$$

Addition rule 2, for events that are not mutually exclusive:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Multiplication rule 1, for independent events:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Multiplication rule 2, for dependent events:

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Formula for conditional probability:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Formula for complementary events:

$$P(\overline{E}) = 1 - P(E) \quad \text{or} \quad P(E) = 1 - P(\overline{E})$$

$$\text{or} \quad P(E) + P(\overline{E}) = 1$$

Fundamental counting rule: In a sequence of n events in which the first one has k_1 possibilities, the second event has k_2 possibilities, the third has k_3 possibilities, etc., the total number possibilities of the sequence will be

$$k_1 \cdot k_2 \cdot k_3 \cdots k_n$$

Permutation rule: The number of permutations of n objects taking r objects at a time when order is important is

$$_nP_r = \frac{n!}{(n-r)!}$$

Combination rule: The number of combinations of r objects selected from n objects when order is not important is

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Review Exercises

1. When a die is rolled, find the probability of getting a
   a. 5
   b. 6
   c. Number less than 5
2. When a card is selected from a deck, find the probability of getting
   a. A club
   b. A face card or a heart
   c. A 6 and a spade
   d. A king
   e. A red card
3. In a survey conducted at a local restaurant during breakfast hours, 20 people preferred orange juice, 16 preferred grapefruit juice, and 9 preferred apple juice with breakfast. If a person is selected at random, find the probability that she or he prefers grapefruit juice.
4. If a die is rolled one time, find these probabilities.
   a. Getting a 5
   b. Getting an odd number
   c. Getting a number less than 3
5. A recent survey indicated that in a town of 1500 households, 850 had cordless telephones. If a household is randomly selected, find the probability that it has a cordless telephone.
6. During a sale at a men's store, 16 white sweaters, 3 red sweaters, 9 blue sweaters, and 7 yellow sweaters were purchased. If a customer is selected at random, find the probability that he bought
   a. A blue sweater
   b. A yellow or a white sweater
   c. A red, a blue, or a yellow sweater
   d. A sweater that was not white
7. At a swimwear store, the managers found that 16 women bought white bathing suits, 4 bought red suits, 3 bought blue suits, and 7 bought yellow suits. If a customer is selected at random, find the probability that she bought
   a. A blue suit
   b. A yellow or a red suit
   c. A white or a yellow or a blue suit
   d. A suit that was not red
8. When two dice are rolled, find the probability of getting
   a. A sum of 5 or 6
   b. A sum greater than 9
   c. A sum less than 4 or greater than 9
   d. A sum that is divisible by 4
   e. A sum of 14
   f. A sum less than 13
9. The probability that a person owns a car is 0.80, that a person owns a boat is 0.30, and that a person owns both a car and a boat is 0.12. Find the probability that a person owns either a boat or a car.
10. There is a 0.39 probability that John will purchase a new car, a 0.73 probability that Mary will purchase a new car, and a 0.36 probability that both will purchase a new car. Find the probability that neither will purchase a new car.
11. A Gallup Poll found that 78% of Americans worry about the quality and healthfulness of their diet. If 5 people are selected at random, find the probability that all 5 worry about the quality and healthfulness of their diet.
    Source: The Book of Odds.
12. Of Americans using library services, 67% borrow books. If 5 patrons are chosen at random, what is the probability that all borrowed books? That none borrowed books?
    Source: American Library Association.
13. Three cards are drawn from an ordinary deck *without* replacement. Find the probability of getting
    a. All black cards
    b. All spades
    c. All queens
14. A coin is tossed and a card is drawn from a deck. Find the probability of getting
    a. A head and a 6
    b. A tail and a red card
    c. A head and a club
15. A box of candy contains 6 chocolate-covered cherries, 3 peppermints, 2 caramels, and 2 strawberry creams. If a piece of candy is selected, find the probability of getting a caramel or a peppermint.
16. A manufacturing company has three factories: X, Y, and Z. The daily output of each is shown here.

Product	Factory X	Factory Y	Factory Z
TVs	18	32	15
Stereos	6	20	13

    If one item is selected at random, find these probabilities.
    a. It was manufactured at factory X or is a stereo.
    b. It was manufactured at factory Y or factory Z.
    c. It is a TV or was manufactured at factory Z.
17. A vaccine has a 90% probability of being effective in preventing a certain disease. The probability of getting the disease if a person is not vaccinated is 50%. In a certain geographic region, 25% of the people get vaccinated. If a person is selected at random, find the probability that he or she will contract the disease.
18. A manufacturer makes three models of a television set, models A, B, and C. A store sells 40% of model A sets, 40% of model B sets, and 20% of model C sets. Of

model A sets, 3% have stereo sound; of model B sets, 7% have stereo sound; and of model C sets, 9% have stereo sound. If a set is sold at random, find the probability that it has stereo sound.

19. The probability that Sue will live on campus and buy a new car is 0.37. If the probability that she will live on campus is 0.73, find the probability that she will buy a new car, given that she lives on campus.

20. The probability that a customer will buy a television set and buy an extended warranty is 0.03. If the probability that a customer will purchase a television set is 0.11, find the probability that the customer will also purchase the extended warranty.

21. Of the members of the Blue River Health Club, 43% have a lifetime membership and exercise regularly (three or more times a week). If 75% of the club members exercise regularly, find the probability that a randomly selected member is a life member, given that he or she exercises regularly.

22. The probability that it snows and the bus arrives late is 0.023. José hears the weather forecast, and there is a 40% chance of snow tomorrow. Find the probability that the bus will be late, given that it snows.

23. At a large factory, the employees were surveyed and classified according to their level of education and whether they smoked. The data are shown in the table.

	Educational level		
Smoking habit	**Not high school graduate**	**High school graduate**	**College graduate**
Smoke	6	14	19
Do not smoke	18	7	25

If an employee is selected at random, find these probabilities.

a. The employee smokes, given that he or she graduated from college.

b. Given that the employee did not graduate from high school, he or she is a smoker.

24. A survey found that 77% of bike riders sometimes ride without a helmet. If 4 bike riders are randomly selected, find the probability that at least one of the riders does not wear a helmet all the time.

Source: USA TODAY.

25. A coin is tossed 5 times. Find the probability of getting at least one tail.

26. The U.S. Department of Health and Human Services reports that 15% of Americans have chronic sinusitis. If 5 people are selected at random, find the probability that at least one has chronic sinusitis.

Source: 100% American.

27. An automobile license plate consists of 3 letters followed by 4 digits. How many different plates can be made if repetitions are allowed? If repetitions are not allowed? If repetitions are allowed in the letters but not in the digits?

28. How many different arrangements of the letters in the word *bread* can be made?

29. How many ways can 3 outfielders and 4 infielders be chosen from 5 outfielders and 7 infielders?

30. How many different ways can 8 computer operators be seated in a row?

31. How many ways can a student select 2 electives from a possible choice of 10 electives?

32. There are 6 Republican, 5 Democrat, and 4 Independent candidates. How many different ways can a committee of 3 Republicans, 2 Democrats, and 1 Independent be selected?

33. How many different computer passwords are possible if each consists of 4 symbols and if the first one must be a letter and the other 3 must be digits?

34. A new employee has a choice of 5 health care plans, 3 retirement plans, and 2 different expense accounts. If a person selects one of each option, how many different options does he or she have?

35. There are 12 students who wish to enroll in a particular course. There are only 4 seats left in the classroom. How many different ways can 4 students be selected to attend the class?

36. A candy store allows customers to select 3 different candies to be packaged and mailed. If there are 13 varieties available, how many possible selections can be made?

37. If a student can select 5 novels from a reading list of 20 for a course in literature, how many different possible ways can this selection be done?

38. If a student can select one of 3 language courses, one of 5 mathematics courses, and one of 4 history courses, how many different schedules can be made?

39. License plates are to be issued with 3 letters followed by 4 single digits. How many such license plates are possible? If the plates are issued at random, what is the probability that the license plate says USA followed by a number that is divisible by 5?

40. A newspaper advertises 5 different movies, 3 plays, and 2 baseball games for the weekend. If a couple selects 3 activities, find the probability that they attend 2 plays and 1 movie.

41. In an office there are 3 secretaries, 4 accountants, and 2 receptionists. If a committee of 3 is to be formed, find the probability that one of each will be selected.

42. For a survey, a subject can be classified as follows:

Gender: male or female

Marital status: single, married, widowed, divorced

Occupation: administration, faculty, staff

Draw a tree diagram for the different ways a person can be classified.

Statistics Today

Would You Bet Your Life?—Revisited

In his book *Probabilities in Everyday Life,* John D. McGervey states that the chance of being killed on any given commercial airline flight is almost 1 in 1 million and that the chance of being killed during a transcontinental auto trip is about 1 in 8000. The corresponding probabilities are $1/1{,}000{,}000 = 0.000001$ as compared to $1/8000 = 0.000125$. Since the second number is 125 times greater than the first number, you have a much higher risk driving than flying across the United States.

Chapter Quiz

Determine whether each statement is true or false. If the statement is false, explain why.

1. Subjective probability has little use in the real world.
2. Classical probability uses a frequency distribution to compute probabilities.
3. In classical probability, all outcomes in the sample space are equally likely.
4. When two events are not mutually exclusive, $P(A \text{ or } B) = P(A) + P(B)$.
5. If two events are dependent, they must have the same probability of occurring.
6. An event and its complement can occur at the same time.
7. The arrangement ABC is the same as BAC for combinations.
8. When objects are arranged in a specific order, the arrangement is called a combination.

Select the best answer.

9. The probability that an event happens is 0.42. What is the probability that the event won't happen?

   a. −0.42 *c.* 0
   b. 0.58 *d.* 1

10. When a meteorologist says that there is a 30% chance of showers, what type of probability is the person using?

    a. Classical *c.* Relative
    b. Empirical *d.* Subjective

11. The sample space for tossing 3 coins consists of how many outcomes?

    a. 2 *c.* 6
    b. 4 *d.* 8

12. The complement of guessing 5 correct answers on a 5-question true/false exam is

    a. Guessing 5 incorrect answers
    b. Guessing at least 1 incorrect answer
    c. Guessing at least 1 correct answer
    d. Guessing no incorrect answers

13. When two dice are rolled, the sample space consists of how many events?

    a. 6 *c.* 36
    b. 12 *d.* 54

14. What is ${}_nP_0$?

    a. 0 *c.* n
    b. 1 *d.* It cannot be determined.

15. What is the number of permutations of 6 different objects taken all together?

    a. 0 *c.* 36
    b. 1 *d.* 720

16. What is 0!?

    a. 0 *c.* Undefined
    b. 1 *d.* 10

17. What is ${}_nC_n$?

    a. 0 *c.* n
    b. 1 *d.* It cannot be determined.

Complete the following statements with the best answer.

18. The set of all possible outcomes of a probability experiment is called the ________.
19. The probability of an event can be any number between and including ________ and ________.
20. If an event cannot occur, its probability is ________.
21. The sum of the probabilities of the events in the sample space is ________.
22. When two events cannot occur at the same time, they are said to be ________.
23. When a card is drawn, find the probability of getting

    a. A jack
    b. A 4
    c. A card less than 6 (an ace is considered above 6)

24. When a card is drawn from a deck, find the probability of getting

a. A diamond
b. A 5 or a heart
c. A 5 and a heart
d. A king
e. A red card

25. At a men's clothing store, 12 men purchased blue golf sweaters, 8 purchased green sweaters, 4 purchased gray sweaters, and 7 bought black sweaters. If a customer is selected at random, find the probability that he purchased

a. A blue sweater
b. A green or gray sweater
c. A green or black or blue sweater
d. A sweater that was not black

26. When 2 dice are rolled, find the probability of getting

a. A sum of 6 or 7
b. A sum greater than 8
c. A sum less than 3 or greater than 8
d. A sum that is divisible by 3
e. A sum of 16
f. A sum less than 11

27. The probability that a person owns a microwave oven is 0.75, that a person owns a compact disk player is 0.25, and that a person owns both a microwave and a CD player is 0.16. Find the probability that a person owns either a microwave or a CD player, but not both.

28. Of the physics graduates of a university, 30% received a starting salary of $30,000 or more. If 5 of the graduates are selected at random, find the probability that all had a starting salary of $30,000 or more.

29. Five cards are drawn from an ordinary deck *without* replacement. Find the probability of getting

a. All red cards
b. All diamonds
c. All aces

30. The probability that Samantha will be accepted by the college of her choice and obtain a scholarship is 0.35. If the probability that she is accepted by the college is 0.65, find the probability that she will obtain a scholarship given that she is accepted by the college.

31. The probability that a customer will buy a car and an extended warranty is 0.16. If the probability that a customer will purchase a car is 0.30, find the probability that the customer will also purchase the extended warranty.

32. Of the members of the Spring Lake Bowling Lanes, 57% have a lifetime membership and bowl regularly (three or more times a week). If 70% of the club members bowl regularly, find the probability that a randomly selected member is a lifetime member, given that he or she bowls regularly.

33. The probability that Mike has to work overtime and it rains is 0.028. Mike hears the weather forecast, and there is a 50% chance of rain. Find the probability that he will have to work overtime, given that it rains.

34. At a large factory, the employees were surveyed and classified according to their level of education and whether they attend a sports event at least once a month. The data are shown in the table.

	Educational level		
Sports event	**High school graduate**	**Two-year college degree**	**Four-year college degree**
Attend	16	20	24
Do not attend	12	19	25

If an employee is selected at random, find the probability that

a. The employee attends sports events regularly, given that he or she graduated from college (2- or 4-year degree)
b. Given that the employee is a high school graduate, he or she does not attend sports events regularly

35. In a certain high-risk group, the chances of a person having suffered a heart attack are 55%. If 6 people are chosen, find the probability that at least 1 will have had a heart attack.

36. A single die is rolled 4 times. Find the probability of getting at least one 5.

37. If 85% of all people have brown eyes and 6 people are selected at random, find the probability that at least 1 of them has brown eyes.

38. How many ways can 5 sopranos and 4 altos be selected from 7 sopranos and 9 altos?

39. How many different ways can 8 speakers be seated on a stage?

40. A soda machine servicer must restock and collect money from 15 machines, each one at a different location. How many ways can she select 4 machines to service in 1 day?

41. One company's ID cards consist of 5 letters followed by 2 digits. How many cards can be made if repetitions are allowed? If repetitions are not allowed?

42. How many different arrangements of the letters in the word *number* can be made?

43. A physics test consists of 25 true/false questions. How many different possible answer keys can be made?

44. How many different ways can 5 cellular telephones be selected from 8 cellular phones?

45. On a lunch counter, there are 3 oranges, 5 apples, and 2 bananas. If 3 pieces of fruit are selected, find the probability that 1 orange, 1 apple, and 1 banana are selected.

46. A cruise director schedules 4 different movies, 2 bridge games, and 3 tennis games for a 2-day period. If a couple selects 3 activities, find the probability that they attend 2 movies and 1 tennis game.

47. At a sorority meeting, there are 6 seniors, 4 juniors, and 2 sophomores. If a committee of 3 is to be formed, find the probability that 1 of each will be selected.

48. For a banquet, a committee can select beef, pork, chicken, or veal; baked potatoes or mashed potatoes; and peas or green beans for a vegetable. Draw a tree diagram for all possible choices of a meat, a potato, and a vegetable.

Critical Thinking Challenges

1. Consider this problem: A con man has 3 coins. One coin has been specially made and has a head on each side. A second coin has been specially made, and on each side it has a tail. Finally, a third coin has a head and a tail on it. All coins are of the same denomination. The con man places the 3 coins in his pocket, selects one, and shows you one side. It is heads. He is willing to bet you even money that it is the two-headed coin. His reasoning is that it can't be the two-tailed coin since a head is showing; therefore, there is a 50-50 chance of it being the two-headed coin. Would you take the bet? (*Hint:* See Exercise 1 in Data Projects.)

2. Chevalier de Méré won money when he bet unsuspecting patrons that in 4 rolls of 1 die, he could get at least one 6, but he lost money when he bet that in 24 rolls of 2 dice, he could get at least a double 6. Using the probability rules, find the probability of each event and explain why he won the majority of the time on the first game but lost the majority of the time when playing the second game. (*Hint:* Find the probabilities of losing each game and subtract from 1.)

3. How many people do you think need to be in a room so that 2 people will have the same birthday (month and day)? You might think it is 366. This would, of course, guarantee it (excluding leap year), but how many people would need to be in a room so that there would be a 90% probability that 2 people would be born on the same day? What about a 50% probability?

   Actually, the number is much smaller than you may think. For example, if you have 50 people in a room, the probability that 2 people will have the same birthday is 97%. If you have 23 people in a room, there is a 50% probability that 2 people were born on the same day!

   The problem can be solved by using the probability rules. It must be assumed that all birthdays are equally likely, but this assumption will have little effect on the answers. The way to find the answer is by using the complementary event rule as P (2 people having the same birthday) = 1 − P (all have different birthdays).

   For example, suppose there were 3 people in the room. The probability that each had a different birthday would be

$$\frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365}=\frac{{}_{365}P_3}{365^3}=0.992$$

   Hence, the probability that at least 2 of the 3 people will have the same birthday will be

$$1-0.992=0.008$$

Hence, for k people, the formula is

$$P(\text{at least 2 people have the same birthday})$$
$$=1-\frac{{}_{365}P_k}{365^k}$$

   Using your calculator, complete the table and verify that for at least a 50% chance of 2 people having the same birthday, 23 or more people will be needed.

Number of people	**Probability that at least 2 have the same birthday**
1	0.000
2	0.003
5	0.027
10	
15	
20	
21	
22	
23	

4. We know that if the probability of an event happening is 100%, then the event is a certainty. Can it be concluded that if there is a 50% chance of contracting a communicable disease through contact with an infected person, there would be a 100% chance of contracting the disease if 2 contacts were made with the infected person? Explain your answer.

Data Projects

1. Make a set of 3 cards—one with a red star on both sides, one with a black star on both sides, and one with a black star on one side and a red star on the other side. With a partner, play the game described in the first Critical Thinking challenge on page 233 one hundred times, and record the results of how many times you win and how many times your partner wins. (*Note:* Do not change options during the 100 trials.)
   a. Do you think the game is fair (i.e., does one person win approximately 50% of the time)?
   b. If you think the game is unfair, explain what the probabilities might be and why.
2. Take a coin and tape a small weight (e.g., part of a paper clip) to one side. Flip the coin 100 times and record the results. Do you think you have changed the probabilities of the results of flipping the coin? Explain.
3. This game is called *Diet Fractions.* Roll 2 dice and use the numbers to make a fraction less than or equal to 1. Player A wins if the fraction cannot be reduced; otherwise, player B wins.
   a. Play the game 100 times and record the results.
   b. Decide if the game is fair or not. Explain why or why not.
   c. Using the sample space for 2 dice, compute the probabilities of winning for player A and for player B. Do these agree with the results obtained in part *a*?

Source: George W. Bright, John G. Harvey, and Margariete Montague Wheeler, "Fair Games, Unfair Games." Chapter 8, *Teaching Statistics and Probability.* NCTM 1981 Yearbook. Reston, Va.: The National Council of Teachers of Mathematics, Inc., 1981, p. 49. Used with permission.

4. Often when playing gambling games or collecting items in cereal boxes, one wonders how long will it be before one achieves a success. For example, suppose there are 6 different types of toys with 1 toy packaged at random in a cereal box. If a person wanted a certain toy, about how many boxes would that person have to buy on average before obtaining that particular toy? Of course, there is a possibility that the particular toy would be in the first box opened or that the person might never obtain the particular toy. These are the extremes.
   a. To find out, simulate the experiment using dice. Start rolling dice until a particular number, say, 3, is obtained, and keep track of how many rolls are necessary. Repeat 100 times. Then find the average.
   b. You may decide to use another number, such as 10 different items. In this case, use 10 playing cards (ace through 10 of diamonds), select a particular card (say, an ace), shuffle the deck each time, deal the cards, and count how many cards are turned over before the ace is obtained. Repeat 100 times, then find the average.
   c. Summarize the findings for both experiments.

Answers to Applying the Concepts

Section 4–2 Tossing a Coin

1. The sample space is the listing of all possible outcomes of the coin toss.
2. The possible outcomes are heads or tails.
3. Classical probability says that a fair coin has a 50-50 chance of coming up heads or tails.
4. The law of large numbers says that as you increase the number of trials, the overall results will approach the theoretical probability. However, since the coin has no "memory," it still has a 50-50 chance of coming up heads or tails on the next toss. Knowing what has already happened should not change your opinion on what will happen on the next toss.
5. The empirical approach to probability is based on running an experiment and looking at the results. You cannot do that at this time.
6. Subjective probabilities could be used if you believe the coin is biased.
7. Answers will vary; however, they should address that a fair coin has a 50-50 chance of coming up heads or tails on the next flip.

Section 4–3 Which Pain Reliever Is Best?

1. There were 192 + 186 + 188 = 566 subjects in the study.
2. The study lasted for 12 weeks.
3. The variables are the type of pain reliever and the side effects.
4. Both variables are qualitative and nominal.
5. The numbers in the table are exact figures.

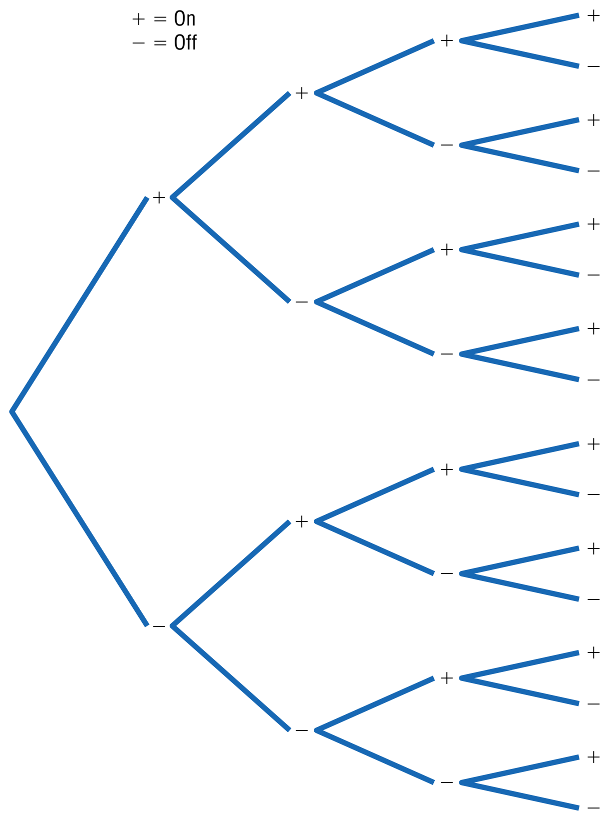
6. The probability that a randomly selected person was receiving a placebo is $192/566 = 0.3392$ (about 34%).
7. The probability that a randomly selected person was receiving a placebo or drug A is $(192 + 186)/566 = 378/566 = 0.6678$ (about 67%). These are mutually exclusive events. The complement is that a randomly selected person was receiving drug B.
8. The probability that a randomly selected person was receiving a placebo or experienced a neurological headache is $(192 + 55 + 72)/566 = 319/566 = 0.5636$ (about 56%).
9. The probability that a randomly selected person was not receiving a placebo or experienced a sinus headache is $(186 + 188)/566 + 11/566 = 385/566 = 0.6802$ (about 68%).

Section 4–4 Guilty or Innocent?

1. The probability of another couple with the same characteristics being in that area is $\frac{1}{12} \cdot \frac{1}{10} \cdot \frac{1}{4} \cdot \frac{1}{11} \cdot \frac{1}{3} \cdot \frac{1}{13} \cdot \frac{1}{100} = \frac{1}{20{,}592{,}000}$, assuming the characteristics are independent of one another.
2. You would use the multiplication rule, since we are looking for the probability of multiple events happening together.
3. We do not know if the characteristics are dependent or independent, but we assumed independence for the calculation in question 1.
4. The probabilities would change if there were dependence among two or more events.
5. Answers will vary. One possible answer is that probabilities can be used to explain how unlikely it is to have a set of events occur at the same time (in this case, how unlikely it is to have another couple with the same characteristics in that area).
6. Answers will vary. One possible answer is that if the only eyewitness was the woman who was mugged and the probabilities are accurate, it seems very unlikely that a couple matching these characteristics would be in that area at that time. This might cause you to convict the couple.
7. Answers will vary. One possible answer is that our probabilities are theoretical and serve a purpose when appropriate, but that court cases are based on much more than impersonal chance.
8. Answers will vary. One possible answer is that juries decide whether or not to convict a defendant if they find evidence "beyond a reasonable doubt" that the person is guilty. In probability terms, this means that if the defendant was actually innocent, then the chance of seeing the events that occurred are so unlikely as to have occurred by chance. Therefore, the jury concludes that the defendant is guilty.

Section 4–5 Garage Door Openers

1. Four on/off switches lead to 16 different settings.



2. With 5 on/off switches, there are $2^5 = 32$ different settings. With 6 on/off switches, there are $2^6 = 64$ different settings. In general, if there are k on/off switches, there are 2^k different settings.
3. With 8 consecutive on/off switches, there are $2^8 = 256$ different settings.
4. It is less likely for someone to be able to open your garage door if you have 8 on/off settings (probability about 0.4%) than if you have 4 on/off switches (probability about 6.0%). Having 8 on/off switches in the opener seems pretty safe.
5. Each key blank could be made into $5^5 = 3125$ possible keys.
6. If there were 420,000 Dodge Caravans sold in the United States, then any one key could start about $420{,}000/3125 = 134.4$, or about 134, different Caravans.
7. Answers will vary.

Section 4–6 Counting Rules and Probability

1. There are five different events: each multiple-choice question is an event.

2. These events are independent.
3. If you guess on 1 question, the probability of getting it correct is 0.20. Thus, if you guess on all 5 questions, the probability of getting all of them correct is $(0.20)^5 = 0.00032$.
4. The probability that a person would guess answer A for a question is 0.20, so the probability that a person would guess answer A for each question is $(0.20)^5 = 0.00032$.
5. There are five different events: each matching question is an event.
6. These are dependent events.
7. The probability of getting them all correct if you are guessing is $\frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{120} = 0.0083$.
8. The difference between the two problems is that we are sampling without replacement in the second problem, so the denominator changes in the event probabilities.