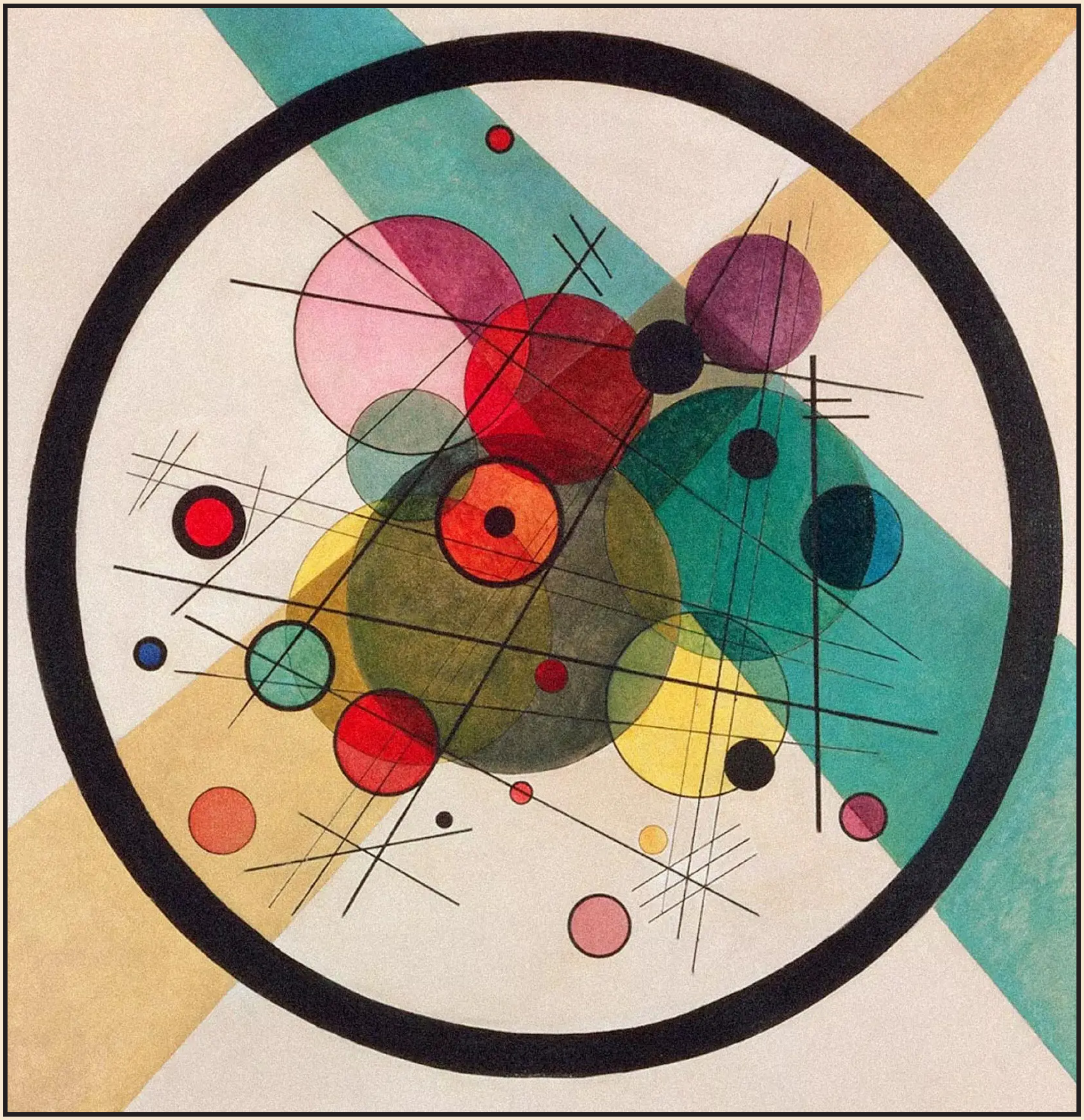
PRIME



Horace Mann School Mathematics Publication

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LETTER from the EDITORS

Dear Readers,

Thank you for picking up a copy of this year's issue of Prime, Horace Mann's premier mathematics publication! Mathematics is the great equalizer and is a part of our daily lives. Prime hopes to generate discussions across a variety of topics in mathematics and demonstrate the vastness of mathematical studies. Our goal is to advance the Horace Mann community's understanding of mathematics and its importance across many aspects of our lives.

In this issue, you'll find a diverse range of topics, such as the history and formation of calculus, applications of math in baseball and politics, and proofs of the infamous Pythagorean Theorem.

We deeply appreciate all of our writers, editors, and Mr. Worrall, our wonderful faculty advisor, all of whom make this publication possible. We thank them all for their care in every article and the issue as a whole.

Finally, we want to thank you all, our readers. Your enthusiasm for mathematics fuels our own. We hope that Prime serves as a source of inspiration and a catalyst for further exploration in the world of mathematics. We urge you all to share the topics, discoveries, and ideas from Prime with others. We hope you enjoy this issue!

Sincerely,
Karolina E. Fic and Sadie Katzenstein
Editors-In-Chief

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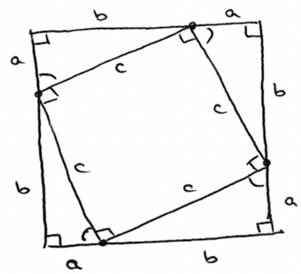
Four Elegant Proofs of the Pythagorean Theorem

By: Kira Lewis

In 9th grade geometry, we all studied the Pythagorean Theorem, but there are more ways to prove it than you might expect. I will cover four especially elegant ones, ranging from well-known to obscure. My personal favorite is Proof 4.

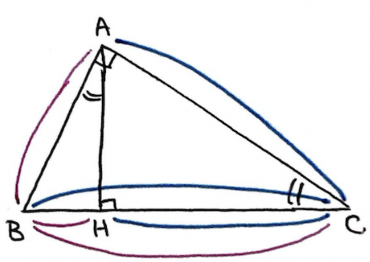
Proof 1: Area



Start with one right triangle, then rotate by 90° and shift it successively until you form a square, as shown below. The total shape has area $(a+b)^2$, the inner square has area c2, and the four triangles together have area $4(\frac{1}{2}ab)$. Therefore, we can write the equation:

$$c^2 + 4(\frac{1}{2}ab) = (a+b)^2$$
$$c^2 + 2ab = a^2 + 2ab + b^2$$
$$c^2 = a^2 + b^2$$

Proof 2: Similarity



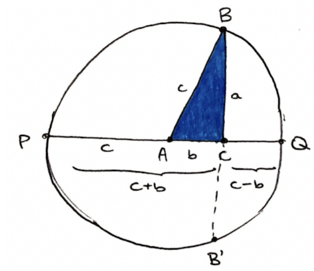
In $\triangle ABC$ with a right angle at A, drop an altitude AH from A. Then, by angle-angle similarity $\triangle ABC$, $\triangle HBA$, and $\triangle HAC$ are all similar.

$$\triangle HBA \sim \triangle ABC \implies \frac{AB}{HB} = \frac{CB}{AB} \implies AB^2 = BC \cdot HB$$
$$\triangle HAC \sim \triangle ABC \implies \frac{AC}{HC} = \frac{BC}{AC} \implies AC^2 = BC \cdot HC$$

Adding these two equations, we get:

$$AB^2 + AC^2 = BC \cdot (HB + HC)$$
$$\text{(Recall } BH + HC = BC\text{)}$$
$$\implies AB^2 + AC^2 = BC^2$$

Proof 3: Power of a Point



What better way to attack a geometry problem than to add some circles? Given the right $\triangle ABC$, construct a circle with center A that passes through B. Then, consider the diameter PQ through C.

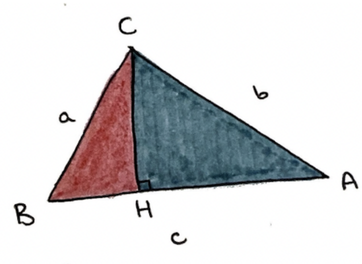
The Power of a Point Theorem states that, when two chords of a circle intersect each other, the following relation holds: $PC \cdot CQ = BC \cdot CB'$

Note that the proof to Power of a Point, which involves showing that triangles $\triangle PCB$ and $\triangle B'QC$ are similar, relies solely on properties of similar triangles, and never the Pythagorean theorem anywhere, so we have no issues with circular reasoning here.

Now, expanding this expression, we get:

$$(c+b)(c-b) = a^2$$
$$c^2 - b^2 = a^2$$
$$a^2 + b^2 = c^2$$

Proof 4: Similarity & Area



Like Proof 2, we start with the right $\triangle ABC$ and altitude CH.

Consider the areas of similar triangles $\triangle BCH$, $\triangle CAH$, and $\triangle ABC$. Remember that the area of a triangle is base times height, so in similar triangles, where the bases and heights are proportional, the area is proportional to the side length squared, if corresponding side lengths are measured. Therefore, we can write:

Area[$\triangle BCH$] is proportional to a^2
Area[$\triangle CAH$] is proportional to b^2
Area[$\triangle ABC$] is proportional to c^2

Area[$\triangle BCH$] + Area[$\triangle CAH$] = Area[$\triangle ABC$], so therefore $a^2 + b^2 = c^2$.

How to Be Irrational

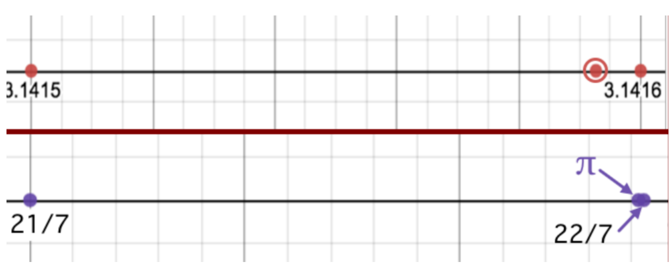
By: Bennett Feng

Let's start by looking at a few irrational numbers: $\sqrt{2} \approx 1.414213562$, $\sqrt{3} \approx 1.732050808$, $e \approx 2.718281828$, and $\pi \approx 3.141592654$. Which one of these numbers do you think is most irrational? While most people think π is the most irrational, this was actually a trick question because there is no real answer. That leads us to another question: how on earth can we quantify irrationality?

An irrational number is a number that cannot be written (is not equal to) a rational $p \times q$ for integers p and q. We can measure the irrationality of a number α according to how well we can approximate α by fractions. This brings us to the concepts of proximity and dexterity, two measures that attempt to capture the essence of how well a rational number approximates an irrational one.

Proximity, as its name suggests, is about how close a rational approximation is to an irrational number. Considering our example number π, we might initially try to measure its irrationality by looking at how closely it can be approximated by rational numbers. For instance, we have $\frac{3}{1}$, $\frac{22}{7}$, and $\frac{31416}{1000}$ as approximations to π, ranked from worst to best proximity. Since any irrational number may be approximated arbitrarily closely by rationals, we can conclude that proximity is not a useful yardstick by which to measure irrationality. Fortunately, some approximations are more dexterous than others.

Unlike proximity, which only considers the closeness of the approximation, dexterity evaluates how skillfully a fraction approximates an irrational number. $\frac{22}{7}$ is an antiquated approximation to π. What's the big deal over $\frac{22}{7}$, anyway? Check it out:



Now let's find a formula that measures the dexterity of an approximation, given $\frac{p}{q}$ as an approximation to an irrational number α.

The formula can be written as $q\left|\frac{p}{q} - \alpha\right|$ or $|p - q\alpha|$, which works even if α is on the right of $\frac{p}{q}$. The term $\left|\frac{p}{q} - \alpha\right|$ represents the absolution difference between the rational approximation and the irrational number, basically the distance between the two. That is the measure of proximity, however, the distance alone isn't sufficient because a fraction with a small denominator might appear to be a good approximation simply due to its low value. To address this, the formula multiplies the distance by the denominator q, which serves as a scale factor. This scale factor ensures that the dexterity value not only considers how close the fraction is to the irrational number but also how precise the approximation is relative to the size of the denominator. The smaller the value of dexterity, the better the approximation. For example, when comparing two approximations of π, such as $\frac{31416}{10000}$ and $\frac{22}{7}$, $\frac{22}{7}$ has a lower dexterity value, making it a more effective approximation. For a change of pace, let's search for more and more dexterous approximations to $\sqrt{3}$.

p	q	dexterity
2	1	0.26795
5	3	0.19615
7	4	0.0718
19	11	0.05256
26	15	0.01924
71	41	0.01408
97	56	0.00515
265	153	0.00377
362	209	0.00138
989	571	0.00101

We see all these values, but what do they mean? The dexterity values seem to be getting closer and closer to 0. To confirm, the dexterity of $\frac{9973081}{5757961}$ is about 0.000001. We hit a dead end similar to when we were looking at proximity that for any irrational number α, there exist rational approximations p and q with arbitrarily small dexterity values. While neither proximity nor dexterity provides a complete picture of irrationality, they both offer valuable perspectives on the complexities of approximation and the intriguing nature of irrational numbers.

Archimedes: The OG Greek Freak

By: Karolina E. Fic

You are in the middle of a math problem and someone is coming at you with a sword. What do you do? Option A: Run for your life. Option B: Continue working on the math problem. This is the story of the death of Archimedes. Archimedes was an Ancient Greek mathematician, believed to have been born in 287 BC in Syracuse, Sicily because he was said to have been 75 years old when he died in 212 BC. During the Roman Siege of Syracuse, a Roman soldier found Archimedes drawing circles in the sand, absorbed in solving a mathematical problem. When the soldier approached him menacingly, Archimedes scolded him saying "μή μου τοὺς κύκλους τάραττε!" ("Do not disturb my circles!"). Despite receiving orders to not harm Archimedes, the Roman soldier, enraged by Archimedes' attitude, killed him with his sword. Clearly Option B was not the most logical choice.

Archimedes is often hailed as "the Father of Mathematics" and thus he is aptly the figure in profile on the obverse of The Fields Medal, the most prestigious award in the field of mathematics. Archimedes contributed immensely to numerous fields of mathematics, most importantly geometry, mechanics, and calculus. His studies laid the groundwork for the concept of limits and integrals, which are the foundations of calculus. Archimedes used methods almost identical to modern integral calculus, which paved the way for later leading mathematicians, such as Isaac Newton and Gottfried Wilhelm Leibniz, who independently developed calculus nearly two millennia later.

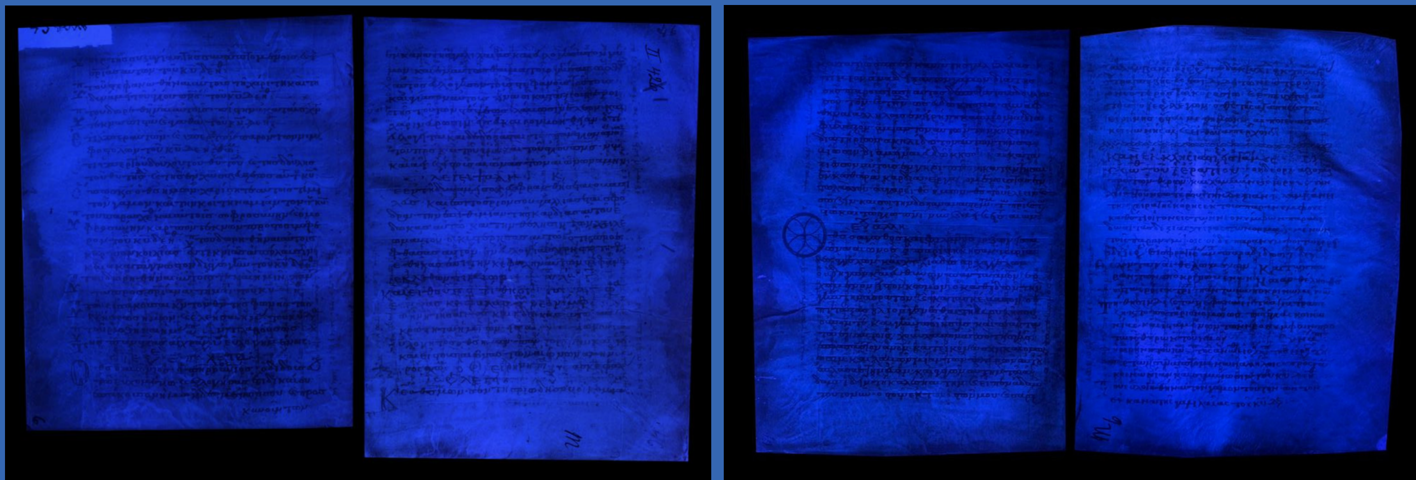
One such example can be found in *The Method of the Mechanical Theorems* (also known as *The Method*), which was considered one of the lost works of Archimedes until 1906, when Johan Ludvig Heiberg rediscovered it in Constantinople within a document called the Archimedes Palimpsest. A palimpsest (from παλίμψηστος, meaning "scraped again") is a manuscript that has been scraped off or rubbed smooth, leaving behind faint traces of the original text, and then re-used by writing over the old text. The Archimedes Palimpsest is a layered text consisting of seven of Archimedes' treatises (including the only known copy of The Method) and a Byzantine prayer book. It was lost numerous times over thousands of years. However, most recently it was loaned to the Walters Art Museum in Baltimore, Maryland, where a team of restorers and scholars have been cleaning, imaging, and transcribing the Archimedes text.

The Method provides incredible insight into Archimedes' approaches, which he used to create many of his later works. A comparison of his other treatises to *The Method* reveals how Archimedes found the areas and volumes of certain figures, such as the figure in his first proposition. Archimedes' approach was to balance elements of the figure against elements of another figure with known measurements. *The Method* takes shape in a letter addressed to his friend Eratosthenes, a Greek polymath and chief librarian at the Library of Alexandria (the center of Greek knowledge in antiquity). In his opening address, Archimedes says that he is now sending the proofs ("τὰς ἀποδείξεις") for the formerly discovered theorems ("τῶν εὑρημένων θεωρημάτων"). Archimedes sent his method to demonstrate how he first arrived at mathematical statements through his keen understanding of mechanics. Later he proved his discoveries geometrically because he believed their proof by *The Method* did not suffice:

"Καὶ γάρ τινα τῶν πρότερόν μοι φανέντων μηχανικῶς ὕστερον γεωμετρικῶς ἀπεδείχθη διὰ τὸ χωρὶς ἀποδείξεως εἶναι τὴν διὰ τούτου τοῦ τρόπου θεωρίαν."

"For also some of the things appearing earlier to me mechanically were later proven geometrically because the theory through this way was without proof."

The first proposition discussed in *The Method* illustrates how Archimedes laid the groundwork for the core concepts of calculus (limits and integrals).



Scans of the Pages of the Archimedes Palimpsest Containing the Introduction of *The Method*

Ancient Greek Text for the First Proposition:

Ἀρχιμήδους Περὶ τῶν μηχανικῶν θεωρημάτων πρὸς Ἐρατοσθένην ἔφοδος.

Ἀρχιμήδης Ἐρατοσθένει εὖ πράττειν.

Ἀπέστειλά σοι πρότερον τῶν εὑρημένων θεωρημάτων ἀναγράψας αὐτῶν τὰς προτάσεις φάμενος εὑρίσκειν ταύτας τὰς ἀποδείξεις, ἃς οὐκ εἶπον ἐπὶ τοῦ παρόντος· ἦσαν δὲ τῶν ἀπεσταλμένων θεωρημάτων αἱ προτάσεις αἵδε· τοῦ μὲν πρώτου· ἐὰν εἰς πρίσμα ὀρθὸν παραλληλόγραμμον ἔχον βάσιν κύλινδρος ἐγγραφῇ τὰς μὲν βάσεις ἔχων ἐν τοῖς ἀπεναντίον παραλληλογράμμοις, τὰς δὲ πλευρὰς ἐπὶ τῶν λοιπῶν τοῦ πρίσματος ἐπιπέδων, καὶ διά τε τοῦ κέντρου τοῦ κύκλου, ὅς ἐστι βάσις τοῦ κυλίνδρου, καὶ μιᾶς πλευρᾶς τοῦ τετραγώνου τοῦ ἐν τῷ κατεναντίον ἐπιπέδῳ ἀχθῇ ἐπίπεδον, τὸ ἀχθὲν ἐπίπεδον ἀποτεμεῖ τμῆμα ἀπὸ τοῦ κυλίνδρου, ὅ ἐστι περιεχόμενον ὑπὸ δύο ἐπιπέδων καὶ ἐπιφανείας κυλίνδρου, ἑνὸς μὲν τοῦ ἀχθέντος, ἑτέρου δέ, ἐν ᾧ ἡ βάσις ἐστὶν τοῦ κυλίνδρου, τῆς δὲ ἐπιφανείας τῆς μεταξὺ τῶν εἰρημένων ἐπιπέδων, τὸ δὲ ἀποτμηθὲν ἀπὸ τοῦ κυλίνδρου τμῆμα ἕκτον μέρος ἐστὶ τοῦ ὅλου πρίσματος.

My Translation for the First Proposition:

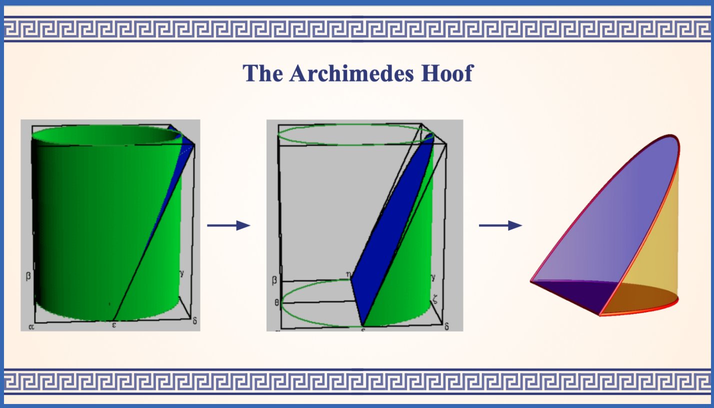
The Method of Archimedes concerning the mechanical theorems to Eratosthenes.

Archimedes wishes Eratosthenes well.

I sent to you earlier, having written [them] up, the propositions of the same discovered theorems, asserting that I would find these proofs, which I did not upon being present [at that time]. And these were the propositions of the sent off theorems: on the one hand of the first [theorem]: If in a straight prism having a parallelogram for a base, a cylinder is inscribed, on the one hand having bases in the opposing parallelograms, and on the other hand [having] the sides upon the remaining planes of the prism; also through both the center of the circle, which is the base of the cylinder, and [through] one side of the square in the opposite plane a plane is drawn; the drawn plane will cut off from the cylinder a section, which is encompassed by two planes and [by] the surface of the cylinder, on the one hand by the one drawn, and on the other hand by the other, in which the base of the cylinder is, and by the surface between the said planes, and the section having been cut off from the cylinder is a sixth part of the whole prism.

The figure that Archimedes is constructing is a cylinder inscribed in a "straight" prism. However, the type of prism is not known until you take into account the constraints: the bases of the cylinder, which by definition are circles, need to be inscribed in the "opposing parallelograms" of the prism. This means the "opposing parallelograms" need to be squares so that the circles are perfectly inscribed within them. Another constraint we have to take into account is that the "sides" of the cylinder, which is the curved surface, need to be "upon the remaining planes of the prism." This means the curved surface of the cylinder needs to be tangent to the four remaining sides of the prism. Now, accounting for all of the constraints, we know the prism is rectangular and the cylinder is perfectly inscribed (i.e. the circular bases of the cylinder are inscribed in the square bases of the prism and the curved surface of the cylinder is tangent to the "remaining planes" of the prism).

Once he establishes this figure, Archimedes then goes on to construct another shape from the cylinder-prism shape. Using a plane, he cuts a section out of the cylinder-prism shape. The plane needs to cut through the center of one of the circles (one of the bases of the cylinder) and through one side of the square in the opposite plane (i.e. the other square base of the prism). Once Archimedes finished the construction of the figure, he states that the plane cuts out a section from the cylinder. The portion of the cylinder cut off by the plane is often referred to as the Archimedes Hoof because of its similar shape.

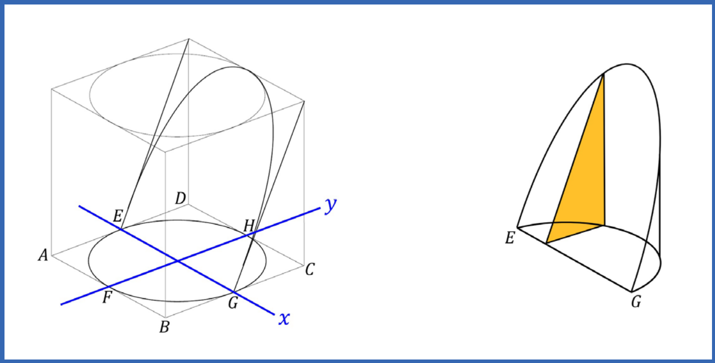


Archimedes claims that the cut portion is 1/6 of the entire prism. This result does not seem obvious at first because one would expect the volume of the cut, which has a curved surface, to involve π. However, the result is a rational number. Remarkably, Archimedes was able to prove this without modern calculus, although his "indivisible" argument for the hoof is strikingly similar and thus is considered the basis of calculus. Archimedes proved this result geometrically. Although a crucial part of the proof is lost because the writing in part of the palimpsest is illegible, the overarching idea remains. Archimedes used parallel slices of an area or volume which are "taken together" to produce the final area or volume. Archimedes' "indivisible" argument for the calculation of the volume of the cylindrical section is very similar to Cavalieri's Principle and the method of indivisibles.

Proof Using Integral Calculus:

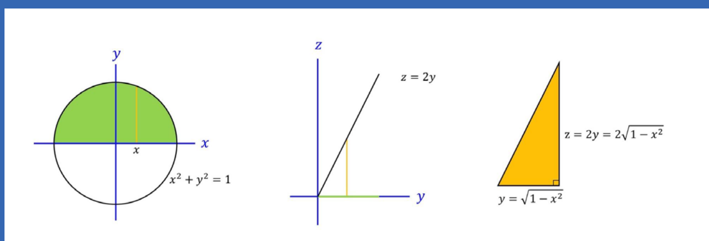
In Calculus, the idea of determining volumes by slicing is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and then take the limit of the Riemann sums to produce an integral for the volume. The volume of the solid can be obtained by integrating the area of the cross section ($A(x)$) from one end of the solid to the other (the interval $[a, b]$): $V = \int_a^b A(x)dx$.

Once we construct the figure established earlier (the cylinder inscribed in the prism with a plane cutting off the hoof section), we need to establish coordinate systems to use the volumes by slicing method and make an appropriate slice (as seen below in orange).



The base of the shape (which is the circle inscribed in the parallelogram) will lie in the $x-y$ coordinate system. In order to make calculations simpler, we want to construct the shape so that the prism is a cube (which still holds with the earlier constraints) with side length 2. Thus, the radius of the circular base will be 1 and the circular base will be represented by the equation $x^2 + y^2 = 1$.

We need to create a third dimension, a z-dimension, to extend upward. Thus, we establish a $y-z$ coordinate system that gives a profile of the slanted surface of the solid (which is the oblique plane passing through the diameter of the circular base that cuts the hoof off from the cylinder). We now need to find the slope of the line of the plane in the $y-z$ coordinate system: since the height of the cube is 2 (which corresponds to Δz), and the radius of the circle is 1 (which corresponds to Δy), the slope of the line of the plane ($\Delta z/\Delta y$) is 2. Therefore, the line is represented by the equation $z = 2y$.



The cross-section is the triangle with one side extending along the y axis, another side extending along the z-axis, and the hypotenuse is the line along the slanted plane (represented by the line $z = 2y$). The side extending along the y axis is equivalent to $\sqrt{1-x^2}$ (by the equation of the circle). The side extending along the z axis is equivalent to $2y$ (by the equation of the slanted line), which is equivalent to $2\sqrt{1-x^2}$. Thus, the area of the cross section is $\frac{1}{2}\sqrt{1-x^2}\cdot 2\sqrt{1-x^2}$ (by the area of a triangle). This is equal to $1-x^2$.

Now we need to evaluate the integral of the cross section over the appropriate interval. Since the radius of the circle is 1 and we need to integrate from one end of the solid to the other, the interval will be $[-1,1]$. Now we can evaluate the integral: $\int_{-1}^{1}(1-x^2)dx$. The result is $x - \frac{1}{3}(x^3)$ evaluated from -1 to 1, which simplifies to 4/3:

$$V = \int_a^b A(x)dx = \int_{-1}^{1}(1-x^2)dx = \left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{4}{3}$$

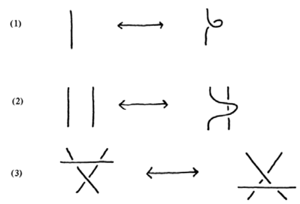
Therefore, the volume of the hoof (which we have shown to be 4/3) is 1/6 the volume of the prism because the volume is 8 (as we constructed the prism to be a cube with side length 2). This matches the value that Archimedes proved without integral calculus.

The Method as a whole and this particular proposition highlights Archimedes' groundbreaking thinking. His thinking contained the germ for understanding the idea of a limit, which is an integral part of calculus (pun intended). Archimedes' idea of comparing areas of cross sections of two different bodies is very similar to Cavalieri's principle. His method is nearly identical to the method of slicing that we use in calculus today. What makes Archimedes so incredible is that he calculated these shapes with mechanics to avoid using integral calculations as integrals and calculus did not exist at the time. His studies mark the beginning of the method of slicing and the theory of integration. Archimedes is truly one of the greatest mathematicians who ever lived. He was more than 2000 years ahead of his time!

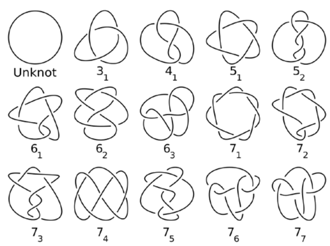
Detangling Knot Theory

By: Gabriela Faybishenko

In our world of rapidly evolving mathematical theories and scientific concepts, there is an intricate and visually captivating field known as knot theory. At first glance, knots seem simple and are used daily: in shoelaces, friendship bracelets, loops, headphone wires, etc. However, underneath their apparent simplicity in everyday life lies a fascinating and firm theory, which mathematicians have studied for centuries. Today, knots are used in highly complex fields of mathematics and many mysterious domains in science.



Reidemeister Moves



Prime Knots

The purpose of knot theory today is to classify and distinguish different types of knots. Mathematicians aim to determine whether two knots are equivalent, in other words, if they can be deformed or manipulated into one another without cutting or passing through themselves. In knot theory, equivalence relies on the set of transformations called Reidmaster moves. In short, Reidmaster moves are certain types of ways to modify knots without changing their fundamental structure. There are three Reidmaster moves that can change the physical characteristics of a knot without changing its arrangement. These include twists, pokes, and slides. Moreover, the concept of knot invariants plays a crucial role in distinguishing knots. Invariants are mathematical properties or quantities that remain unchanged under deformations allowed by Reidemeister moves. These invariants act as "fingerprints" for knots, aiding mathematicians in classifying and understanding their intricacies. However, like numbers that we use in everyday life, there are such things as "prime knots." Prime knots are knots that cannot be undone. Reidmaster moves might change how the knot might appear; however, the fundamental structure will not change. As in prime knots, no matter how many Reidmaster moves are performed, there is no possible way for a prime knot to be undone. In mathematics, the most simple knot is known as an "unknot" (or "trivial knot") which is a circle. Such simple structures are known as knots because there is no beginning or end to this figure, and by definition, this is a knot.

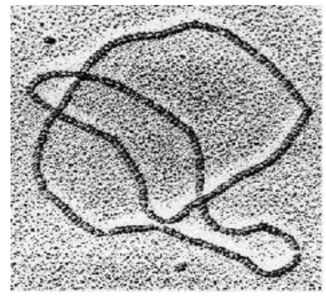
While knot theory today revolves around mathematics, its origin of importance extends far beyond. Thousands of years ago, knots were not only used for decoration, but they also had hundreds of other underlying meanings. The first knots were found in Chinese artwork dating several centuries BC. In Tibetan Buddhism, endless knots first appeared. Throughout time, knots have been used in many cultures to signify strength and unity because they do not have an end or a beginning. The Incas used knots to create quipus, which consisted of horizontal strings or a wooden block from which knotted and colored strings would hang. Archeologists believe that quipus were generally used to depict fairy tales and poetry. The color, amount of knots per string, location, sequence, and way in which they were woven all had unique meanings.

Quipu from the Incas

Another famous use of knots is the famous story in Greek Mythology called "The Gordian Knot." In this story, there was a country that did not have a king, and an oracle decreed that the next man to enter the city with an ox-cart should become king. A man named Gordius arrived with his wife and an ox-cart, and surely he became king. In gratitude to the gods, he dedicated his ox-cart to Zeus, and tied it with a highly intricate knot, later known as the Gordian Knot. Another oracle foretold that the person who untied the knot would one day rule all of Asia. For many years, nobody could untie or even loosen the knot, until one day in the 4th Century BC, a young man approached the ox-cart, and after understanding there was no loose end to untie the knot. He took out his sword and sliced the knot in half. That man was later known as Alexander the Great.

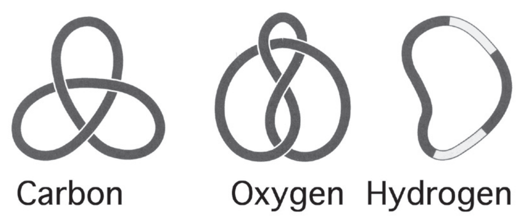
Jean-Simon Berthelemy, Alexander the Great cuts the Gordian Knot

While knots originate from art and necessary tools to survive, the application of knot theory extends far beyond complex forms of mathematics and science. Rosalind Franklin, James Watson, and Francis Crick uncovered the shape of DNA (double helix) in 1953, and it was not until the 1980s that scientists discovered that DNA strands could become tangled, similar to knots. When separated strands of DNA twist onto themselves, a knot forms. The discovery that knots were found in DNA inspired many scientists to further analyze and study knots as well as knot theory, not only because they could be found in genes, but also because if they could be found in DNA, there must be an infinite number of places where knots could be right in front of our eyes. Physicists have also explored knot theory's implications in understanding the behavior of subatomic particles, the properties of polymers, and the dynamics of string theory in theoretical physics.



Knotted DNA

Knot theory, with its visually captivating subject matter and deep mathematical underpinnings, stands as a blueprint for the beauty and complexity of mathematics. From the study of simple knots to the exploration of intricate ones, mathematicians delve into a world that transcends physical tangibility, uncovering connections that reach into diverse realms of science and art. As research in knot theory progresses, its implications continue to expand, offering insights into the nature of entanglement and connectivity that permeate our universe.



Knots in Elements

Gaussian Integers

By: Sadie Katzenstein

Carl friedrich gauss was a German mathematician, geodesist,[1] and physicist who has been referred to as the "Prince of Mathematicians" due to his significant contributions to many mathematical and scientific fields. Most notably, Gauss published the second and third complete proofs of the fundamental theorem of algebra, developed the theories of binary and ternary quadratic forms, invented the fast Fourier transform algorithm (crucial to the discovery of the dwarf planet Ceres), introduced the Gaussian gravitational constant, and derived the method of least squares, which all sciences use to minimize measurement error.

Gauss's work in number theory stands out among his countless contributions to mathematics. In the early 19th century, he provided a more nuanced understanding of mathematical structures by revolutionizing our understanding of integers with Gaussian integers. A Gaussian integer is a complex number of the form $a + bi$, where a and b are integers, and i is the imaginary number $i = \sqrt{-1}$, or $i^2 = -1$. Gauss's invention of Gaussian integers was a deliberate extension of the number system to accommodate solutions to equations that could not be expressed solely in terms of real numbers. They were crucial in understanding the importance of complex numbers in solving mathematical problems and in describing phenomena previously inaccessible with traditional integers, such as the behavior of electrical circuits.

Similar to other integers, when standard operations such as addition, subtraction, and multiplication are applied to Gaussian integers, they form an integral domain ($\mathbb{Z}[i]$). One of the most distinguishing factors of an integral domain is that there are no non-zero elements whose product is zero, meaning there are no zero divisors.[2] More specifically, Gaussian integers form a Euclidean domain, which is a specific type of integral domain that is equipped with a Euclidean function. This function allows Gaussian integers to perform similarly to integers. Gaussian integers also have Euclidean division, the ability to divide one element by another and get a quotient and some notion of a remainder, and an Euclidean algorithm, an efficient method for computing the greatest common divisor (GCD) of two numbers.

There is no natural linear ordering of complex numbers that preserves the properties of a total order. This means that, unlike integers, the set of Gaussian integers $\mathbb{Z}[i]$ does not have a total ordering that respects arithmetic or is compatible with their algebraic structure. So, any two Gaussian integers are not necessarily comparable (greater than, less than, or equal to each other) with respect to their ordering relationship.

Gaussian numbers are essential to understanding various concepts in mathematics and science. For example, Gaussian integers are vital for analyzing wave behavior, including the propagation of light waves in optics. They also contribute to the mathematical framework of quantum mechanics to represent quantum states. Additionally, Gaussian integers' unique factorization properties are useful in constructing error detection and correction codes in telecommunications.

Gauss's proofs, theorems, and other discoveries have become essential tools in diverse fields, leaving an enduring legacy that continues to shape how we understand and apply mathematics in our technologically advanced world.

[1]Someone who measures and monitors the Earth's size, shape, and gravity field to determine the exact coordinates of any point on Earth and how that point will move over time.

[2]Two numbers that are not zero but their product is zero.

The Math Behind WAR

By: Zach Shwartz

What is WAR? WAR, known as Wins Above Replacement, is an innovative statistic used to compare baseball players based on their performance. The basic idea behind WAR is a calculation that estimates how many more wins a team would achieve with that player rather than a "replacement-level" player. WAR is one of the most complicated statistics to be calculated in baseball. Many sites calculate WAR differently due to the loose structure of its formula. Still, they all follow this standard:

WAR = (Batting Runs + Baserunning Runs + Fielding Runs + Positional Adjustment + League Adjustment + Replacement Runs) / Runs Per Win

The formula itself does not seem complicated when looking at it, but each component is challenging to calculate. WAR considers all the different aspects of being a good baseball player. These statistics evolve, just like technology. Statistics are applied to players worldwide, attempting to quantify how well players do, so teams can determine whether they want a player.

Many statistics are misleading; for example, take two players, one with 20 stolen bases and one with 10. Some can say the person with 20 stolen bases is the better stealer, but what if you were told the person who stole 20 bases attempted to steal a base 40 times, and the person who has 10 stolen bases attempted 10 times? With this new statistic, you can determine the second player was a better stealer due to a higher success rate. Coaches across the country use statistics to value players. For example, Billy Beane, a former General Manager of the Oakland Athletics, had a strategy called "Moneyball." Using the Moneyball strategy, Beane focused on using modern statistical analysis to identify undervalued players, as the Oakland Athletics did not have a high payroll (amount of money a team spends per year to pay players). One of the key statistics Beane looked at was OBP (On Base Percentage). While Batting Average takes into account only hits, OBP includes walks and is calculated by the percentage a player gets on base, regardless of the circumstance.

When evaluating players, it is important to use the correct statistics to determine a player's value. WAR is a prime example of how statistics evolve, exemplified by how complicated it is to calculate. Statistics in the past did not take into account fielding and base running; rather, they just used hitting statistics as the batting average. Statistics is the math behind any sport, applied throughout the world. In this article, I will explain how WAR is calculated and the implications each aspect has within WAR.

Batting Runs are the amount of offensive runs a player contributes to a team. Batting Runs is calculated by the formula:

Batting Runs = wRAA + (lgR/PA – (PF × lgR/PA)) × PA + (lgR/PA – (AL or NL non-pitcher wRC/PA)) × PA

wRAA stands for Weighted Runs Above Average, IgR stands for team runs per 162 games, PA stands for Plate Appearances, PF stands for Park Factor, and wRC stands for Weighted Runs Created. wRAA is significant for calculating batting runs as it measures the amount of offensive runs a player contributes to a team compared to the average player. PF is used in this formula because no baseball stadium is the same. While sports like basketball and football have regulation-sized fields, baseball fields vary from very short fences to very far fences and high altitudes. PF helps adjust the statistics to take into account that every stadium is different; therefore, players playing in certain stadiums would not have an unfair advantage. wRC attempts to value a player's offensive value relative to the rest of the league. wRC helps evaluate multiple players from different years. For example, if you wanted to compare a player from the 1940s to a player today, a simple statistic like batting average does not take into account how the player compares to the rest of the league. Back in 1940, it was easier to hit a pitch due to the recent advancements in technology and medicine; therefore, a statistic like wRC provides a way to determine a player's offensive value relative to how the rest of the league is doing.

Baserunning Runs are the amount of runs a player's base running ability adds to the team's total runs. Baserunning runs include steals and how well a player runs the bases after getting a hit. The formula for Baserunning Runs is:

Baserunning Runs = UBR + wSB + wGDP

UBR stands for Ultimate Base Running, which measures how well a player runs the bases when not stealing. UBR is important for determining how someone's baserunning impacts a team since baserunning is not all about stolen bases. For example, a player who scores from second on a single while another player on the same play would stay at third, the player who scores is more valuable to the team. wSB stands for weighted Stolen Base runs, and it estimates how many runs a player contributes to a team by stealing bases, compared to an average player. This is the baserunning counterpart to wRAA. wSB helps compare players from different years as the statistic measures stolen bases relative to the league. In 2023, steals were up 40% from 2022, as new rules such as limited pickoff moves and bigger bases helped encourage more stolen bases. A player with 10 stolen bases in 2022 would be compared to a player with 14 stolen bases in 2023.

Fielding Runs are calculated much differently than batting and baserunning runs. While batting and baserunning runs are calculated by a formula, fielding runs do not have one. Instead, sites use video analysis to judge one's fielding impact on their team. Fielding is one-half of baseball, and within WAR, the goal is to determine how many more wins a team would have with a player rather than using a replacement player. WAR is unique since no statistic before it calculated batting, baserunning, and fielding together.

Positional Adjustment is necessary as some positions are harder to play than others. This helps defense become a greater factor within WAR and is calculated by the formula:

Positional Adjustment = ((Innings Played/9) / 162) × position run value

Position run value varies based on the site since people value positions differently. Usually, sites have similar position values; for example, a designated hitter would have the lowest positional value, and a catcher would have the highest. Positions like shortstop and centerfield will have higher positional values as those positions are more involved in every play and are harder to play.

League Adjustment is a way to even out players who play in different leagues. The MLB comprises two leagues' therefore, one league will have a higher RAA (runs above average) than the other. League adjustment is a small correction to make it so each league's RAA average is zero. League Adjustment is calculated by:

League Adjustment = ((-1) × (lgBatting Runs + lgBase Running Runs + lgFielding Runs + lgPositional Adjustment) / lgPA) × PA

Replacement Runs are important to WAR as all the components so far for calculating WAR concern the league average. WAR is the measure of wins a team will have with a player rather than using a replacement-level player, not an average player. The concept of replacement runs is that there is 1,000 WAR (all of the WAR added up from every player in the MLB) to go around the league, with 57% going to the batters and 43% to the pitchers. Using a formula to determine the amount allocated for a replacement player's plate appearances, we can determine the baseline value of a replacement player.

Runs Per Win determines how many runs are needed to achieve a win. The formula for Runs Per Win is:

Runs Per Win = 9*(MLB Runs Scored / MLB Innings Pitched)*1.5 + 3

All the components within WAR calculate and adjust how many runs a player provides to a team relative to a replacement-level player. There is no set-in-stone number of runs that determines a win, so this formula is a way to determine how many runs should equal a win based on how the MLB has been doing that year.

Statistics are critical to the evaluation of players for teams, as scouts cannot watch every time a player plays. Statistics simplify how well a player is doing; therefore, statistics like WAR are very useful. WAR takes into account hitting, baserunning, and fielding, the three main components of baseball. Statistics evolve, and there will likely be new ones that are more efficient at evaluating players.

The Game of Hackenbush By: Maya Rangarajan

Hackenbush is a two-player game where participants take turns eliminating edges from a preset graph. The objective for each player is to strategically maneuver the opponent into a position where they cannot remove any more edges. A player wins by successfully placing their opponent in a situation with no remaining legal moves.

There are numerous Hackenbush variations. Green (or Monochromatic) Hackenbush is where all segments are green, and both players can eliminate any segment. In Red-Blue Hackenbush, the focus of this article, one player is assigned Red and the other Blue. Red moves by cutting a red segment, and blue by cutting a blue segment. The cut segment is deleted together with any other segments that are no longer connected to the ground. Red-Blue-Green Hackenbush has red, green, and blue segments. The blue player can choose any blue or green segments, while the red player can choose any red or green segments. Although these seem like simple changes in the playing board, the effect on determining the best playing option is drastically different.

Hackenbush meets all the requirements of a finite, combinatorial game. The game is deterministic: there is no randomization mechanism, such as flipping a coin or rolling a die. There is perfect information: each player knows all the information about the state of the game, and nothing is hidden. There is a guaranteed outcome: the first player to fulfill the winning condition is the winner, leaving no possibility of a tie or draw. The game is also finite: there is no sequence of moves that will lead to an infinite game.

Who wins? Assuming each player makes optimal moves, the winner of the game can be determined before the first move is made. In this essay, I will examine how this is possible. Starting out with an easy example, take a look at Figure 1 below. Who will win?

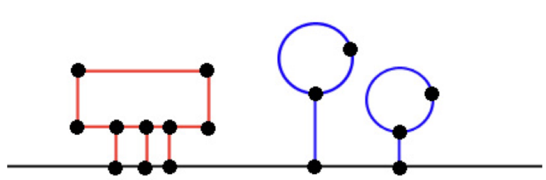


Figure 1

The red has 10 edges compared to the blue's collective 6. If Red removes the edges of the figure from top to bottom, they can ensure there will be at least $10 - 6 = 4$ edges left when blue runs out of edges. With no interdependence, each edge is worth one point, and the difference in the number of edges between red and blue determines the winner. If red has more edges, red will win, irrespective of who moves first.

What about the game in Figure 2, in which case the moves are not interdependent and there is no difference in the number of edges? Both players have identical figures to remove, so who wins?

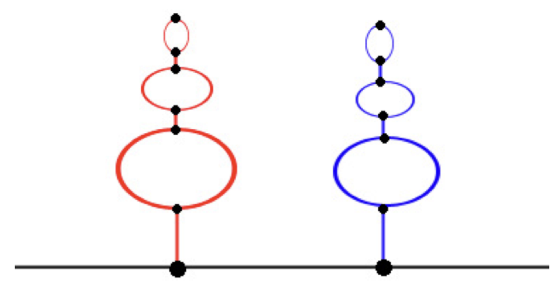


Figure 2

Whoever starts will have to cut some edge of the picture. The other player then can adopt a mimicking strategy and simply copy whatever the player who starts first has just done on their own picture, and thus be assured of always having a move left before the game ends. This strategy of mirroring the moves of the first player is known as the Tweedledee and Tweedledum strategy. So, under optimal play, the player who starts will always lose this game. Such games, where the first player to move always loses, are called zero games.

More generally, if the red and blue edges do not interact, in other words, if cutting a red edge can never cause a blue one to be removed and vice-versa, then the result of a game played optimally with m such red edges and n such blue edges depends only on the value of $n - m$. If $n - m$ is positive, blue is certain to win. If $n - m$ is negative, red is certain to win. If it is zero, the player who goes second wins.

Things get more interesting when we consider graphs that involve interdependence. What is the value of the game below in Figure 3? If blue removes the edge from, say, the figure on the extreme left, one of the red edges will be disconnected from the ground, causing red to lose that edge. Playing it out, we realize this is a zero game. As the single red edge to the right is worth -1 from a blue perspective, it grants red with one extra move, and the game is zero value, so we can conclude each of the two identical positions to the left of the red edge are worth $+\frac{1}{2}$. Note: the convention is to always provide values from the perspective of the blue player.

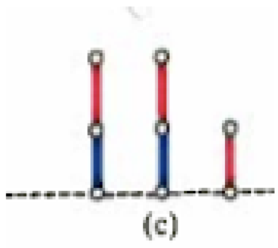


Figure 3

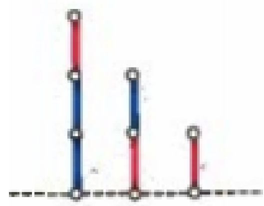


Figure 4

Take another zero value game depicted below in Figure 4. We know the far right position is worth -1. The middle position is worth $-\frac{1}{2}$, which we arrive at by flipping the parity of the middle position in Figure 3 to reflect the fact the positions of red and blue are flipped. So the two right positions are worth $-1+(-\frac{1}{2}) = -1\frac{1}{2}$. Given it is a zero sum game, the far left position must be worth $+1\frac{1}{2}$.

Extending the idea of the value of positions, we can ascertain the value of any game. First, let's represent a game with a formal notation. For any game G, we can write $G = \{l_0, l_1, l_2, \ldots \mid r_0, r_1, r_2, \ldots\}$ where Blue can move to a game worth $l_0, l_1, l_2, \ldots$ if Blue were to move first and Red can move to a game worth $r_0, r_1, r_2, \ldots$, if Red were to move first. The game in Figure 1 can be expressed as $\{-5, -7 \mid -3\}$. As explained previously, Blue starts with 6 edges to Red's 10. Blue moving first, can remove an edge that is not grounded or one of the grounded edges. The former will leave Blue with 5 edges to Red's 10, i.e., a score of -5, and the latter will leave Blue with 3 edges to Red's 10, i.e., a score of -7. Clearly, Blue will move to -5, the higher of the two values. Similarly, Red has many moves possible moving first, but the most advantageous move will be to remove the topmost edge, leading Red with 9 edges to Blue's 6 or a score of -3 (remember, the convention is to track scores from Blue's perspective). We can shorten the notation by only using the largest value for blue and the smallest value for red, as each player will move to the largest value possible and make the optimal move, collapsing the representation of the game to $\{-5 \mid -3\}$. Since this notation describes a game just as well as an actual picture (as both describe the optimal moves for our players), we can turn the formal notation into a single number using the Simplicity Rule. To do this, let us define a few terms:

$$v(G) = \text{value of any Hackenbush position}$$
$$b = \text{the largest value of } l_0, l_1, l_2, \ldots$$
$$r = \text{the smallest value of } r_0, r_1, r_2, \ldots$$

The Simplicity Rule states that $v(G)$ is equal to x, where x is the simplest number that fits into G. It is determined as follows:

1. Find an integer x closest to zero, such that $b < x < r$. If no such integer exists, then ...
2. Find a rational number x, such that $b < x < r$ and whose denominator is the smallest possible power of two.

Using this rule, we can see $v(G)$ for the game in Figure 1 is the integer x closes to zero such that $-5 < x < -3$, i.e., -4.

See the table below for illustrative values of b and r, and the corresponding value of x.

b	r	x
2	8	3
-5	-3	-4
0	1	$\frac{1}{2}$
0.25	$\frac{11}{16}$	$\frac{3}{8}$
-5	$2\frac{5}{8}$	0

Based on the value of G, the game's outcome can be predicted a priori.

If $v(G) > 0$, Blue wins
If $v(G) < 0$, Red wins
If $v(G) = 0$, the player who goes second wins

In the book "Winning Ways for your Mathematical Plays" by Berlekamp, Conway, and Guy, the binary tree (Figure 5) is provided as a tool for efficiently identifying the simplest numbers. To determine the value of a game, we navigate the tree until we find the most straightforward number that corresponds to the game. Using the Simplest Rule principle and this binary tree, we can assign a number to any Red-Blue Hackenbush image and determine the winner of any game before it even starts!

Computational Efficiency of Sorting Algorithms

By: Sophie Willer-Buchardi

Computational efficiency is based on two factors: the time complexity and space complexity of an algorithm. This article will focus on the time complexity of different sorting algorithms. Time complexity is the amount of time an algorithm takes, as a function of the length of the input. This is normally expressed through Big-O notation, which describes the worst-case, i.e. the longest it could take for an algorithm to complete. Some classes of time complexity, listed in order of efficiency, include constant time $O(1)$ (the time an algorithm takes is not dependent on the input), logarithmic time $O(\log n)$ (the time an algorithm takes grows logarithmically based on the input), linear time $O(n)$ (the time an algorithm takes grows linearly based on the input), quadratic time $O(n^2)$ (the time an algorithm takes grows quadratically based on the input), and exponential time $O(2n)$ (the time an algorithm takes grows exponentially based on the input).

To determine an algorithm's time complexity, one must identify its different operations and how often they are executed in relation to the input, i.e., analyze loops, recursion, etc.

Bubble sort has an interesting time complexity. Bubble sort, also known as sinking sort, iterates through every element in an array, comparing the element to the values of those next to it, swapping it if needed. It repeatedly loops through the array until sorted. In the worst case, bubble sort has to iterate through the array n times, as it has to iterate through the array, swapping elements to sort them. Each comparison is constant time, and this is repeated for each element in the array, i.e. n times. Therefore, the time complexity is $O(n^2)$, and in the worst case, the algorithm loops n times, and each loop is $O(n)$.

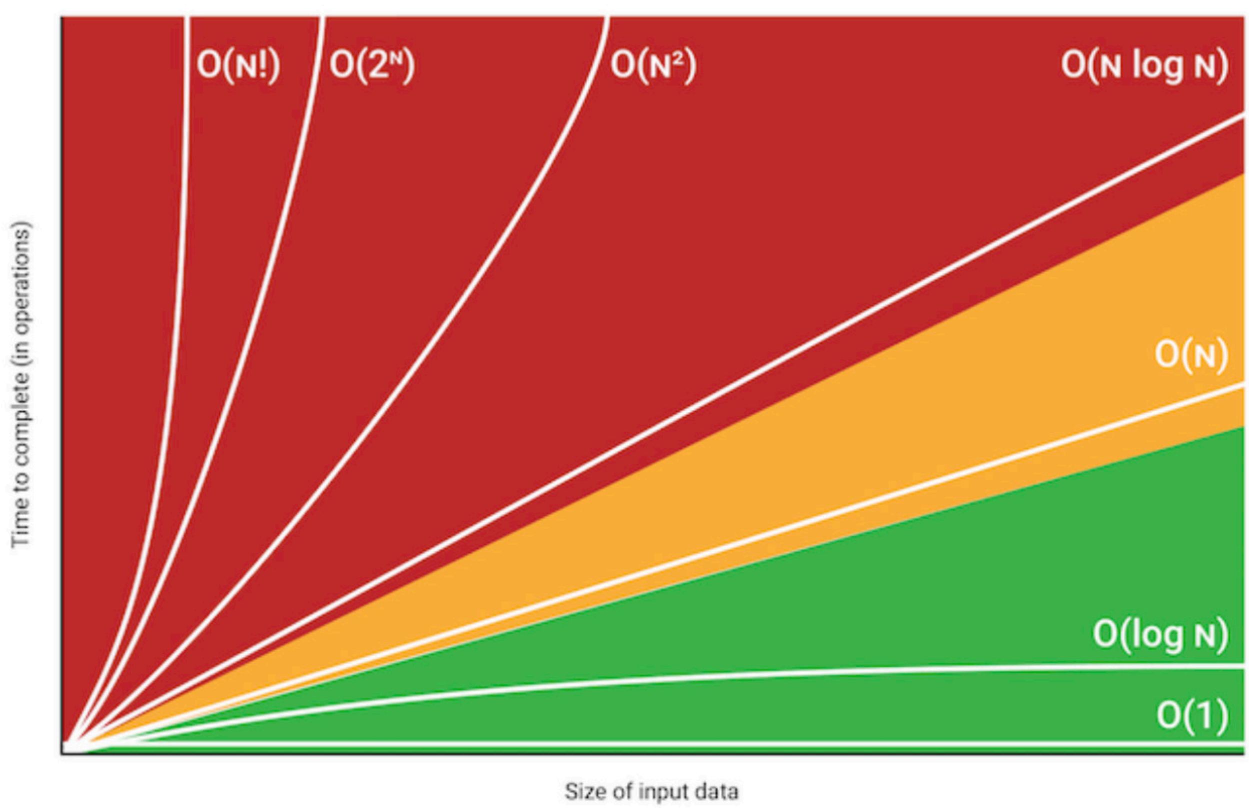
Similarly, modified bubble sort is bubble sort with a condition that checks if the array is already sorted prior to being inputted. This only affects the time complexity for the best case of bubble sort as, unless the input is already sorted, the algorithm is identical to traditional bubble sort, so the time complexity is still $O(n^2)$.

Another sorting algorithm is selection sort. Selection sort iterates through an array, placing the smallest element at the front of the array. It repeats this process with the unsorted elements until the array is sorted. Similarly to bubble sort, selection sort is also $O(n^2)$ as it iterates through the array up to n times, and each iteration is $O(n)$.

Insertion sort removes one element at a time and then iterates through the array, placing the element in the proper location within a sorted subarray. It repeats this process for every element until the entire list is sorted. Although this algorithm's best case is $O(n)$, the worst case is the same as selection sort and bubble sort as it involves iterating through the array, $O(n)$, for each element, so it is also $O(n^2)$.

Now, let's look at an algorithm with a different time complexity, merge sort. Merge sort continuously divides an array in half until it can not be divided further, meaning it reaches a single element array, which is always sorted, and then merges and sorts the sorted subarrays into a single sorted array. Finding the division point is $O(1)$, as this is not dependent on the input size. Merge sort divides the array into log n levels. At each level, it also iterates through the array to sort it, which is $O(n)$, as previously discussed. Since it repeats this sorting $\log n$ times and the sorting at each level takes $O(n)$, the merge sort has a time complexity of $O(n \log n)$.

Quicksort works by selecting a "pivot" element in an array and partitions the other elements into subarrays, one with elements less than the pivot and one with elements greater. It then recursively sorts the subarrays repeating this process, eventually leading to a sorted array. Assuming the partitioning is relatively balanced, i.e. the two subarrays are similar sizes, there are $\log n$ levels. Similarly to merge sort, the work done at each level is $O(n)$. Therefore, the best and average case is $O(n \log n)$, as on average the partitions will be relatively balanced. If the partitioning is not balanced and each "pivot" element is either the greatest or lowest element, there will be n levels as each subarray is the previous array without the pivot elements. Therefore, the worst case is $O(n^2)$, as the work done on each level is $O(n)$ and will be done n times.



Linearity of Expectation

By: Arda Altintepe

Linearity of expectation is a powerful tool that is widely used in computer science and statistics to analyze algorithms or generate probabilities. In this article, I will discuss what it is, how it is proved, and one application.

To understand concepts like linearity of expectation, it is important to know the meaning of an expected value. An expected value is a theoretical weighted average of a data set or a single numerical value in which the data tends to. For instance, if you had a die with sides labeled 1-6, the probability of it landing on each side is $\frac{1}{6}$. The value of each side of the die will be a possible value for something called a discrete random variable, which we will call X. In this case, the expected value would be the sum of each number on the die (or value of X) multiplied by its probability:

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 3.5$$

Thus, the expected value of X is 3.5. However, if you decided to change the side “6” of the die into a “2”, doubling the probability of 2, we would get this expected value:

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{2}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 2.83$$

More generally, the expected value of any set of data with x representing each value of the random variable X , such as the one above, can be modeled as follows:

$$E[X] = \sum_x xP(X = x)$$

Note: $P(X = x)$ represents the probability of the value of the random variable X when it is set to x.

Linearity of Expectation: By definition, linearity of expectation states that the expected value of the sum of random variables equals the sum of the individual expected values of each variable:

$$E[X + Y] = E[X] + E[Y]$$

Proof: Firstly, we have to create an equation for $E[X + Y]$ using the definition of the expected value formula. However, since we are creating the expected value of $X + Y$, where both are random variables, we have to add the sums of all values $(X + Y)$ for each value x and y multiplied by their joint probabilities:

$$E[X + Y] = \sum_x \sum_y (x + y)P(X = x, Y = y)$$

Next, if we expand the expression by distributing the x and y, we get:

$$E[X + Y] = \sum_x \sum_y xP(X = x, Y = y) + \sum_x \sum_y yP(X = x, Y = y)$$

In summation, if you have a sum with a constant that is unaffiliated with the summation, you can factor it out. For example, if you have this summation,

$$\sum_{i=1}^{n} ax_i = ax_1 + ax_2 + \ldots + ax_n$$

and the a remains constant, you can factor it out:

$$\sum_{i=1}^{n} ax_i = a\sum_{i=1}^{n} x_i$$

Back to our proof. Since the x term is constant in the summation for all y, and the y term is constant in the summation for all x, we can factor out x and y accordingly:

$$E[X + Y] = \sum_x x \sum_y xP(X = x, Y = y) + \sum_x y \sum_y P(X = x, Y = y)$$

Now take the part:

$$\sum_y P(X = x, Y = y)$$

Since this is a joint probability, meaning the probabilities of the values are multiplied, we can rewrite this as:

$$\sum_y P(X = x) \cdot P(Y = y)$$

Notice that $P(X = x)$ is unaffiliated with the summation of y. Using the same rule as before, we can factor it out:

$$P(X = x)\sum_y P(Y = y)$$

Now, the expression represents $P(X = x)$ multiplied by the sum of the probabilities of all y values. However, the sum of all probability always equals one. Thus, the sum of all the probabilities for the random variable Y is equivalent to one:

$$\sum_y P(X = x, Y = y) = P(X = x) \cdot 1 = P(X = x)$$

This step is why linearity of expectation works for dependent values as well. Do not forget that the same logic applies to the sum of all x values:

$$\sum_x P(X = x, Y = y) = P(Y = y)$$

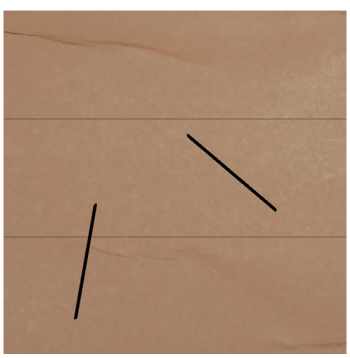
Plugging this all back into the proof, we have:

$$E[X+Y] = \sum_x xP(X = x) + \sum_y yP(Y = y)$$

Look familiar? If not, go back to the formula for the expected value. Using that formula, we can now finish the proof:

$$E[X+Y] = E[X] + E[Y]$$

Applications: One of the most famous applications of linearity of expectation is Buffon's Needle Problem. The Buffon Needle Problem states if a needle is dropped on parallel pieces of paper or wood (we will use wood), what is the probability that the needle will hit one of the lines between the two pieces of wood? Also, we will assume that the needle is the same length as the width of each wood plank.



Firstly, since the needle is the same size as the width of each wood plank, we know that it cannot cross the line more than once. Therefore, if we use a random variable X to represent the number of lines the needle crosses, X can only be two values: 0 and 1. Now, what if there were two needles? Call them X_1 and X_2. By linearity of expectation:

$$E[X_1 + X_2] = E[X_1] + E[X_2]$$

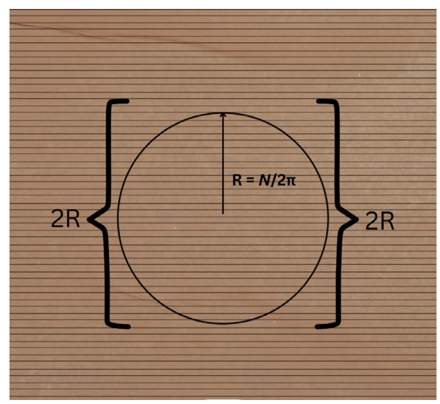
Since we know that the expected values of X_1 and X_2 must be equal because they are the same needle, we can group them together under one random variable X. By linearity of expectation, their expected value will not change on how they are arranged or if they are dependent:

$$E[X_1 + X_2] = 2E[X]$$

Hopefully, you start to see a general trend. If not, here is the expected value for three needles:

$$E[X_1 + X_2 + X_3] = E[X_1] + E[X_2] + E[X_3] = 3E[X]$$

A more general formula we can conclude is $E[X_1 + X_2 + \ldots + X_N] = N \times E[X]$ Now, say we have N needles, and have N be a big enough number to create a circle out of the needles. This way, we can best visualize the crossings. Note that the bigger N is, the better the approximation to a real circle. Since we have N needles and the length of each needle is 1, the circumference of our "circle" will be N. Thus, our radius will be $\frac{N}{2\pi}$. Assuming the width of each plank is 1, this means that for each side of our circle, we will have the diameter of crossings, or $2R$ crossings, so the total number of crossings will be $4R$.



(Imagine the circle to be made out many tiny straight lines or needles)

Since $R = \frac{N}{2\pi}$, $4R = 4\frac{N}{2\pi} = \frac{2N}{\pi}$. However, from before, we know that the number of crossings with N needles, from linearity of expectation, is equivalent to $N \times E[X]$. Thus, $\frac{2N}{\pi} = N \times E[X]$. Finally, solving for the expected value of X, we are left with $E[X] = \frac{2}{\pi} \approx 0.637$.

Since our random variable X could only be 0 or 1, our expected value is equivalent to the probability of a needle crossing the wood plank. Therefore the answer is approximately 64%.

The Buffon Needle Problem is commonly solved using calculus. However, we could still solve it using linearity of expectation, assuming the needle and width of the wood plank were equal. Linearity of expectation may seem "obvious" or intuitive. Even so, because it still works for dependent variables, it is commonly used to analyze complex random algorithms, solve proofs with non-independent variables, and in many fields of computer science and theoretical computer science.

Let's quickly look at a simple example that uses dependent discrete random variables. A teacher is giving back exam papers but forgets to look at the names of each paper. There are n students in the class. What is the expected value of students who correctly got their papers back? Notice that if a student gets the paper of another student, the other student has no chance of getting the correct paper. Therefore, we are dealing with dependent variables.

Let X be the total number of students who correctly get their own paper back. Let X_i represent 0 or 1, depending on if each student correctly got their paper back or not. Using linearity of expectation, we get: $E[X] = E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$. Similarly to the Buffon Needle Problem, we know that each student's chances are equal, so we can generate this similar equation: $E[X] = n \times E[X_1]$. The expected value of X_1 is equal to the probability of a student receiving their paper, as there are only two possible values of the random variable: 0 and 1. Therefore $E[X1] = \frac{1}{n}$. Finally, plugging that in, we get $E[X] = n \times \frac{1}{n} = 1$. One student is expected to receive their paper back correctly. Thus, you can see how much using linearity of expectation simplified the seemingly daunting fact that the variables were dependent.

Throughout the proof and while solving the problems above, we calculated expected values using discrete random variables and their sums. But what if those variables were no longer discrete but continuous? For instance, what if we were solving for the expected value of something like time? Would linearity of expectation still apply?

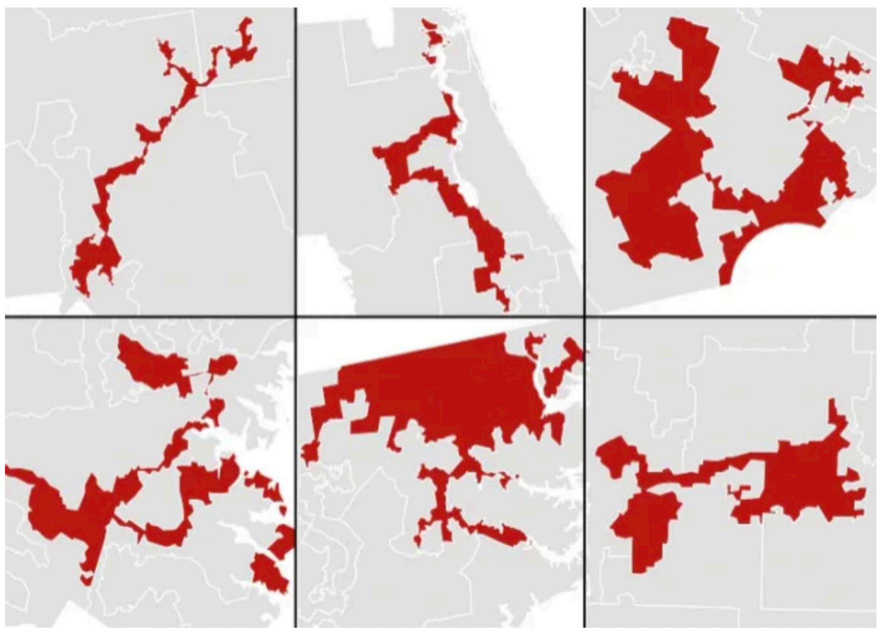
Well, to answer that question, let's first understand what continuous variables are and how to calculate their expected values; say that we have a function on the time it takes for a random person to wash their hands along with the likelihood. In this case, it is possible to approximate the expected time value by creating likelihood intervals under the graph and then assigning a discrete random variable to take on each value. Then, you would just sum all the values for the time it takes to wash your hands multiplied by their probabilities for each interval. With this method, the smaller the interval, the better the approximation. However, with a continuous variable like time, there is a more accurate way. If you let the width of the interval approach 0 and the number of intervals approaches infinity, you would be left with this integral equation:

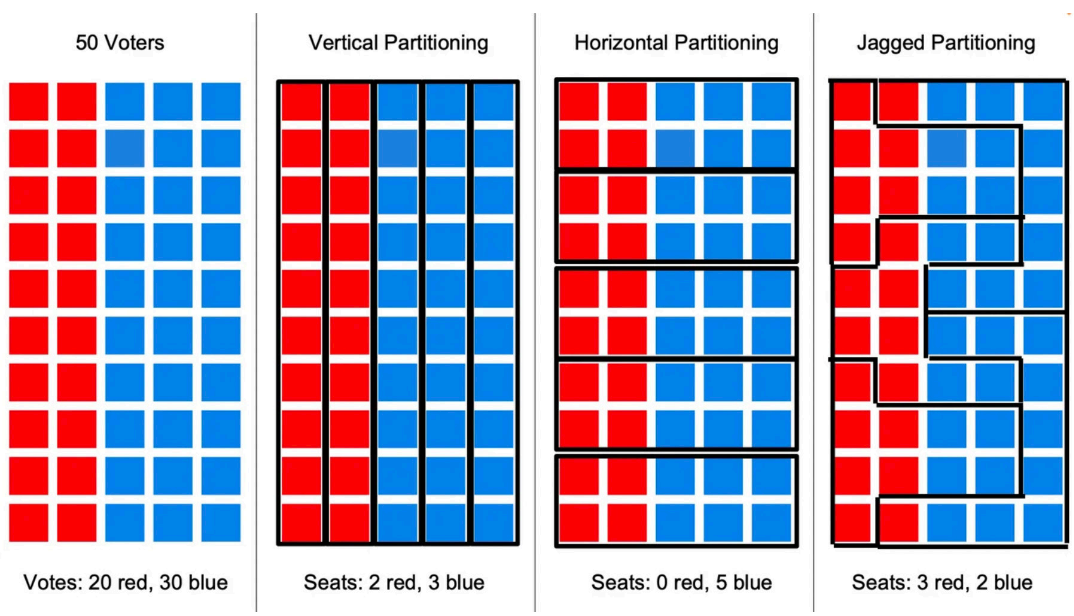
$$E[X] = \int_{-\infty}^{\infty} xL(X = x)dx$$

In this case, $L(X = x)$ essentially represents $f(x)$ since the likelihood is the y-axis. For linearity of expectation, the proof structure with continuous variables is essentially the exact same as before. You still create an equation with the integral above for both X and Y, and then factor out the value x and y and cancel the terms that are equivalent to 1. So, in short, the answer is yes. Linearity of expectation does also work with continuous random variables.

The Math of Gerrymandering

By: Meghan Mantravadi





In most u.s. states, legislatures decide on the division of electoral districts about every 10 years (after a census). For the House of Representatives, these districts, newly drawn by population, determine how many seats each state gets, influencing their representation in the body. Repeatedly, the incumbent parties are suspected of using this constitutional redistricting practice to their advantage. This practice of gerrymandering, the manipulation of congressional districts to favor one constituency or political party, has a long history in the US, which can be traced back to 19th century Massachusetts, where a cartoonist at the time noticed that one of the districts resembled a salamander and thus coined the expression gerrymandering.

The division of electoral districts, which many politicians use to justify gerrymandering, is not an easy task. Each state follows its own set of guidelines in this regard. Overall, a district should have nearly equal numbers of voters, be contiguous, not discriminate against ethnic groups, not cross county lines, and follow natural boundary lines, such as rivers. However, setting rules for equitable districting is not so simple. Mathematicians today are racking their brains over the question and arming themselves with computing power to deal with it.

Let's take a look at some basic examples of how gerrymanders work - in the arrays below, there are 50 voters with two different political affiliations (20 red and 30 blue). However, with different types of partitioning, there are different outcomes, regardless of which is in the majority. (See Figure 1).

In vertical and horizontal partitioning, blue stays the majority. However, in jagged partitioning, red wins the most seats despite not being the majority. Outside of these neatly drawn arrays, gerrymanders thus often take bizarre shapes. (See Figure 2).

The efficiency gap is a metric that is used to evaluate the fairness of electoral districts in a two-party system. It establishes which political party has the most wasted votes. To illustrate this, we can return to the initial example of 50 voters (20 for red and 30 for blue) and calculate the efficiency gap for the various divisions. When all boundaries are drawn vertically in the first case, the first and second districts (from the left) each have 10 red votes, wasting four each. In contrast, the third, fourth, and fifth districts each have ten blue votes, four of which are also wasted. The efficiency gap is thus $|(2\times4)-(3\times4)|/50 = 2/25 = 0.08$ (the vertical bars indicate absolute value).

Each district in the second division is equal: blue always wins by six votes out of ten. As a result, none of the blue votes are wasted, whereas all of the red votes are. The efficiency gap is $20/50 = 0.4$, which is much greater than in the first division. The third example is the most intriguing: the two districts where blue wins 9 to 1 each have a three-point blue advantage. In each of the three winning red districts, four blue votes are wasted, for a total of $(2 \times 3) + (3 \times 4) = 18$ surplus blue votes. In comparison, only two red votes were wasted. This yields an efficiency gap of $(18-2)/50 = 8/25 = 0.32$.

How Math Can Help Detect and Limit Gerrymandering

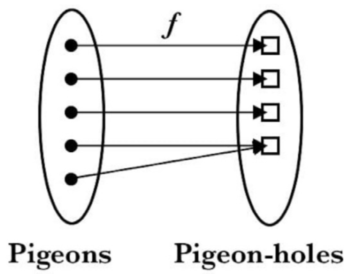
Tufts University's Moon Duchin believes that, in addition to looking at numerical distributions, another approach to spotting gerrymandering may lie in its geometry. Duchin is looking for new ideas regarding what types of district shapes might make fair maps. For instance, by drawing lines across a state based on population density and then folding along those lines as if the state were a piece of origami, she is investigating whether the "curvature" of the resulting shape can reveal gerrymandering.

It has long been assumed that odd-looking, sprawling regions with long, winding perimeters are signs of unfair mapmaking. However, to quantify what "odd-looking" really means, political scientists have developed more than thirty different scores that can be used to assess gerrymanders. Curvature may be useful in this situation. Curvature can be positive, zero, or negative in pure mathematics. A sphere is an example of a positive curvature shape, with all of its curves bending in the same direction. Negative curvature, on the other hand, occurs when a shape curves in opposite directions. Duchin's results indicate that negative curvature produces sprawling regions with high perimeters, both of which are classic symptoms of gerrymandering. In other words, negative curvature could be a sign of a gerrymander.

Pigeonhole Principle

By: Olivia Xu

The pigeonhole principle, also known as Dirichlet's box principle or Dirichlet's drawer principle, is a fundamental mathematical concept that emerged in 1624. It might seem complex, but it is actually quite simple. The Pigeonhole Principle states that if there are n items put into m containers and $n > m$, then at least one container must contain more than one item.



A Simple Illustration of the Pigeonhole Principle

By the principle, we can conclude that if there are 366 people in a room, at least two people share the same birthday (unless it is a leap year). This phenomenon is because there are 365 days in a year, and if 365 people have different birthdays, the 366th and final person will have to share a birthday with any of the 365.

The Pigeonhole Principle can be applied to various problems. For instance, consider proving that when selecting any 7 whole numbers, there is always a triple a, b, c, that all differ from each other by a multiple of 3. Any whole number has a remainder of 0,1, or 2 when divided by 3. No matter which 7 numbers are chosen, they each have one of these 3 remainders, so by the Pigeonhole Principle, there must be at least 3 numbers with the same remainder. If they have the same remainder when divided by 3, then their difference is divisible by 3 as well. Moreover, this principle is evident even in everyday tasks. For example, organizing files on a computer: having more files than folders means that at least one folder will contain multiple files.

The Pigeonhole Principle might seem like a straightforward mathematical concept, but its influence extends far beyond numbers and equations. Overlaps and repetitions are simply inevitable and an aspect of our everyday lives.

Challenge: *Suppose that 101 positive integers are arranged in a circle. The sum of all the numbers is 300. Prove that you can always choose a consecutive sequence of numbers that sums up to 200.*

Solution: Label all the numbers around the circle $a_1, a_2, \ldots, a_{101}$. The sums $s_k = a_1 \ldots + a_k$ for $1 \leq k \leq 101$. Since there are 101 possible sums, there must be at least 2 sums, s_i and s_{i+j}, that have the same remainder when divided by 100, meaning that they end in the same 2 digits. The difference between these two sums must be 100 or 200 (300 is not possible because the sum of all numbers is 300). $s_{i+j} - s_i = a_{i+1} + \ldots + a_{i+j}$. If the difference between $s_{i+j} - s_i$ is 200, then we are done. And if the difference is 100, then the remaining consecutive numbers must sum to 200.

Infinitely Many Proofs of Infinitely Many Primes

By: Alicia Li

The existence of infinitely many primes is a well-known mathematical fact: how many proofs do you know that support this fact? I propose there to be infinitely many.

The Classic Proof: Proof by Contradiction

Assume the contrary, that there are finitely many primes: $p_1, p_2, \ldots, p_n$. Consider their product plus one, and call it N: $N = (p_1 \times p_2 \times \ldots \times p_n) + 1$. This number is not divisible by any of our primes p_1 through p_n, since they leave a remainder of 1 when divided by each of them. Thus, N is a new prime number, and we can always repeat this process, meaning there are infinitely many prime numbers. Q.E.D.

For the following proofs, we introduce a notation in number theory: modular arithmetic, which is an arithmetic system in which only remainders are considered. If an integer a leaves a remainder of r when divided by m, we define $a \equiv r \pmod{m}$, said as "a is congruent to r modular m." This is also often referred to as a being a $mk+r$ number, for a positive integer k. For example, 17 leaves a remainder of 1 when divided by 4, so $17 \equiv 1 \pmod{4}$, and 17 is a $4k+1$ number (more specifically, it's a $4k+1$ prime).

The $1 \pmod{4}$ prime Proof:

We prove that there are infinitely many primes of the form $4k+1$, where k is an integer, also using a contradiction approach.

Let's first prove a lemma: given an odd prime p, if there is a perfect square congruent to $-1 \pmod{p}$, p is a $1 \pmod{4}$ prime.

We know there exists a square $x2 \equiv -1 \pmod{p}$. Since p is odd, $p-1$ is even, so $\frac{p-1}{2}$ is a positive integer (assuming p is a non-negative integer). Consider x^{p-1}. This is equal to $(x^2)^{\frac{p-1}{2}}$, which is equal to $(-1)^{\frac{p-1}{2}} \pmod{p}$, from our earlier assumption. We know p isn't a factor of x, so by Fermat's Little Theorem, $x^{p-1} \equiv 1 \pmod{p}$. This means $(x^2)^{\frac{p-1}{2}} \equiv 1 (modp)$, and combining this with $(x^2)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}}$, $\frac{p-1}{2}$ is even, meaning it can be represented as $2k$. Thus, $p = 4k+1$.

We can now proceed with our proof. Suppose there are finitely many $4k+1$ primes $p_1, p_2, \ldots, p_n$. Let's set $N = 4(p_1 \times p_2 \times \ldots \times p_n)^2 + 1$ and $M = p_1 \times p_2 \times \ldots \times p_n$. N is $1 \pmod{4}$ and larger than all our listed $4k+1$ primes p_i, so N must be composite, aka not prime. N is not a multiple of 2 or any p_i, so it must have a prime divisor q that is $4k+3$.

We have that $N = 4M^2 + 1 \equiv 0 \pmod{q}$, or $4M^2 \equiv -1 \pmod{q}$. Since $4M^2 = (2M)^2$, we have found a perfect square congruent to $-1 \pmod{q}$. We also know q is odd, meaning q has to be a $1 \pmod{4}$ prime by our lemma. This gives us a contradiction since we previously deduced that q is a $3 \pmod{4}$ prime. Q.E.D.

The $3 \pmod{4}$ prime Proof:

Assume we have a finite collection of primes (not equal to 3) $p_1, p_2, \ldots, p_n \equiv 3 \pmod{4}$. Let $N = 4p_1p_2 \ldots p_n + 3$. Since odd primes are either $1 \pmod{4}$ or $3 \pmod{4}$, and the product of $1 \pmod{4}$ primes can never yield a $3 \pmod{4}$ number, N must have a $3 \pmod{4}$ divisor q. However, N is not a multiple of $p_1, p_2, \ldots, p_n$, so q must be a new $3 \pmod{4}$ prime. Thus, in any finite list of $4k+3$ primes, we can always find a $4k+3$ prime not on the list, so there must be infinitely many $4k+3$ primes. Q.E.D.

The Infinite Proof:

Let's take two relatively prime positive integers A and B. Consider $A+B$. $A+B$ isn't divisible by A, so A and $A+B$ are relatively prime. We continue: $A(A+B)+B$ is coprime to, meaning shares no factors with, both A and $A+B$, and so on. This process of taking the product of all our coprime numbers and adding B can be continued to generate infinitely many coprime numbers. Each of these numbers will have distinct prime factors. Thus, there are infinitely many primes.

There are infinitely many ways to choose relatively prime A and B, which gives us infinitely many ways of proving infinitely many primes! Q.E.D.

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