Summer Assignment **AP Calculus BC**

June 2024

Dear Student:
Your AP Calculus Exam is tentatively scheduled for Monday, May 12, 2025. That may seem like a long way away, but it leaves us less than 150 school days of class next year to prepare. With snow days, half days, exams, and who knows what else, we can only expect 135-140 actual days of class before the exam.

This summer assignment is designed to provide a review of algebraic and trigonometric concepts that were covered during AP Pre-Calculus. By covering this material before school begins in the fall, we will be able to continue and expand our study of *differential calculus* (the study of derivatives) on the **first day** of school.

You may work with another student, or students, and you may e-mail me for help. However, any evidence of copying will result in failure of the assignment (a test grade) and may constitute grounds for removal from the class. The important thing is that YOU understand the work.

My e-mail address is: cwhite@springfieldschools.com
Don't be shy!!!! Seek help or e-mail me if you need help or would like further explanations. Responses will not be automatic; so do NOT wait until the last minute to email for help.

Due Date for Assignments—The assignments are due on Thursday, August 29, 2024 at noon. Please drop them off in my mailbox in the main office at JDHS. If you need to mail your work, the address is:

Jonathan Dayton High School
Attn: Mr. White
139 Mountain Avenue
Springfield, NJ 07081

Hard copy is greatly preferred, but you may also scan and e-mail your work (please be sure that it is legible) to cwhite@springfieldschools.com.

The maximum possible grade for a late assignment is a C. If you know now that you will need an extension due to summer travel, please contact me ASAP.

******Remember the goal is for YOU to understand the material thoroughly so that we can focus on the calculus material immediately when the school year begins.**
Failure to turn in the assignment will result in you having to drop the course. If you are not willing to work on your own, to hand in material on time or you cannot demonstrate an understanding of the material on this assignment, you will not succeed in this course.

Algebra/Trigonometry Reference Sheet

This sheet captures and briefly demonstrates algebra and trigonometry concepts that you should be familiar with already. It may help with some of the problems on the assignment.

Negative Exponents: An expression involving a negative exponent can be rewritten as a fraction by moving the exponentiated term to the opposite side of a fraction and making the exponent positive.

Examples: a) $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$ b) $x^{-2} = \frac{1}{x^2}$

Rational Exponents: An expression involving a rational (fractional) exponent can be rewritten as a radical with the denominator of the fraction as the root number (index) and the numerator as a power either inside or outside the radical.

Example: a) $16^{3/4} = \sqrt[4]{16}^{\,3} = 2^3 = 8$

Evaluating Natural Logs: A natural log is equal to the power that you must raise e to in order to get the expression inside of the natural log.

Examples: a) $\ln e = 1$ because $e^1 = e$ b) $\ln 1 = 0$ because $e^0 = 1$
c) $\ln e^5 = 5$ because $e^5 = e^5$ d) $\ln \sqrt{e} = \frac{1}{2}$ because $e^{1/2} = \sqrt{e}$

Trig values: You need to know your trig values. This chart can help.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	Undef.

Factoring: You need to know the basics of factoring. It does not matter to me what method you use, you just need to know how to do it.

Examples: a) $x^2 - 9x + 14 = (x-2)(x-7)$ b) $2x^2 - 5x - 3 = (2x+1)(x-3)$
Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
Sum/Difference of Cubes: $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

READ EACH SECTION AND STUDY THE EXAMPLE PROBLEMS. Complete the Problem Sets at the end of each section on separate paper. Show all work. Please complete the problems in order.

Topic 1: Limits. There are usually 2 topics—but topic 2 has been excluded this year.
Section 1.1: Evaluating limits using direct substitution.

In a nutshell, limits describe what happens to the value of a function (the *y*-value) as the *x*-value input into the function draws closer and closer to a certain value. Limits can be either what's known as *sided limits*, meaning that they describe what happens to the *y*-value as the *x*-values being used approach a certain value from either the negative side or from the positive side, or *general limits* which describe what happens to the *y*-value as *x* approaches a certain value regardless of the direction. Here are a couple of examples:

<u>Example 1</u>: Consider the function $f(x)=x^2$. By generating a table of values, find the sided limits

$\lim\limits_{x\to 3^-} x^2$ and $\lim\limits_{x\to 3^+} x^2$

To find this limit, we will have to observe what happens to the value of x^2 as we use *x*-values that get progressively closer to 3 but from the negative side (using only values less than 3).

x	2	2.5	2.9	2.99	2.999
$f(x)$	4	6.25	8.41	8.9401	8.994001

From the values in the table, it should be clear that as *x* approaches 3 (using values less than 3) the value of x^2 is getting closer to 9, so $\lim\limits_{x\to 3^-} x^2 = 9$

To find this limit, we will have to observe what happens to the value of x^2 as we use *x*-values that get progressively closer to 3 but from the positive side (using only values greater than 3).

x	4	3.5	3.1	3.01	3.001
$f(x)$	16	12.25	9.61	9.0601	9.006001

From the values in the table, it should be clear that as *x* approaches 3 (using values greater than 3) the value of x^2 is getting closer to 9, so $\lim\limits_{x\to 3^+} x^2 = 9$

Since the sided limits $\lim\limits_{x\to 3^-} x^2 = \lim\limits_{x\to 3^+} x^2 = 9$, we say that the general limit $\lim\limits_{x\to 3} x^2$ = 9.

Example 2: Consider the function $g(x)=\frac{1}{x}$. By generating a table of values, find the sided limits.

$$\lim_{x\to 0^-}\frac{1}{x} \qquad \text{and} \qquad \lim_{x\to 0^+}\frac{1}{x}$$

To find this limit, we will have to observe what happens to the value of $\frac{1}{x}$ as we use x-values that get progressively closer to 0 but from the negative side (using only values less than 0).

x	-1	-0.5	-0.1	-0.01	-0.001
$f(x)$	-1	-2	-10	-100	-1000

From the values in the table, it should be clear that as x approaches 0 (using values less than 0) the value of $\frac{1}{x}$ is getting progressively more negative, so $\lim_{x\to 0^-}\frac{1}{x}=-\infty$

To find this limit, we will have to observe what happens to the value of $\frac{1}{x}$ as we use x-values that get progressively closer to 0 but from the positive side (using only values greater than 0).

x	1	0.5	0.1	0.01	0.001
$f(x)$	1	2	10	100	1000

From the values in the table, it should be clear that as x approaches 0 (using values greater than 0) the value of $\frac{1}{x}$ is getting progressively more positive, so $\lim_{x\to 0^+}\frac{1}{x}=+\infty$

Since the sided limits $\lim_{x\to 0^-}\frac{1}{x}\neq\lim_{x\to 0^+}\frac{1}{x}$, we say that the general limit $\lim_{x\to 0}\frac{1}{x}$ does not exist.

As you may have noticed, finding limits using tables of values is not a particularly efficient process, especially if the Great Calculator Apocalypse is upon us (which it occasionally will be). You also may have noticed, especially in Example 1, the value of a limit can most effectively and efficiently be determined by simple substitution. In Example 1, we found $\lim_{x\to 3}x^2=9$ by finding the sided limits and showing that they were equal. However, you will notice that if we had simply substituted the 3 into the expression x^2, we also would have arrived at the correct answer of $\lim_{x\to 3}x^2=9$. In Example 2, substituting 0 into the expression $\frac{1}{x}$, would have resulted in an answer of $\frac{1}{0}$, which is a strong indication that the limit does not exist. Any result of substitution that yields an expression that has a zero in the denominator (with the extremely important exception of $\frac{0}{0}$ that will be addressed later) indicates that the limit is either $\infty,-\infty$ or does not exist.

Direct substitution is the easiest way to evaluate most limits. Be careful when evaluating limits related to piecewise functions—the limits can be a little tricky given the inconsistent nature of piecewise functions.

<u>Example 3</u>: Consider the following function:

$$f(x)=\begin{cases}3x-1; x<1\\ x^2+1; 1\le x\le 3\\ 5-2x; x>3\end{cases}$$

Find the limits $\lim_{x\to 1} f(x)$, $\lim_{x\to 3} f(x)$ and $\lim_{x\to 5} f(x)$.

At $x = 1$, the behavior of the piecewise function changes. To the left of $x = 1$, the function behaves as $3x-1$, and to the right of $x = 1$, the function behaves as x^2+1. Direct substitution can still be used to find the sided limits, but consideration must be given to the changing behavior of the function at $x = 1$: $\lim_{x\to 1^-} f(x)=3(1)-1=2$ and $\lim_{x\to 1^+} f(x)=(1)^2+1=2$. Since both of the sided limits are equal to 2, we can say that $\lim_{x\to 1} f(x)=2$.

At $x = 3$, the behavior of the piecewise function changes again. To the left of $x = 3$, the function behaves as x^2+1 and to the right of $x = 3$, the function behaves as $5-2x$. Using direct substitution, we find that $\lim_{x\to 3^-} f(x)=(3)^2+1=10$ and $\lim_{x\to 3^+} f(x)=5-2(3)=-1$. Since the sided limits are not equal, $\lim_{x\to 3} f(x)$ does not exist.

At $x = 5$, the behavior of the piecewise function does not change. On both sides of $x = 5$, the function behaves as $5 - 2x$, so the limit can simply be found by directly substituting into the function: $\lim_{x\to 5} f(x)=5-2(5)=-5$.

Problem Set 1.1: Evaluating Limits Using Direct Substitution.
Find the *<u>exact</u>* *value of the following limits. These should be done without a calculator. Decimal approximations will be considered incorrect* (The not-so-subtle message here is know your trig values).

1. $\lim_{x\to 2}\left(2x^3-x^2-5x+1\right)$
2. $\lim_{x\to -1}\frac{2x+5}{x^2+3}$
3. $\lim_{x\to \frac{\pi}{4}}\left(\sin^2 x+\tan x\right)$
4. $\lim_{x\to \frac{\pi}{3}}\left(\sec x-\sin x\right)$
5. $\lim_{x\to 2}\left(\frac{1}{x}+\ln e^x\right)$
6. $\lim_{x\to 4}\left(3x^{3/2}-2x^2+1\right)$
7. Consider the piecewise function below and find the limits:

$$f(x)=\begin{cases} x^2+1;\, x<0 \\ \cos(\pi x);\, 0\le x\le 2 \\ x-3;\, x>2 \end{cases}$$

a. $\lim_{x\to 0} f(x)$

b. $\lim_{x\to 2} f(x)$

c. $\lim_{x\to 4} f(x)$

Section 1.2: Continuity.

Continuity is a pre-calculus topic, but it is expected that you understand it in AP Calculus, and it does (although rarely) appear on the AP Exam. You know from pre-calc that a function can be proven to be continuous at *x* = *a* if the following three things are true:

- $f(a)$ exists (meaning that if you plug in *a* for *x*, you get a number back)
- $\lim_{x\to a} f(x)$ exists (meaning that the sided limits as *x* approaches *a* are equal)
- $f(a)=\lim_{x\to a} f(x)$ (the answers to the first two steps are equal to each other)

If a function is not continuous—there are three types of discontinuities that are commonly referenced on the AP Exam:

- **Infinite discontinuities**: An infinite discontinuity is a vertical asymptote. Infinite discontinuities occur when $f(a)$ does not exist in either the original function or in the reduced form of the original function.
- **Jump discontinuities**: A jump discontinuity occurs when the *y*-value of a function abruptly changes (or jumps) from one value to another at a certain *x*-value. Jump discontinuities occur typically when the second step in the continuity test above fails—when $\lim_{x\to a^-} f(x)\ne \lim_{x\to a^+} f(x)$
- **Point discontinuities**: These are also known as removable discontinuities. They occur either when $f(a)$ does not exist in the original function but does exist once the function is reduced, or when the third step of the test fails—when $f(a)$ exists and $\lim_{x\to a} f(x)$ exits, but $f(a)\ne \lim_{x\to a} f(x)$

Example 1: Determine if the function $f(x)=\dfrac{x-4}{x^2+2x-24}$ is continuous at

i) $x = -6$
ii) $x = 4$

i) Does $f(-6)$ exist? $f(-6)=\dfrac{-6-4}{(-6)^2+2(-6)-24}=\dfrac{-10}{0}$. Since $\dfrac{-10}{0}$ is undefined, $f(-6)$ does not exist. This fails the first step of the continuity test, so $f(x)$ is not continuous at $x = -6$. See the discussion below part ii regarding the type of discontinuity at $x = -6$.

ii) Does $f(4)$ exist? $f(4)=\dfrac{4-4}{4^2+2(4)-24}=\dfrac{0}{0}$. Since $\dfrac{0}{0}$ is undefined, $f(4)$ does not exist. This fails the first step of the continuity test, so $f(x)$ is not continuous at $x = 4$.

Types of discontinuities in Example 1: When a continuity test fails step 1, it indicates the presence of either an infinite or a point discontinuity. With a rational function such as $f(x)$ above, try to simplify the function by factoring and reducing as shown below:

$$f(x)=\frac{x-4}{x^2+2x-24}=\frac{x-4}{(x+6)(x-4)}=\frac{1}{x+6}$$

i) Using the reduced expression $f(x)=\dfrac{1}{x+6}$, $f(-6)=\dfrac{1}{0}$ which is still undefined. When $f(a)$ is still undefined even in the reduced function, this indicates an ***infinite discontinuity*** at $x = -6$.

ii) Using the reduced expression $f(x)=\dfrac{1}{x+6}$, $f(4)=\dfrac{1}{10}$. When a certain value of x results in an undefined expression in the original function, but a valid, numeric answer in the reduced expression, this indicates a ***point discontinuity*** at $x = 4$. Remember that point discontinuities can also be called ***removable discontinuities***. You can be asked to rewrite $f(x)$ as a piecewise function where the point discontinuity has been removed. The rewrite is simple—the function is equal to the original function for all values of x other than those that make the denominator zero, but equal to the result of plugging the x-value of the point discontinuity into the reduced function at the x-value of the point discontinuity. For our function $f(x)$, the new function would be: $f(x)=\begin{cases} \dfrac{x-4}{x^2+2x-24}; x\neq 4,-6 \\ \dfrac{1}{10} \quad ; x=4 \end{cases}$

Example 2: For the piecewise function

$$f(x)=\begin{cases} 3-x & ; x<0 \\ x^2+1 & ; 0\leq x<3 \\ 6 & ; x=3 \\ 4x-2 & ; 3<x<5 \\ x^2-7 & ; x\geq 5 \end{cases}$$

determine if f is continuous at i) $x = 0$; ii) $x = 3$ and iii) $x = 5$

i) Does $f(0)$ exist? Yes, $f(0)=(0)^2+1=1$ using the second piece of the piecewise function (which is used when $0\leq x\leq 3$)

Does $\lim\limits_{x\to 0} f(x)$ exist? Testing the sided limits, we find by using direct substitution into the first piece of the piecewise function that $\lim\limits_{x\to 0^-} f(x)=3$ and by direct substitution into the second piece of the piecewise function that $\lim\limits_{x\to 0^+} f(x)=1$. Since $\lim\limits_{x\to 0^-} f(x)\neq \lim\limits_{x\to 0^+} f(x)$, the general limit $\lim\limits_{x\to 0} f(x)$ does not exist, so the function is not continuous at $x = 0$. Since the continuity test failed at step 2, this indicates a ***jump discontinuity*** at $x = 0$.

ii) Does $f(3)$ exist? Yes, $f(3)=6$ using the third piece of the piecewise function.

Does $\lim\limits_{x\to 3} f(x)$ exist? Testing the sided limits, we find by using direct substitution into the second piece of the piecewise function that $\lim\limits_{x\to 3^-} f(x)=10$ and by direct substitution into the fourth piece of the piecewise function that $\lim\limits_{x\to 3^+} f(x)=10$. Since $\lim\limits_{x\to 3^-} f(x)$ and $\lim\limits_{x\to 3^+} f(x)$ are both equal to 10, $\lim\limits_{x\to 3} f(x)=10$

Does $f(3)=\lim\limits_{x\to 3} f(x)$? No, $6\neq 10$ so $f(x)$ is not continuous at $x = 3$. Since the continuity test failed at step 3, this indicates a *point discontinuity* at $x = 3$. If asked to remove the point discontinuity, simply change the 6 in the third piece of the piecewise function to 10.

iii) Does $f(5)$ exist? Yes, $f(5)=(5)^2-7=18$ using the last piece of the piecewise function.

Does $\lim_{x\to 5} f(x)$ exist? Testing the sided limits, we find by using direct substitution into the fourth piece of the piecewise function that $\lim_{x\to 5^-} f(x)=18$ and by direct substitution into the last piece of the piecewise function that $\lim_{x\to 5^+} f(x)=18$. Since $\lim_{x\to 5^-} f(x)$ and $\lim_{x\to 5^+} f(x)$ are both equal to 18, $\lim_{x\to 5} f(x)=18$

Does $f(5)=\lim_{x\to 5} f(x)$? Yes, 18 = 18 so since all three steps of the continuity test pass at x = 5, $f(x)$ is continuous at x = 5.

Problem Set 1.2: Continuity.

For problems #1-2, determine if $f(x)$ is continuous at the given values of x. Support your work and justify your answers using the 3-step continuity test. If you find that the function is not continuous, state the type of discontinuity (infinite, jump or point).

1.

$$f(x)=\begin{cases}\sqrt{x}-3; & x<1\\ 2x-4; & 1\le x\le 5\\ x-1; & x>5\end{cases}$$

a. Continuous at x = 1? b. Continuous at x = 5?

2. $f(x)=\dfrac{x^2+5x-14}{x^2+8x+7}$ a. Continuous at x = -1? b. Continuous at x = -7

3. Find the value at which $f(x)=\dfrac{x^2-3x-18}{x+3}$ is discontinuous. Show how you know there is a point discontinuity at this x value. Then rewrite $f(x)$ as a piecewise function such that the discontinuity has been removed.

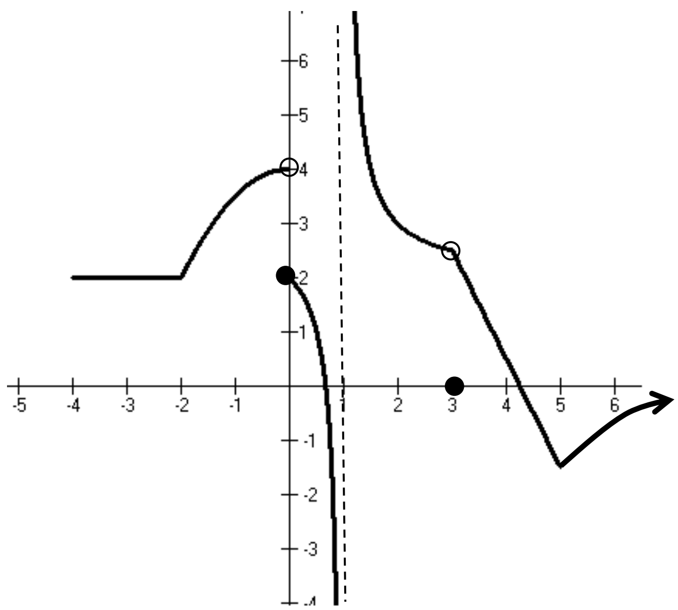
4. Sketch the graph of a function that has a point discontinuity at x = -1, a jump discontinuity at x = 1 and an infinite discontinuity at x = 5.

5. Generate the equation of a rational function that has a point discontinuity at x = 3 and an infinite discontinuity at x = -9.

Section 1.3: Evaluating limits from graphs.

Evaluating limits from graphs is fairly simple. When looking for left-sided limits (limits as $x \to a^-$), read the graph from left to right. When looking for right-sided limits (limits as $x \to a^+$), read the graph from right to left. When looking for general limits (as $x \to a$), compare the sided limits to see if they are the same. Finding general limits from a graph presents a good opportunity to review the different types of discontinuities—if there is a jump discontinuity or an infinite discontinuity at a certain value of *x*, then the general limit of the function does not exist at that *x*-value. This is *not* true for point discontinuities—remember limits do not care what the *y*-value of a function is at a certain point, just near that point, and with point discontinuities, the function's behavior on both sides of the discontinuity should be consistent. Consider the graph and limits below:

Example 1:



Find the limits (answers are given and explained below):

a. $\lim_{x\to -2^-} f(x)$ b. $\lim_{x\to -2^+} f(x)$ c. $\lim_{x\to -2} f(x)$

d. $\lim_{x\to 0^-} f(x)$ e. $\lim_{x\to 0^+} f(x)$ f. $\lim_{x\to 0} f(x)$

g. $\lim_{x\to 1^-} f(x)$ h. $\lim_{x\to 1^+} f(x)$ i. $\lim_{x\to 1} f(x)$

j. $\lim_{x\to 3^-} f(x)$ k. $\lim_{x\to 3^+} f(x)$ l. $\lim_{x\to 3} f(x)$

m. $\lim_{x\to \infty} f(x)$

Solutions

a. As you approach, *x* = -2 from the left, the function maintains a constant value of 2, so $\lim_{x\to -2^-} f(x) = 2$.
b. As you approach, *x* = -2 from the right, the function decreases towards 2, so $\lim_{x\to -2^+} f(x) = 2$
c. Since the sided limits are equal, $\lim_{x\to -2} f(x) = 2$.
d. Tracing along the graph towards *x* = 0 (the *y*-axis) from the left, the function is increasing towards 4, so $\lim_{x\to 0^-} f(x) = 4$. Notice that there is an open circle at the point (0, 4) indicating that $f(0) \neq 4$. Again, this does not effect the value of the limit because limits only care what happens to the *y*-value near the given *x*-value.

e. Tracing along the graph towards $x = 0$ (the y-axis) from the right, the function is increasing towards 2, so $\lim_{x \to 0^+} f(x) = 2$.

f. Since the sided limits are different, $\lim_{x \to 0} f(x)$ does not exist. Notice that there is a jump discontinuity on the graph. This always occurs if the sided limits are different (but are not infinite)

g. As you approach $x = 1$ from the left, the function decreases without bound towards the vertical asymptote at $x = 1$ so $\lim_{x \to 1^-} f(x) = -\infty$

h. As you approach $x = 1$ from the right, the function increases without bound towards the vertical asymptote at $x = 1$ so $\lim_{x \to 1^+} f(x) = \infty$

i. Since the sided limits are different, $\lim_{x \to 1} f(x)$ does not exist. Notice that there is an infinite discontinuity (vertical asymptote) on the graph at $x = 1$. This always occurs if the sided limits at a certain x-value are infinite.

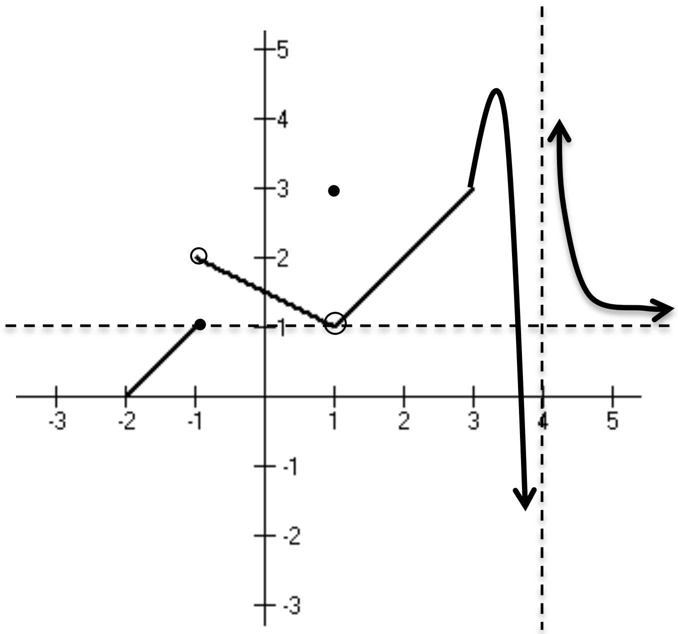
j. As you approach, $x = 3$ from the left, the function decreases towards 2.5 (best guess), so we will estimate that $\lim_{x \to 3^-} f(x) = 2.5$

k. As you approach, $x = 3$ from the right, the function increases towards 2.5 (best guess), so we will estimate that $\lim_{x \to 3^+} f(x) = 2.5$

l. Since the sided limits are equal, $\lim_{x \to 3} f(x) = 2.5$. Notice the point discontinuity—even though $f(3) \neq 2.5$, the point discontinuity does not affect the value of the limit.

m. Limits as $x \to \infty$ or $x \to -\infty$ describe horizontal asymptotes. Horizontal asymptotes are unlike vertical asymptotes in that they may be crossed. Horizontal asymptotes define the end behavior of this function. Here, as we move x towards infinity (to the far right edge of the graph), the value of the function appears to be flattening out at 0 (approaching the x-axis). Therefore we say that $\lim_{x \to \infty} f(x) = 0$

Problem Set 1.3: Evaluating Limits from Graphs.



Based on the graph of $f(x)$ at left, find the following limits.

a) $\lim_{x\to -1^-} f(x)$ b) $\lim_{x\to -1^+} f(x)$ c) $\lim_{x\to -1} f(x)$

d) $\lim_{x\to 1^-} f(x)$ e) $\lim_{x\to 1^+} f(x)$ f) $\lim_{x\to 1} f(x)$

g) $\lim_{x\to -2^-} f(x)$ h) $\lim_{x\to -2^+} f(x)$ i) $\lim_{x\to -2} f(x)$

j) $\lim_{x\to 4^-} f(x)$ k) $\lim_{x\to 4^+} f(x)$ l) $\lim_{x\to 4} f(x)$

m) $\lim_{x\to \infty} f(x)$ (think about g it's trickyish)

Section 1.4: Limits as $x\to\infty$ ***and*** $x\to-\infty$

As mentioned in the previous section, limits as $x\to\infty$ and $x\to-\infty$ give the end behavior of a function. **Keep in mind—horizontal asymptotes also give end behavior. If you are asked to find horizontal asymptotes, just find the limits as** $x\to\infty$ **and** $x\to-\infty$**!** Since direct substitution is not really an option (you can't really plug ∞ into a mathematical expression—it's not a number), we need another method for determining the values of these limits. For many types of functions, polynomials for example, the limits will simply trend toward either ∞ or $-\infty$

<u>Example 1</u>: Find $\lim_{x\to\infty}\left(4x^3-7x^2+x+2\right)$

As the value of *x* gets progressively large, the value of $4x^3$ dominates the value of the function. As *x* gets larger, the overall value of the function gets increasingly large, so $\lim_{x\to\infty}\left(4x^3-7x^2+x+2\right)=\infty$.

Example 1 should seem relatively intuitive. Think of a cubic graph with a positive lead coefficient—you know that as you move toward the right side of the graph, the graph goes up indefinitely towards positive infinity. The limit as $x\to\infty$ simply describes the end behavior of the cubic function on the far right side of the graph.

Rational functions (fractions) provide a more interesting study for limits as $x\to\infty$ and $x\to-\infty$. When attempting to deduce the value of these limits when examining a rational function, the best approach is to compare the relative sizes of the numerator and the denominator of the fraction. Ask yourself the question, which gets bigger faster? To help with this analysis, use the mnemonic device Tom Likes Pizza Every Friday Night[1].

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[1] © Michaela Del Viscovo and Mallory Clinton, 2015.

In the mnemonic, the first letter stands for a different type of function (this is also useful when considering convergence and divergence of infinite series).

T—Trigonometric functions (actually just $\sin x$ and $\cos x$). These just bounce back and forth between -1 and +1, so they don't actually get big at all as *x* increases.
L—Logarithmic functions (generally $\ln x$ or some variant thereof). These increase very slowly as the value of *x* increases.
P—Polynomial functions (things like $\sqrt{x}, x^2, x^{100}$, etc.). These increase much faster than log functions, and the bigger the exponent, the faster the increase. If you are examining a rational function that involves a polynomial in both the numerator and denominator, the largest power wins. If the largest power is the same in both the numerator and the denominator, the limit is determined by the lead coefficients.
E—Exponential functions (things like $2^x, e^x, 10^x$, etc.). Exponential functions get large very fast. When comparing two exponential functions, generally speaking (as long as the exponents are of the same degree), the larger the base, the faster the increase.
F—Factorials (*x*!). These typically don't occur in limits because they are not continuous functions—they tend to appear frequently in the analysis of infinite series. They get big *really* fast.
N—n^n (or x^x). This is a singular expression—both the base and the exponent are variables. This also doesn't come up much, but when it does, it beats pretty much anything.

Consider some basic functions above when *x* = 10
T—sin 10 = -0.544 (very small)
L—ln 10 = 2.303 (not that much bigger)
P—10^2 = 100 (eh, still not very impressive)
E—$2^{10} = 1024$ (ok, that's kinda big)
F—10! = 3,628,800 (now we're cooking with gas)
N—$10^{10} = 10{,}000{,}000{,}000$ (yup—that's big. And that's only using *x* = 10)

The basic rules here are as follows:

- If the numerator wins (gets bigger faster), the limit is either $+\infty$ or $-\infty$. To determine which one, figure out if as *x* gets really big, both numerator and denominator are both positive or both negative (in which case the limit is $+\infty$), or if the numerator and denominator go in opposite directions (in which case the limit is $-\infty$).

  <u>Examples 2 & 3</u>:

  $$\lim_{x\to\infty}\frac{4x^3+1}{\ln x}=\infty$$

  Polynomial in the numerator and log in the denominator. Numerator wins, and both will be positive, so the limit is $+\infty$

  $$\lim_{x\to\infty}\frac{1-2^x}{x^{20}}$$

  Exponential in the numerator and polynomial in denominator. Numerator wins, but it gets very negative while the denominator is positive, so the limit is $-\infty$

- If the denominator wins (gets bigger faster), the limit is 0. Not a lot of ambiguity here—these are easy.

  Example 4:
  $\lim_{x\to\infty}\frac{\ln\left(x^2\right)}{3^x-1}=0$ The exponential in the denominator grows much faster than the logarithm in the numerator. The denominator wins. The limit is 0.

- If you have an exponential over an exponential, the base decides it.

  Example 5:
  $\lim_{x\to\infty}\frac{2^x}{3^{x-1}+1}=0$ The exponential in the denominator has a base of 3 and the exponential in the exponential in the numerator has a base of 2. Since 3 > 2, the denominator wins. There are exceptions to this—if the numerator was 2^{x^2}, that would actually win, but problems such as this rarely come up.

- If you have a polynomial over a polynomial and the largest powers in the numerator and denominator are different, the larger power wins. If the largest powers are the same, the limit is determined by the lead coefficients.

  Examples 6 & 7:

  $\lim_{x\to\infty}\frac{4x^3+1}{x^2-10}=\infty$

  Polynomial in both the numerator and denominator. Numerator wins because power is greater. Both are getting bigger so the limit is $+\infty$

  $\lim_{x\to\infty}\frac{x^2-3x+10}{2x^2+x+6}=\frac{1}{2}$

  Largest powers on top and bottom are both 2. The limit is thus equal to the value of the lead coefficients.

Usually, the limits as $x\to\infty$ and $x\to-\infty$ are the same—most graphs only have one horizontal asymptote (remember—these are the limits that must be calculated to find horizontal asymptotes). However, there are exceptions and you have to carefully consider each problem. Consider the problem from the 2008 AP Exam below.

Example 8: What are all the horizontal asymptotes of the graph of $y=\frac{5+2^x}{1-2^x}$ in the *xy*-plane?

To answer the question, we must find $\lim_{x\to\infty}\frac{5+2^x}{1-2^x}$ and $\lim_{x\to-\infty}\frac{5+2^x}{1-2^x}$

For $x \to \infty$, the 5 and the 1 don't really matter. The numerator and denominator are going to get equally big, but in opposite directions. Think about plugging in a "biggish" number like 10. You'd get $\frac{1029}{-1023}$. The numerator and denominator are basically the same value but with opposite signs, so $\lim_{x \to \infty} \frac{5+2^x}{1-2^x} = -1$.

For $x \to -\infty$, the exponentials actually become insignificant. 2 to a very large negative power is 1 over 2 to a very large power, and 1 over a very large number is basically zero. Therefore $\lim_{x \to -\infty} \frac{5+2^x}{1-2^x} = \frac{5+0}{1-0} = 5$

The function has two horizontal asymptotes: $y = -1$ and $y = 5$.

Problem Set 1.4: *Limits as* $x \to \infty$ *and* $x \to -\infty$.

1. Find the following limits:

a) $\lim_{x \to \infty} \frac{\sin x}{x^2+1}$ b) $\lim_{x \to \infty} \frac{8x^3+x}{4x-3x^3}$ c) $\lim_{x \to \infty} \frac{x^2+\ln x}{2-3^x}$

d) $\lim_{x \to \infty} x^2 e^{-x}$ e) $\lim_{x \to \infty} \frac{1-5^x}{3^{x+2}}$ f) $\lim_{x \to \infty} \frac{x^{2000}}{x^x+2}$

2. Find all horizontal asymptotes of the following functions

a. $f(x) = \frac{3x^4+2x^2-9x+1}{4x^4-7x^3+6x^2-10x+2}$

b. $g(x) = \frac{1}{2x^2} + 3$

c. $h(x) = \frac{3^x+x^2}{3^x-2x^2}$

Section 1.5: What happens when $\lim_{x\to a} f(x) = \frac{0}{0}$?

Somewhat disappointingly, calculus books and exam makers like to ask students to find values of limits such as $\lim_{x\to 5}\frac{x-5}{x^2-25}$ where direct substitution yields an answer of $\frac{0}{0}$. Even more disappointingly, this does not mean automatically that the limit does not exist. If you attempt to evaluate a limit by direct substitution, and you find that the result is $\frac{0}{0}$, there are a variety of algebraic methods including factoring, multiplying by a conjugate expression, finding common denominators, and expanding binomials that you can use to simplify the expression for $f(x)$ so that after simplifying, direct substitution yields the actual value of the limit.

<u>Example 1</u>: Simplifying by factoring. Find $\lim_{x\to 5}\frac{x-5}{x^2-25}$.

You know a variety of factoring methods—difference of squares, sum and difference of cubes, basic trinomial factoring and factoring by grouping—all of which can potentially be useful when finding limits where direct substitution returns $\frac{0}{0}$. In this example, the denominator is a difference of squares. Factor and simplify the expression $\frac{x-5}{x^2-25}$.

$$\lim_{x\to 5}\frac{x-5}{x^2-25} = \lim_{x\to 5}\frac{x-5}{(x-5)(x+5)} = \lim_{x\to 5}\frac{1}{x+5}$$

Once you have simplified, direct substitution can be used: $\lim_{x\to 5}\frac{x-5}{x^2-25} = \lim_{x\to 5}\frac{1}{x+5} = \frac{1}{10}$

<u>Example 2</u>: Simplifying by expanding a binomial. Find $\lim_{x\to 0}\frac{(x+2)^3-8}{x}$.

Expanding binomials is most easily done if you have some knowledge Pascal's Triangle. Pascal's Triangle is formed as shown below

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

and so on, where the numbers in each row begin and end with 1, and all the middle numbers are found by adding the two numbers from the row above. When creating the binomial expansion of $\left(a+b\right)^n$, you create a series of terms, each comprised of three elements: a coefficient (which is taken from Pascal's Triangle—always use the row where the exponent you start with is the second number in the row), *a* to a power, and *b* to a power. In the first term of the expansion, the power on *a* starts at *n* and counts down in successive terms, and the power on *b* starts at 0 and counts up in successive terms. Following these directions, $\left(x+2\right)^3$ expands as: $1\left(x\right)^3\left(2\right)^0+3\left(x\right)^2\left(2\right)^1+3\left(x\right)^1\left(2\right)^2+1\left(x\right)^0\left(2\right)^3=$ $x^3+6x^2+12x+8$. This expansion can be then used to simplify the expression:

$$\lim_{x\to 0}\frac{\left(x+2\right)^3-8}{x}=\lim_{x\to 0}\frac{x^3+6x^2+12x+8-8}{x}=\lim_{x\to 0}\frac{x^3+6x^2+12x}{x}=\lim_{x\to 0}\frac{x\left(x^2+6x+12\right)}{x}=\lim_{x\to 0}\left(x^2+6x+12\right)$$

You can now use direct substitution to find that $\lim_{x\to 0}\frac{\left(x+2\right)^3-8}{x}=\lim_{x\to 0}\left(x^2+6x+12\right)=12$

<u>Example 3</u>: Simplifying by finding a common denominator. Find $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}$

In this problem, you need to find a common denominator for the two fractions in the numerator, so that you can add them together. Once that is done you can divide by the *x* in the denominator (multiply by $\frac{1}{x}$) and simplify .

$$\lim_{x\to 0}\frac{\left(\frac{2}{2}\right)\frac{1}{2+x}-\frac{1}{2}\left(\frac{2+x}{2+x}\right)}{x}=\lim_{x\to 0}\frac{\frac{2}{2\left(2+x\right)}-\frac{2+x}{2\left(2+x\right)}}{x}=\lim_{x\to 0}\frac{\frac{-x}{2\left(2+x\right)}}{x}=\lim_{x\to 0}\frac{-x}{2\left(2+x\right)}\cdot\frac{1}{x}=\lim_{x\to 0}\frac{-1}{2\left(2+x\right)}$$

Be careful in between the second and third steps above! When subtracting, you have to distribute the negative—many people forget to do this and wind up with the third step as $\lim_{x\to 0}\frac{\frac{x}{2\left(2+x\right)}}{x}$, which is going to give you the wrong answer. Once correctly simplified, use direct substitution: $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}=\lim_{x\to 0}\frac{-1}{2\left(2+x\right)}=-\frac{1}{4}$

Example 4: Simplify by multiplying by a conjugate expression. Find $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

This method is typically used when you are dealing with a rational expression that also includes a radical. Multiply the numerator and the denominator by the conjugate of the numerator: $\sqrt{1+x}+1$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}\cdot\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}=\lim_{x\to 0}\frac{1+x+\sqrt{1+x}-\sqrt{1+x}-1}{x\left(\sqrt{1+x}+1\right)}=\lim_{x\to 0}\frac{x}{x\left(\sqrt{1+x}+1\right)}=\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}$$

The limit can now be evaluated using direct substitution: $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}=\frac{1}{2}$

Problem Set 1.5: *What happens when* $\lim_{x\to a}f\left(x\right)=\frac{0}{0}$?

Find the limits below. Refer back to the examples if you get stuck!

1. $\lim_{x\to 0}\frac{\left(x-3\right)^4-81}{x}$

2. $\lim_{x\to 8}\frac{x^2-13x+40}{x-8}$

3. $\lim_{x\to -6}\frac{2x^2+13x+6}{x^2-36}$

4. $\lim_{x\to 4}\frac{x^2+x-20}{x^3-64}$

5. $\lim_{x\to -2}\frac{x^2-4}{x^3+8}$

6. $\lim_{x\to 10}\frac{\sqrt{x-6}-2}{x-10}$

7. $\lim_{x\to 0}\frac{\frac{1}{x+5}-\frac{1}{5}}{x}$

8. $\lim_{x\to 2}\frac{3x^2-4x-4}{2x^2-8}$

9. $\lim_{x\to 4}\frac{\left(x+2\right)^2-9x}{x-4}$

10. $\lim_{x\to 0}\frac{e^x-e^{2x}}{1-e^x}$

11. $\lim_{x\to 2}\frac{x-2}{\sqrt{x}-\sqrt{4-x}}$

12. $\lim_{x\to 0}\frac{\frac{1}{\left(x+2\right)^2}-\frac{1}{4}}{x}$