Part 1: Lines and Linear Functions

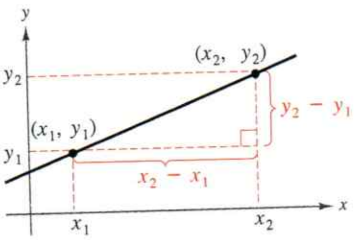


FIGURE 36
$\Delta y = y_2 - y_1 =$ change in y
$\Delta x = x_2 - x_1 =$ change in x

The Slope of a Line

The **slope** of a nonvertical line is a measure of the number of units a line rises (or falls) vertically for each unit of horizontal change from left to right. Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure 36. As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (Δ is the Greek uppercase letter *delta,* and the symbols Δy and Δx are read "delta y" and "delta x.")

Definition of the Slope of a Line

The **slope** m.of a nonvertical line passing through the points (x_1, y_1) and (x_2, y_2) is

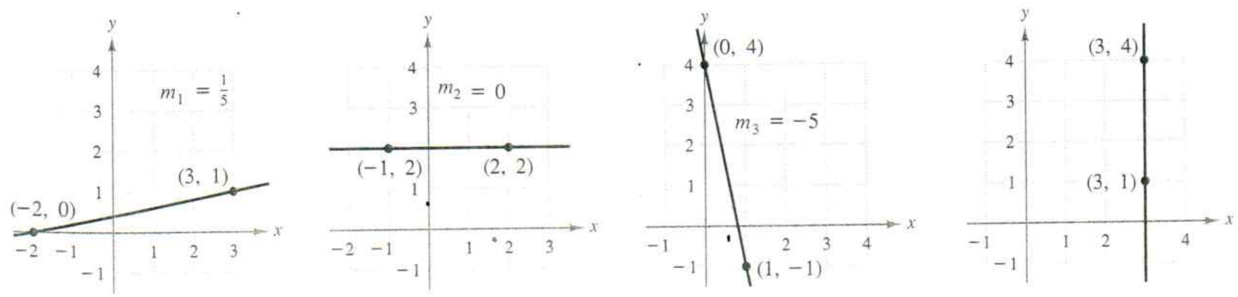
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \qquad x_1 \neq x_2.$$

REMARK When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

It does not matter in which order you subtract *as long as* you are consistent and both "subtracted coordinates" come from the same point.

Figure 37 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an "undefined slope."



If m is positive, then the line rises.

If m is zero, then the line is horizontal.

If m is negative, then the line falls.

If the line is vertical, then the slope is undefined.

FIGURE 37

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope. This can be verified from the similar triangles shown in Figure 38. (Recall that the ratios of corresponding sides of similar triangles are equal.)

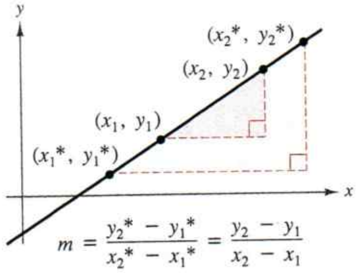


FIGURE 38
Any two points on a line can be used to determine its slope.

You can write an equation of a line if you know the slope of the line and the coordinates of one point on the line. Suppose the slope is m and the point is (x_1, y_1). If (x, y) is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation, involving the two variables x and y, can be rewritten in the form $y - y_1 = m(x - x_1)$, which is called the **point-slope equation of a line.**

Point-Slope Equation of a Line

An equation of the line with slope m passing through the point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1).$$

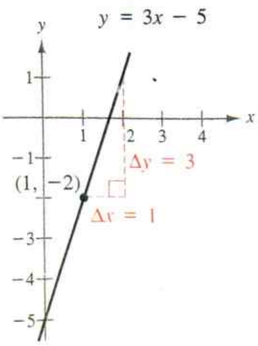


FIGURE 39
Point-slope equation of a line.

EXAMPLE 1 Finding an Equation of a Line

Find an equation of the line that has a slope of 3 and passes through the point $(1, -2)$.

Solution

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute -2 for y_1, 1 for x_1, and 3 for m
$y + 2 = 3x - 3$	Simplify
$y = 3x - 5$	Solve for y

(See Figure 39.)

The slope of a line can be interpreted as either a *ratio* or a *rate*. If the x-axis and y-axis have the same unit of measure, then the slope has no units and is a **ratio.** If the x-axis and y-axis have different units of measure, then the slope is a rate or **rate of change.** As you study this text, you will encounter applications involving both interpretations of slope.

Sketching the Graph of a Line

In Section 2, we mentioned that many problems in analytic geometry can be classified in two basic categories: (1) Given a graph, what is its equation? and (2) Given an equation, what is its graph? The point-slope equation of a line can be used to solve problems in the first category. However, this form is not especially useful for solving problems in the second category. The form that is better suited to sketching the graph of a line is the **slope-intercept** form for the equation of a line.

The Slope-Intercept Equation of a Line

The graph of the linear equation

$$y = mx + b$$

is a line having a *slope* of m and a *y-intercept* at $(0, b)$.

EXAMPLE 3 Sketching Lines in the Plane

Sketch the graphs of the equations.

a. $y = 2x + 1$ **b.** $y = 2$ **c.** $3y + x - 6 = 0$

Solution

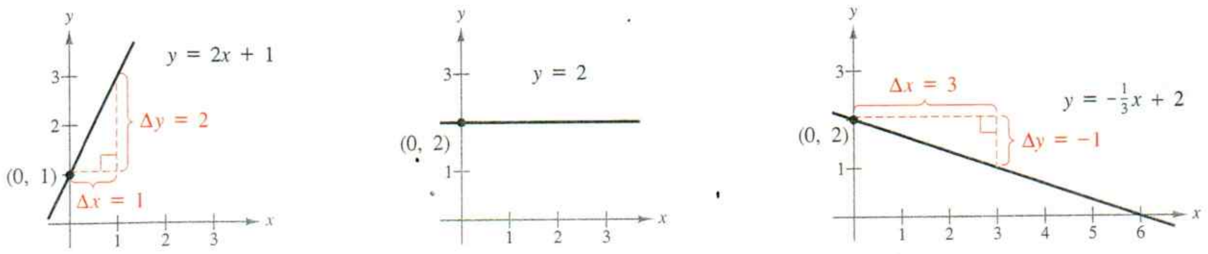
a. Because $b = 1$, the y-intercept is $(0, 1)$. Because the slope is $m = 2$, you know that the line rises 2 units for each unit it moves to the right, as shown in Figure 42(a).

b. Because $b = 2$, the y-intercept is $(0, 2)$. Because the slope is $m = 0$, you know that the line is horizontal, as shown in Figure 42(b).

c. Begin by writing the equation in slope-intercept form.

$$3y + x - 6 = 0 \quad \text{Original equation}$$
$$3y = -x + 6 \quad \text{Isolate } y\text{-term on the left}$$
$$y = -\frac{1}{3}x + 2 \quad \text{Slope-intercept form}$$

In this form, you can see that the y-intercept is $(0, 2)$ and the slope is $m = -\frac{1}{3}$. This means that the line falls 1 unit for every 3 units it moves to the right, as shown in Figure 42(c).



(a) $m = 2$; line rises. (b) $m = 0$; line is horizontal. (c) $m = -\frac{1}{3}$; line falls.

FIGURE 42

Because the slope of a vertical line is not defined, its equation cannot be written in the slope-intercept form. However, the equation of *any* line can written in the **general form**

$$Ax + By + C = 0$$

where A and B are not *both* zero. For instance, the vertical line given by $x = a$ can be represented by the general form $x - a = 0$.

Summary of Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Point-slope form: $y - y_1 = m(x - x_1)$
5. Slope-intercept form: $y = mx + b$

REMARK Three points are **collinear** if they lie on the same line.

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure 43.

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal.
2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is, if and only if

$$m_1 = -\frac{1}{m_2}.$$

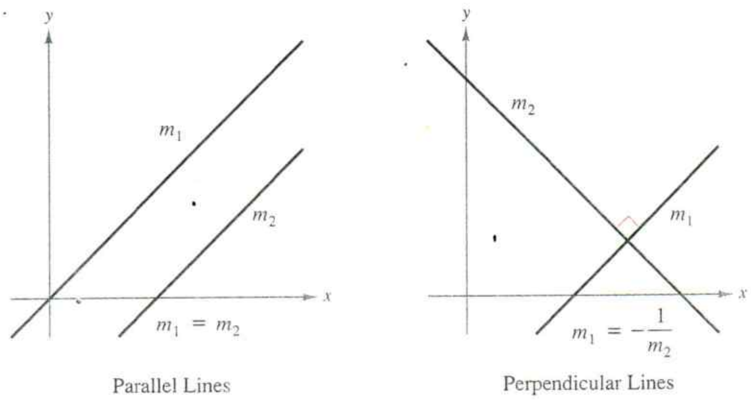


FIGURE 43

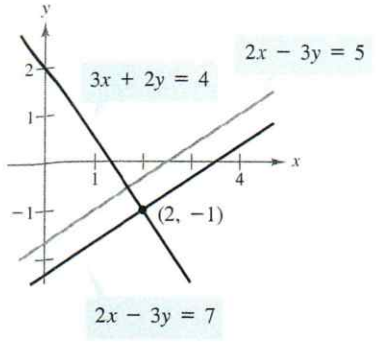


FIGURE 44
Lines parallel and perpendicular to $2x - 3y = 5$.

EXAMPLE 4 Finding Parallel and Perpendicular Lines

Find the general forms of the equations of the lines that pass through the point $(2, -1)$ and are

a. parallel to the line $2x - 3y = 5$
b. perpendicular to the line $2x - 3y = 5$

as shown in Figure 44.

Solution By writing the linear equation $2x - 3y = 5$ in slope-intercept form $y = \frac{2}{3}x - \frac{5}{3}$, you can see that the given line has a slope of $m = \frac{2}{3}$.

a. The line through $(2, -1)$ that is parallel to the given line also has a slope of $\frac{2}{3}$.

$$y - (-1) = \frac{2}{3}(x - 2) \qquad \text{Point-slope form}$$
$$3(y + 1) = 2(x - 2) \qquad \text{Simplify}$$
$$2x - 3y - 7 = 0 \qquad \text{General form}$$

Note the similarity to the original equation.

b. Using the negative reciprocal of the slope of the given line, you can determine that the slope of a line perpendicular to the given line is $-\frac{3}{2}$. Therefore, the line through the point $(2, -1)$ that is perpendicular to the given line has the following equation.

$$y - (-1) = -\frac{3}{2}(x - 2) \qquad \text{Point-slope form}$$
$$2(y + 1) = -3(x - 2) \qquad \text{Simplify}$$
$$3x + 2y - 4 = 0 \qquad \text{General form}$$

Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y) where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r. In this case r is the independent variable and A is the dependent variable.

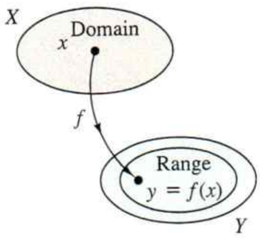


FIGURE 47
A real-valued function f of a real variable.

Definition of a Real-Valued Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function** f **of a real variable** x from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$. The **range** of f is a subset of Y and consists of all images of numbers in X (see Figure 47).

Functions can be specified in a variety of ways. In this text, however, we will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation $x + 2y = 1$ defines y, the dependent variable, as a function of x, the independent variable. To **evaluate** this function (that is, to find the y-value that corresponds to a given x-value), it is convenient to isolate y on the left side of the equation.

$$y = \frac{1}{2}(1 - x)$$

Using f as the name of the function, you can write this equation as

$$f(x) = \frac{1}{2}(1 - x). \quad \text{Function notation}$$

Function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling you that x is the independent variable and that the function itself is "f." The notation $f(x)$ is read "f of x." Function notation allows you to be less wordy. Instead of asking "What is the value of y that corresponds to $x = 3$?" you can ask "What is $f(3)$?"

The Domain and Range of a Function

The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined. For example, the function given by

$$f(x) = \frac{1}{x^2 - 4}, \qquad 4 \le x \le 5$$

has an explicitly defined domain given by $\{x: 4 \le x \le 5\}$. On the other hand, the function given by

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain which is the set $\{x: x \ne \pm 2\}$.

EXAMPLE 2 Finding the Domain and Range of a Function

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range, observe that $f(x) = \sqrt{x - 1}$ is never negative. Moreover, as x takes on the various values in the domain, $f(x)$ takes on all nonnegative values. Thus, the range is the interval $[0, \infty)$, as indicated in Figure 48.

b. The domain of the function

$$f(x) = \sqrt{x^2 - x - 6}$$

is the set of all x-values such that $x^2 - x - 6 \ge 0$. Using the techniques shown in Example 4 in Section 1, you can conclude that the domain is

$$(-\infty, -2] \cup [3, \infty).$$

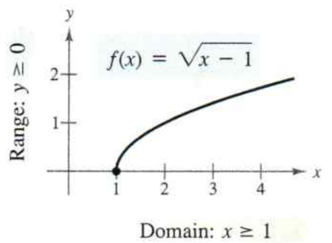


FIGURE 48
The domain of $f(x)$ is $[1, \infty)$ and the range is $[0, \infty)$.

EXAMPLE 3 A Function Defined by More than One Equation

Determine the domain and range of the following function.

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ \sqrt{x - 1}, & \text{if } x \ge 1 \end{cases}$$

Solution Because f is defined for $x < 1$ and $x \ge 1$, the domain is the entire set of real numbers. On the portion of the domain for which $x \ge 1$, the function behaves as in Example 2. For $x < 1$, the values of $1 - x$ are positive. Therefore, the range of the function is the interval

$[0, \infty)$. Range

(See Figure 49.)

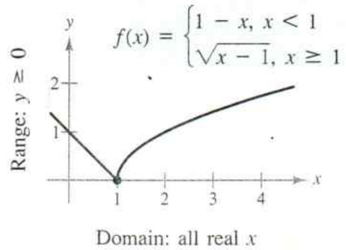


FIGURE 49
The domain of $f(x)$ is $(-\infty, \infty)$ and the range is $[0, \infty)$.

A function from X to Y is **one-to-one** if to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function given in Example 2a is one-to-one, whereas the function given in Example 3 is not one-to-one. If the range consists of all of Y, then the function is called **onto.**

In an equation that defines a function, the role of the variable x is simply that of a placeholder. For instance, the function given by

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\quad) = 2(\quad)^2 - 4(\quad) + 1$$

where parentheses are used instead of x. To evaluate $f(-2)$, simply place -2 in each set of parentheses.

$$\begin{aligned} f(-2) &= 2(-2)^2 - 4(-2) + 1 \\ &= 2(4) + 8 + 1 \\ &= 17 \end{aligned}$$

REMARK Although f is often used as a convenient function name and x as the independent variable, you can use other symbols. For instance, the following equations all define the same function.

$$f(x) = x^2 - 4x + 7, \qquad f(t) = t^2 - 4t + 7, \qquad g(s) = s^2 - 4s + 7$$

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = x^2 + 7$, evaluate the following.

a. $f(3a)$ **b.** $f(b - 1)$ **c.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0$

Solution

a. $f(3a) = (3a)^2 + 7$ Replace x with $3a$

$= 9a^2 + 7$ Simplify

b. $f(b - 1) = (b - 1)^2 + 7$ Replace x with $b - 1$

$= b^2 - 2b + 1 + 7$ Expand binomial

$= b^2 - 2b + 8$ Simplify

c.

$$\begin{aligned} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{[(x + \Delta x)^2 + 7] - (x^2 + 7)}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} \\ &= 2x + \Delta x, \qquad \Delta x \neq 0 \end{aligned}$$

REMARK The expression in Example 1c is called a difference quotient and has special significance in calculus.

Combinations of Functions

Two functions can be combined in various ways to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the following functions.

$$f(x) + g(x) = (2x - 3) + (x^2 + 1) = x^2 + 2x - 2 \quad \text{Sum}$$
$$f(x) - g(x) = (2x - 3) - (x^2 + 1) = -x^2 + 2x - 4 \quad \text{Difference}$$
$$f(x)g(x) = (2x - 3)(x^2 + 1) = 2x^3 - 3x^2 + 2x - 3 \quad \text{Product}$$
$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 + 1} \quad \text{Quotient}$$

You can combine two functions in yet another way to form a **composite function.**

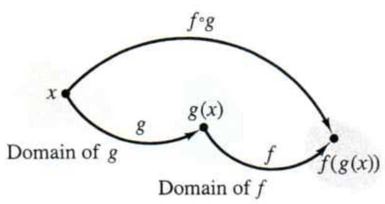


FIGURE 56
The domain of the composite function $f \circ g$.

Definition of Composite Function

Let f and g be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure 56).

The composite of f with g may not be equal to the composite of g with f.

EXAMPLE 4 Finding Composites of Functions

Given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, find

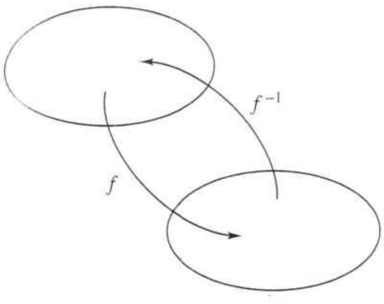
a. $f \circ g$ **b.** $g \circ f$.

Solution

a. $(f \circ g)(x) = f(g(x)) = 2(g(x)) - 3 = 2(x^2 + 1) - 3$
$= 2x^2 - 1$

b. $(g \circ f)(x) = g(f(x)) = (f(x))^2 + 1 = (2x - 3)^2 + 1$
$= 4x^2 - 12x + 10$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$.



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 5.10

Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 5.10. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x$$

EXPLORATION

Finding Inverse Functions Explain how to "undo" each of the following functions. Then use your explanation to write the inverse function of f.

a. $f(x) = x - 5$

b. $f(x) = 6x$

c. $f(x) = \dfrac{x}{2}$

d. $f(x) = 3x + 2$

e. $f(x) = x^3$

f. $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

Definition of Inverse Function

A function g is the **inverse function** of the function f if

$f(g(x)) = x$ for each x in the domain of g

and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

NOTE Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of -1 as a superscript. That is, in general, $f^{-1}(x) \neq 1/f(x)$.

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but if it does, the inverse function is unique (see Exercise 99).

You can think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. Use the definition of an inverse function to check the following.

$f(x) = x + c$ and $f^{-1}(x) = x - c$ are inverse functions of each other.

$f(x) = cx$ and $f^{-1}(x) = \dfrac{x}{c}$, $c \neq 0$, are inverse functions of each other.

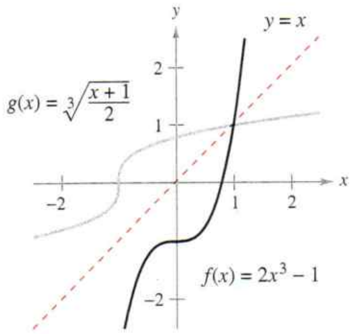
EXAMPLE 1 Verifying Inverse Functions

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x. The composition of f with g is given by

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$



f and *g* are inverse functions of each other.
Figure 5.11

The composition of g with f is given by

$$\begin{aligned} g(f(x)) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x. \end{aligned}$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 5.11).

STUDY TIP In Example 1, try comparing the functions f and g verbally.

For f: First cube x, then multiply by 2, then subtract 1.

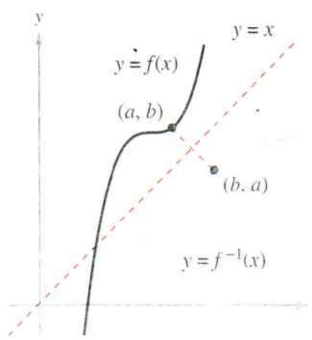
For g: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"?

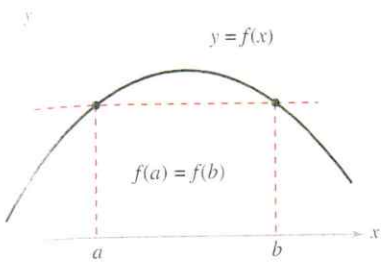
In Figure 5.11, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the following theorem.

THEOREM 5.6 Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
Figure 5.12



If a horizontal line intersects the graph of f twice, then f is not one-to-one.
Figure 5.13

Existence of an Inverse Function

Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 5.13). The following theorem formally states why the horizontal line test is valid. (Recall from Section 3.3 that a function is *strictly monotonic* if it is either increasing on its entire domain or decreasing on its entire domain.)

> **THEOREM 5.7 The Existence of an Inverse Function**
>
> 1. A function has an inverse function if and only if it is one-to-one.
> 2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Proof To prove the second part of the theorem, recall from Section P.3 that f is one-to-one if for x_1 and x_2 in its domain

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

The *contrapositive* of this implication is logically equivalent and states that

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Now, choose x_1 and x_2 in the domain of f. If $x_1 \neq x_2$, then, because f is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \qquad \text{or} \qquad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, f is one-to-one on the interval. The proof of the first part of the theorem is left as an exercise (see Exercise 100).

EXAMPLE 2 The Existence of an Inverse Function

Which of the functions has an inverse function?

a. $f(x) = x^3 + x - 1$ **b.** $f(x) = x^3 - x + 1$

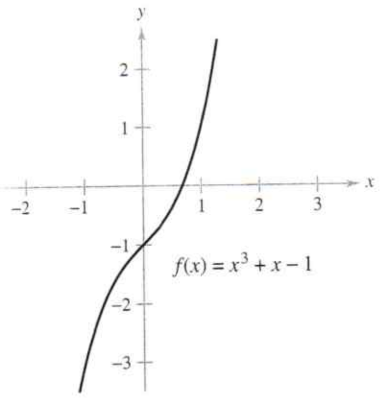
Solution

a. From the graph of f shown in Figure 5.14(a), it appears that f is increasing over its entire domain. To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of x. So, f is strictly monotonic and it must have an inverse function.

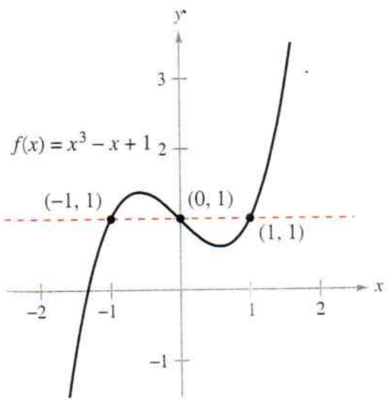
b. From the graph of f shown in Figure 5.14(b), you can see that the function does not pass the horizontal line test. In other words, it is not one-to-one. For instance, f has the same value when $x = -1, 0,$ and 1.

$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

So, by Theorem 5.7, f does not have an inverse function.



(a) Because f is increasing over its entire domain, it has an inverse function.



(b) Because f is not one-to-one, it does not have an inverse function.
Figure 5.14

NOTE Often it is easier to prove that a function *has* an inverse function than to find the inverse function. For instance, it would be difficult algebraically to find the inverse function of the function in Example 2(a).

The following guidelines suggest a procedure for finding an inverse function.

Guidelines for Finding an Inverse Function

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} to be the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

EXAMPLE 3 Finding an Inverse Function

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$

Solution The function has an inverse function because it is increasing on its entire domain (see Figure 5.15). To find an equation for the inverse function, let $y = f(x)$ and solve for x in terms of y.

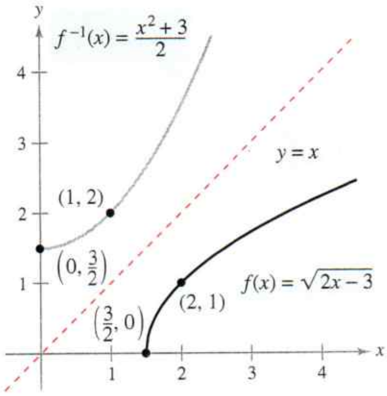
$$\sqrt{2x-3} = y \qquad \text{Let } y = f(x).$$

$$2x - 3 = y^2 \qquad \text{Square each side.}$$

$$x = \frac{y^2+3}{2} \qquad \text{Solve for } x.$$

$$y = \frac{x^2+3}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$f^{-1}(x) = \frac{x^2+3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$



The domain of f^{-1}, $[0, \infty)$, is the range of f.
Figure 5.15

The domain of f^{-1} is the range of f, which is $[0, \infty)$. You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x-3}\right)^2 + 3}{2} = \frac{2x - 3 + 3}{2} = x, \quad x \geq \frac{3}{2}$$

NOTE Remember that any letter can be used to represent the independent variable. So,

$$f^{-1}(y) = \frac{y^2+3}{2}$$

$$f^{-1}(x) = \frac{x^2+3}{2}$$

$$f^{-1}(s) = \frac{s^2+3}{2}$$

all represent the same function.

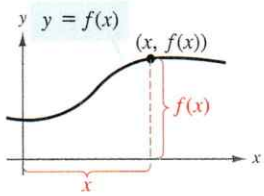


FIGURE 50
The graph of a function.

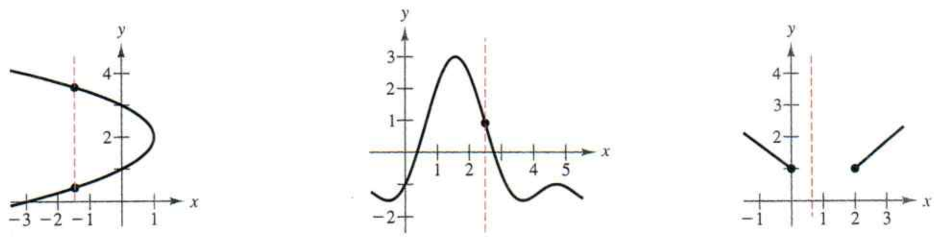
The Graph of a Function

The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure 50, note that

$$x = \text{the directed distance from the } y\text{-axis}$$

$$f(x) = \text{the directed distance from the } x\text{-axis.}$$

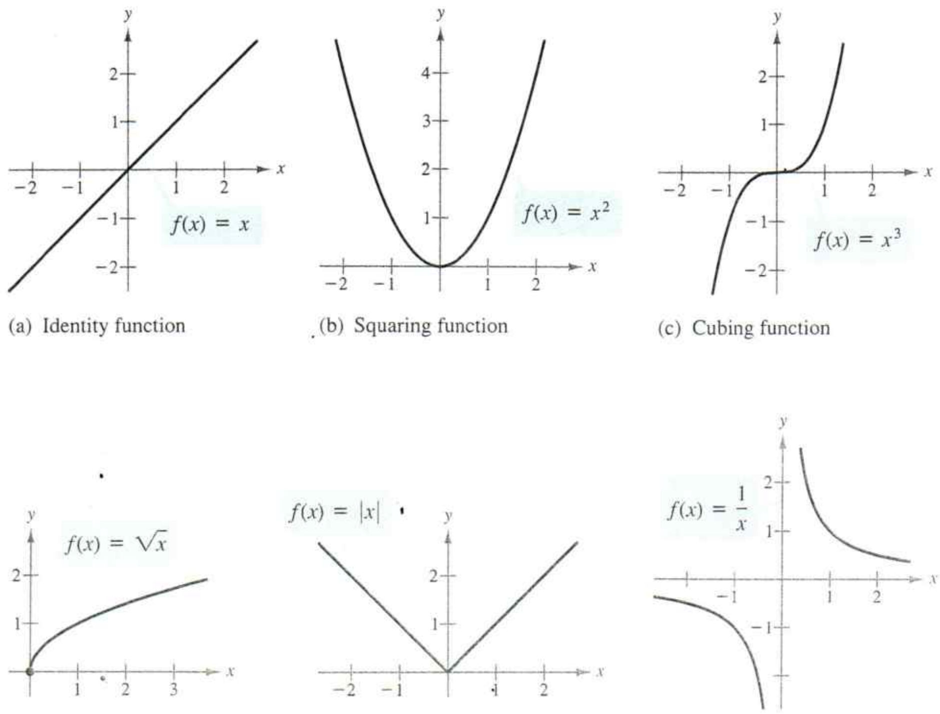
A vertical line can intersect the graph of a function of x at most *once.* This observation provides a convenient visual test (called the **vertical line test**) for functions of x. For example, in Figure 51(a), you can see that the graph does not define y as a function of x because a vertical line intersects the graph twice.



(a) Not a function of x (b) A function of x (c) A function of x

FIGURE 51
The vertical line test for functions.

Figure 52 shows the graphs of six basic functions. You should be able to recognize these graphs.



(a) Identity function (b) Squaring function (c) Cubing function

(d) Square root function (e) Absolute value function (f) Rational function

FIGURE 52
The graphs of six basic functions.

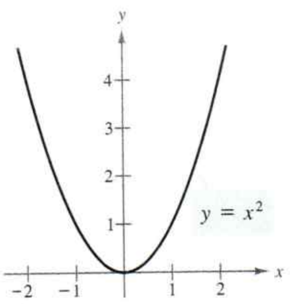
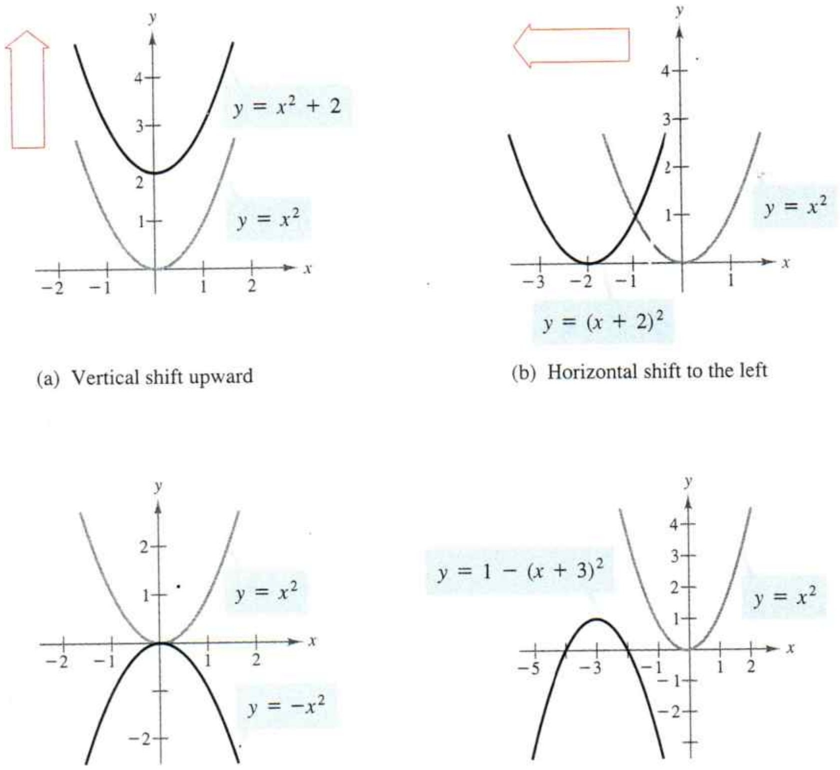


FIGURE 53
The original graph.

Transformations of Functions

Some families of graphs have the same basic shape. For example, consider the graph of $y = x^2$, as shown in Figure 53. Now compare this graph with those shown in Figure 54.



(a) Vertical shift upward

(b) Horizontal shift to the left

(c) Reflection

(d) Shift to the left, reflect, and then shift upward

FIGURE 54
Transformations of the graph of $y = x^2$.

Each of the graphs in Figure 54 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections. Function notation lends itself well to describing transformations of graphs in the plane.

Basic Types of Transformations ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$

DEFINITION Exponential Function

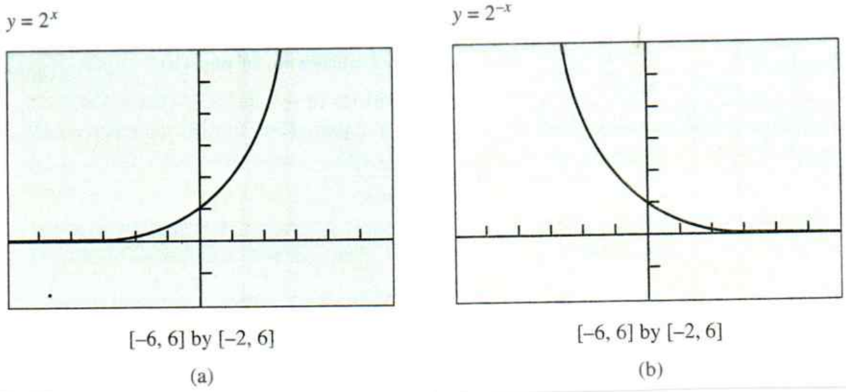
Let a be a positive real number other than 1. The function

$$f(x) = a^x$$

is the **exponential function with base *a*.**

The domain of $f(x) = a^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. If $a > 1$, the graph of f looks like the graph of $y = 2^x$ in Figure 1.22a. If $0 < a < 1$, the graph of f looks like the graph of $y = 2^{-x}$ in Figure 1.22b.

Figure 1.22 A graph of (a) $y = 2^x$ and (b) $y = 2^{-x}$.



EXAMPLE 1 Graphing an Exponential Function

Graph the function $y = 2(3^x) - 4$. State its domain and range.

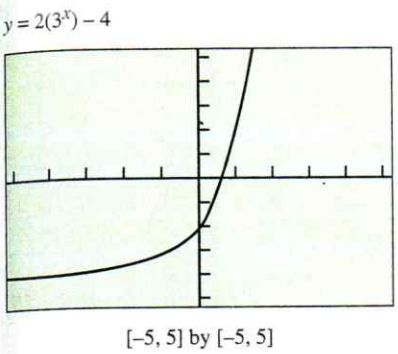


Figure 1.23 The graph of $y = 2(3^x) - 4$. (Example 1)

SOLUTION

Figure 1.23 shows the graph of the function y. It appears that the domain is $(-\infty, \infty)$. The range is $(-4, \infty)$ because $2(3^x) > 0$ for all x.

Exponential functions obey the rules for exponents.

Rules for Exponents

If $a > 0$ and $b > 0$, the following hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \cdot b^x = (ab)^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

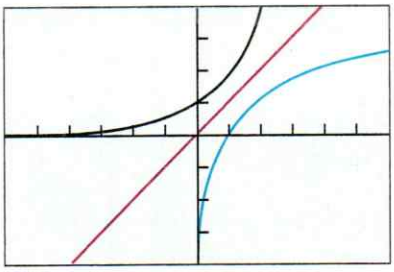
Logarithmic Functions

If a is any positive real number other than 1, the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *base a logarithm function.*

DEFINITION Base *a* Logarithm Function

The **base *a* logarithm function** $y = \log_a x$ is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x.



[–6, 6] by [–4, 4]

Figure 1.37 The graphs of $y = 2^x$ $(x_1 = t, y_1 = 2^t)$, its inverse $y = \log_2 x$ $(x_2 = 2^t, y_2 = t)$, and $y = x$ $(x_3 = t, y_3 = t)$.

Because we have no technique for solving for x in terms of y in the equation $y = a^x$, we do not have an explicit formula for the logarithm function as a function of x. However, the graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$, or by using parametric graphing (Figure 1.37).

Logarithms with base e and base 10 are so important in applications that calculators have special keys for them. They also have their own special notation and names:

$$\log_e x = \ln x,$$
$$\log_{10} x = \log x$$

The function $y = \ln x$ is called the **natural logarithm function** and $y = \log x$ is often called the **common logarithm function.**

Properties of Logarithms

Because a^x and $\log_a x$ are inverses of each other, composing them in either order gives the identity function. This gives two useful properties.

Inverse Properties for a^x and $\log_a x$

1. Base a: $a^{\log_a x} = x$; $\log_a a^x = x$, $a > 1, x > 0$
2. Base e: $e^{\ln x} = x$, $\ln e^x = x$, $x > 0$

These properties help us with the solution of equations that contain logarithms and exponential functions.

EXAMPLE 4 Using the Inverse Properties

Solve for x: (a) $\ln x = 3t + 5$ (b) $e^{2x} = 10$

SOLUTION

(a) $\ln x = 3t + 5$

$e^{\ln x} = e^{3t+5}$ Exponentiate both sides.

$x = e^{3t+5}$ Inverse Property

(b) $e^{2x} = 10$

$\ln e^{2x} = \ln 10$ Take logarithms of both sides.

$2x = \ln 10$ Inverse Property

$x = \frac{1}{2} \ln 10 \approx 1.15$

The logarithm function has the following useful arithmetic properties.

Properties of Logarithms

For any real numbers $x > 0$ and $y > 0$,

1. *Product Rule:* $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:* $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Power Rule:* $\log_a x^y = y \log_a x$

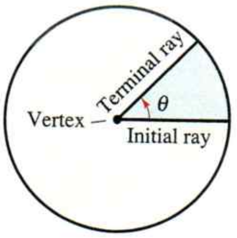


FIGURE 58
Standard position of an angle.

Angles and Degree Measure

An **angle** has three parts: an **initial ray,** a **terminal ray,** and a **vertex** (the point of intersection of the two rays), as shown in Figure 58. An angle is in **standard position** if its initial ray coincides with the positive x-axis and its vertex is at the origin. We assume that you are familiar with the degree measure of an angle.* It is common practice to use θ (the Greek lowercase letter *theta*) to represent both an angle and its measure. Angles between 0° and 90° are **acute** and angles between 90° and 180° are **obtuse.**

Positive angles are measured *counterclockwise,* and negative angles are measured *clockwise.* For instance, Figure 59 shows an angle whose measure is $-45°$. You cannot assign a measure to an angle by simply knowing where its initial and terminal rays are located. To measure an angle, you must also know how the terminal ray was revolved. For example, Figure 59 shows that the angle measuring $-45°$ has the same terminal ray as the angle measuring 315°. Such angles are **coterminal.** In general, if θ is any angle, then

$$\theta + n(360), \qquad n \text{ is a nonzero integer}$$

is coterminal with θ.

An angle that is larger than 360° is one whose terminal ray has been revolved more than one full revolution counterclockwise, as shown in Figure 47. You can form an angle whose measure is less than $-360°$ by revolving a terminal ray more than one full revolution clockwise.

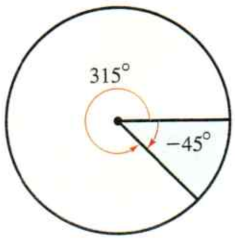


FIGURE 59
Coterminal angles

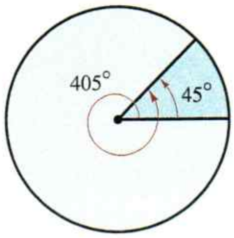


FIGURE 60
Coterminal angles

REMARK It is common to use the symbol θ to refer to both an *angle* and its *measure.* For instance, in Figure 60, you can write the measure of the smaller angle as $\theta = 45°$.

Radian Measure

To assign a radian measure to an angle θ, consider θ to be a central angle of a circle of radius 1, as shown in Figure 61. The **radian measure** of θ is then defined to be the length of the arc of the sector. Because the circumference of a circle is $2\pi r$, the circumference of a **unit circle** (of radius 1) is 2π. This implies that the radian measure of an angle measuring $360°$ is 2π. In other words, $360° = 2\pi$ radians.

Using radian measure for θ, the length s of a circular arc of radius r is $s = r\theta$, as shown in Figure 62.

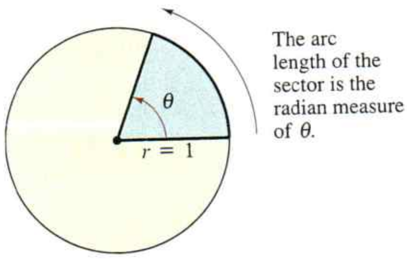


FIGURE 61
Unit circle

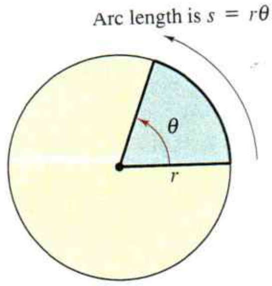


FIGURE 62
Circle of radius r

You should know the conversions of the common angles shown in Figure 63. For other angles, use the fact that $180°$ is equal to π radians.

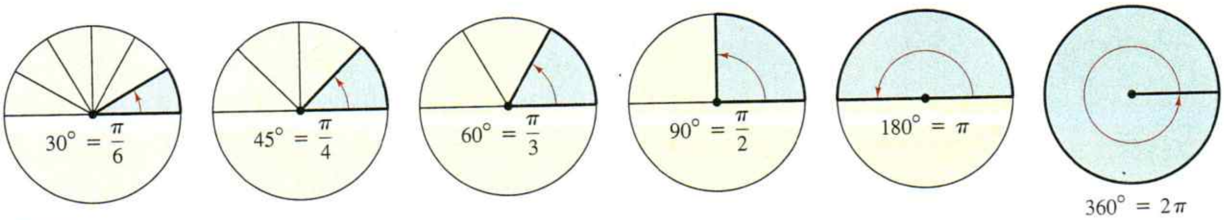


FIGURE 63
Radian and degree measure for several common angles.

EXAMPLE 1 Conversions Between Degrees and Radians

a. $40° = (40\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = \dfrac{2\pi}{9}$ radians

b. $-270° = (-270\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = -\dfrac{3\pi}{2}$ radians

c. $-\dfrac{\pi}{2}$ radians $= \left(-\dfrac{\pi}{2}\ \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\cancel{\pi}\ \cancel{\text{rad}}}\right) = -90°$

d. $\dfrac{9\pi}{2}$ radians $= \left(\dfrac{9\pi}{2}\ \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\pi\ \cancel{\text{rad}}}\right) = 810°$

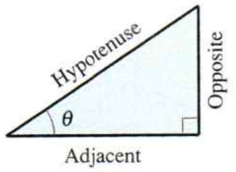


FIGURE 64
Sides of a right triangle.

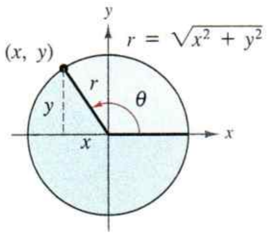


FIGURE 65
An angle in standard position.

The Trigonometric Functions

There are two common approaches to the study of trigonometry. In one, the trigonometric functions are defined as ratios of two sides of a right triangle. In the other, these functions are defined in terms of a point on the terminal side of an angle in standard position. We define the six trigonometric functions, **sine, cosine, tangent, cotangent, secant,** and **cosecant** (abbreviated as sin, cos, etc.), from both viewpoints.

Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \frac{\pi}{2}$ (see Figure 64).

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \cos\theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \tan\theta = \frac{\text{opp.}}{\text{adj.}}$$

$$\csc\theta = \frac{\text{hyp.}}{\text{opp.}} \qquad \sec\theta = \frac{\text{hyp.}}{\text{adj.}} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}}$$

Circular function definitions, where θ is any angle (see Figure 65).

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

The following trigonometric identities are direct consequences of the definitions. (ϕ is the Greek letter *phi.*)

Trigonometric Identities [*Note that* $\sin^2\theta$ *is used to represent* $(\sin\theta)^2$.]

Pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Reduction formulas:

$$\sin(-\theta) = -\sin\theta \qquad \sin\theta = -\sin(\theta - \pi)$$
$$\cos(-\theta) = \cos\theta \qquad \cos\theta = -\cos(\theta - \pi)$$
$$\tan(-\theta) = -\tan\theta \qquad \tan\theta = \tan(\theta - \pi)$$

Sum or difference of two angles:

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Half-angle formulas:

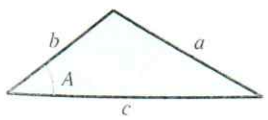
$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$
$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

Double angle formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$
$$= \cos^2\theta - \sin^2\theta$$

Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Law of Cosines

Reciprocal identities:

$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

Quotient identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Evaluating Trigonometric Functions

There are two ways to evaluate trigonometric functions: (1) decimal approximations with a calculator (or a table of trigonometric values) and (2) exact evaluations using trigonometric identities and formulas from geometry. When using a calculator to evaluate a trigonometric function, remember to set the calculator to the appropriate mode—degree mode or radian mode.

EXAMPLE 2 Exact Evaluation of Trigonometric Functions

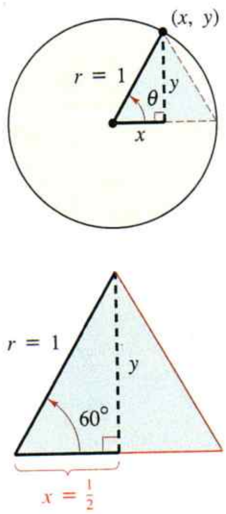


FIGURE 66
The angle $\pi/3$ in standard position.

Evaluate the sine, cosine, and tangent of $\dfrac{\pi}{3}$.

Solution Begin by drawing the angle $\theta = \pi/3$ in the standard position, as shown in Figure 66. Then, because $60^\circ = \pi/3$ radians, you can draw an equilateral triangle with sides of length 1 and θ as one of its angles. Because the altitude of this triangle bisects its base, you know that $x = \frac{1}{2}$. Using the Pythagorean Theorem, you obtain

$$y = \sqrt{r^2 - x^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Now, knowing the values of x, y, and r, you can write the following.

$$\sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2}$$

$$\tan\frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

REMARK All angles in the remainder of this text are measured in radians unless stated otherwise. For example, when we write sin 3, we mean the sine of three radians, and when we write sin 3°, we mean the sine of three degrees.

The degree and radian measures of several common angles are given in Table 4, along with the corresponding values of the sine, cosine, and tangent (see Figure 67).

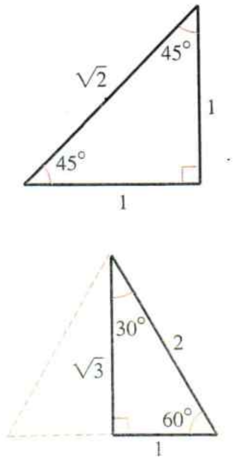


FIGURE 67
Common angles.

TABLE 4
Common First Quadrant Angles

Degrees	0	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

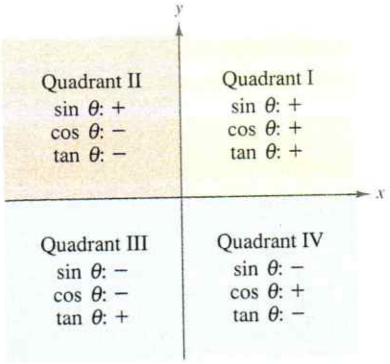


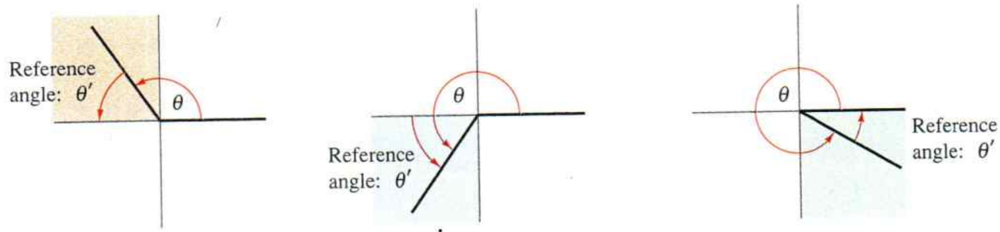
FIGURE 68
Quadrant signs for trigonometric functions.

The quadrant signs of the sine, cosine, and tangent functions are shown in Figure 68. To extend the use of Table 4 to angles in quadrants other than the first quadrant, you can use the concept of a **reference angle** (see Figure 69), with the appropriate quadrant sign. For instance, the reference angle for $3\pi/4$ is $\pi/4$, and because the sine is positive in the second quadrant, you can write

$$\sin\frac{3\pi}{4} = +\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Similarly, because the reference angle for 330° is 30°, and the tangent is negative in the fourth quadrant, you can write

$$\tan 330° = -\tan 30° = -\frac{\sqrt{3}}{3}.$$



Quadrant II
$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)

Quadrant III
$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)

Quadrant IV
$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

FIGURE 69

EXAMPLE 3 Trigonometric Identities and Calculators

Evaluate the trigonometric expression.

a. $\sin\left(-\frac{\pi}{3}\right)$ **b.** $\sec 60°$ **c.** $\cos(1.2)$

Solution

a. Using the reduction formula $\sin(-\theta) = -\sin\theta$, you can write

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

b. Using the reciprocal identity $\sec\theta = 1/\cos\theta$, you can write

$$\sec 60° = \frac{1}{\cos 60°} = \frac{1}{1/2} = 2.$$

c. Using a calculator, you can obtain

$$\cos(1.2) \approx 0.3624.$$

Remember that 1.2 is given in *radian* measure. Consequently, your calculator must be set in radian mode.

Solving Trigonometric Equations

How would you solve the equation $\sin\theta = 0$? You know that $\theta = 0$ is one solution, but this is not the only solution. Any one of the following values of θ is also a solution.

$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$

You can write this infinite solution set as $\{n\pi : n \text{ is an integer}\}$.

EXAMPLE 4 Solving a Trigonometric Equation

Solve the equation

$$\sin\theta = -\frac{\sqrt{3}}{2}.$$

Solution To solve the equation, you should consider that the sine is negative in Quadrants III and IV and that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

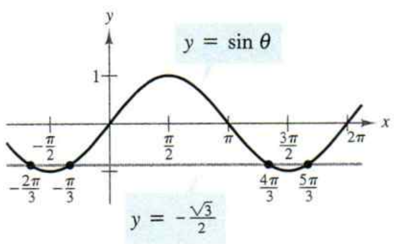


FIGURE 70

Solution points of $\sin\theta = \frac{-\sqrt{3}}{2}$.

Thus, you are seeking values of θ in the third and fourth quadrants that have a reference angle of $\pi/3$. In the interval $[0, 2\pi]$, the two angles fitting these criteria are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{and} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

By adding integer multiples of 2π to each of these solutions, you obtain the following general solution.

$$\theta = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2n\pi, \qquad \text{where } n \text{ is an integer.}$$

(See Figure 70.)

EXAMPLE 5 Solving a Trigonometric Equation

Solve $\cos 2\theta = 2 - 3\sin\theta$, where $0 \le \theta \le 2\pi$.

Solution Using the double angle identity $\cos 2\theta = 1 - 2\sin^2\theta$, you can rewrite the equation as follows.

$$\begin{aligned} \cos 2\theta &= 2 - 3\sin\theta && \text{Given equation} \\ 1 - 2\sin^2\theta &= 2 - 3\sin\theta && \text{Trigonometric identity} \\ 0 &= 2\sin^2\theta - 3\sin\theta + 1 && \text{Quadratic form} \\ 0 &= (2\sin\theta - 1)(\sin\theta - 1) && \text{Factor} \end{aligned}$$

If $2\sin\theta - 1 = 0$, then $\sin\theta = 1/2$ and $\theta = \pi/6$ or $\theta = 5\pi/6$. If $\sin\theta - 1 = 0$, then $\sin\theta = 1$ and $\theta = \pi/2$. Thus, for $0 \le \theta \le 2\pi$, there are three solutions.

$$\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \text{and} \quad \frac{\pi}{2}$$

Graphs of Trigonometric Functions

A function f is **periodic** if there exists a nonzero number p such that $f(x + p) = f(x)$ for all x in the domain of f. The smallest such positive value of p (if it exists) is the **period** of f. The sine, cosine, secant, and cosecant functions each have a period of 2π, and the other two trigonometric functions have a period of π, as shown in Figure 71.

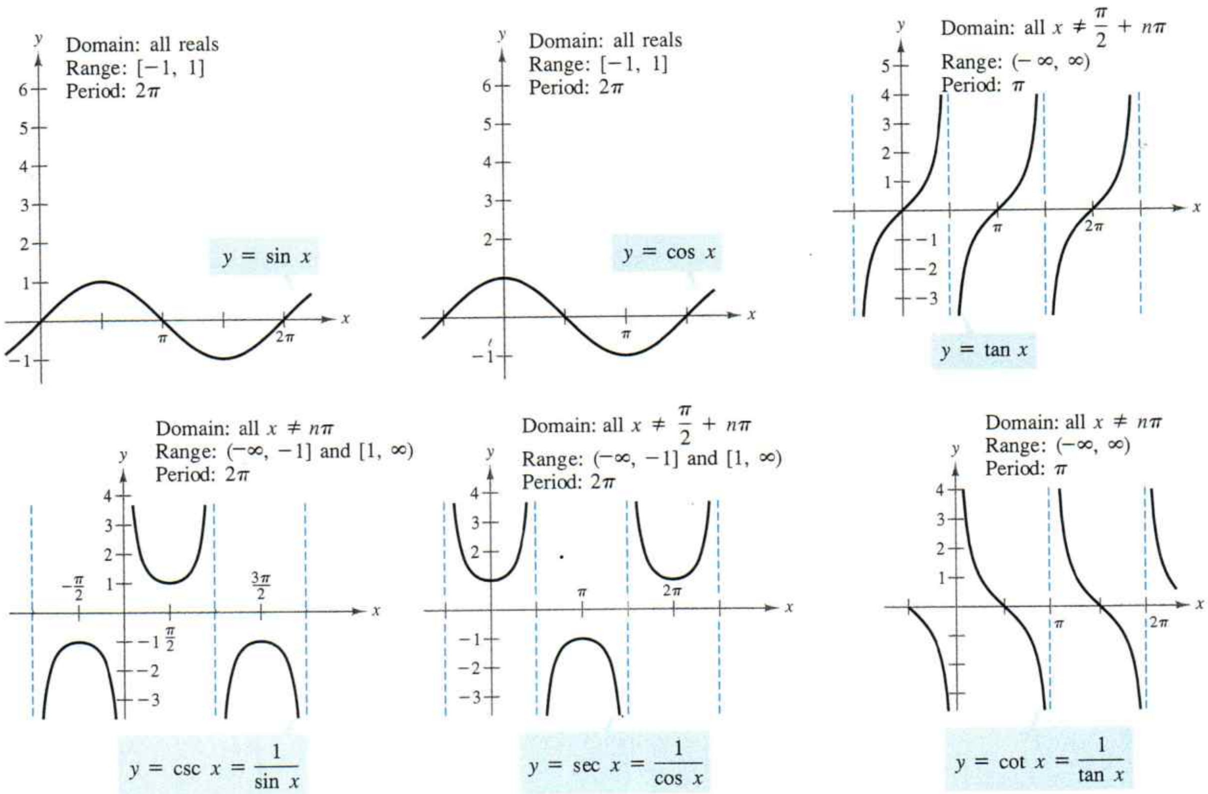


FIGURE 71
The graphs of the six trigonometric functions.

Note in Figure 71 that the maximum value of $\sin x$ and $\cos x$ is 1 and the minimum value is -1. The graphs of the functions $y = a \sin bx$ and $y = a\cos bx$ oscillate between $-a$ and a, and hence have an **amplitude** of $|a|$. Furthermore, because $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, it follows that the functions $y = a\sin bx$ and $y = a\cos bx$ each have a period of $2\pi/|b|$. Table 5 summarizes the amplitudes and periods for some types of trigonometric functions.

TABLE 5

Function	*Period*	*Amplitude*
$y = a\sin bx$ or $y = a\cos bx$	$\dfrac{2\pi}{\lvert b\rvert}$	$\lvert a\rvert$
$y = a\tan bx$ or $y = a\cot bx$	$\dfrac{\pi}{\lvert b\rvert}$	Not applicable
$y = a\sec bx$ or $y = a\csc bx$	$\dfrac{2\pi}{\lvert b\rvert}$	Not applicable

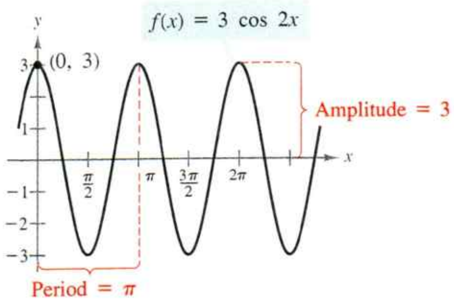


FIGURE 72

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Sketch the graph of $f(x) = 3\cos 2x$.

Solution The graph of $f(x) = 3\cos 2x$ has an amplitude of 3 and a period of $2\pi/2 = \pi$. Using the basic shape of the graph of the cosine function, sketch one period of the function on the interval $[0, \pi]$, using the following pattern.

Maximum: $(0, 3)$ Minimum: $\left(\frac{\pi}{2}, -3\right)$ Maximum: $[\pi, 3]$

By continuing this pattern, you can sketch several cycles of the graph, as shown in Figure 72.

The discussion of horizontal shifts, vertical shifts, and reflections given in Section 5 can be applied to the graphs of trigonometric functions, as illustrated in Example 7.

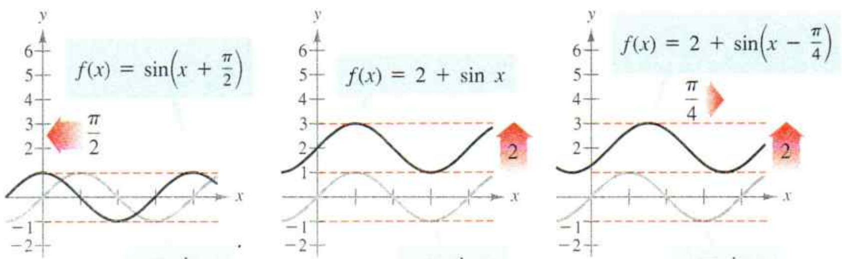
EXAMPLE 7 Shifts of Graphs of Trigonometric Functions

Sketch the graph of the following functions.

a. $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ **b.** $f(x) = 2 + \sin x$ **c.** $f(x) = 2 + \sin\left(x - \frac{\pi}{4}\right)$

Solution

a. To sketch the graph of $f(x) = \sin(x + \pi/2)$, shift the graph of $y = \sin x$ to the left $\pi/2$ units, as shown in Figure 73(a).
b. To sketch the graph of $f(x) = 2 + \sin x$, shift the graph of $y = \sin x$ up 2 units, as shown in Figure 73(b).
c. To sketch the graph of $f(x) = 2 + \sin(x - \pi/4)$, shift the graph of $y = \sin x$ up 2 units and to the right $\pi/4$ units, as shown in Figure 73(c).



(a) Horizontal shift to the left (b) Vertical shift upward (c) Horizontal and vertical shift

FIGURE 73
Transformations of the graph of $y = \sin x$.

These resources were taken from the following sources:

Finney, R. L., Demana, F. D., Waits, B. K., & Kennedy, D. (2012). *Calculus: Graphical, Numerical, Algebraic* (4th ed.). Boston: Prentice Hall.

Larson, R. Hostetler, R. P., & Edwards, B. H. (2006). *Calculus of a Single Variable* (8th ed.). Boston: Houghton Mifflin.