# SUMMER REVIEW TOPICS
# FOR
# AP CALCULUS

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Class: Ap Calculus

The Advanced Placement Calculus course is meant for the most capable mathematical minds and for talented students who challenge each other in the pursuit of excellence. The course demands analytical reasoning, as well as skilled and disciplined study habits appropriate for continued success at the college level.

The following topics are prerequisites for this course they must be thoroughly reviewed during the summer:

Function and Graphs
- a. Linear Functions and their properties
- b. Absolute Value Functions and their properties
- c. Polynomial Functions and their properties
- d. Rational Functions and graphing
- e. Exponential and Logorithmic Functions and their properties

Trigonometry
- a. Trigonometric Functions and their graphing
- b. Compositions of Functions and graphing
- c. Application of Trigonometry

Analytical Trigonometry
- a. Application and Verification of Fundamental Identities
- b. Solving Trigonometric Equations
- c. Sum and Difference Identities
- d. Multiple Angle Identities

Application of Analytical Trigonometry
- a. Law of Sines
- b. Law of Cosines
- c. Vectors
- d. DeMoivre's Theorem and $n^{th}$Roots
- e. Trigonometric Form of Complex Numbers

Systems of Linear Equations in more than two variables
  Matrices and Determinants
  The Binomial Theorem
  Sequence and Sigma Notation

Conics
- a. Parabolas
- b. Ellipses
- c. Hyperbole

A comprehensive test will be given in the first week of snhool to evaluate the students performance as needed in this course.

44. Let $f(x) = (x^3 + 1)/x$. Show that

$$\lim_{x \to \pm\infty} [f(x) - x^2] = 0$$

This shows that the graph of $f$ approaches the graph of $y = x^2$, and we say that the curve $y = f(x)$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of $f$.

45. Discuss the asymptotic behavior of $f(x) = (x^4 + 1)/x$ in the same manner as in Exercise 44. Then use your results to help sketch the graph of $f$.

46. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketch its graph without going through the curve-sketching procedure of this section.

## 3.7 GRAPHING WITH CALCULUS AND CALCULATORS

If you have not already read Section 3 of Review and Preview, you should do so now. In particular, it explains how to avoid some of the pitfalls of graphing devices by choosing appropriate viewing rectangles.

The method we used to sketch curves in the preceding section was a culmination of much of our study of differential calculus. The graph was the final object that we produced. In this section our point of view is completely different. Here we *start* with a graph produced by a graphing calculator or computer and then we refine it. We use calculus to make sure that we reveal all the important aspects of the curve. And with the use of graphing devices we can tackle curves that would be far too complicated to consider without technology. The theme is the *interaction* between calculus and calculators.

EXAMPLE 1 Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of $f'$ and $f''$ to locate all maximum and minimum points and intervals of concavity correct to one decimal place.

SOLUTION If we specify a domain but not a range, many graphing devices will deduce a suitable range from the values computed. Figure 1 shows the plot from one such device if we specify that $-5 \leqslant x \leqslant 5$. Although this viewing rectangle is useful for showing that the asymptotic behavior (or end behavior) is the same as for $y = 2x^6$, it is obviously hiding some finer detail. So we change to the viewing rectangle $[-3, 2]$ by $[-50, 100]$ shown in Figure 2.

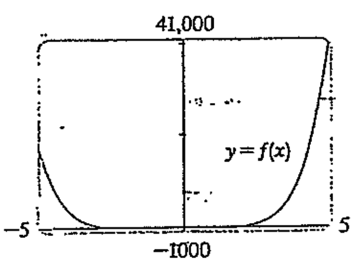


FIGURE 1

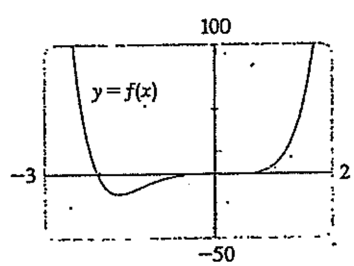


FIGURE 2

From this graph it appears that there is an absolute minimum value of about $-15.3$ when $x \approx -1.6$ (by using the cursor) and $f$ is decreasing on $(-\infty, -1.6)$ and increasing on $(-1.6, \infty)$. Also there appears to be a horizontal tangent at the origin and inflection points when $x = 0$ and when $x$ is somewhere between $-2$ and $-1$.

Now let us try to confirm these impressions using calculus. We differentiate and get

$$f'(x) = 12x^5 + 15x^4 + 9x^2 - 4x \qquad f''(x) = 60x^4 + 60x^3 + 18x - 4$$

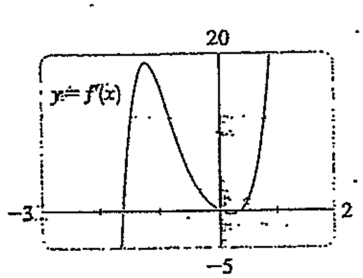


FIGURE 3

When we graph $f'$ in Figure 3 we see that $f'(x)$ changes from negative to positive when $x \approx -1.6$; this confirms (by the First Derivative Test) the minimum value that

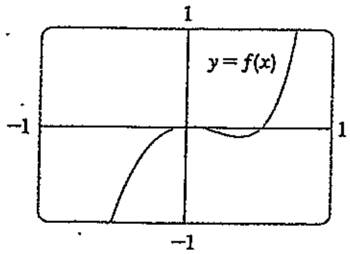


FIGURE 4

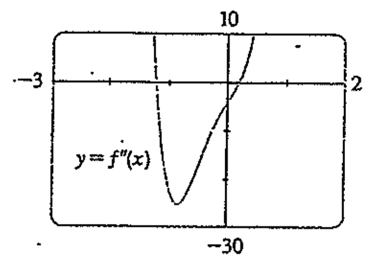


FIGURE 5

we found earlier. But, perhaps to our surprise, we also notice that $f'(x)$ changes from positive to negative when $x = 0$ and from negative to positive when $x \approx 0.35$. This means that $f$ has a local maximum at 0 and a local minimum when $x \approx 0.35$, but these were hidden in Figure 2. Indeed if we now zoom in toward the origin in Figure 4, we see what we missed before: a local maximum value of 0 when $x = 0$ and a local minimum value of about $-0.1$ when $x \approx 0.35$.

What about concavity and inflection points? From Figures 2 and 4 there appear to be inflection points when $x$ is a little to the left of $-1$ and when $x$ is a little to the right of 0. But it is difficult to determine inflection points from the graph of $f$, so we graph the second derivative $f''$ in Figure 5. We see that $f''$ changes from positive to negative when $x \approx -1.2$ and from negative to positive when $x \approx 0.2$. So, correct to one decimal place, $f$ is concave upward on $(-\infty, -1.2)$ and $(0.2, \infty)$ and concave downward on $(-1.2, 0.2)$. The inflection points are $(-1.2, -9.8)$ and $(0.2, -0.05)$.

We have discovered that no single graph reveals all the important features of this polynomial. But Figures 2 and 4, when taken together, do provide an accurate picture. ∎

EXAMPLE 2 Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that contains all the important features of the function. Estimate the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

SOLUTION Figure 6, produced by a computer with automatic scaling, is a disaster. Some graphing calculators use $[-10, 10]$ by $[-10, 10]$ as the default viewing rectangle, so let's try it. We get the graph shown in Figure 7; it's a major improvement.

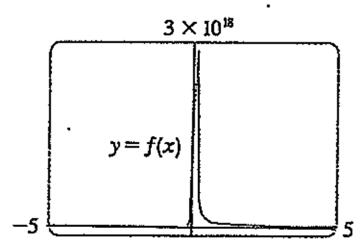


FIGURE 6

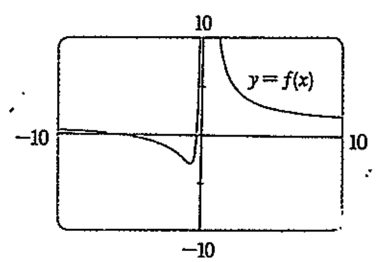


FIGURE 7

The $y$-axis appears to be a vertical asymptote and indeed it is because

$$\lim_{x \to 0} \frac{x^2 + 7x + 3}{x^2} = \infty$$

Figure 7 also allows us to estimate the $x$-intercepts: about $-0.5$ and $-6.5$. The exact values are obtained by using the quadratic formula to solve the equation $x^2 + 7x + 3 = 0$; we get $x = (-7 \pm \sqrt{37})/2$.

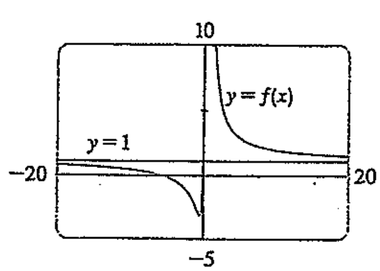


FIGURE 8

To get a better look at horizontal asymptotes we change to the viewing rectangle $[-20, 20]$ by $[-5, 10]$ in Figure 8. It appears that $y = 1$ is the horizontal asymptote and this is easily confirmed:

$$\lim_{x \to \pm\infty} \frac{x^2 + 7x + 3}{x^2} = \lim_{x \to \pm\infty} \left(1 + \frac{7}{x} + \frac{3}{x^2}\right) = 1$$

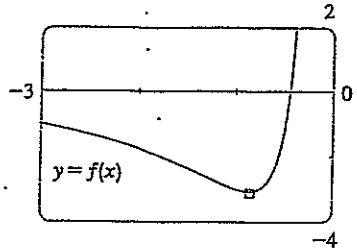


FIGURE 9

To estimate the minimum value we zoom in to the viewing rectangle $[-3, 0]$ by $[-4, 2]$ in Figure 9. The cursor indicates that the absolute minimum value is about $-3.1$ when $x \approx -0.9$ and we see that the function decreases on $(-\infty, -0.9)$ and $(0, \infty)$ and increases on $(-0.9, 0)$. The exact values are obtained by differentiating:

$$f'(x) = -\frac{7}{x^2} - \frac{6}{x^3} = -\frac{7x + 6}{x^3}$$

This shows that $f'(x) > 0$ when $-\frac{6}{7} < x < 0$ and $f'(x) < 0$ when $x < -\frac{6}{7}$ and when $x > 0$. The exact minimum value is $f(-\frac{6}{7}) = -\frac{111}{36} \approx -3.08$.

Figure 9 also shows that an inflection point occurs somewhere between $x = -1$ and $x = -2$. We could estimate it much more accurately using the graph of the second derivative, but in this case it is just as easy to find exact values. Since

$$f''(x) = \frac{14}{x^3} + \frac{18}{x^4} = 2\frac{7x + 9}{x^4}$$

we see that $f''(x) > 0$ when $x > -\frac{9}{7}$ $(x \neq 0)$. So $f$ is concave upward on $(-\frac{9}{7}, 0)$ and $(0, \infty)$ and concave downward on $(-\infty, -\frac{9}{7})$. The inflection point is $(-\frac{9}{7}, -\frac{71}{27})$.

The analysis using the first two derivatives shows that Figures 7 and 8 display all the major aspects of the curve. ■

EXAMPLE 3 Graph the function $f(x) = \dfrac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4}$.

SOLUTION Drawing on our experience with a rational function in Example 2, let's start by graphing $f$ in the viewing rectangle $[-10, 10]$ by $[-10, 10]$. From Figure 10 we have the feeling that we are going to have to zoom in to see some finer detail and also to zoom out to see the larger picture. But, as a guide to intelligent zooming, let's first take a close look at the expression for $f(x)$. Because of the factors $(x - 2)^2$ and $(x - 4)^4$ in the denominator we expect $x = 2$ and $x = 4$ to be the vertical asymptotes. Indeed

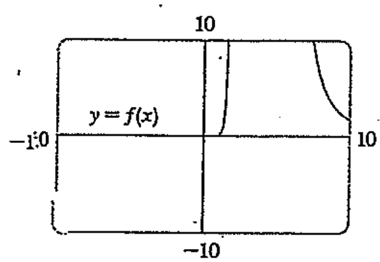


FIGURE 10

$$\lim_{x \to 2} \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \infty \qquad \text{and} \qquad \lim_{x \to 4} \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \infty$$

To find the horizontal asymptotes we divide numerator and denominator by $x^6$:

$$\frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \frac{\dfrac{1}{x}\left(1 + \dfrac{1}{x}\right)^3}{\left(1 - \dfrac{2}{x}\right)^2\left(1 - \dfrac{4}{x}\right)^4} \to 0 \quad \text{as} \quad x \to \pm\infty$$

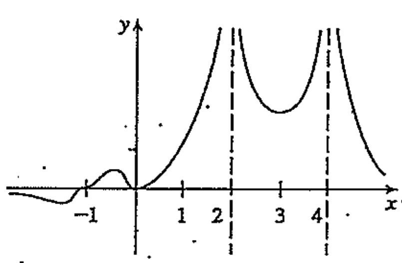


FIGURE 11

so the $x$-axis is the horizontal asymptote.

It is also very useful to consider the behavior of the graph near the $x$-intercepts using an analysis like that in Example 9 in Section 3.5. Since $x^2$ is positive, $f(x)$ does not change sign at 0 and so its graph doesn't cross the $x$-axis at 0. But, because of the factor $(x + 1)^3$, the graph does cross the $x$-axis at $-1$ and has a horizontal tangent there. Putting all this information together, but without using derivatives, we see that the curve has to look something like the one in Figure 11.

Now that we know what to look for, we zoom in (several times) to produce the graphs in Figures 12 and 13 and zoom out (several times) to get Figure 14.

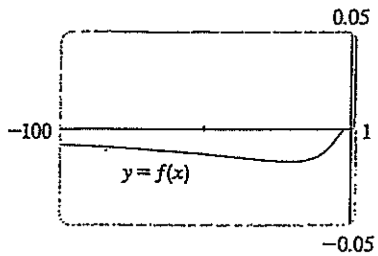


FIGURE 12

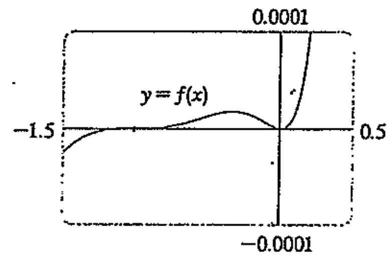


FIGURE 13

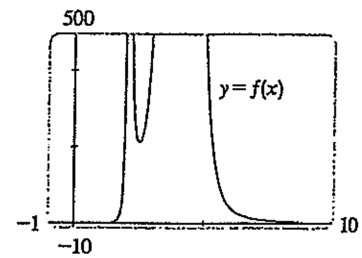


FIGURE 14

We can read from these graphs that the absolute minimum is about $-0.02$ and occurs when $x \approx -20$. There is also a local maximum $\approx 0.00002$ when $x \approx -0.3$ and a local minimum $\approx 211$ when $x \approx 2.5$. These graphs also show two inflection points near $-5$ and $-1$ and two between $-1$ and 0. To estimate the inflection points closely we would need to graph $f''$, but to compute $f''$ by hand is an unreasonable chore. If you have a computer algebra system, then it is easy (see Exercise 13).

We have seen that, for this particular function, *three* graphs (Figures 12, 13, and 14) are necessary to convey all the useful information. The only way to display all these features of the function on a single graph is to draw it by hand. Despite the exaggerations and distortions, Figure 11 does manage to summarize the essential nature of the function. ∎

The family of functions

$$f(x) = \sin(x + \sin cx)$$

where $c$ is a constant, occurs in applications to frequency modulation (FM) synthesis. A sine wave is modulated by a wave with a different frequency ($\sin cx$). The case where $c = 2$ is studied in Example 4. Exercise 15 explores another special case.

EXAMPLE 4 Graph the function $f(x) = \sin(x + \sin 2x)$. For $0 \leqslant x \leqslant \pi$, locate all maximum and minimum values, intervals of increase and decrease, and inflection points correct to one decimal place.

SOLUTION We first note that $f$ is periodic with period $2\pi$. Also, $f$ is odd and $|f(x)| \leqslant 1$ for all $x$. So the choice of a viewing rectangle is not a problem for this function: we start with $[0, \pi]$ by $[-1.1, 1.1]$ (see Figure 15). It appears that there are three local maximum values and two local minimum values in that window. To confirm this and locate them more accurately, we calculate that

$$f'(x) = \cos(x + \sin 2x) \cdot (1 + 2\cos 2x)$$

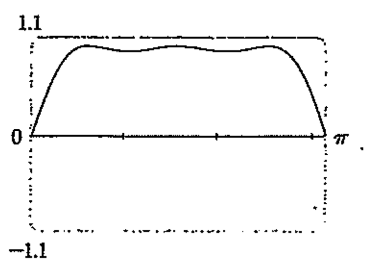


FIGURE 15

and graph both $f$ and $f'$ in Figure 16. Using zoom-in and the First Derivative Test, we find the following values to one decimal place.

| | |
|---|---|
| Intervals of increase: | $(0, 0.6), (1.0, 1.6), (2.1, 2.5)$ |
| Intervals of decrease: | $(0.6, 1.0), (1.6, 2.1), (2.5, \pi)$ |
| Local maximum values: | $f(0.6) \approx 1, f(1.6) \approx 1, f(2.5) \approx 1$ |
| Local minimum values: | $f(1.0) \approx 0.94, f(2.1) \approx 0.94$ |

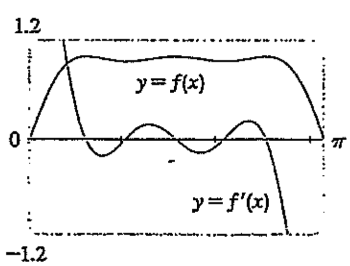


FIGURE 16

The second derivative is

$$f''(x) = -(1 + 2\cos 2x)^2 \sin(x + \sin 2x) - 4\sin 2x \cos(x + \sin 2x)$$

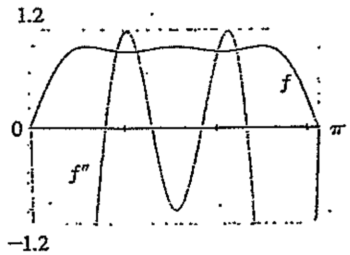


FIGURE 17

Graphing both $f$ and $f''$ in Figure 17, we obtain the following approximate values:

| | |
|---|---|
| Concave upward on: | $(0.8, 1.3), (1.8, 2.3)$ |
| Concave downward on: | $(0, 0.8), (1.3, 1.8), (2.3, \pi)$ |
| Inflection points: | $(0, 0), (0.8, 0.97), (1.3, 0.97), (1.8, 0.97), (2.3, 0.97)$ |

Having checked that Figure 15 does indeed represent $f$ accurately for $0 \leq x \leq \pi$, we can state that the extended graph in Figure 18 represents $f$ accurately for $-2\pi \leq x \leq 2\pi$.

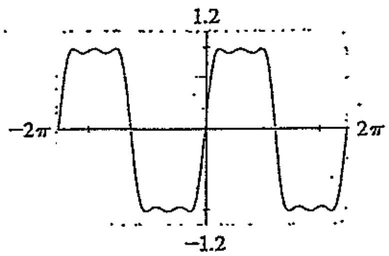


FIGURE 18

NOTE: Examples 1 and 4 would have been almost impossible to do without a graphing calculator or computer. For instance, in Example 1 we would have had to use Newton's method twice to find the roots of $f'(x) = 0$ and then twice more to solve $f''(x) = 0$. In Example 4 we would have had to use Newton's method a total of nine times.

Our final example is concerned with *families* of functions. This means that the functions in the family are related to each other by a formula that contains one or more arbitrary constants. Each value of the constant gives rise to a member of the family and the idea is to see how the graph of the function changes as the constant changes.

EXAMPLE 5 How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as $c$ varies?

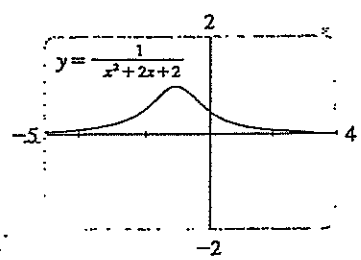


FIGURE 19
$c = 2$

SOLUTION The graphs in Figures 19 and 20 (the special cases $c = 2$ and $c = -2$) show two very different-looking curves. Before drawing any more graphs, let's see what members of this family have in common. Since

$$\lim_{x \to \pm\infty} \frac{1}{x^2 + 2x + c} = 0$$

for any value of $c$, they all have the $x$-axis as a horizontal asymptote. A vertical asymptote will occur when $x^2 + 2x + c = 0$. Solving this quadratic equation, we get $x = -1 \pm \sqrt{1 - c}$. When $c > 1$, there is no vertical asymptote (as in Figure 19). When $c = 1$ the graph has a single vertical asymptote $x = -1$ because

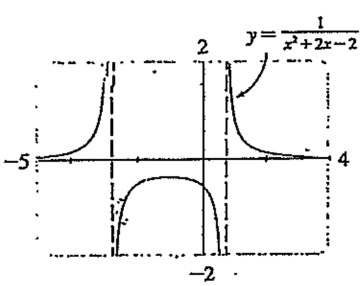


FIGURE 20
$c = -2$

$$\lim_{x \to -1} \frac{1}{x^2 + 2x + 1} = \lim_{x \to -1} \frac{1}{(x + 1)^2} = \infty$$

When $c < 1$ there are two vertical asymptotes: $x = -1 + \sqrt{1 - c}$ and $x = -1 - \sqrt{1 - c}$ (as in Figure 20).

Now we compute the derivative:

$$f'(x) = -\frac{2x + 2}{(x^2 + 2x + c)^2}$$

This shows that $f'(x) = 0$ when $x = -1$ (if $c \neq 1$), $f'(x) > 0$ when $x < -1$, and $f'(x) < 0$ when $x > -1$. For $c \geq 1$ this means that $f$ increases on $(-\infty, -1)$ and decreases on $(-1, \infty)$. For $c > 1$, there is an absolute maximum value $f(-1) = 1/(c - 1)$. For $c < 1$, $f(-1) = 1/(c - 1)$ is a local maximum value and the intervals of increase and decrease are interrupted at the vertical asymptotes.

Figure 21 is a "slide show" displaying five members of the family, all graphed in the viewing rectangle $[-5, 4]$ by $[-2, 2]$. As predicted, $c = 1$ is the value at which a transition takes place from two vertical asymptotes to one, and then to none. As $c$ increases from 1, we see that the maximum point becomes lower; this is explained by the fact that $1/(c-1) \to 0$ as $c \to \infty$. As $c$ decreases from 1, the vertical asymptotes become more widely separated because the distance between them is $2\sqrt{1-c}$, which becomes large as $c \to -\infty$. Again the maximum point approaches the $x$-axis because $1/(c-1) \to 0$ as $c \to -\infty$.

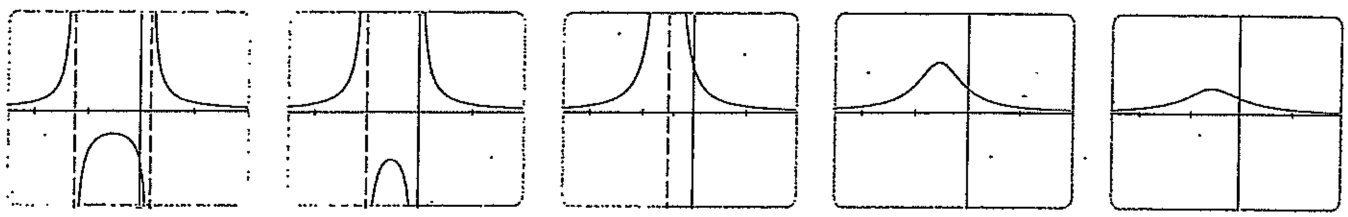


FIGURE 21
The family of functions
$f(x) = \dfrac{1}{x^2 + 2x + c}$

There is clearly no inflection point when $c \leqslant 1$. For $c > 1$ we calculate that

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

and deduce that inflection points occur when $x = -1 \pm \sqrt{3(c-1)}/3$. So the inflection points become more spread out as $c$ increases and this seems plausible from the last two parts of Figure 21. ◼

## EXERCISES 3.7

1–6 ◼ Produce graphs of $f$ that reveal all the important aspects of the curve. In particular, you should use graphs of $f'$ and $f''$ to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

1. $f(x) = 4x^4 - 7x^2 + 4x + 6$
2. $f(x) = 8x^5 + 45x^4 + 80x^3 + 90x^2 + 200x$
3. $f(x) = \sqrt[3]{x^2 - 3x - 5}$
4. $f(x) = \dfrac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2}$
5. $f(x) = x^2 \sin x, \ -7 \leqslant x \leqslant 7$
6. $f(x) = \sin x + \frac{1}{3}\sin 3x$

7–10 ◼ Produce graphs of $f$ that reveal all the important aspects of the curve. Estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points, and use calculus to find these quantities exactly.

7. $f(x) = 8x^3 - 3x^2 - 10$
8. $f(x) = \dfrac{x^2 + 11x - 20}{x^2}$
9. $f(x) = x\sqrt{9 - x^2}$
10. $f(x) = x - 2\sin x, \ -2\pi \leqslant x \leqslant 2\pi$

11–12 ◼ Sketch the graph by hand using asymptotes and intercepts, but not derivatives. Then use your sketch as a guide to producing graphs (with a graphing device) that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

11. $f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$
12. $f(x) = \dfrac{10x(x-1)^4}{(x-2)^3(x+1)^2}$

CAS 13. If $f$ is the function considered in Example 3, use a computer algebra system to calculate $f'$ and then graph it to confirm that all the maximum and minimum values are as given in the example. Calculate $f''$ and use it to estimate the intervals of concavity and inflection points.

CAS 14. If $f$ is the function of Exercise 12, find $f'$ and $f''$ and use their graphs to estimate the intervals of increase and decrease and concavity of $f$.

15. In Example 4 we considered a member of the family of functions $f(x) = \sin(x + \sin cx)$ that occur in FM synthesis. Here we investigate the function with $c = 3$. Start by graphing $f$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$. How many local maximum points do you see? The graph has more than are visible to the naked eye. To discover the hidden maximum and minimum points you will need to examine the graph of $f'$ very carefully. In