NCDPI Unpacked Content

Discrete Math for Computer Science

Aligned to $\mathrm{NC}^2\mathrm{ML}$’s Instructional Framework Recommended Order

2022 Alignment

North Carolina Department of Public Instruction

INSTRUCTIONAL SUPPORT TOOLS

FOR ACHIEVING NEW STANDARDS

Discrete Mathematics for Computer Science ● Unpacked Contents
For the new Standard Course of Study that will be effective in all North Carolina schools in the 2020-21 School Year.

This document is designed to help North Carolina educators teach **Discrete Mathematics for Computer Science** Standard Course of Study. NCDPI staff are continually updating and improving these tools to better serve teachers and districts.

What is the purpose of this document?
The purpose of this document is to increase student achievement by ensuring educators understand the expectations of the new standards. This document may also be used to facilitate discussion among teachers and curriculum staff and to encourage coherence in the sequence, pacing, and units of study for grade-level curricula. This document, along with on-going professional development, is one of many resources used to understand and teach the NC SCOS.

What is in the document?
This document includes a detailed clarification of each standard in the grade level along with a *sample* of questions or directions that may be used during the instructional sequence to determine whether students are meeting the learning objective outlined by the standard. These items are included to support classroom instruction and are not intended to reflect summative assessment items. The examples included may not fully address the scope of the standard. The document also includes a table of contents of the standards organized by domain with hyperlinks to assist in navigating the electronic version of this instructional support tool.

How do I send Feedback?
Link for: Feedback for NC's Unpacking Documents.
We will use your input to refine our unpacking of the standards. Thank You!

Just want the standards alone?
Link to: North Carolina Mathematics Standards

Discrete Mathematics for Computer Science Standards

Number & Quantity	Functions	Statistics & Probability	Graph Theory	Logic
DCS.N.1 *Apply operations with matrices and vectors.* DCS.N.1.1 DCS.N.1.2 DCS.N.1.3	***DCS.F.1*** *Apply recursively-defined relationships to solve problems.* DCS.F.1.1 DCS.F.1.2 DCS.F.1.3 DCS.F.1.4 DCS.F.1.5	***DCS.SP.1*** *Apply combinatorics concepts to solve problems.* DCS.SP.1.1 DCS.SP.1.2	***DCS.GT.1*** *Understand graph theory to model relationships and solve problems.* DCS.GT.1.1 DCS.GT.1.2 DCS.GT.1.3	***DCS.L.1*** *Evaluate mathematical logic to model and solve problems.* DCS.L.1.1 DCS.L.1.2 DCS.L.1.3 DCS.L.1.4
DCS.N.2 *Understand matrices to solve problems.* DCS.N.2.1 DCS.N.2.2 DCS.N.2.3 DCS.N.2.4			***DCS.GT.2*** *Apply graph theory to solve problems.* DCS.GT.2.1 DCS.GT.2.2 DCS.GT.2.3 DCS.GT.2.4	
DCS.N.3 *Understand set theory to solve problems.* DCS.N.3.1 DCS.N.3.2 DCS.N.3.3 DCS.N.3.4				
DCS.N. 4 *Understand statements related to number theory and set theory.* DCS.N.4.1 DCS.N.4.2 DCS.N.4.3 DCS.N.4.4				

Standards for Mathematical Practice

Practice	Explanation and Example
Make sense of problems and persevere in solving them.	In Discrete Mathematics for Computer Science (DCS), students solve real world problems through the application of theory and algorithmic thinking. Students seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking themselves, “What is the most efficient way to solve the problem?”, “Does this make sense?”, and “Can I solve the problem in a different way?”
Reason abstractly and quantitatively.	In DCS, students represent a wide variety of real world contexts through the use of matrices, sets, diagrams, vertex-edge graphs and tables. They examine patterns in their processes. Students contextualize to understand the meaning of the number or variable as related to the problem and decontextualize to manipulate symbolic representations by applying properties of operations.
Construct viable arguments and critique the reasoning of others.	In DCS, students construct arguments using verbal or written explanations accompanied by matrices, expressions, equations, graphs, and tables. They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. They pose questions like “How did you get that?”, “Why is that true?” “Does that always work?” They explain their thinking to others and respond to others’ thinking.
Model with mathematics.	In DCS, students model problem situations symbolically, graphically, tabularly, and contextually. Students form expressions, equations, or inequalities from real world contexts and connect symbolic and graphical representations. Students solve systems of equations using matrices. Students use vertex-edge graphs to represent data and describe associations between variables. Students need many opportunities to connect and explain the connections between the different representations. They should be able to use all of these representations as appropriate to a problem context.
Use appropriate tools strategically.	Students consider available tools when solving a mathematical problem and decide when certain tools might be helpful. For instance, students in DCS may create a minimum spanning tree to create an optimal schedule. Like hand tools, it is essential for students to know when and how to use the many algorithms that are a part of this course. It is essential for students to make a choice between methods that produce optimal solutions and those that can be solved efficiently.
Attend to precision.	In DCS, students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Students use appropriate terminology when referring to the matrices, vectors, set and number theory, graph theory, and logical reasoning.
Look for and make use of structure.	Students routinely seek patterns or structures to model and solve problems. In DCS, students apply properties to generate equivalent expressions that involve matrices, sets, recursion, and logic statements.
Look for and express regularity in repeated reasoning.	In DCS, students use repeated reasoning to understand algorithms and make generalizations about patterns. Students use iterative processes to determine the GCF and LCM between any two numbers and use a variety of algorithms to solve problems in a context.
Use strategies and procedures flexibly.	Students make a choice between methods and algorithms that produce optimal solutions and those that can be solved efficiently in DCS. It is essential for a student to not only know an algorithm, but to know when best to use that particular algorithm. Students should be comfortable solving a problem, using the same or different algorithms, and producing different solutions that are valid.
Reflect on mistakes and misconceptions.	In DCS, it is essential for students to reflect upon mistakes and misconceptions. Mistakes are often the cornerstone of learning. Successful students in this course will reflect upon their own thinking and learning to maximize their potential to find optimal solutions efficiently.

Discrete Math for Computer Science Instructional Blueprint

Unit	Concept	Duration	Content Standards	Document Pages
1	Building Classroom Community	2 – 3 Days	**N/A**	8 – 10
2	Combinatorics	1 – 2 Weeks	**DCS.SP.1.1**	11 – 14
			DCS.SP.1.2	
3	Graph Theory	3 – 4 Weeks	**DCS.GT.1.1**	15 – 27
			DCS.GT.1.2	
			DCS.GT.1.3	
			DCS.GT.2.1	
			DCS.GT.2.2	
			DCS.GT.2.3	
			DCS.GT.2.4	
4	Matrices	2 – 3 Weeks	**DCS.N.1.1**	28 – 38
			DCS.N.1.2	
			DCS.N.1.3	
			DCS.N.2.1	
			DCS.N.2.2	
			DCS.N.2.3	
			DCS.N.2.4	

Discrete Math for Computer Science Instructional Blueprint				
Unit	**Concept**	**Duration**	**Content Standards**	**Document Pages**
5	Logic	2 Weeks	**DCS.L.1.1** **DCS.L.1.2** **DCS.L.1.3** **DCS.L.1.4**	39 – 45
6	Set Theory	2 Weeks	**DCS.N.3.1** **DCS.N.3.2** **DCS.N.3.3** **DCS.N.3.4** **DCS.N.4.3**	46 – 53
7	Number Theory	1 – 2 Weeks	**DCS.N.4.1** **DCS.N.4.2** **DCS.N.4.4**	54 – 59
8	Recursively Defined Functions	2 – 3 Weeks	**DCS.F.1.1** **DCS.F.1.2** **DCS.F.1.3** **DCS.F.1.4** **DCS.F.1.5**	60 – 66

Online Resources

- Mathematics | NC DPI NCDPI K-12 Mathematics Site.
- https://www.nc2ml.org/ (North Carolina Collaborative for Mathematics Learning, i.e. NC^{2}ML) - NC network of support for teachers. Provides resources, the ability to share best practices, and develop mathematical mindsets.

<table>
<tr><th colspan="9">Domain</th><th>Conceptual Category</th></tr>
<tr><th>K</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>HS</th></tr>
<tr><td>Counting and Cardinality</td><td colspan="5"></td><td colspan="2">Ratio and Proportions</td><td>Functions</td><td>Algebra</td></tr>
<tr><td colspan="6">Operations and Algebraic Thinking</td><td colspan="3">Expressions and Equations</td><td>Functions</td></tr>
<tr><td colspan="6">Number and Operations Base Ten</td><td colspan="3" rowspan="2">The Number System</td><td rowspan="2">Number and Quantity</td></tr>
<tr><td colspan="3"></td><td colspan="3">Number and Operations Fractions</td></tr>
<tr><td colspan="6">Measurement and Data</td><td colspan="3">Statistics and Probability</td><td>Statistics and Probability</td></tr>
<tr><td colspan="9">Geometry</td><td>Geometry</td></tr>
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Unit 1

Building Classroom Community

Unit 1: Building Classroom Community

Source: NC^{2}ML Retrieved from: https://www.nc2ml.org/wp-content/uploads/2020/04/Discrete-Mathematics-for-Computer-Science-Instructional-FrameworkFINAL.pdf

Content:

It is recommended that the first week of the school year be spent engaging students with open-ended mathematics problems designed to support the students' growth mindset. This first week is also an opportune time for setting up the classroom expectations and norms for collaborating with classmates and participating in whole class discussions.

What is the Mathematics?

- Develop mathematicians with positive attitudes about their ability to do mathematics by:
  - Creating opportunities to develop an appreciation for mistakes
  - Seeing mistakes as opportunities to learn
  - Teaching students to take responsibility for their learning
- Develop mathematicians who respect others by:
  - Demonstrating acceptance, appreciation, and curiosity for different ideas and approaches
  - Establishing procedures and norms for productive mathematical discourse
  - Consider other solution paths
- Develop mathematicians with a mindset for problem solving by:
  - Encouraging student authority and autonomy when problem solving
  - Emphasizing questioning, understanding, and reasoning about math, not just doing math for the correct answer
  - Asking follow-up questions when students are both right and wrong
  - Allowing students to engage in productive struggle and moving them along by questioning, not telling

Important Considerations:

For success, significant time should be spent setting up the classroom culture. This includes:

- Developing classroom norms for communication (ex: non-verbal signals, listening and speaking expectations, talk moves for math discussions)
- Developing math routines
- Setting various expectations for the structure of the math block (ex: expectations for whole class instruction, cooperative learning, independent learning, effective integration of technology, etc.)
- Math discourse needs explicit modeling and practice. This includes students:
  - Sharing their thinking
  - Actively listening to the ideas of others
  - Connecting to others' ideas
  - Asking questions to clarify understanding

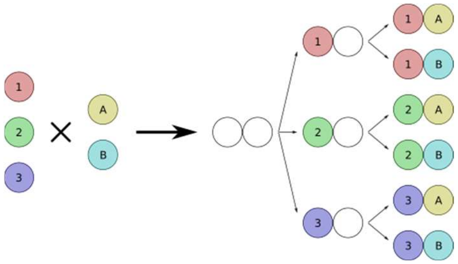
- Mathematical norms: http://www.youcubed.org/wp-content/uploads/Positive-ClassroomNorms2.pdf

Resources (Open Access):

- See https://www.youcubed.org/ for suggested activities on building classroom community

Unit 2

Combinatorics



Unit 2: Combinatorics

Source: NC^{2}ML Retrieved from: https://www.nc2ml.org/wp-content/uploads/2020/04/Discrete-Mathematics-for-Computer-Science-Instructional-FrameworkFINAL.pdf

Content Standard(s):

DCS.SP.1 Apply combinatorics concepts to solve problems.

DCS.SP.1.1

Implement the Fundamental Counting Principle to solve problems.

DCS.SP.1.2

Implement procedures to calculate a permutation or combination.

What is the Mathematics?

- Students expand their understanding of counting possibilities of outcomes. It is important to develop the rationale for why these counting procedures are useful by having the students begin by creating a list of possibilities to demonstrate the difficulty in ensuring an exhaustive list. Students can also create tree diagrams to visualize the possible outcomes which they are familiar with from Math 2. {How many 3-digit lottery tickets can be created using only the numbers 1,2, and 3?}, {How many ways are there to get exactly 2 heads when flipping a coin 3 times?}
- It is important to include scenarios that require the use of multiple counting methods. {A team of students to represent a school will be chosen from each of the four classes: Freshmen, Sophomore, Junior, and Seniors. There are 3 Freshmen candidates, 4 Sophomore candidates, 5 Junior candidates, and 10 senior candidates. There will be 2 students from each class chosen and there will be 1 senior captain and 1 senior co-captain of the team. How many teams can be created?}

Vocabulary:

Combination
Counting
Factorial
Fundamental Counting Principle
Permutation

Important Considerations:

- This unit follows the Building Classroom Community because it is accessible to all students. There are numerous applications across mathematics and computer science, including graph theory, probability, coding, and cryptography. In addition, the study of combinatorics provides many opportunities to engage in the standards of mathematical practice.
- This unit lays a foundation for the course and introduces ideas that should be used as tools in future units. For example, combinations are useful when determining the number of edges in a complete graph with n vertices in the Graph Theory Unit. In the Set Theory unit, combinatorics can be used to determine the total number of possible elements in a set. Recursion and recursive thinking can be introduced when exploring the Fundamental Counting Principle, i.e., 5! = 5*4!

Global Perspectives:

Combinatorics is an essential part of Computer Science. Graph Theory, the study of objects and connections, is a branch of combinatorics. Combinatorics are useful for enumerating all possibilities, an essential part of brute force algorithms like Kruskal's and Prim's algorithms in graph theory. Combinatorial reasoning is an essential part of the study of algorithms, analyzing data structures, and optimization problems.

Resources (Open Access):

- http://walkinginmathland.weebly.com/teaching-math-blog/wendy (Walking in Mathland Blog)
- http://discrete.openmathbooks.org/dmoi3/ch_counting.html Unit One
- https://drive.google.com/open?id=13kfh5CKqwQn88t0B9rE-hQBTJyntlh1V

Unit 2 Unpacking

Source: NC DPI Discrete Mathematics for Computer Science Unpacking Documents. Retrieved from https://www.dpi.nc.gov/nc-discrete-mathematics-computer-science-unpacking-rev-june-2022

Standard: DCS.SP.1.1

DCS.SP.1.1 Implement the Fundamental Counting Principle to solve problems.

Clarification	**Checking for Understanding**
The foundation for this standard was laid in 7th grade, during the exploration of the probability of compound events (NC.7.SP.8). Students should be able to use the Fundamental Counting Principle, also called Multiplication Principal, to solve problems in a context.	**Indicator**: You take a survey with five "yes" or "no" answers. How many different ways could you complete the survey? _Answer: 2 * 2 * 2 * 2 * 2 = 32._ **Indicator**: For the school lunch line, each student has a choice between 2 main dishes, 3 side dishes and 2 drinks. If a student must choose one of each, how many combinations of the lunch are possible? _Answer: 12 combinations are possible_

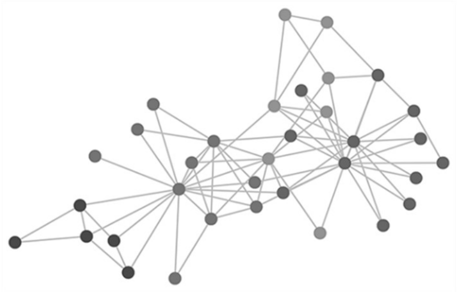
Standard: DCS.SP.1.2

DCS.SP.1.2 Implement procedures to calculate a permutation or combination.

Clarification	**Checking for Understanding**
Students should be able to use permutations and combinations to solve problems in a context. The number of different permutation or combinations possible can be found using the formulas, $P_k^n = \frac{n!}{(n-k)!}$ or $C_k^n = \frac{n!}{k!(n-k)!}$, with *n* being the objects available with *k* objects being selected at a time. In this course, the focus should remain on properly interpreting the context to determine the correct procedure or combination of procedures needed to find the solution.	**Indicator**: In how many ways can a group of 5 members be formed by selecting 3 boys out of 6 and 2 girls out of 5? *Answer:* $C(6,3) \cdot C(5,2) = 20 \cdot 10 = 200$ **Indicator**: How many more ways can 10 juniors running for the positions of president, vice president, secretary, and treasurer be selected when compared to 12 sophomores running for 5 identical positions of class representative? *Answer:* $P(10,4) - C(12,5) = 4248$

Unit 3

Graph Theory



Unit 3: Graph Theory

Source: NC^{2}ML Retrieved from: https://www.nc2ml.org/wp-content/uploads/2020/04/Discrete-Mathematics-for-Computer-Science-Instructional-FrameworkFINAL.pdf

Content Standard(s):

DCS.GT.1 Understand graph theory to model relationships and solve problems.

DCS.GT.1.1

Represent real world situations with a vertex-edge graph, adjacency matrix, and vertex-edge table.

DCS.GT.1.2

Test graphs and digraphs for Euler paths, Euler circuits, Hamiltonian paths, or Hamiltonian circuits.

DCS.GT.1.3

Interpret a complete digraph to determine rank.

DCS.GT.2 Apply graph theory to solve problems.

DCS.GT.2.1

Implement critical path analysis algorithms to determine the minimum project time.

DCS.GT.2.2

Implement the brute force method, the nearest-neighbor algorithm, and the cheapest-link algorithm to find solutions to a Traveling Salesperson Problem.

DCS.GT.2.3

Implement vertex-coloring techniques to solve problems.

DCS.GT.2.4

Implement Kruskal and Prim's algorithms to determine the weight of the minimum spanning tree of a connected graph.

What is the Mathematics?

Students will learn to use vertex-edge graphs to create visual models for real world situations. The intent of this unit is to introduce students to central concepts of graph theory and use those ideas to model and solve a range of problems. It provides a context for algorithmic thinking, a focus of Discrete Math for Computer Science. This unit allows students multiple opportunities to explore and "play" with mathematics. It is important that the tasks that are chosen support student exploration. Social networks provide relevant connections for students.

Vocabulary:

Adjacency matrix
Adjacent
Bipartite graphs
Brute force method
Cheapest Link Algorithm
Chromatic number
Circuits
Complete
Connected
Critical Path
Cycle
Degree/valence
Digraphs
Earliest start time/Latest start time
Edges
Euler circuits and paths
Hamiltonian Circuits and paths
Kruskal's Algorithm
Loops
Markov Chains
Minimum spanning tree
Nearest Neighbor Algorithm
Node
Paths
Planar
Prim's Algorithm
Tournament graph
Traveling Salesperson Problem
Trees

Important Considerations:
• DCS.GT.1.1 should be taught first in order for students to develop the concepts of vertices and edges. Unlike the geometries that most students have experienced before, it is important to make explicit that the “intersection” of two or more edges does not imply a vertex. Social Networking is a useful context for understanding graph theory and for seeing its relevance. • GT 1.2, 1.3, and 2.2 should be clustered together. While Euler and Hamiltonian circuits and paths may be taught separately in any order, 1.3 and 2.2 are specific applications of Hamiltonian circuits. • GT1.1 and GT1.3 (determining rank from a digraph) refer to tournament graphs, where each vertex represents a competitor and each directed edge represents the result of the competition between the connected competitors. For example, if competitor A defeated competitor B, the edge would flow from competitor A ro competitor B. From the tournament graph, students should be able to interpret the ranking of the competitors in relation to each other. • Applications of graph theory can be completed in any order after the foundation has been built: GT 2.1, 2.3, 2.4 (critical path analysis, vertex-coloring, and minimum spanning trees). • There is a strong connection between graph theory and matrices. Graphs can be represented as a matrix, where one row and one column of a matrix represents one vertex of the graph. If two vertices, say A and B, are connected, it is represented by a 1 in the Ath row, Bth column of the matrix. If A and B are not connected, it is represented by a 0. • Brute Force algorithms (e.g. Kruskal, Prim) provide a connection to combinatorics.
Global Perspectives:
Graph theory provides an introduction to algorithmic thinking which should be viewed as one of the overall foci of Discrete Mathematics for Computer Science. While the unit focuses on some specific graph theory terminology and ideas, algorithmic thinking and decision making are metacognitive concepts that should be made explicit to students. Computer scientists use graph theory and the accompanying thinking skills in their work daily. Graphs are central in understanding and solving problems related to planning and scheduling, modeling networks, data organization, and to analyze algorithms.
Resources (Open Access):
• Seven Bridges of Konigsberg (“Math is Fun”) • TedEd Video: Seven Bridges of Konigsberg • “Four Is Not Enough”: https://www.quantamagazine.org/the-numbers-and-geometry-behind-amath-coloring-puzzle-20180618/ • Open Text: Unit 4 • NCTM Graph Creator • Overview of TSP/Intro to TSP • https://www.sciencealert.com/amateur-solves-decades-old-maths-problem-about-colours-thatcan-never-touch-hadwiger-nelson-problem

Unit 3 Unpacking

Source: NC DPI Discrete Mathematics for Computer Science Unpacking Documents. Retrieved from https://www.dpi.nc.gov/nc-discrete-mathematics-computer-science-unpacking-rev-june-2022

Standard: DCS.GT.1.1

DCS.GT.1.1 Represent real world situations with a vertex-edge graph, adjacency matrix, and vertex-edge table.

Clarification

Students should be able to create vertex-edge graphs, adjacency matrices, and vertex-edge tables from a context.

A vertex-edge graph consists of points (vertices) and connections between the points (edges) that represent a context.
Students may be asked to create several types of vertex-edge graphs, including a tournament graph. See GT.1.3 for more information and interpreting tournament graphs.

Two vertices are considered adjacent if there exists an edge between them. In an adjacency matrix, adjacent vertices are represented with a 1. No adjacent vertices are represented with a 0.

A vertex-edge table consists of a table which displays the number of edges from each vertex. The number of edges from a vertex is also known as the degree of that vertex. Vertex-edge tables can include listing the adjacent vertices of each vertex. In other sources, this may be known as an edge table.

Checking for Understanding

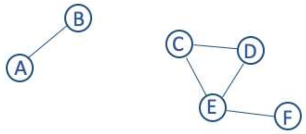
Indicator: The table shows a list of students who are Facebook "friends."

a. Create a vertex edge graph that represents the information in the table.
b. Construct an adjacency matrix to represent the information in the table.

Student	Is friends with:
Anna	Beth
Chun	Dalton
Dalton	Edwin
Edwin	Chun
Edwin	Felicia

Answers:

a. *Sample answer.*



b.

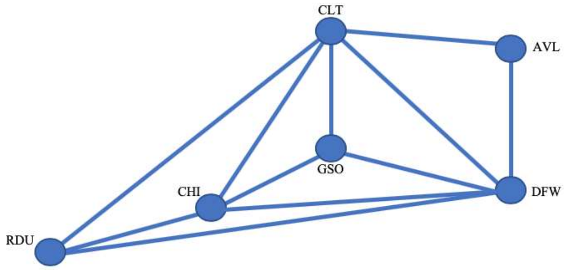
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Indicator: Use the airport abbreviations for the following cities:
Asheville (AVL), Charlotte (CLT), Chicago (CHI), Dallas (DFW), Greensboro (GSO), and Raleigh (RDU)
American Airlines has direct flights between the following airports:
CLT & AVL, CLT & DFW, CLT & GSO, CLT & CHI, CLT & RDU, AVL & DFW, GSO & DFW, GSO & CHI, RDU & DFW, RDU & CHI, CHI & DFW

a. Create a vertex-edge graph to represent the direct flights between the cities.
b. Create an adjacency matrix to represent the direct flights between the cities.
c. Create a vertex-edge table to represent the direct flights between the cities.

Answers:

a.



b.

$$\begin{array}{c|cccccc} & AVL & CLT & CHI & DFW & GSO & RDU \\ \hline AVL & 0 & 1 & 0 & 1 & 0 & 0 \\ CLT & 1 & 0 & 1 & 1 & 1 & 1 \\ CHI & 0 & 1 & 0 & 1 & 1 & 1 \\ DFW & 1 & 1 & 1 & 0 & 1 & 1 \\ GSO & 0 & 1 & 1 & 1 & 0 & 0 \\ RDU & 0 & 1 & 1 & 1 & 0 & 0 \end{array}$$

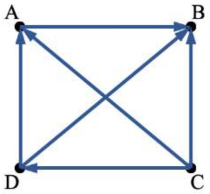
c.

Vertex	Edges
AVL	2
CLT	5
CHI	4
DFW	5
GSO	3
RDU	3

Indicator: Create a tournament graph to represent the round robin tournament represented below:

Game	AB	AC	AD	BC	BD	CD
Winner	A	C	D	C	D	C

Answer:



Standard: DCS.GT.1.2

DCS.GT.1.2 Test graphs and digraphs for Euler paths, Euler circuits, Hamiltonian paths, or Hamiltonian circuits.

Clarification

Students evaluate graphs and digraphs to determine if they meet the criteria for being an Euler path, Euler circuit, Hamiltonian path, or Hamiltonian circuit. A digraph is a directed graph in which the flow is from one vertex to another vertex represented by arrows.

Euler Circuits and Paths

While Euler circuits and paths can be found through trying random paths, as the graphs and digraphs become more complex, testing can become cumbersome. Students should know how to use the degree of the vertices to make a quick determination. The degree of a vertex is the number of edges that have the vertex as an endpoint. To be an Euler circuit, the degree of all vertices of a connected graph must be even. To be an Euler path, the connected graph has exactly two vertices with an odd degree.

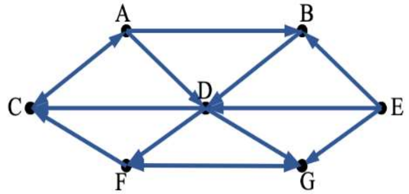
Hamiltonian Circuits and Paths

Unlike Euler paths and circuits, there are no simple tests to determine if a graph or digraph is a Hamiltonian circuit or path. Some theorems do exist for certain categories of graphs. Knowing and using those theorems is not an expectation of this course.

Understanding a Hamiltonian circuit is essential to Traveling Salesperson Problems which are addressed in GT.2.2.

Checking for Understanding

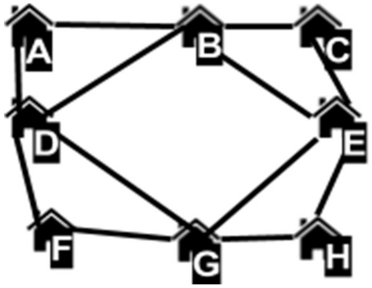
Indicator: A group of 7 Discrete math students are working on a project and plan to share information via text messages. The digraph below shows the 7 students' communication outside of class. Does this represent a Hamiltonian circuit? Explain.



Answer:

No, a Hamiltonian Circuit does not exist as all directions are away from E, so even if the path started at E, it could never return to E.

Indicator: The graph below shows a small neighborhood of homes along with roads connecting them.



a. Is it possible for the person in house B to travel to each house to hand out flyers for her missing pet pig and return back home without passing by the same house twice? If so, name the circuit, and if not, explain.
b. The latest snow storm has dumped over a foot of snow. Is it possible for the person in House F to plow every road exactly once without traveling over the same road twice? If so, name the circuit and if not, explain.

Answers:

a. Yes, it is possible. One example is B-C-E-H-G-F-D-A-B
b. Yes, it is possible since every vertex has an even degree. One example is F-D B-E-G-D-A B-C-E H-G-F

Indicator: Create a tournament graph to represent the round robin tournament below and list all possible Hamiltonian paths.

Game	AB	AC	AD	BC	BD	CD
Winner	A	C	D	C	D	C

Answer:

See GT1.1 for the tournament graph. There is only one possible path, C - D - A - B

Standard: DCS.GT.1.3

DCS.GT.1.3 Interpret a complete digraph to determine rank.

Clarification

In this objective, students examine a tournament graph to determine the ranking of participants in a tournament. A tournament graph is a directed complete graph in which a result of the interaction is represented by the direction of the arrow. The base of the arrow starts with the "winner" and points to the "loser."

In a complete graph, all vertices are connected to all other vertices by an edge. The context of a tournament graph does not have to be limited to a tournament.

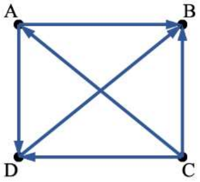
From the tournament graph, students should be able to interpret the ranking of the competitors in relation to each other.

Students may be asked to create a tournament graph to reflect rankings (DCS.GT.1.1).

Note: It is not the expectation of this standard for students to determine the rank of a digraph. The rank of a digraph is a distinct and more complex concept.

Checking for Understanding

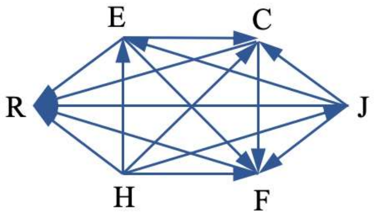
Indicator: The graph represents the results of a round-robin tournament held by the robotics club at North High School. Determine the final rankings for teams A, B, C, and D.



Answer:

Highest to lowest: C - A - D - B

Indicator: The graph represents the bowling championship tournament for the following high school teams: Riverdale, Edwin, Johnson, Curry, Hamilton, and Franklin.



a. Who was the champion?

b. How many wins did each team achieve?

Answers:

a. Hamilton

b. Riverdale 0, Edwin 3, Johnson 4, Curry 2, Hamilton 5, and Franklin 1.

Standard: DCS.GT.2.1

DCS.GT.2.1 Implement critical path analysis algorithms to determine the minimum project time.

Clarification

Checking for Understanding

Students are expected to complete a critical path analysis to find the minimum project time. The critical path of a project is the critical path from start to end. Processing time is the sum of the time to complete all steps in a path. For the critical path, the processing time is also known as the critical time or the minimum project time.

A systematic method to find the critical path is using the Backflow Algorithm. In this method:

1. Beginning at the End, work backward to find the critical time, the longest total processing time, for each vertex.
2. Record the critical time for each vertex.
3. Find and highlight the path that creates the critical time from the Start.

Note: How the processing time and critical time are represented in a graph can vary.

The Critical Path Algorithm

1. Find the critical times using the backflow algorithm.
2. Create a critical-time priority list critical times.
3. Create schedule

Only the first step is required to find the minimum project time, as the minimum project time is the critical time from the Start.

The other steps in the Critical Path Algorithm can be added as time allows or as an extension.

Students can be asked to find the minimum project time of an already completed schedule, as long as all prerequisite information is given.

Note: Names and methods of representation critical paths can vary depending on the source.

Indicator: Create a graph to represent the project listed.

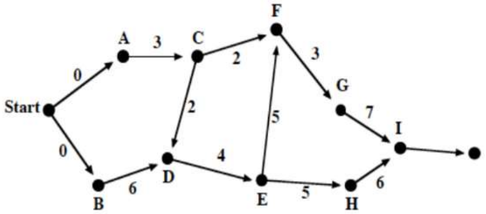
a. What is the earliest start time for Task G?
b. What is the latest start time for Task B?
c. Name the critical path.
d. What is the minimum project time required?

Task	Time	Prerequisite
A	3	none
B	4	none
C	2	A
D	3	B, C
E	5	D
F	2	B
G	1	E,F

Answers:

a. 13
b. 1
c. Start-A-C-D-E-G-Finish
d. 14

Indicator: According to the graph shown below:

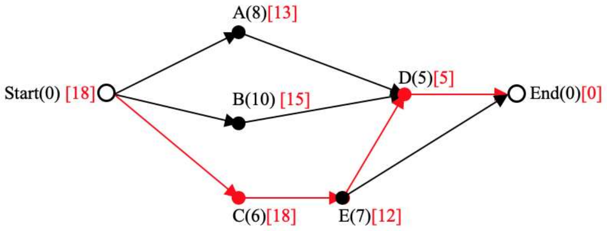


a. How many prerequisites does F have?
b. What is the earliest start time for E?
c. What is the minimum project time?
d. What is the critical path?

Answers:

a. 5
b. 9
c. 24
d. Start - B - D - E - F - G - I - End

Indicator: A backflow algorithm has been applied to a potential project and is shown in the graph below. The units are in hours.



a. What is the processing time and the critical time for task E?
b. How was the critical time for E calculated?
c. What is the minimum project time for this project?
d. What does the minimum project time tell us?

Answers:

a. *The processing time is 7 hours and the critical time is 12 hours.*
b. *The critical time was calculated using the longest total time from E to the End. This goes from E-D-End, which is 7+5=12 hrs.*
c. *18 hrs*
d. *The minimum project time tells us the minimum amount of time possible to complete all tasks in the project.*

Standard: DCS.GT.2.2

DCS.GT.2.2 Implement the brute force method, the nearest-neighbor algorithm, and the cheapest-link algorithm to find solutions to a Traveling Salesperson Problem.

Clarification

Traveling Salesperson Problems (TSP) are a common category of problem in which the goal is to determine the most efficient pathway. TPS problems involve a variety of context other than a traveling salesperson.
Students should know that all Traveling Salesperson problems involve Hamiltonian circuits.

Brute Force method:

1. List all Hamiltonian circuits.
2. Calculate the total weight of each circuit.
3. Choose an optimal circuit.

Checking for Understanding

Indicator: Kayana decided to do a driving tour of 4 college campuses to determine which school(s) she is most interested in attending. The table below shows the distances between each university.

	ASU	NCSU	UNCA	UNCW
ASU	--------	186	86.5	317
NCSU	186	--------	246	129
UNCA	86.5	246	--------	334

Students should know that the brute force method produces the most optimal path. However, the time to complete the brute force method increases greatly for each additional vertex.

<u>Nearest-Neighbor algorithm</u>:

1. From a starting vertex choose the edge with the lowest edge weight (nearest neighbor).
2. From that vertex chose the next edge with the lowest weight to go to the next vertex and continue in this pattern.
3. From the last vertex, travel back to the starting vertex.

In this course, the starting vertex will be provided when students are asked to complete this algorithm. The repetitive nearest-neighbor algorithm is not an expectation of this standard.

<u>Cheapest-Link algorithm</u>:

a. Pick the edge with the smallest weight (cheapest) and mark it.
b. Pick the next edge with the smallest weight and mark it.
c. Continue in the process of marking the next smallest weight edge, skipping any edge that would either:
    a. close the circuit or
    b. create any vertex with three edges coming out of a single vertex.
d. Connect the last vertices to close the circuit.
e. If needed, reorder the path to start from a particular vertex based on context.

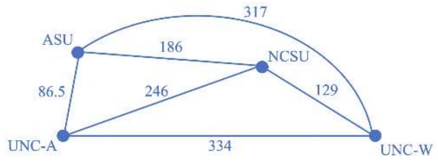
Other methods and algorithms are not an expectation of this course.

UNCW	317	129	334	--------

a. Draw a weighted graph using this data.
b. What is the total distance Kayana will travel using the Nearest Neighbor Algorithm, if she begins and ends her travel at Appalachian State University (ASU)?
c. What is the total distance Kayana will travel using the Cheapest Link algorithm, if she begins and ends at ASU?
d. What is the optimal solution? Draw a tree diagram to show all possible routes.

Answers:

a.

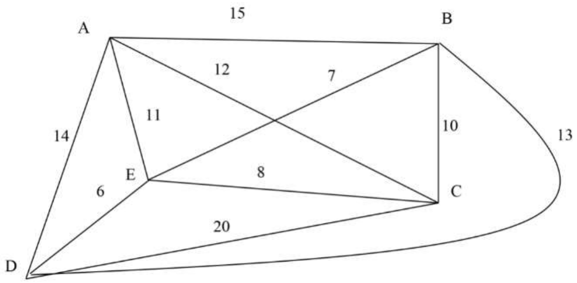


b. (ASU-UNCA)+(UNCA-NCSU)+(NCSU-UNCW)+(UNCW-ASU)
86.5+246+129+317
778.5 miles

c. (ASU-NCSU)+(NCSU-UNCW)+(UNCW-UNCA)+(UNCA-ASU)
186+129+334+86.5
735.5 miles

d. The optimal path forms a quadrilateral. (ASU-UNCA)+(UNCA-UNCW)+(UNCW-NCSU)+(NCSU-ASU)
86.5+334+129+186
735.5 miles

Indicator: Mr. Deal sells horse feed and must travel to four farms today and then return home. Since he pays for his own gas, he wants to ensure that the distance he travels in total will be the least possible distance.



a. If his home is at **A**, find the best way for him to travel using the Brute Force Method and then the Nearest Neighbor Method.

b. Which method is always guaranteed to give you the shortest route? Why is it not always used?

Answers:

a. Brute force:
A-D-E-B-C-A, 49 units
Nearest Neighbor: A-E-D-B-C-A, 52 units

b. The Brute force method guarantees the shortest route because it looks at all possibilities. This also makes it an inefficient method.

Standard: DCS.GT.2.3

DCS.GT.2.3 Implement vertex-coloring techniques to solve problems.

Clarification

Vertex coloring is a way of labeling vertices to solve problems involving constraints.

The goal of vertex coloring is to find the minimum number of colors needed. This number is known as the chromatic number.

The events represented by the same color vertex can occur at the same time, as there are no constraints between those events. This makes vertex coloring useful when creating schedules and maps, clustering, data mining, networking, and determining resource allocation.

In this course, vertex coloring problems should remain simple enough to be completed by hand, without the use of advanced algorithms.

Checking for Understanding

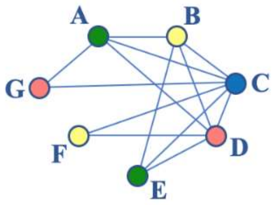
Indicator: It is time for Mrs. Blevins to create a testing schedule for Discrete High School. The difficulty is that some students have multiple tested subjects, so they must be scheduled for different days. The table shows which tests must be scheduled separately.

Test	Cannot be tested with
A	B, C, D, G
B	A, C, D, E
C	A, B, D, G, E
D	A, B, C, E, F
E	B, C, D
F	C, D
G	A, C

Draw a vertex-edge graph to represent this situation and find the chromatic number. What does the chromatic number mean in this situation?

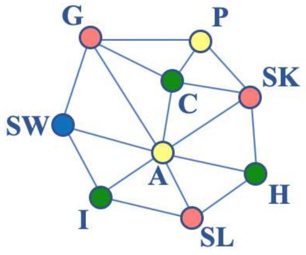
Answer:

The chromatic number is 4, which means Mrs. Blevins must create 4 different testing schedules. Sample graph provided.



Indicator: Represent the following map as a vertex-edge graph. Use the graph to determine the chromatic number.

Answer:
Chromatic Number = 4. Sample graph provided.



Standard: DCS.GT.2.4

DCS.GT.2.4 Implement Kruskal and Prim's algorithms to determine the weight of the minimum spanning tree of a connected graph.

Clarification

For this objective, students will determine the weight of a minimum spanning tree.

Kruskal's algorithm:

- Ensure the graph is a spanning tree
- List the edges in order from shortest to longest, breaking ties arbitrarily
- Mark the first edge on the list
- Mark the next edge on the list, unless it would form cycle
- Repeat the last step until n-1 edges have been marked.

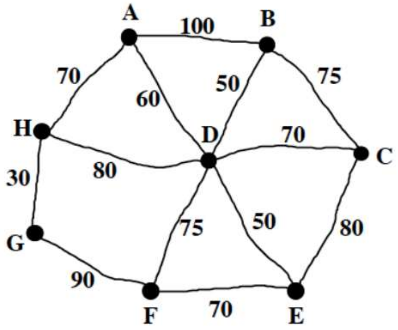
The marked edges and vertices are the minimum spanning tree and that weight can be calculated.

Prim's algorithm:

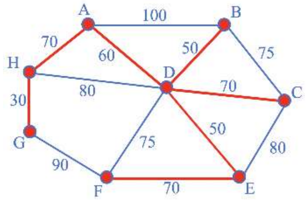
- Starting with the edge with the least weight, mark that edge and circle the vertices. Ties are broken arbitrarily.

Checking for Understanding

Formative Check: Use Kruskal's algorithm to find the minimum spanning tree.



Answer:



Indicator: Project: Distribution Centers
Using a map of North Carolina:

- Find and label 8 cities from the various regions of the state
- Create and label a vertex-edge graph that shows the distance needed to travel between your cities. At most, you can connect one city to four other cities.

- Find the edge with the least weight that contains one circled vertex and one uncircled vertex. Mark that line and circle the uncircled vertex.
- Repeat the previous step until all vertices are circled.

The marked edges and vertices are the minimum spanning tree and that weight can be calculated.

Students should know that the algorithms can produce different minimum spanning trees. Other algorithms are not an expectation of this course.

- All of the cities will have a distribution center and be connected in a distribution network.
- Your goal is to:
  - Identify the most efficient distribution network,
  - Determine the city where the items being distributed will be produced,
  - Determine how many drivers will be needed at each distribution center.
- Restrictions for distribution centers and drivers:
  - Due to travel restriction agreements with the drivers, a driver can only connect to one adjacent city/distribution center.
  - Mileage is the key factor in determining a cost efficient network for your company.
  - It is also more efficient for your company to have as many drivers as possible live in the same city.

Answer:

Student answers can vary. In creating an efficient distribution network, the students should determine that they need to find the minimum spanning tree. The method used is at the discretion of the students. The city with the most connections in the minimum spanning tree (MST) is probably the most efficient city to produce the item. The number of drivers can be organized in various ways, keeping in line with the restriction listed in the problem.

Unit 4

Matrices

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & \dots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \end{array}$$

Unit 4: Matrices

Source: NC^{2}ML Retrieved from: https://www.nc2ml.org/wp-content/uploads/2020/04/Discrete-Mathematics-for-Computer-Science-Instructional-FrameworkFINAL.pdf

Content Standard(s):

DCS.N.1 Apply operations with matrices and vectors.

DCS.N.1.1

Implement procedures of addition, subtraction, multiplication, and scalar multiplication on matrices.

DCS.N.1.2

Implement procedures of addition, subtraction, and scalar multiplication on vectors.

DCS.N.1.3

Implement procedures to find the inverse of a matrix.

DCS.N.2 Understand matrices to solve problems.

DCS.N.2.1

Organize data into matrices to solve problems.

DCS.N.2.2

Interpret solutions found using matrix operations including Leslie Models and Markov Chains, in context.

DCS.N.2.3

Represent a system of equations as a matrix equation.

DCS.N.2.4

Use inverse matrices to solve a system of equations with technology.

What is the Mathematics?

- In this unit, students learn to use matrices to organize information and solve problems from a context. Labeling the rows and columns of a matrix and referring to particular elements of a matrix support students in maintaining the connection between a matrix and context.
- Students should explore what properties of operations for matrices are similar and different from the real numbers. Remaining grounded in a context is important for students to develop an understanding of matrix multiplication. Examining the number of dimensions of two matrices and developing a rule to see if their product exists should follow from an understanding of the dimensions represented by rows and columns of the matrices. Make explicit for students that unlike multiplication of real numbers, matrix multiplication is noncommutative.
- To determine if a matrix has an inverse, students will find its determinant to see if its value is nonzero. Students should use appropriate technology to calculate determinants of square matrices larger than a 3x3.
- Using inverse matrices, students will learn to represent and solve a system of equations using matrix equations.
- Matrices are used to represent vectors and model phenomena in the physical, natural, and social sciences. For example, Markov Chains use transition matrices to represent probabilities of transitioning among a finite number of states and predict stable state vectors after a sequence of possible events. Leslie models are useful for modeling population growth and also use transition matrices to represent the proportion of members of a certain age group that survive and enter the next age group.

Vocabulary:

Column
Determinant
Dimensions
Elements
Identity Matrix
Inverse Matrix
Leslie Model
Markov Chain
Matrix Operations
Row
Scalar
Transpose
Vector

Important Considerations:

- DCS.N.2.1 and DCS.N.1.1 introduce the usefulness of matrices for organizing data and operating with data to solve problems and should come first.
- For DCS.N.1.3, students should build conceptual understanding of inverse matrices for 2x2 matrices and build procedural fluency with technology for larger matrices.
- DCS.N.2.2, DCS.N.2.3, & DCS.N.2.4 are uses of matrices to solve problems. Students learn to use matrix equations to solve systems of linear equations, complementing approaches students learned in Grade 8 and NC Math 1. Markov Chains and the Leslie model use matrix equations to solve problems in a variety of contexts in the physical, life, and social sciences.
- Students should add and subtract vectors written in the form $ai + bj$ as well as with matrices. Students are not expected to perform other vector operations.

Global Perspectives:

Matrices are essential tools for computer scientists. Matrices are used to organize and retrieve data, model networks, detecting errors in data transmission, and in algorithms like Google's original page rank algorithm, compression techniques, and mining data. Matrices are also central to computer graphics, animation, and face recognition technologies.

Resources (Open Access):

- Cryptography: Harry Potter
- The Leslie Model
- Article: Train Delays and Markov Chains Matrix Multiplication Tasks
- http://news.mit.edu/2013/explained-matrices-1206

Unit 4 Unpacking

Source: NC DPI Discrete Mathematics for Computer Science Unpacking Documents. Retrieved from https://www.dpi.nc.gov/nc-discrete-mathematics-computer-science-unpacking-rev-june-2022

Standard: DCS.N.1.1

DCS.N.1.1 Implement procedures of addition, subtraction, multiplication, and scalar multiplication on matrices.

Clarification

Students have not been introduced to matrices in previous courses. However, the concept of a matrix is similar to two-way tables seen in 8th grade.

In the objective, students are expected to add, subtract, and multiply matrices in a context. This context can include, but is not limited to, real-world problems, the transformation of points on a coordinate plane, Leslie Models and Markov Chains. Technology should be used as deemed appropriate by the teacher.

This objective works in conjunction with DCS.N.2.1 as students must decide how to organize data from the context before performing matrix operations.
For students to make this decision, students must:

- understand matrix operations,
- be able to identify elements within a matrix and corresponding elements between matrices,
- name a matrix using rows and columns to determine the feasibility of operations, and
- know which properties of operation apply to matrices.

For both Leslie Models and Markov Chains, students should be able to organize the data into the appropriate matrices (DCS.N.2.1). In alignment with this standard, students evaluate matrices to solve problems involving Leslie Models and Markov chains.

For all evaluated matrix expressions, students will also interpret solutions in context (DCS.N.2.2).

Checking for Understanding

Formative Check: Without the aid of technology, what is element e23of the product of G · H?

$$G = \begin{bmatrix} 3 & 2 \\ 5 & 0.5 \end{bmatrix} \qquad H = \begin{bmatrix} 7 & -1 & 0.5 \\ -5 & -3.5 & 4 \end{bmatrix}$$

Answer: 4.5

Indicator: A few local companies donate spirit items which can be sold along with the items made by the Booster Club at games. JJ's Sporting Goods donates 100 hats and 100 pennants in September and 125 hats and 75 pennants in October. Friendly Fred's Foods donates 105 hats and 125 pennants in September and 110 hats and 100 pennants in October. Use matrices to show many items are available each month from both sources?

Answer: Student matrices may be organized differently.

$$\begin{array}{ccc} & \text{Hats} & \text{Pennants} \\ \text{Sept} & 205 & 225 \\ \text{Oct} & 235 & 175 \end{array}$$

Indicator: Riverside High School Booster Club is running the concession stands at a JV Football game.

- Booster club sells cans of soda for $1, a slice of pizza for $3, a chicken sandwich for $5, and candy for $1.50.
- The home side concession stand sold 160 cans of soda, 82 slices of pizza, 103 chicken sandwiches, and 72 pieces of candy.
- The visitors side concession stand sold 102 cans of soda, 65 slices of pizza, 52 chicken sandwiches, and 37 pieces of candy.
- The booster club purchased the soda for $.25 per can, each slice of pizza for $1.5, each chicken sandwich for $3.29, and each piece of candy for $.70.
- The home side concession was stocked with 200 cans of soda, 96 slices of pizza, 120 chicken sandwiches, and 150 pieces of candy.
- The visitors side concession stand was stocked with 120 cans of soda, 80 slices of pizza, 80 chicken sandwiches, and 100 pieces of candy.

Using matrices, determine the booster club's profit for the JV game.
Answer: $464.49

Standard: DCS.N.1.2

DCS.N.1.2 Implement procedures of addition, subtraction, and scalar multiplication on vectors.

Clarification

This is the student's introduction to vectors. In a science context, a vector often represents a force being applied in a certain direction. As this objective requires students to work with vectors in a context, it is important for students to properly interpret the components of a vector and distinguish a vector from coordinate points.

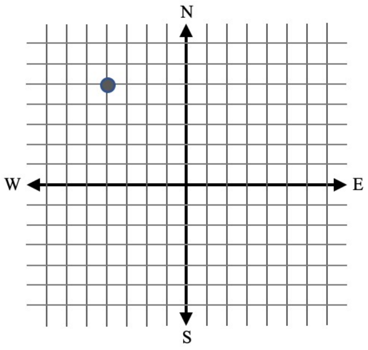
The use of vectors can be connected to the students' previous work with translations from 8th grade and NC Math 2.

Only the component form of a vector will be used in this course. Students are expected to translate the vector into a matrix and use matrix operations to solve problems.

A vector can be defined differently, depending on the context. For this standard, the definition of a vector is aligned to the mathematical and physical science definition. In computer programming, graphics, and security, the term vector has distinct but related definitions.

Checking for Understanding

Indicator: A boat is traveling in water with a current that is flowing on average 2 mph directly west. The boat is traveling slowly on the vector $< 5, -3 >$ in mph. The graph, with a scale of 1 mile, indicates the boat's current position.



a. Using matrices, write an expression that represents the position of the boat in n hours.

b. Where will the boat be in 2 hours?

Answer:

a. *Sample answer:* $\begin{bmatrix}-4\\5\end{bmatrix}+n\begin{bmatrix}5\\-3\end{bmatrix}-n\begin{bmatrix}2\\0\end{bmatrix}$

b. $(2, -1)$

Indicator: The image of a triangle is A'(-2,1), B'(0,-1), and C'(-3,2). Triangle ABC was translated by $\boldsymbol{j} < -2, 3 >$. What are the coordinates of the vertices of the pre-image?

Answer: $A(0, -2)$, $B(2, -4)$, $C(-1, -1)$

Standard: DCS.N.1.3

DCS.N.1.3 Implement procedures to find the inverse of a matrix.

Clarification

In the past, students have worked with additive and multiplicative inverses when solving equations.

Students should be able to use the determinant to identify if the inverse of a matrix is possible. Students should be familiar with using technology to find the inverse of larger matrices.

The expectation of this standard is that students will discern the need to find the inverse matrix from a context.
Technology should be used as deemed appropriate by the teacher.

Checking for Understanding

Indicator: You receive the name of a famous piece of art hidden in the coded message: 21, 0, 53, 2, 11, 3, 24, 1.

You know the original coding matrix, shown here. $\begin{bmatrix}-3 & 5\\-1 & 2\end{bmatrix}$
What matrix would be used to decode the message?

Answer: $\begin{bmatrix}-2 & 5\\-1 & 3\end{bmatrix}$

Indicator: During a soccer season, referees are paid different rates for different types of games. There are three types of games in a typical season: non-conference, conference, and playoff games. There are two referees for each game and schools only have to pay for home games. The information below is from three high schools with the same pay scale for referees.

High School	Home Games Nonconference	Home Games Conference	Playoff Games	Total Pay for Soccer Referees
Green River	5	7	2	$1806
Blue Creek	3	8	1	$1570
Black Lake	6	6	0	$1476

a. Write a matrix equation based on the context.
b. What is the inverse matrix you would use to solve the matrix equation?

Answers:

a.

$$\begin{bmatrix} 5 & 7 & 2 \\ 3 & 8 & 1 \\ 6 & 6 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1806 \\ 1570 \\ 1476 \end{bmatrix}$$

b.

$$\begin{bmatrix} \frac{1}{8} & -\frac{1}{4} & \frac{3}{16} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{48} \\ \frac{5}{8} & -\frac{1}{4} & -\frac{19}{48} \end{bmatrix}$$

Standard: DCS.N.2.1

DCS.N.2.1 Organize data into matrices to solve problems.

Clarification

In 8th grade, students organize data and solve problems using two-way tables. While the terminology of matrices will be new to students, the organizational structure is not. As with two-way tables, there are a variety of ways to organize data into a matrix, therefore, it is essential that students sufficiently label their data.

An essential part of the decision of how to organize data into matrices is the understanding of the requirements for matrix operations. In problems with multiple matrices that require matrix operations, students must understand that the choice of how to organize one matrix often then determines the organization of the other matrices in order to complete the operations.
Students should also know how to organize data into matrices for Leslie Models and Markov Chains.

Checking for Understanding

Indicator: Riverside High School Booster Club is running the concession stands at a JV Football game.

- Booster club sells cans of soda for $1, a slice of pizza for $3, a chicken sandwich for $5, and candy for $1.50.
- The home side concession stand sold 160 cans of soda, 82 slices of pizza, 103 chicken sandwiches, and 72 pieces of candy.
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- The booster club purchased the soda for $.25 per can, each slice of pizza for $1.5, each chicken sandwich for $3.29, and each piece of candy for $.70.
- The home side concession was stocked with 200 cans of soda, 96 slices of pizza, 120 chicken sandwiches, and 150 pieces of candy.
- The visitors side concession stand was stocked with 120 cans of soda, 80 slices of pizza, 80 chicken sandwiches, and 100 pieces of candy.

Leslie Models

Leslie Models are used to model population growth using age distributions with corresponding birth and survival rates.

The Leslie model is $P_k = P_0 \cdot L^k$, where P_0 is the initial population matrix, L is the Leslie matrix, and k is the number of cycles.

For this standard, students are expected to organize data into the initial population matrix and Leslie matrix.

Markov Chains

Markov Chains are used situations that involve a finite number of states that change over time.

A Markov chain is $D_k = D_0 \cdot T^k$, where D_0 is the initial-state matrix, T is the transition matrix, and k is the number of transitions.

For this standard, students are expected to organize data into the initial-state matrix and transition matrix.

For this course, there are no set limits for the size of matrices or the number of matrices created to solve a problem.

Setup a matrix expression that represents the booster club's profits for the JV football game.

Answer:

Student matrices may be organized differently. In general, when written as one expression: $P(H_S + V_S) - C(H_I + V_I)$, where P represents the price of the item, H_S represents the items sold on the home side, V_S represents the items sold on the visitor side, C represents the cost of the items, H_I represents the items place in inventory on the home side, and V_I represents the items placed in inventory on the visitor side. Students will need to align the matrices and elements so they can perform the matrix operations.

Indicator: A video store owner has found the probability that a customer who rented a movie today will rent a movie tomorrow is 35%. The probability that a customer who did not rent a movie today but will rent tomorrow is 10%.

Write a transition matrix that represents this information.

Answer:

$$\begin{bmatrix} .35 & .65 \\ .1 & .9 \end{bmatrix}$$

Indicator: A population of laboratory animals models the following birth and survival rates. Write the Leslie matrix that would be used to model population growth.

Age	Birthrate	Survival Rate
0-3	0.3	0.9
3-6	0.9	0.7
6-9	0.9	0.8
9-12	0.6	0

Months	0-3	3-6	6-9	9-12
Number	10	11	8	4

Answer:

$$\begin{bmatrix} 0.3 & 0.9 & 0.9 & 0.6 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$$

Standard: DCS.N.2.2

DCS.N.2.2 Interpret solutions found using matrix operations including Leslie Models and Markov Chains, in context.

Clarification	**Checking for Understanding**

Students should be able to interpret the solutions of matrix operations based on the context of the problem. This can include, interpret the matrix as a whole, interpreting elements within the solution, and interpreting any intermediate steps taken in the solving process. For Leslie Models and Markov Chains, students should be able to: • Organize the data into the appropriate matrices (DCS.N.2.1) and • Evaluate the matrices (DCS.N.1.1). The expectation of this standard is that students will be able to interpret the results of the Leslie Model and Markov chains.	**Indicator**: Riverside High School Booster Club is running the concession stands at a JV Football game. • Booster club sells cans of soda for \$1, a slice of pizza for \$3, a chicken sandwich for \$5, and candy for \$1.50. • The home side concession stand sold 160 cans of soda, 82 slices of pizza, 103 chicken sandwiches, and 72 pieces of candy. • The visitors side concession stand sold 102 cans of soda, 65 slices of pizza, 52 chicken sandwiches, and 37 pieces of candy. • The booster club purchased the soda for \$.25 per can, each slice of pizza for \$1.5, each chicken sandwich for \$3.29, and each piece of candy for \$.70. • The home side concession was stocked with 200 cans of soda, 96 slices of pizza, 120 chicken sandwiches, and 150 pieces of candy. • The visitors side concession stand was stocked with 120 cans of soda, 80 slices of pizza, 80 chicken sandwiches, and 100 pieces of candy. a. In the context of this problem, what does \$129.29 represent? b. How much money did the booster club spend on stocking pizza for both concessions stands? *Answers:* *a. The booster club's profit from the visitor's concession.* *b. \$264* **Indicator**: A video store owner has found that the probability that a customer who rented a movie today will rent a movie tomorrow is 35%. The probability that a customer who did not rent today but will rent tomorrow is 10%. a. If 853 out of his 8745 customers rented a movie on Monday night, how many customers can he expect to rent a movie on Tuesday? b. About how many of his customers can be expected to rent a movie three weeks from Monday and explain how you can know this without calculating each day? *Answers:* a. Tuesday will have 1088 renters. b. Three weeks from Monday will have 1166 renters.

Indicator: A population of laboratory animals models the following birth and survival rates.

Age	Birthrate	Survival Rate
0-3	0.3	0.9
3-6	0.9	0.7
6-9	0.9	0.8
9-12	0.6	0

The following table displays the initial population:

Months	0-3	3-6	6-9	9-12
Number	10	11	8	4

a. How many newborn animals were there after four cycles?
b. What is the total population after two cycles?
c. When will the population reach 800?
d. What is the long-term growth rate?

Answers:

a. 47
b. 58
c. During the 11th cycle
d. About 34%

Standard: DCS.N.2.3

DCS.N.2.3 Represent a system of equations as a matrix equation.

Clarification	**Checking for Understanding**
Students will use a matrix equation to represent a system of equations from a context. The number of variables and equations are not limited.	**Indicator**: Represent the following system as a matrix equation. $\begin{cases} 5x - y = 7 \\ 2x + 3y = -1 \end{cases}$ *Answer:* $\begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$ **Indicator**: During a soccer season, referees are paid different rates for different types of games. There are three types of games in a typical season: non-conference, conference, and playoff games. There are two referees for each game and schools only have to pay for home games. The information below is from three high schools with the same pay scale for referees.

Write a matrix equation based on the context.

High School	Home Games Nonconference	Home Games Conference	Playoff Games	Total Pay for Soccer Referees
Green River	5	7	2	$1806
Blue Creek	3	8	1	$1570
Black Lake	6	6	0	$1476

Answer:

$$\begin{bmatrix} 5 & 7 & 2 \\ 3 & 8 & 1 \\ 6 & 6 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1806 \\ 1570 \\ 1476 \end{bmatrix}$$

Standard: DCS.N.2.4

DCS.N.2.4 Use inverse matrices to solve a system of equations with technology.

Clarification

Students will use technology and inverse matrices to solve systems of equations from a context. The number of variables and equations in the system is not limited.

Checking for Understanding

Indicator: It is the end of the semester. You and two friends cannot remember how your teacher weighted your grades to determine your final score. You and your friend have received the following information. Determine how much each of the categories are weighted.

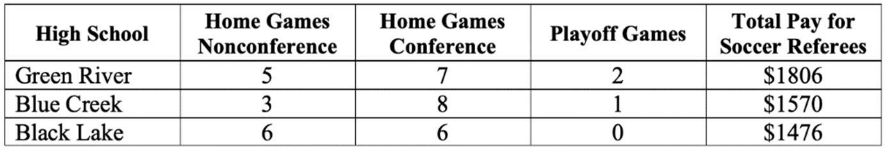
Quizzes	Tests	Exam	Final
95	92	91	92.34

Quizzes	Tests	Exam	Final
72	83	86	81.62

Quizzes	Tests	Exam	Final
87	72	65	73.3

Answer:
Quizzes - .18, Tests - .62, Exam - .2

Indicator: During a soccer season, referees are paid different rates for different types of games. There are three types of games in a typical season: non-conference, conference, and playoff games. There are two referees for each game and schools only have to pay for home games. The information below is from three high schools with the same pay scale for referees.



High School	Home Games Nonconference	Home Games Conference	Playoff Games	Total Pay for Soccer Referees
Green River	5	7	2	$1806
Blue Creek	3	8	1	$1570
Black Lake	6	6	0	$1476

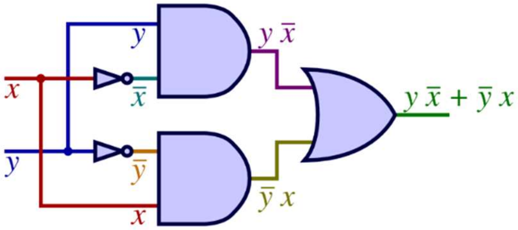
How much is each referee paid for each type of game?

Answer:

Nonconference- $110, Conference - $136, and Playoff - $152

Unit 5

Logic



Unit 5: Logic

Source: NC^{2}ML Retrieved from: https://www.nc2ml.org/wp-content/uploads/2020/04/Discrete-Mathematics-for-Computer-Science-Instructional-FrameworkFINAL.pdf

Content Standard(s):

DCS.L.1 Evaluate mathematical logic to model and solve problems.

DCS.L.1.1

Construct truth tables that encode the truth and falsity of two or more statements.

DCS.L.1.2

Critique logic arguments (e.g., determine if a statement is valid or whether an argument is a tautology or contradiction).

DCS.L.1.3

Check 1s and 0s to determine whether a statement is true or false using Boolean logic circuits.

DCS.L.1.4

Judge whether two statements are logically equivalent using truth tables.

What is the Mathematics?

Mathematical logic is the foundation of many areas of mathematics and computer science. In this unit, students investigate the truthfulness of a proposition by examining its assumptions and justifications. Truth tables are useful for determining the validity of an argument by enumerating all possible combinations of its premises (up to three in this course – p, q, and r). They examine conditional (if-then) and biconditional (if and only if) statements, evaluate more complex arguments using the laws of syllogism and detachment, and demonstrate the logical equivalence of two statements by showing they have identical truth tables. They also learn and use Boolean Algebra, a branch of algebra where variables have only two possible values, 0 (false) and 1 (true), and operations such as conjunction (and), disjunction (or), and negation (not) relating them. Boolean logic circuits represent functions that map one or more Boolean variables as inputs through a gate that represents a Boolean operator to a single output. In addition to having direct applications to electric circuits, this representation is also useful in verifying the truthfulness of statements.

Vocabulary:

Argument
Assertion
Biconditional
Boolean gates
Conclusion
Conditional
Consequent
Contradiction
Contrapositive
Converse
Deduction
Disjunction
Exclusion & inclusion
Fallacy
False
If and only if
Implication
Inference
Invalid
Inverse
Law of Detachment
Law of Syllogism
Negation
Premise
Proposition
Tautology
Truth
Validity

Important Considerations:

- You can support students in learning to construct truth tables (DCS.L.1.1) by beginning with a context that has meaning for them and evaluating simple claims. Abstraction is an important goal in this unit but should not come at the expense of context.
- As students encounter and critique more complex arguments (DCS.L.1.2), they should recognize the converse, inverse, and contrapositives of conditional statements, biconditional statements.
- (DCS.L.1.4) Students can examine logical arguments to identify contradictions and tautologies as well as verify that two statements are logically equivalent by showing that their truth tables are identical.
- Boolean logic circuits (DCS.L.1.3) are another representation used to examine the truthfulness of statements. In this representation, logical operators (and, or, not) are represented by different shapes known as gates. Each assumption is represented by a Boolean variable with values of 0 and 1 as an input that a gate maps to a single Boolean variable output.

Global Perspectives:

Mathematical logic is the underlying principle of coding. Boolean logic gates and circuits encode statements and create conditions in electrical engineering and computers. Understanding Boolean logic creates more powerful search criteria when using search engines such as Google, Bing, and DuckDuckGo. Evaluating logical arguments is an increasingly important skill for being an informed participant in a democracy. For example, logic is useful in determining the truthfulness of statements made by political candidates, pundits, and political broadcasters.

Resources (Open Access:)

- Online Stanford Textbook
- Open Text: Unit Three
- Stanford Intro to Logic Course for Secondary Education

Unit 5 Unpacking

Source: NC DPI Discrete Mathematics for Computer Science Unpacking Document. Retrieved from https://www.dpi.nc.gov/nc-discrete-mathematics-computer-science-unpacking-rev-june-2022

Standard: DCS.L.1.1

DCS.L.1.1 Construct truth tables that encode the truth and falsity of two or more statements.

Clarification

In this course, students are expected to construct truth tables. In other objectives, students will use truth tables as a tool to solve problems or justify claims.

Student should be able to interpret and use the following logic symbols:

$\neg p$
The negation of p
“not p”

$\sim p$
the negation of p
“not p”

$p \wedge q$
conjunction of p and q
“p and q”

$p \vee q$
disjunction of p and q
“p or q”

$p \oplus q$
exclusive disjunction of
p or q
“p XOR q”

$p \rightarrow q$
p implies q
“If p then q”

For this course, the problems will be limited to a total of 3 arguments (premises).

Checking for Understanding

Indicator: Construct a truth table for the statement:

~P⋀(P→Q)

Answer:

P	**Q**	**~P**	**(P→Q)**	**~P⋀(P→Q)**
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Indicator: Construct a truth table for the following scenario.
You are told that if you are passing your classes, finished your chores, and did not get grounded, then you can go out with your friends for the weekend.

Answer:

Sample answer. Let P be for passing grades, C for completed chores, and G for being grounded.

$(P \wedge C) \wedge {\sim}T$

P	C	$P \wedge C$		$P \wedge C$	G	${\sim}G$	$(P \wedge C) \wedge {\sim}G$
T	T	T		T	T	F	F
T	F	F		T	F	T	T
F	T	F		F	T	F	F
F	F	F		F	F	T	F
				F	T	F	F
				F	F	T	F
				F	T	F	F
				F	F	T	F

Standard: DCS.L.1.2

DCS.L.1.2 Critique logic arguments (e.g. determine if a statement is valid or whether an argument is a tautology or contradiction).

Clarification

In critiquing logical arguments, students should be able to determine the truthfulness of an argument and any claims made in the justification of that argument. Students should be able to suggest corrections, connections, or next steps as needed.

As part of determining the truth of an argument, students should be familiar with:

- the law of syllogism,
- the law of detachment,
- tautologies, assertions that are true in all interpretations, and
- contradictions, assertions that are incompatible or incongruous.

To evaluate logic statements, students should also be able to understand and use the following logic terms:

- converse
- inverse
- contrapositive
- “if and only if”

Checking for Understanding

Indicator: Determine the validity of the following statement:
If you invest in Corporation X, then you get rich. You didn't invest in Corporation X. Therefore, you didn't get rich.

Answer:

- *Let p be the statement "You invest in Corporation X."*
- *Let q be the statement "You get rich."*
- *Then the argument has this symbolic form:*

$$\begin{array}{c} p \rightarrow q \\ {\sim}p \\ \hline \therefore {\sim}q \end{array}$$

- *Interpret the truth table.*

		premise	premise	conclusion
p	q	$p \rightarrow q$	${\sim}p$	${\sim}q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Notice that in the third row, the conclusion is FALSE while both premises are TRUE. This tells us that the argument is INVALID.

Indicator: Show that the following logical argument is a tautology.

$$(P \rightarrow Q) \vee (Q \rightarrow P)$$

Answer:

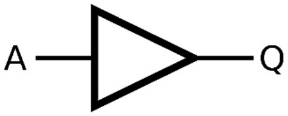
P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Standard: DCS.L.1.3

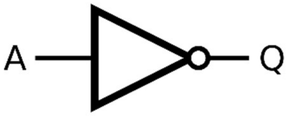
DCS.L.1.3 Check 1s and 0s to determine whether a statement is true or false using Boolean logic circuits.

Clarification

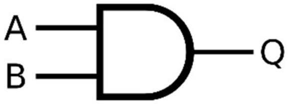
Students should be able to interpret and use symbols used in Boolean logic circuits, composed of logic gates, to determine the truthfulness of statements. Students should be able to interpret and use the following symbols:



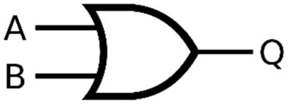
Buffer Gate
Input A
Output A



NOT Gate
Input A
Output ~A

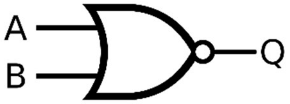


AND Gate
Inputs A,B
Output A⋀B

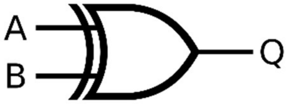


OR Gate
Inputs A,B
Output A⋁B

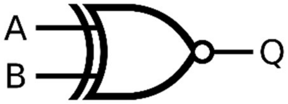
NAND Gate
Inputs A,B
Output ~(A⋀B)



NOR Gate
Inputs A,B
Output ~(A⋁B)



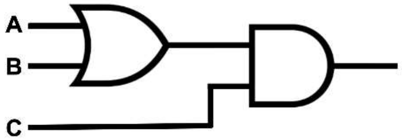
XOR Gate
Inputs A,B
Output A⨁B



XNOR Gate
Inputs A,B
Output ~(A⨁B)

Checking for Understanding

Indicator: If A = 0 and B = 1 and C=0, find whether the following statement is true (1) or false (0) and explain:



Answer:

The first gate is A OR B. With A being off (false) and B being on (true), A OR B is on (true).

The second gate is (A OR B) AND C. With A OR B being on (true) and C being off (false), the second gate is off (false).

Students should interpret a “1” as being true and a “0” as being false. Students should be able to check the truthfulness of statements that include multiple gates.

Standard: DCS.L.1.4

DCS.L.1.4 Judge whether two statements are logically equivalent using truth tables.

Clarification

Two logic statements are equivalent if the inputs and resulting outputs are the same. For this objective, students will use truth tables to demonstrate the claim of equivalency.

Checking for Understanding

Indicator: Use a truth table to prove the following statement.

$$p \to q = (q \vee {\sim} p)$$

Answer:

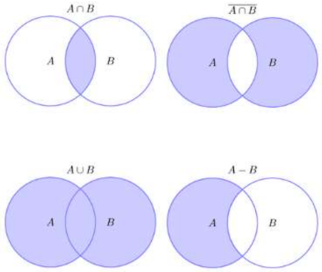
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	${\sim}p$	$q \vee {\sim}p$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Since the all sets of inputs return the same outputs, respectively, these two statements are logically equivalent.

Unit 6

Set Theory



Unit 6: Set Theory

Source: NC^{2}ML Retrieved from: https://www.nc2ml.org/wp-content/uploads/2020/04/Discrete-Mathematics-for-Computer-Science-Instructional-FrameworkFINAL.pdf

Content Standard(s):

DCS.N.3 Understand set theory to solve problems.

DCS.N.3.1

Recognize sets, subsets, and proper subsets.

DCS.N.3.2

Implement set operations to find unions, intersections, complements and set differences with multiple sets.

DCS.N.3.3

Represent properties and relationships among sets using Venn diagrams.

DCS.N.3.4

Interpret Venn diagrams to solve problems.

DCS.N. 4 Understand statements related to number theory and set theory.

DCS.N.4.3

Conclude that sets are equal using the properties of set operations.

What is the Mathematics?

Students have studied some ideas from set theory in the context of data and probability with Venn diagram representations in 7th grade and NC Math 2 (NC.M2.S-CP.6-8). This unit builds upon these prior understandings and experiences to scaffold a more formal understanding of sets and set operations. Students will use Venn diagrams that represent 2 or 3 sets to understand the meanings of sets, subsets, and proper subsets, their properties, relationships among them, and operations on them (unions, intersections, complements, and differences). Along with Venn diagrams, they use set properties to investigate if sets are equivalent.

Vocabulary:

Complement S^c

Element ($\in or \notin$)

Empty Set { }

Intersection (∩)

Null Set ∅

Proper Subset (⊂)

Set Difference (−)

Subset (⊆)

Union (∪)

Universal Set U

Venn Diagram

Important Considerations:

- Students have had formal and informal experiences with sets prior to this course. Students should recognize subsets and proper subsets (DMC.N.3.1) of finite and infinite sets. For example, students should recognize the real numbers and the imaginary numbers proper subsets of the complex numbers and the natural numbers as a subset of the set of positive integers greater than 0.

- Venn diagrams are useful representations of sets, and students should use them to investigate set properties, operations on two or more sets, and the relationships among them (DMC.N.3.2-4).
- Students should operate on sets in contexts that are relevant and of interest to them (). They should explore set properties and relationships and use them to solve problems (DCS.N.3.3) and show that two sets are equal (DCS.N.4.3)

Global Perspectives:

Set theory is a fundamental idea of computer science. Computer scientists use set theory as a tool to reason formally about computation the objects of computation. Sets are the underlying mathematical tool for many common computing functions like matching and searching. Set theory is used to design data structures that are efficient and conserve storage space, network security systems, and is the basis for programming languages.

Resources (Open Access):

- Open Text: Unit 0

Unit 6 Unpacking

Source: NC DPI Discrete Mathematics for Computer Science Unpacking Documents. Retrieved from https://www.dpi.nc.gov/nc-discrete-mathematics-computer-science-unpacking-rev-june-2022

Standard: DCS.N.3.1

DCS.N.3.1 Recognize sets, subsets, and proper subsets.

Clarification

Since middle school students have been informally working with the concept of a set. In this course, students are formalizing the definition and notation of sets, subsets, and proper subsets.

Students should be able to interpret set notation to identify subsets and proper subsets. Student should be able to recognize the following set notation symbols:

{ }	∅	∈	∉
empty set*	null set*	"is an element in"	"is not an element in"
|	N	Z	Q
"such that"	natural numbers	integers	rational numbers
R	⊆	⊂	U
real numbers	subset	proper subset	universal set

** In set theory, the empty set and null set have the same meaning.*

Checking for Understanding

Indicator: Let $B = \{n \in Z \mid n^2 < 20\}$.

a. Describe B using the listing method.

b. List subsets of {a, b, c}.

c. List subsets of {a, b, c, d}.

Answer:

- *a. {-4, -3, -2, -1, 0, 1, 2, 3, 4}*
- *b. ∅, {a}, {b}, {c}, {a,b}, {b,c}, {a,c}, {a,b,c}*
- *c. ∅, {a}, {b}, {c}, {d}, {a,b}, {b,c}, {a,c}, {a,d}, {b,d}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, {a,b,c,d}*

Standard: DCS.N.3.2

DCS.N.3.2 Implement set operations to find unions, intersections, complements and set differences with multiple sets.

Clarification

Students were introduced to unions, intersections, complements in NC Math 2 in the context of describing a sample space. In this course, students will be asked to work with multiple sets and to find set differences.

Student should be able to recognize the following set operation symbols:

Checking for Understanding

Formative Check: Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{2,4,6,8,10\}$, $B = \{1,3,5,7,9\}$, and $C = \{1,2,3,4\}$.

Describe each of the following sets using the listing method.

a. $(A \cup B) \cap C$ b. $A \cap B \cap C$ c. A^c

d. $(A \cup C)'$ e. $A \backslash C$ f. $C - B$

Answer:

a. {1,2,3,4} *b. ∅* *c. {1,3,5,7,9}*

d. {5,7,9} *e. {6,8,10}* *f. {2,4}*

∪ union	∩ intersection	A' complement of set A
A^c complement of set A	$A - B$ set difference, “A cut down by B”	$A\backslash B$ set difference, “A cut down by B”

Please note that some set operations have multiple common symbols. Set difference can also be called the relative difference. For example, $A - B$ can be called the relative difference between A and B.

Indicator: In a recent survey, 278 people were asked if they had visited 3 popular restaurants: Macburger (MB), Pizza House (PH), and Dairy Blast (DB). Below are the survey results.

MB	PH	DB	MB & PH	MB & DB	PH & DB	MB, PH, & DB
112	103	94	39	34	40	18

a. How many people did not visit any of the restaurants?
b. How many people only visited Dairy Blast?
c. Explain where in this problem you see examples of unions, intersections, complements and set differences.

Answer:

a. 64
b. 38
c. sample answers, MB & PH is an example of an intersection, visited only DB is set difference of the union of MB & PH

Standard: DCS.N.3.3

DCS.N.3.3 Represent properties and relationships among sets using Venn diagrams.

Clarification

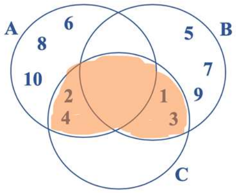
Students use Venn diagrams to represent the relationships between sets. Venn diagrams are often used to visually represent set operations.
Students can use Venn Diagrams to explore the properties of set operations.
Students should know where to represent all of the elements in the universal set.

Checking for Understanding

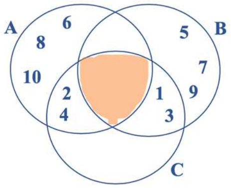
Indicator: Let *U*= {1,2,3,4,5,6,7,8,9,10}, *A*= {2,4,6,8,10}, *B*= {1,3,5,7,9}, and *C*= {1,2,3,4}.
Describe each of the following set operations using Venn diagrams.

a. $(A \cup B) \cap C$
b. $A \cap B \cap C$
c. A^c
d. $(A \cup C)'$
e. $A\backslash C$
f. $C - B$

Answers:



a.



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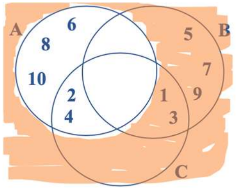

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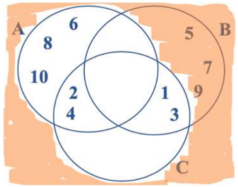
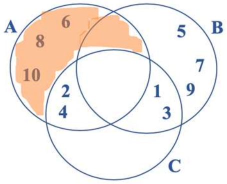
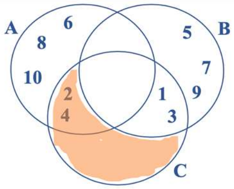
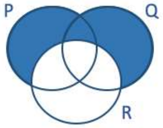
d. 

e. 

f. 

Indicator: Create and shade the portion of the Venn diagram that illustrates the following set: $(P \cup Q) \cap R^c$

Answer:

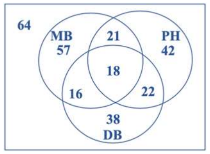


Indicator: In a recent survey, 278 people were asked if they had visited 3 popular restaurants: Macburger (MB), Pizza House (PH), and Dairy Blast (DB). Below are the survey results.

MB	PH	DB	MB & PH	MB & DB	PH & DB	MB, PH, & DB
112	103	94	39	34	40	18

Draw a Venn Diagram to represent this situation. Find any information not given.

Answer:



Standard: DCS.N.3.4

DCS.N.3.4 Interpret Venn diagrams to solve problems.

Clarification

Venn diagrams can be an important tool to organize data from sets.
This organizational tool can assist in solving problems.

Checking for Understanding

Indicator: In a recent survey, 278 people were asked if they had visited 3 popular restaurants: Macburger (MB), Pizza House (PH), and Dairy Blast (DB). Below are the survey results.

MB	PH	DB	MB & PH	MB & DB	PH & DB	MB, PH, & DB
112	103	94	39	34	40	18

Using a Venn Diagram, answer the following questions.

a. How many people did not visit any of the restaurants?
b. How many people only visited Dairy Blast?
c. How many people visited Macburger and Pizza House but not Dairy Blast?

Answers:

a. 64
b. 38
c. 21

Standard: DCS.N.4.3

DCS.N.4.3 Conclude that sets are equal using the properties of set operations.

Clarification

Students may use a variety of tools to determine the equality of sets, including the listing method, Venn diagram, and logical statements. Sets are equivalent when the sets contain the same elements.
Students are expected to know the basic properties of set operations for unions and intersections.

Checking for Understanding

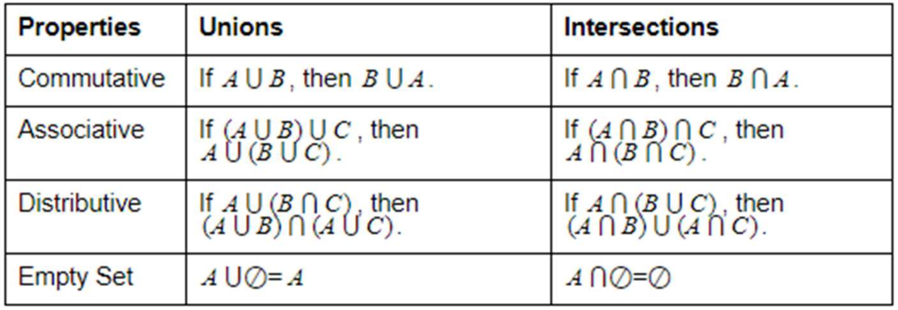
Indicator: Make a conjecture about the following:

a. the number of subsets of a set with five elements.
b. the number of subsets of a set with n elements.

Answers:

a. 2^{5}*subsets*
b. 2^{n}*subsets*

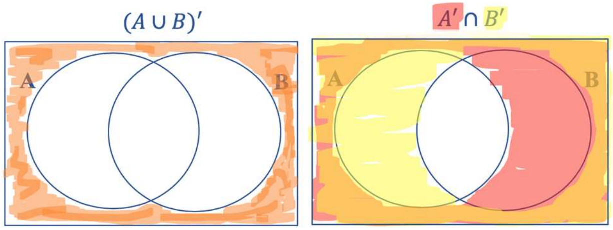
Indicator: Provide a justification for the following:
If A and B are sets, then $(A \cup B)' = A' \cap B'$



Properties	Unions	Intersections
Commutative	If $A \cup B$, then $B \cup A$.	If $A \cap B$, then $B \cap A$.
Associative	If $(A \cup B) \cup C$, then $A \cup (B \cup C)$.	If $(A \cap B) \cap C$, then $A \cap (B \cap C)$.
Distributive	If $A \cup (B \cap C)$, then $(A \cup B) \cap (A \cup C)$.	If $A \cap (B \cup C)$, then $(A \cap B) \cup (A \cap C)$.
Empty Set	$A \cup \emptyset = A$	$A \cap \emptyset = \emptyset$

Answer:

Using Venn Diagrams to show equivalence.

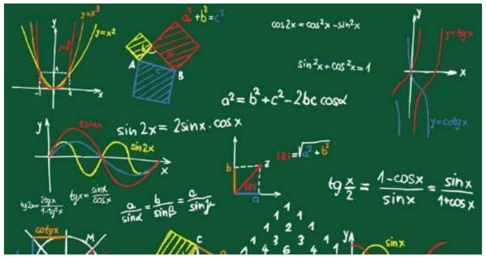


$(A \cup B)'$

$A' \cap B'$

Unit 7

Number Theory



Unit 7: Number Theory

Source: $\mathrm{NC}^2\mathrm{ML}$ Retrieved from: https://www.nc2ml.org/wp-content/uploads/2020/04/Discrete-Mathematics-for-Computer-Science-Instructional-FrameworkFINAL.pdf

Content Standard(s):

DCS.N. 4 Understand statements related to number theory and set theory.

DCS.N.4.1

Use the Euclidean Algorithm to determine greatest common factor and least common multiple.

DCS.N.4.2

Use the Fundamental Theorem of Arithmetic to solve problems.

DCS.N.4.4

Explain theorems related to greatest common factor, least common multiple, even numbers, odd numbers, prime numbers, and composite numbers.

What is the Mathematics?

This unit builds on students' experiences with numbers since the elementary grades, their understanding of sets, and knowledge of real number operations to explore some aspects of number theory relevant to computer science and engage in algorithmic thinking. IN this course, students will explore methods for determining the GCF and LCM, including the Euclidean algorithm. They will also investigate and generalize properties of numbers from patterns to classify sets of numbers (e.g., even, odd, prime, and composite) and the set of numbers that result from arithmetic operations on them (e.g. the sum of two odd numbers is even, the product of two odd numbers is odd. Using the Fundamental Theorem of Arithmetic, students will solve problems by representing any integer as a unique product of prime numbers.

Vocabulary:

Euclidean Algorithm	Fundamental Theorem of Arithmetic

Important Considerations:

- This unit is not intended to be a deep dive into Number Theory. Rather, it is intended to be an introduction to ideas central to number theory ideas that are useful in computer science.
- Students should investigate the Euclidean algorithm and using snap cubes or an area model with graph paper to show the process of finding the greatest common divisor of two numbers and justifying the result is the greatest (DCS.N.4.1).
- The Fundamental Theorem of Arithmetic (DCS.N.4.2) shows that every integer can be uniquely expressed as the product of prime numbers. Students should use this result and skills involving exponents to solve problems involving multiplication, division, even and odd, prime composite, perfect squares, and perfect cubes.

- Students should explore and explain algorithms for finding, and theorems about, the greatest common factors, least common multiples, even and odd numbers, and prime and composite numbers. They should develop solid arguments using words, figures, or algebra to explain why each works (DCS.N.4.4)

Global Perspectives:

Computer scientists use Number Theory when developing new algorithms and data structures. Results from number theory offer insight into underlying relationships among numbers which often simplifies algorithms and makes them more efficient. Number theory plays a fundamental role in cryptography, from encryption algorithms to cryptocurrencies.

Resources (Open Access):

- Open Text: Unit Five

Unit 7 Unpacking

Source: NC DPI Discrete Mathematics for Computer Science Unpacking Documents. Retrieved from https://www.dpi.nc.gov/nc-discrete-mathematics-computer-science-unpacking-rev-june-2022

Standard: DCS.N.4.1

DCS.N.4.1 Use the Euclidean Algorithm to determine the greatest common factor and least common multiple.

Clarification

Students were taught GCF and LCM, using prime factorization and other methods in 6th grade. In middle school, students relied on their intuition and experience to guide the process. In this standard, students will learn an iterative process, the Euclidean Algorithm, to find the GCF and LCM. This can serve as a point of comparison between human intuition and iterative processes needed in computer science and programming.

Students should have an understanding of the mathematical reasoning behind the Euclidean Algorithm and finding the LCM. (See DCS.N.4.4)

Euclidean Algorithm

1. Find the gcf of *a* and *b*, letting *a* be the largest number. Notated as gcf(*a*,*b*). The largest possible factor of *a* and *b*, would be *b*. If $(a) \div b$ has a remainder of zero, *b* is the gcf. If the remainder of $(a) \div b$ is not zero, then we know that $a = n \cdot b + r_1$, where *n* is an integer and r_1 is the remainder.
2. If $b \div r_1$ has a remainder of zero, r_1 is the gcf. If the remainder is not zero, then we know that $b = n \cdot r_1 + r_2$, where *n* is a number and r_2 is the remainder.
3. If $r_1 \div r_2$ has a remainder of zero, r_2 is the gcf. If the remainder is not zero, then we know that $r_1 = n \cdot r_2 + r_3$, where *n* is a number and r_3 is the remainder.
4. Repeat this pattern until the remainder is 0. The divisor of the expression that produced the remainder of 0 is the gcf(*a*,*b*)

Using the Euclidean Algorithm to find the GCF, the LMC a and b can be found using the gcf(a, b). $lcm(a, b) = \frac{a \cdot b}{gcf(a,b)}$

Students should be able to recognize other mathematical terms that should also be interpreted as GCF or LCM, such as greatest common divisor or least common denominator.

Checking for Understanding

Indicator: Use the Euclidean algorithm to find the greatest common factor of 420 and 130.

Answer:

1. *gcf (420, 130): 420 ÷ 130= 3 r30*
2. *gcf (130, 30): 130 ÷ 30= 4 r10*
3. *gcf (30, 10): 30 ÷ 10= 3 r0*
   Since 10 divides 30, then gcf(420,130)=10.

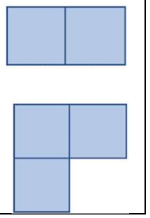
Indicator: Use the Euclidean algorithm to find the least common multiple of 420 and 130.

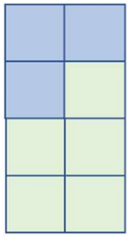
Answer:

Since gcf(420,130)=10, then lcm(420,130)=420 ·130 ÷ 10=5460

<table>
<tr><th colspan="2">Standard: DCS.N.4.2</th></tr>
<tr><td colspan="2">DCS.N.4.2 Use the Fundamental Theorem of Arithmetic to solve problems.</td></tr>
<tr><td>Clarification</td><td>Checking for Understanding</td></tr>
<tr><td>Students began working with this concept when writing the prime factorization of a number and found the GCF and LCM in 6th grade.
In this course, students will use their knowledge of the Fundamental Theorem of Arithmetic to solve various problems. Some of these problems may include:
• Multiplication and Division
• Determining if a number is: odd, even, prime, composite, perfect square, perfect cube
Students should have an understanding of these relationships. (See DCS.N.4.4)</td><td>Indicator: Use the Fundamental Theorem of Arithmetic to find the least positive integer <i>n</i> such that $(2^5)(3)(5^2)(7^5)(n)$ is a perfect square. Write the resulting product as a perfect square. Explain.
<i>Answer:</i>
<i>To be a perfect square, all of the exponents must be divisible by 2.</i>
$n = 2 \cdot 3 \cdot 7 = 42 - n$<i>, must contain the “missing” prime factors necessary to make the exponents divisible by 2.</i>
$(2^5)(3)(5^2)(7^5)(42)$ - <i>substitute 42 for n</i>
$(2^6)(3^2)(5^2)(7^6)$ - rewrite the expression as a product of prime factors
$((2^3)(3^1)(5^1)(7^3))^2$- rewrite the expression as a perfect square
$41{,}160^2$- evaluate the product of prime factors</td></tr>
<tr><th colspan="2">Standard: DCS.N.4.4</th></tr>
<tr><td colspan="2">DCS.N.4.4 Explain theorems related to greatest common factor, least common multiple, even numbers, odd numbers, prime numbers, and composite numbers.</td></tr>
<tr><td>Clarification</td><td>Checking for Understanding</td></tr>
<tr><td>The expectation of this standard is that students will be able to informally explain the mathematical reasoning behind theorems related to the topics listed and how they relate to a problem being solved. For example, students should not only be able to explain the Fundamental Theorem of Arithmetic, but would also be able to explain how the student uses the theorem to solve a problem.

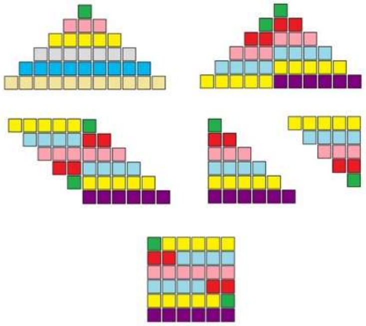
The theorems targeted for students' explanations should be those that are used in other standards throughout this course, such as those seen in:
• DCS.N.4.1 - GCF, LCM
• DCS.N.4.2 - Even, odd, prime, and composite numbers</td><td>Indicator: Suppose x is an odd integer and y is an odd integer. Is $x + y$ an even or odd integer? Provide a justification.
<i>Answer: Sample answers below</i>
<i><u>Algebraic approach</u></i>
<i>If</i> $2n$ <i>is an even number, then</i> $2n + 1$ <i>is an odd number.</i>
$(2n_1 + 1) + (2n_2 + 1)$<i>, adding two odd numbers</i>
$2n_1 + 2n_2 + 2$<i>, rewriting using the properties of operations</i>
$2(n_1 + n_2 + 1)$<i>, rewriting the expression by factoring a two.</i>

<i>This shows that the sum of two odd numbers is a multiple of 2, which is the definition of an even number.</i>
<i><u>Drawing Approach</u></i>
<i>If a number is represented as objects in two columns, an even number will always have a complete row and an odd number will always have one row with only one object. (See to the right.)</i>
</td></tr>
</table>

	When adding two odd numbers, each number has one row with one object. As the numbers are added, the two rows with one object will be combined to make a complete row. With all rows having two objects, the sum represents an even number. 

Unit 8

Recursion



Unit 8: Recursion

Source: NC^2ML Retrieved from: https://www.nc2ml.org/wp-content/uploads/2020/04/Discrete-Mathematics-for-Computer-Science-Instructional-FrameworkFINAL.pdf

Content Standard(s):

DCS.F.1 Apply recursively-defined relationships to solve problems.

DCS.F.1.1

Implement procedures to find the nth term in an arithmetic or geometric sequence using spreadsheets.

DCS.F.1.2

Represent the sum of a sequence using sigma notation.

DCS.F.1.3

Implement procedures to find the sum of a finite sequence.

DCS.F.1.4

Implement procedures to find the sum of an infinite sequence and determine if the series converges or diverges.

DCS.F.1.5

Interpret the solutions to arithmetic and geometric sequences and series problems, in context.

What is the Mathematics?

In this unit, students extend their knowledge of recursively defined arithmetic and geometric sequences from NC Math 1 using technology (e.g. spreadsheets) to find the nth term of an arithmetic and geometric sequence. They will develop an understanding of methods and formulas for computing sums of finite arithmetic and geometric series, use them to solve problems, and represent series in sigma notation. They will determine if an infinite geometric series is convergent or divergent and find the sum if it converges.

Vocabulary:

Arithmetic sequence Arithmetic series Closed form/Explicit form Convergent	Divergent Geometric sequence Geometric series

Important Considerations:

- DCS.F.1.1 extends students' understanding of arithmetic and geometric sequences as linear and exponential functions. In this course, students learn to use technology to find the nth term of sequences. Technology can assist students in making connections between arithmetic and geometric growth by comparing graphical, tabular, and algebraic representations of a sequence. These representations are useful in developing intuition about convergent and divergent series (DCS.F.1.4).

- Students should know that a series is the sum of a number of terms of a sequence and understand the reasoning behind common formulas for arithmetic and geometric series (DCS.F.1.3). For example, recognizing the average value of pairs of terms in an arithmetic sequence leads to the formula for the sum of an arithmetic sequence. Another line of reasoning is that one can form a sequence of partial sums and the nth term of this sequence is equal to the sum of the first n terms of the series. For example, the series 1 + 3 + 5 + … + $(2n + 1)$ is equivalent to the sequence of partial sums 1, 4, 9, …n_2. Using this fact, the sum of the first n terms of the series 1 + 3 + 5 + … + $(2n + 1)$ = n_2.
- Just as NC Math 1 students begin with Next-Now notation but move to more formal ways of notating recursively defined sequences, students in this course should learn a more formal notation for series. Assist students in recognizing underlying structure and expressing it using sigma notation (DCS.F.1.2). Sigma notation is new for students, and it is to highlight and discuss when it is appropriate to use 0 or 1 as initial value. With sigma notation, make explicit that the index counts the number of items in the set and not the number of intervals between the first and last element. It is also important to make the distinction between indexing the term number in a series and the natural numbers themselves.
- Students should have opportunities to develop understandings of the ideas in this unit as well as solve problems and interpret solutions in meaningful contexts (DCS.F.1.5).

Global Perspectives:

In computer science, recursion is a concept where solutions to a problem rely on solutions to smaller instances of the same problem. Computer scientists use recursion in when creating models and simulations, when solving complex computational problems, defining dynamic data structures, developing algorithms, and generating computer graphics. Recursive thinking involves decomposing a large task into smaller processes to simplify calculations and improve efficiency.

Resources (Open Access):

- Handshake Problem/Handshake Problem Reflections
- Sequences Puzzle
- Open Text: Unit 3

Unit 8 Unpacking

Source: NC DPI Discrete Mathematics for Computer Science Unpacking Documents. Retrieved from https://www.dpi.nc.gov/nc-discrete-mathematics-computer-science-unpacking-rev-june-2022

Standard: DCS.F.1.1

DCS.F.1.1 Implement procedures to find the *nth* term in an arithmetic or geometric sequence using spreadsheets.

Clarification

This mathematical content of this standard has been covered in previous high school math courses.

The expectation of this standard is that students solve arithmetic and geometric problems using a spreadsheet. The spreadsheet application used is at the discretion of the instructor. For some students, this may require instruction on the use of the chosen spreadsheet application.

Checking for Understanding

Formative Check: Given two terms in a geometric sequence, find the 8th term and the recursion formula.

$$a_5 = 768 \text{ and } a_2 = 12$$

Answers:

$a_8 = 49,125$

$a_n = 4 \cdot a_{n-1}$

Indicator: Given a term in an arithmetic sequence and the common difference, find the recursive formula and the three terms in the sequence after the last one given.

$$a_{22} = -44 \text{ and } d = -2$$

Answers:

$a_{23} = -46$

$a_{24} = -48$

$a_{25} = -50$

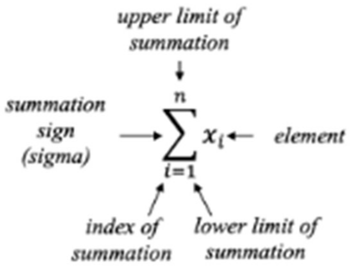
$a_n = a_{n-1} - 2$

Standard: DCS.F.1.2

DCS.F.1.2 Represent the sum of a sequence using sigma notation.

Clarification

Students should be able to write a summation using sigma notation.



Checking for Understanding

Indicator: Use sigma notation to represent the sum of the first 10 squares, starting with 1.

Answer:

$$\sum_{x=1}^{10} x^2$$

Indicator: Use sigma notation to represent the 7 elements of a series. The elements of the series are the difference of three times an element of set *a* and 1.

Students should be able to distinguish when the index will be used directly in the calculation of the sum and when the index is a reference to an element of a set.	*Answer:* $\sum_{i=1}^{7}(3a_i - 1)$

Standard: DCS.F.1.3

DCS.F.1.3 Implement procedures to find the sum of a finite sequence.

Clarification	Checking for Understanding
Students are expected to find the sum of a finite sequence. Students should be able to distinguish between a sequence and a series. For this standard, students may be asked to find the sum of a sequence or series. Students should be able to interpret the use of ellipsis, …, in a mathematical context. Students may use the formulas for sum of finite arithmetic and geometric sequences.	**Formative Check**: Find the sum of the given geometric or arithmetic sequence or series. a. $\sum_{n=1}^{11} 4\left(-\frac{1}{3}\right)^{n-1}$ b. $5,15,45,\ldots,98415$ c. $b = \{3,9,12,21\}$ $\sum_{i=2}^{3}\frac{1}{3}{b^2}_i$ *Answers:* *a. about 3* *b. 147,620* *c. 75* **Indicator**: Sarah puts \$300 in a savings account that earns 5.25% compounded annually. a. Write a recursive relation for the situation. b. During what year will Sarah double her money? *Answers:* *a.* $a_n = 1.0525 \cdot a_{n-1}$ *b. During the middle of the 13th year*

Standard: DCS.F.1.4

DCS.F.1.4 Implement procedures to find the sum of an infinite sequence and determine if the series converges or diverges.

Clarification	Checking for Understanding
For this standard, students may be asked to find the sum of a sequence or series. Students should be able to interpret the use of ellipsis, …, in a mathematical context.	**Formative Check**: Find the sum of the series.

Students should be able to use partial sums to verify whether a sequence is convergent or divergent. If the sequence converges, students should be able to hypothesize the value the sum approaches. Students may use the formula for the sum of infinite geometric sequences.	a. $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$ b. $5+20+80+320+1280+\ldots$ *Answers:* *a. 2* *b. ∞* **Indicator**: Explain how you know if the following series converges or diverges. If it does converge, find the sum. $\sum_{n=1}^{\infty}3(.75)^{n-1}$ *Answer:* *Since $\lvert r\rvert < 1$, each successive element will be smaller than the previous until at a point at which the magnitude of the element will no longer change the sum. This series will converge at 12.* **Indicator**: A tennis ball is launched from the ground. It travels a total of 47 feet from the launch to the first contact with the ground. In each successive bounce, the tennis ball will travel 53% of the previous bounce. How far will the tennis ball travel in total? *Answer: 100 ft*

Standard: DCS.F.1.5

DCS.F.1.5 Interpret the solutions to arithmetic and geometric sequences and series problems, in context.

Clarification	**Checking for Understanding**
Students should be able to describe the solution to problems involving sequence and series problems using the context of the problem.	**Indicator**: An auditorium has 18 seats in the first row. Each successive row has two additional seats. The last row has 84 seats. a. Write a relation for the number of seats in each row. b. What information can you obtain from that relation?

	Answers: *a.* $s_r = s_{r-1} + 2 \text{ or } s = 2(r-1) + 18$ *b. the number of seats in any row, the total number of seats or rows in the auditorium* **Indicator**: A tennis ball and super ball are launched from the ground. Each ball travels to a height of 100 ft. The height a tennis ball travels during each successive bounce is reduced by 47%. The height a super ball travels during each successive bounce is reduced by 15%. Jon is 6 ft tall. After how many bounces will each ball’s height be less than Jon’s height? *Answer:* *Tennis ball - 5 bounces, super ball - 18 bounces*

TOOL 2

HESS COGNITIVE RIGOR MATRIX (MATH-SCIENCE CRM):

Applying Webb's Depth-of-Knowledge Levels to Bloom's Cognitive Process Dimensions

Revised Bloom's Taxonomy	Webb's DOK Level 1 Recall & Reproduction	Webb's DOK Level 2 Skills & Concepts	Webb's DOK Level 3 Strategic Thinking/Reasoning	Webb's DOK Level 4 Extended Thinking
Remember Retrieve knowledge from long-term memory, recognize, recall, locate, identify	o Recall, observe, & recognize facts, principles, properties o Recall/ identify conversions among representations or numbers (e.g., customary and metric measures)	Use these Hess CRM curricular examples with most mathematics or science assignments or assessments.		
Understand Construct meaning, clarify, paraphrase, represent, translate, illustrate, give examples, classify, categorize, summarize, generalize, infer a logical conclusion), predict, compare/contrast, match like ideas, explain, construct models	o Evaluate an expression o Locate points on a grid or number on number line o Solve a one-step problem o Represent math relationships in words, pictures, or symbols o Read, write, compare decimals in scientific notation	o Specify and explain relationships (e.g., non-examples/examples; cause-effect) o Make and record observations o Explain steps followed o Summarize results or concepts o Make basic inferences or logical predictions from data/observations o Use models /diagrams to represent or explain mathematical concepts o Make and explain estimates	o Use concepts to solve non-routine problems o Explain, generalize, or connect ideas using supporting evidence o Make and justify conjectures o Explain thinking/reasoning when more than one solution or approach is possible o Explain phenomena in terms of concepts	o Relate mathematical or scientific concepts to other content areas, other domains, or other concepts o Develop generalizations of the results obtained and the strategies used (from investigation or readings) and apply them to new problem situations
Apply Carry out or use a procedure in a given situation; carry out (apply to a familiar task), or use (apply) to an unfamiliar task	o Follow simple procedures (recipe-type directions) o Calculate, measure, apply a rule (e.g., rounding) o Apply algorithm or formula (e.g., area, perimeter) o Solve linear equations o Make conversions among representations or numbers, or within and between customary and metric measures	o Select a procedure according to criteria and perform it o Solve routine problem applying multiple concepts or decision points o Retrieve information from a table, graph, or figure and use it solve a problem requiring multiple steps o Translate between tables, graphs, words, and symbolic notations (e.g., graph data from a table) o Construct models given criteria	o Design investigation for a specific purpose or research question o Conduct a designed investigation o Use concepts to solve non-routine problems o Use & show reasoning, planning, and evidence o Translate between problem & symbolic notation when not a direct translation	o Select or devise approach among many alternatives to solve a problem o Conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze Break into constituent parts, determine how parts relate, differentiate between relevant-irrelevant, distinguish, focus, select, organize, outline, find coherence, deconstruct	o Retrieve information from a table or graph to answer a question o Identify whether specific information is contained in graphic representations (e.g., table, graph, T-chart, diagram) o Identify a pattern/trend	o Categorize, classify materials, data, figures based on characteristics o Organize or order data o Compare/ contrast figures or data o Select appropriate graph and organize & display data o Interpret data from a simple graph o Extend a pattern	o Compare information within or across data sets or texts o Analyze and draw conclusions from data, citing evidence o Generalize a pattern o Interpret data from complex graph o Analyze similarities/differences between procedures or solutions	o Analyze multiple sources of evidence o Analyze complex/abstract themes o Gather, analyze, and evaluate information
Evaluate Make judgments based on criteria, check, detect inconsistencies or fallacies, judge, critique	"UG"–unsubstantiated generalizations = stating an opinion without providing any support for it!		o Cite evidence and develop a logical argument for concepts or solutions o Describe, compare, and contrast solution methods o Verify reasonableness of results	o Gather, analyze, & evaluate information to draw conclusions o Apply understanding in a novel way, provide argument or justification for the application
Create Reorganize elements into new patterns/structures, generate, hypothesize, design, plan, produce	o Brainstorm ideas, concepts, or perspectives related to a topic	o Generate conjectures or hypotheses based on observations or prior knowledge and experience	o Synthesize information within one data set, source, or text o Formulate an original problem given a situation o Develop a scientific/mathematical model for a complex situation	o Synthesize information across multiple sources or texts o Design a mathematical model to inform and solve a practical or abstract situation