Math League SCASD 2020-21

Meet #5

Algebra

Self-study Packet

Problem Categories for this Meet (in addition to topics of earlier meets):

1. Mystery: Problem solving
2. Geometry: Solid Geometry (Volume and Surface Area)
3. Number Theory: Set Theory and Venn Diagrams
4. Arithmetic: Combinatorics and Probability
5. **Algebra: Solving Quadratics with Rational Solutions, including word problems**

Important things you need to know about **ALGEBRA:**

Solving quadratics with rational solutions, including word problems

- If xy = 0, then x = 0 or y = 0. This is called the Zero Product Property

- If (x – 3) (x + 2) = 0, then x – 3 = 0 *or* x + 2 = 0. The solutions to this problem are x =3 and x = -2

- When a graph crosses the x-axis, y = 0.

- To multiply binomials, such as (x – 4) (x + 6), we can use the distributive property. A mnemonic is **FOIL**. Foil means multiply the **F**irst, **O**utside, **I**nside, and **L**ast Terms.

  $(x – 4) (x + 6) = x^2 + 6x – 4x – 24 = x^2 + 2x – 24$

- You should notice that in the above example, the -4 and 6 add to equal 2 and multiply to equal -24. Use this knowledge to work backward to factor a trinomial.

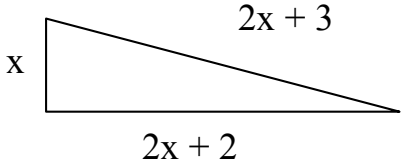
- Factor $x^2 – 7x + 12$ (Think: What are two numbers that multiply to equal 12 and add to equal -7? -3 and -4) So, $x^2 – 7x + 12 = (x – 3) (x – 4)$

- If $x^2 – 7x + 12 = 0$, then $(x – 3) (x – 4) = 0$, so x = 3 or x = 4

- The quadratic formula can also be used to solve quadratic equations.
  If $Ax^2 + Bx + C = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: How many units are in the shortest length of the right triangle below?



By Pythagorean Theorem, $x^2 + (2x + 2)^2 = (2x + 3)^2$
So, $x^2 + (2x + 2)(2x + 2) = (2x + 3)(2x + 3)$
Using FOIL, $x^2 + 4x^2 + 4x + 4x + 4 = 4x^2 + 6x = 6x + 9$
Combining like terms, $5x^2 + 8x + 4 = 4x^2 + 12x + 9$
Subtracting the right side to get everything on the left, we get $x^2 – 4x – 5 = 0$
So $(x – 5)(x+1) = 0$, Therefore, x = 5 or x = -1
A side length cannot be negative, so x must be 5.

- ✓ Check. If x = 5, 2x + 2 = 12 and 2x + 3 = 13. $5^2 + 12^2 = 13^2$

Category 5
Algebra ***Calculator Meet***
Meet #5 - April, 2019

1) There are two values of N that make the following quadratic equation true. What is the sum of those two values of N?

$$(N + 4)(N - 9) = 0$$

2) There are two values of W that make the following quadratic equation true. What is the positive difference, or the absolute value of the difference, between those two values of W?

$$10W = W^2 - 24$$

3) A rocket is launched vertically from ground level at an initial velocity (starting speed) of 128 feet per second. For how many seconds is the rocket at least 112 feet above ground level?

Use the quadratic equation $y = gt^2 + vt + h$

where **g = - 16 feet/second/second, the constant of gravity at the surface of the Earth,**
t is the time in seconds that the rocket is in the air,
v is the initial velocity,
h is the initial height of the rocket in feet, and
y is the height in feet of the rocket at any time t seconds.

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 5
Algebra
Meet #5 - April, 2019

Answers	
1)	5
2)	14
3)	6

1) If (N + 4)(N - 9) = 0, then either N + 4 = 0 or N - 9 = 0, then N = - 4 or N = 9. The sum of these solutions is -4 + 9, or 5.

2)

$$10W = W^2 - 24$$
$$0 = W^2 - 10W - 24$$
$$0 = (W-12)(W+2)$$

So, either W = 12 or W = -2. The positive difference between them is 12 - (-2), or 14.

3) Substitute: G = - 16; V = 128; Y = 112; H = 0.

$112 = (-16)(T^2) + 128T + 0$ **Use the substitutions listed above.**

$0 = -16(T^2) + 128T - 112$ **Subtract 112 from both sides.**

$0 = T^2 - 8T + 7$ **Divide both sides by - 16.**

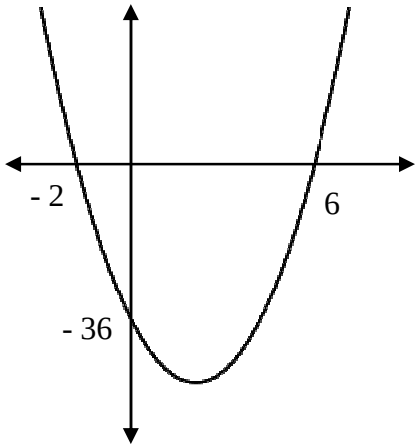
$0 = (T-1)(T-7)$ **Factor.**

T = 1 or T = 7. Therefore, the rocket was at or above 112 feet above the ground from 1 second until 7 seconds into the flight, so the rocket was in flight during the interval $1 \le T \le 7$ or the difference 7 - 1, or 6 seconds.

Category 5
Algebra ***Calculator Meet***
Meet #5 - April, 2017

1) If (X + 7) (5X - 2) = ax^2+bx+c, then what is the value of a + b + c ?

2) A quadratic equation is graphed so that the x-intercepts are (- 2, 0) and (6, 0) while the y-intercept is (0, - 36). When x = 10, then what is the value of y ?



3) The quadratic equation $H=-4.9t^2+vt+h$ approximates the height in feet, H, that an object will attain after t seconds when launched vertically from a height of h meters with an initial upward velocity (starting velocity, or speed) of v meters per second. Once the object reaches its maximum height, gravity will draw the object back toward Earth. A candlepin bowling ball is tossed vertically into the air from a platform that is 58.8 meters above the water at an initial upward velocity of 19.6 meters per second.
How many seconds after the launch did it take for the bowling ball to hit the water ?

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 5
Algebra
Meet #5 - April, 2017

Answers	
1)	24
2)	144
3)	6

1) Multiply the two binomials to determine the values of a, b, and c:

$(x+7)(5x-2) = 5x^2 + 35x - 2x - 14 = 5x^2 + 33x - 14$

So, a = 5, b = 33, and c = - 14.

a + b + c = 5 + 33 + (- 14) = 24

2) First find the value of a (the coefficient of the first term) by using the data that the two x-intercepts yield the factors (x + 2) (x - 6) and then substituting the value of a third known point for x and y:

$y = a(x+2)(x-6)$

$-36 = a(0+2)(0-6)$

$-36 = -12a$

$a = 3$

Now substitute 10 for x: $y = 3(10+2)(10-6)$, **so** $y = 144$.

3) $H = -4.9t^2 + vt + h$ **the given formula**

$H = -4.9t^2 + 19.6t + 58.8$ **Substitute 19.6 for v and 58.8 for h.**

$0 = -4.9t^2 + 19.6t + 58.8$ **Set the equation to zero, the height of the water.**

$0 = t^2 - 4t - 12$ **Divide both sides by - 4.9.**

$0 = (t-6)(t+2)$ **Factor.**

t = - 2 or t = 6 **Solve for t.**

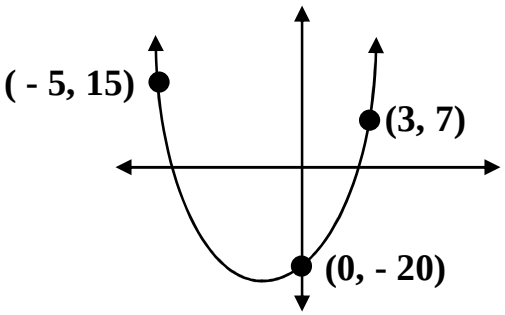
t = - 2 is extraneous, as it occurred prior to the launch, so we use t = 6.

Therefore, it took 6 seconds for the bowling ball to hit the water.

Category 5
Algebra
Meet #5 - March, 2015
Calculator meet

1) If (N + 3)(N - 7) = 0, then what is the average of the two possible values of N that make this quadratic equation true?

2) The quadratic equation $Y = aX^2 + bX + c$, when graphed, is a parabola that passes through the points (3, 7) and (-5, 15) and has a Y-intercept of (0, -20), as shown. What is the value of a + b + c ?



3) A rocket is launched vertically from ground level at an initial velocity (starting speed) of 128 feet per second. For how many seconds is the rocket at least 112 feet above ground level? Use the quadratic equation where g = - 16 feet/second/second, the constant of gravity at the surface of the Earth, t is the time in seconds that the rocket is in the air, v is the initial velocity, h is the initial height of the rocket in feet, and y is the height in feet of the rocket at any time t seconds.

$$y = gt^2 + vt + h$$

. . . . . .

ANSWERS

1) _____

2) _____

3) _____

Jonas Salk announced his development of the polio vaccine on March 26, 1953 . . . "It is always with excitement that I wake up in the morning, wondering what my intuition will toss up to me, like gifts from the sea. I work with it and rely on it. It's my partner."

Solutions to Category 5
Algebra
Meet #5 - March, 2015

Answers	
1)	2
2)	- 15
3)	6

1) If (N + 3)(N - 7) = 0, then either N + 3 = 0 or N - 7 = 0, then N = - 3 or N = 7. The average of these solutions is (- 3 + 7) / 2, or 2.

2) One possible strategy:

1: substitute the X and Y coordinates of each of the known points into the general quadratic equation,
2: solve the resulting system to find the values of a, b, and c, and then
3: find the sum a + b + c.

for (3, 7): $7 = a(3^2) + b(3) + c$ **. . . or, simplified,** $7 = 9a + 3b + c$

for (- 5, 15): $15 = a((-5)^2) + b(-5) + c$ **. . . or, simplified,** $15 = 25a - 5b + c$

for (0, - 20): $-20 = a((0)^2) + b(0) + c$ **. . . or, simplified,** $-20 = c$.

Now substitute - 20 for c in the first two equations, yielding
7 = 9a + 3b - 20 and 15 = 25a - 5b - 20
Simplifying: 27 = 9a + 3b and 35 = 25a - 5b
Divide both sides of the first equation by 3 and both sides of the second equation by 5, yielding: 9 = 3a + b and 7 = 5a - b. Adding the two equations yields: 16 = 8a, so, a = 2 and then b = 3. So, a + b + c = - 15.

3) Substitute: G = - 16; V = 128; Y = 112; H = 0.

$112 = (-16)(T^2) + 128T + 0$	**Use the substitutions listed above.**
$0 = -16(T^2) + 128T - 112$	**Subtract 112 from both sides.**
$0 = T^2 - 8T + 7$	**Divide both sides by - 16.**
$0 = (T - 1)(T - 7)$	**Factor.**

T = 1 or T = 7. Therefore, the rocket was at or above 112 feet above the ground from 1 second until 7 seconds into the flight, so the rocket was in flight for the difference 7 - 1, or 6 seconds.

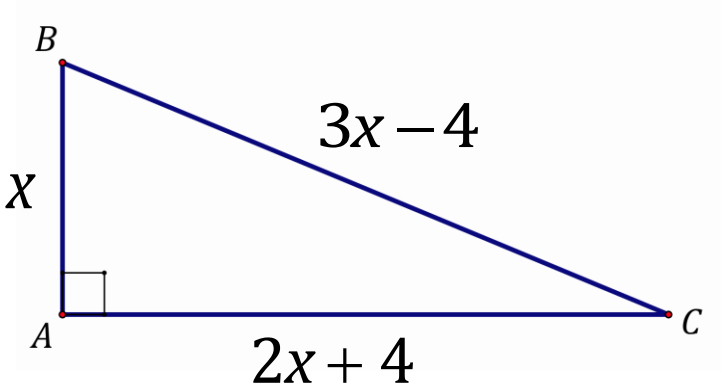
Category 5
Algebra
Meet #5, March/April 2013

You may use a calculator.

1. Terry is trying to graph a particular quadratic equation on a coordinate system. She found that the vertex of the parabola is at the point (–2.5, 90.25) and that the parabola crosses the *x*-axis at (7, 0). Find the coordinates of the other point where the parabola crosses the *x*-axis.

2. The sum of a number and its reciprocal is $2\frac{1}{30}$. What is the number if it is less than 1?

3. Triangle ABC below is a right triangle with a right angle at vertex A. How many square units are there in the area triangle ABC?



Answers

1. (_____,_____)
2. _____
3. _____ sq. units

Solutions to Category 5
Algebra
Meet #5, March/April 2013

Answers
1. (–12, 0)
2. $\frac{5}{6}$
3. 120 sq. units

1. We should not be put off by the large *y* value of the vertex. The important part is the *x* value of the vertex, which is –2.5. The parabola has a vertical line of symmetry where *x* = –2.5. We know that one root of the equation is at (7, 0), which is 9.5 units to the right of (–2.5, 0). The other root must be 9.5 units to the left of (–2.5, 0), which is **(–12, 0)**. Note: The points where the parabola crosses the *x* axis are also known as the "roots" of the equation.

2. If we call the unknown number *x* and its reciprocal 1/*x*, we get the equation $x+\frac{1}{x}=2\frac{1}{30}$, which is equivalent to $x+\frac{1}{x}=\frac{61}{30}$. If we multiply both sides of the equation by 30*x*, we get $30x^2+30=61x$. This is a quadratic equation, so we set it equal to zero as: $30x^2-61x+30=0$. This is not easily factored, but can be solved using the quadratic formula.
A more experienced mathlete might call the unknown number a/b and its reciprocal b/a. The sum is thus $\frac{a}{b}+\frac{b}{a}=\frac{a^2}{ab}+\frac{b^2}{ab}=\frac{a^2+b^2}{ab}$, which reveals much more about the structure of the problem. Our sum is 61/30, so we need to find two factors of 30 such that the sum of their squares is 61. The obvious numbers to try are 5 and 6, and indeed $\frac{5}{6}+\frac{6}{5}=\frac{25}{30}+\frac{36}{30}=\frac{61}{30}$.
The number must be **5/6**.

3. We can use the Pythagorean Theorem to solve for *x* as shown at right. The value zero doesn't make sense, so *x* must be 10 units. The length of the other leg must be 2 × 10 + 4 = 24 units. (The hypotenuse is 3 × 10 – 4 = 26 units.) The area of the triangle is thus 10 × 24 ÷ 2 = **120 square units**.

$$x^2+(2x+4)^2=(3x-4)^2$$
$$x^2+4x^2+16x+16=9x^2-24x+16$$
$$-4x^2+40x=0$$
$$-4x(x-10)=0$$
$$x=0 \text{ or } x=10$$

Category 5 – Algebra

1. If we add 4 inches to a square's length, and shorten its width by 4 inches, we get a new rectangle whose area is 75% of the original square's area.
   How many inches are in the length of the square's side?

2. One of the solutions to the equation $x^2 + B \cdot x - 8 = 0$ is $x = 2$.
   What is the value of the other solution?

3. The product of the solutions of the equation $x^2 - 9x + C = 0$, is twice as much as their sum. What is the larger of the two solutions?

Answers

1. _____

2. _____

3. _____

Solutions to Category 5 - Algebra

Answers	
1.	8
2.	-4
3.	6

1. If we call the square's length x, then we can write:
   $(x+4)\cdot(x-4) = x^2 - 16 = \frac{3}{4}\cdot x^2$
   and so $x^2 = 64$ and $x = 8\ inches$. *Since we're looking for a length, we're interested only in the positive solution.*

2. Since we know that $x = 2$ is a solution, we can use this value in the original equation to get $2^2 + B\cdot 2 - 8 = 0$ and solve to get $B = 2$.
   Now the original equation is known to be $x^2 + 2\cdot x - 8 = 0$ and the other solution for this is $x = -4$.

3. In the general case of a quadratic equation $A\cdot x^2 + B\cdot x + C = 0$ we know that the sum of solutions is $-B$, and their product is $A\cdot C$. In our case $A = 1$ and $B = -9$, and so the sum of solutions is $+9$, and their product is $18 = C$.
   Our equation then is $x^2 - 9\cdot x + 18 = (x-3)\cdot(x-6) = 0$.
   The solutions are of course $x = 3$ and $x = 6$.

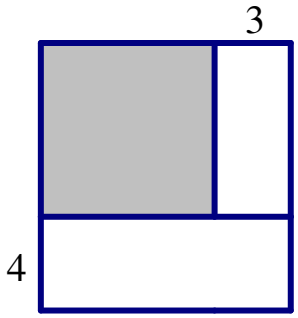
You may use a calculator today.

Category 5
Algebra
Meet #5, March 2009

1. What is the positive difference between the two solutions to the equation below?

$$3x^2 + 12x - 31 = 32$$

2. The diagram below is a large rectangle which is made up of a square and 2 rectangles as shown. The area of the entire figure is 756. What is the area of the square?



3. The difference between a positive number and $3\frac{1}{3}$ times its reciprocal is equal to $1\frac{1}{6}$. What is the number? Express your answer as a common fraction.

Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #5, March 2009

Answers

1. 10

2. 576

3. $\frac{5}{2}$

1. $3x^2 + 12x - 31 = 32$

$3x^2 + 12x - 63 = 0$

Dividing both sides of the equation by 3 gives us:

$x^2 + 4x - 21 = 0$

$(x + 7)(x - 3) = 0$

$x + 7 = 0$ or $x - 3 = 0$

$x = -7$ or $x = 3$

The positive difference between two solutions is $3 - (-7) = 10$

2. If we call the side of the square x, the square has area x^2, the rectangle to the right of the square has area $3x$ and the area of the rectangle below the square is $4(x + 3) = 4x + 12$.

The total area of the three shapes is $x^2 + 3x + 4x + 12 = x^2 + 7x + 12 = 756$.

$x^2 + 7x - 744 = 0$

$(x + 31)(x - 24) = 0$

$x + 31 = 0$ or $x - 24 = 0$

$x = -31$ or $x = 24$

Since the side length of the square cannot be negative, the side of the square is 24 and the area of the square is $24^2 = 576$.

3. $x - 3\frac{1}{3}\left(\frac{1}{x}\right) = 1\frac{1}{6}$

$x - \frac{10}{3}\left(\frac{1}{x}\right) = \frac{7}{6}$

$x - \frac{10}{3x} = \frac{7}{6}$

Multiplying both sides of the equation by $6x$ gives us:

$6x^2 - 20 = 7x$

$6x^2 - 7x - 20 = 0$

$(2x - 5)(3x + 4) = 0$

$2x - 5 = 0$ or $3x + 4 = 0$

$2x = 5$ or $3x = -4$

$x = \frac{5}{2}$ or $x = -\frac{4}{3}$

Since we know x is positive, it must be $\frac{5}{2}$.