

Math League SCASD

2020- 21

Meet #5

Arithmetic

Self-study Packet

Problem Categories for this Meet (in addition to topics of earlier meets):

1. Mystery: Problem solving
2. Geometry: Solid Geometry (Volume and Surface Area)
3. Number Theory: Set Theory and Venn Diagrams
- 4. Arithmetic: Combinatorics and Probability**
5. Algebra: Solving Quadratics with Rational Solutions, including word problems

Important things you need to know about **ARITHMETIC:** *Probability and Combinatorics*

- To find the probability of compound events, multiply their individual probabilities. For example if you flip a coin and roll a number cube, the probability that you would land heads and roll a 4 is $P(\text{heads}) \cdot P(4) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$
- A chart is also a useful way to find probabilities that cannot be solved through straightforward methods.
For Example: *You roll two die, numbered 1-6. What is the probability that the sum of your two die will be a 7 or an 8?*

↙ +	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

← Roll of Die 1

Roll of Die 2 ↗

There are six 7's and five 8's. Of the 36 total possible ways to roll, 13 are either 7's or 8's. The probability is 13/36.

- To find the number of ways things can be ordered, multiply the number of choices for each spot. For example if you want to know how many different ways 10 people can come in first, second, and third place, multiply 10 by 9 by 8, because there are 10 options for first, then 9 left for second, then 8 left for third. There are 720 ways. This is called a "permutation".
- If order is not important, such as in the problem "How many different groups of three people can be chosen from eight people?" you can find the number of different orders three people can be chosen from 8 (by multiplying 8 by 7 by 6) and then dividing by the number of ways the 3 people can be grouped (3 by 2 by 1). This would give $336 \div 6 = 56$ ways. This is called a "combination".

Category 4

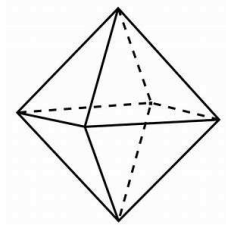
Arithmetic

Meet #5 - April, 2019

Calculator meet

1) Frank was carrying the 12 pages of a yearly calendar but accidentally dropped all 12 sheets. He gathered them up off the floor. What is the probability that the sheet on top was from a month starting with the letter "J?" Express your answer as a common (simplified) fraction.

2) Two octahedral dice, with faces numbered 1, 2, 3, 4, 5, 6, 7, and 8 are rolled. What is the probability that the sum of the numbers showing on the bottom surfaces of the two dice is less than ten? Express your answer as a common (simplified) fraction.



3) The letters of the word EXCELLENT are arranged to form other "words," with a "word" defined as any 9-letter arrangement of the letters in the word EXCELLENT. How many different words are possible, including the original word?

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 4
Arithmetic
Meet #5 - April, 2019

- 1) Of the 12 months, three begin with the letter "J", namely January, June, and July. Hence, $\frac{3}{12}$ of the calendar months begin with the letter "J," or $\frac{1}{4}$ in simplified form.
- 2) The outcomes of the two octahedral dice are arranged along the top row and left-side column, with the sums of the two bottom faces in the interior of the grid:

	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	11
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

Sums less than 10 lie along diagonals that contain the sums 2, 3, 4, 5, 6, 7, 8, and 9. There are 36 such sums of the 64 total sums. The probability of those 36 sums is $\frac{36}{64}$, or $\frac{9}{16}$ as a common (simplified) fraction.

- 3) If all nine letters were different, there would be $9!$ possible arrangements of those letters. But since some of those letters repeat, we must divide by the number of repetitions as follows, accounting for the three Es and two Ls:

$$\frac{9!}{(3!)(2!)} = \frac{(9)(8)(7)(6)(5)(4)(3)(2)(1)}{[(3)(2)(1)][(2)(1)]} = (9)(8)(7)(5)(4)(3)(1) = 30,240$$

Therefore, there are 30,240 ways that the letters in the word EXCELLENT can be arranged to form different "words."

<u>Answers</u>	
1)	$\frac{1}{4}$
2)	$\frac{9}{16}$
3)	30,240

Solutions to Category 4

Arithmetic

Meet #5 - April, 2017

1) The total number of utensils is $17 + 24 + 19 = 60$.

The probability of selecting a teaspoon is

$$\frac{24}{60} = \frac{12}{30} = \frac{6}{15} = \frac{2}{5}$$

2) ${}_{12}C_3 = 220$.

3) The probability of choosing two of the same color
= (probability of two raspberry) + (probability of
two cherry) + (probability of two melon)

$$= \left(\frac{6}{15} \times \frac{5}{14} \right) + \left(\frac{4}{15} \times \frac{3}{14} \right) + \left(\frac{5}{15} \times \frac{4}{14} \right)$$

$$= \left(\frac{30}{210} \right) + \left(\frac{12}{210} \right) + \left(\frac{20}{210} \right)$$

$$= \frac{62}{210} = \frac{31}{105}$$

Answers

1) $\frac{2}{5}$

2) 220

3) $\frac{31}{105}$

**Solutions to Category 4
Arithmetic
Meet #5 - March, 2015**

Answers

1) 120

2) $\frac{1}{3}$

3) 34 (%)

1) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

2) The total number of Skittles is $17 + 28 + 31 + 8 = 84$.

$$\frac{\text{red}}{\text{total}} = \frac{28}{84} = \frac{14}{42} = \frac{7}{21} = \frac{1}{3}$$

3) Use the "choose" or "combination" function, since there is no importance as to the order in which the heads land.

$$\text{number of (4 heads)} = {}^6C_4 = \frac{6!}{4!(6-4)!} = 15$$

$$\text{number of (5 heads)} = {}^6C_5 = \frac{6!}{5!(6-5)!} = 6$$

$$\text{number of (6 heads)} = {}^6C_6 = \frac{6!}{6!(6-6)!} = 1$$

These numbers can be found in the 7th row of Pascal's Triangle:

1 6 15 20 15 6 1 where the sum is 64.

The total number of possible outcomes for tossing six coins is 2^6 , or 64.

$$\text{So, the probability of getting four or more heads} = \frac{15+6+1}{64} = \frac{22}{64} = 0.34375.$$

Rounding to the nearest whole percent yields 34%.

Category 4
Arithmetic
Meet #5, March/April 2013

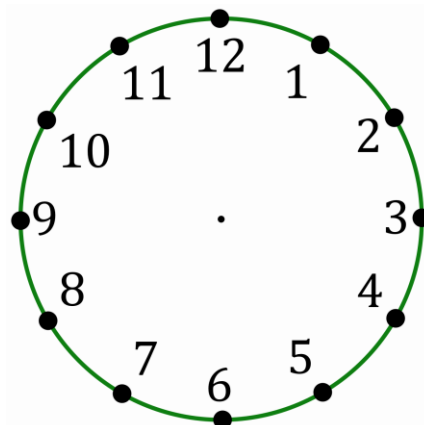
You may use a calculator.

1. There are 3 red marbles and 1 blue marble in a bag. If 3 marbles are chosen at random, what is the probability that all three marbles are red? Express your answer as a common fraction.

2. Twenty students signed up for a weekend ski trip, but only 16 students can go. How many ways can 16 of the 20 students be chosen to go on the ski trip?

3. Cheryl created a game in which the pieces move around on the hours of a clock. Each turn, a player flips a coin and rolls two standard dice. If the coin lands on heads, he or she moves clockwise the number of spaces indicated by the sum of the numbers on the dice. If the coin lands on tails, he or she moves counterclockwise the number of spaces indicated by the sum of the numbers on the dice. If a player is on hour 10 at some point in the game, what is the probability he or she will advance to hour 2 on the next move?

Answers	
1.	_____
2.	_____ ways
3.	_____



Solutions to Category 4
Arithmetic
Meet #5, March/April 2013

1. If we choose three marbles from the bag, only one marble will be left behind. The only way to get three red marbles is to leave the one blue marble behind. The probability is thus $\frac{1}{4}$.

2. If 16 students are to be chosen for the ski trip, then exactly 4 students won't get to go. It is easier to think about choosing the 4 students who will be left out. There are 20 possible students to choose first, followed by 19 to choose second, then 18, then 17, which would be $20 \times 19 \times 18 \times 17 = 116,280$ ways to choose, except that we don't care about the order in which a group of four students were chosen. The same group of four can be chosen in $4 \times 3 \times 2 \times 1 = 24$ different ways. In fact, the number 116,280 counts *every* group of four 24 times, so we divide by 24 to get **4845 ways** that four people can be eliminated (and thus 16 people chosen) for the ski trip. Some students will simply use the nCr function in their calculator, which computes as follows:

$${}_{20}C_{16} = \frac{20!}{16!(20-16)} = \frac{20 \times 19 \times 18 \times 17 \times 16!}{16! \times 4 \times 3 \times 2 \times 1} = 5 \times 19 \times 3 \times 17 = 4845$$

3. There are two ways to get from hour 10 to hour 2. The player might go clockwise by 4 hours or counterclockwise by 8 hours. To go clockwise by 4 hours, the player must flip heads and roll a sum of 4. The probability of this happening is $\frac{1}{2} \times \frac{3}{36} = \frac{3}{72}$. To go counterclockwise by 8 hours, the player must flip tails and roll a sum of 8. The probability of this happening is $\frac{1}{2} \times \frac{5}{36} = \frac{5}{72}$. Adding these two cases together, we calculate a probability of $\frac{3}{72} + \frac{5}{72} = \frac{8}{72} = \frac{1}{9}$.

Answers

1. $\frac{1}{4}$

2. 4845 ways

3. $\frac{1}{9}$

Category 4 – Arithmetic

1. If Larry tosses a fair coin four times, what is the probability he'll get the same result each time? *Express your answer as a common fraction.*

2. How many possible passwords can there be that match the listed criteria?
 - a. The password must be either 2 or 3 characters long.
 - b. The password may contain only letters and numbers.
 - c. The password is case-sensitive (So 'Math' is not identical to 'MATH').
(The same character can be used more than once in a password).

3. Three players bring two balls each to a tennis practice. When the practice is over, each player picks two balls at random. What is the probability that each player ended up with the same two balls she came with?
Express your answer as a common fraction.

Answers	
1.	_____
2.	_____
3.	_____

Solutions to Category 4 – Arithmetic

<u>Answers</u>	
1.	$\frac{1}{8}$
2.	242172
3.	$\frac{1}{90}$

1. Each individual toss has two possible outcomes (Head / Tail), and so four consecutive tosses have 2^4 possible outcomes.

Two of these fit our condition (four consecutive tails, and four consecutive heads), so the probability is $\frac{2}{2^4} = \frac{1}{8}$

2. The characters allowed are the 26 UPPER-CASE letters, the 26 lower-case letters, and the 10 digits, so a total of 62 characters.

There are therefore 62^2 passwords made up of two characters, and 62^3 made up of three, for a total of $62^2 + 62^3 = 242,172$ passwords.

3. This seems quite difficult. If we label the players A, B, C and number the balls 1,2,3,4,5,6, then the *random picking at the end is equivalent to* arranging the balls in some order, and giving the first two to player A , the next two to player B , etc. And so the question becomes: If we arrange the six balls in a random order, what is the probability that balls #1, #2 are in the first two positions, balls #3, #4, are in the third and fourth positions etc.

There are of course $6!$ permutations for the balls, and $2^3 = 8$ of them match our criteria [Arrangements of the sort $\{(1,2), (3,4), (5,6)\}$ where we don't care about the internal order in each pair, as long as the pairs are in the correct order. Each pair has two permutations, hence 2^3].

The probability then is $\frac{2^3}{6!} = \frac{8}{720} = \frac{1}{90}$

Category 4
Arithmetic
Meet #5, March 2009

1. The coach at the Instant Messaging Legion of Extraordinary Mathematicians had 13 students tryout for the IMLEM team. The coach must choose 10 of them to compete at the upcoming meet. How many different groups of 10 students can the coach choose from the 13 students?
2. Buggy was at the penny candy store where they have two big barrels of mixed candy. One of the barrels contains candy which costs 3, 5, 6, 7, 8 or 13 cents each. The other barrel contains candy which costs 2, 4, 5, 6, 8 or 11 cents. If Buggy chooses 1 piece of candy from each barrel, what is the probability that the total cost is an even number of cents? Express your answer as a common fraction.
3. How many different 5-digit zip codes are possible if the only restriction is that no two consecutive digits can be the same?

Answers	
1.	_____
2.	_____
3.	_____

Solutions to Category 4
 Arithmetic
 Meet #5, March 2009

Answers

1. 286

2. $\frac{4}{9}$

3. 65610

1. Choosing 10 of the 13 students is the same as choosing 3 of the students to not be on the team. Choosing 3 of 13 can be found as a combination by: ${}_{13}C_3 = \frac{13!}{(13-3)! \cdot 3!} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1} = 286$ ways to choose 3 students not on the team and 10 on the team.

2. The table below shows the possible sums of the costs of the two pieces of candy. There are 36 possible ways to pick the two pieces of candy and 16 of them have even total costs. That's $\frac{16}{36} = \frac{4}{9}$.

+	3	5	6	7	8	13
2	5	7	8	9	10	15
4	7	9	10	11	12	17
5	8	10	11	12	13	18
6	9	11	12	13	14	19
8	11	13	14	15	16	21
11	14	16	17	18	19	24

Alternatively we can use case work to figure this one out. In order for the sum of two whole numbers to be even, either both numbers are even or both are odd. If both numbers are even there is a $\frac{2}{6}$ chance the candy from the first barrel has an even price and a $\frac{4}{6}$ chance the candy in the second barrel has an even price, so the probability of both candies having an even price is $\frac{2}{6} \times \frac{4}{6} = \frac{8}{36}$. Similarly, the probability that both have odd number prices is $\frac{4}{6} \times \frac{2}{6} = \frac{8}{36}$. The probability of either of those happening is $\frac{8}{36} + \frac{8}{36} = \frac{16}{36} = \frac{4}{9}$.

3. The first digit of the zip code could be any of the 10 digits. The second digit could not be the same as the first digit so there are 9 choices. The third digit cannot be the same as the second, but it could be the same as the first, so 9 choices again. Similarly there are 9 choices for the 4th and 5th digits. That's a total of $10 \times 9 \times 9 \times 9 \times 9 = 65610$ choices.