Math League SCASD 2020-21

Meet #5

Number Theory

Self-study Packet

Problem Categories for this Meet (in addition to topics of earlier meets):

1. Mystery: Problem solving
2. Geometry: Solid Geometry (Volume and Surface Area)
3. **Number Theory: Set Theory and Venn Diagrams**
4. Arithmetic: Combinatorics and Probability
5. Algebra: Solving Quadratics with Rational Solutions, including word problems

Important Information you need to know about NUMBER THEORY:
Set Theory, Venn Diagrams

The symbol $\cap$ stands for intersection. If you see the notation $A \cap B$, it means all the elements that are in Set A **AND** Set B.

The symbol $\cup$ stands for union. If you see the notation $A \cup B$, it means all the elements that are in Set A **OR** Set B.

For example:

$$\{2,3,5,7\} \cap \{2,4,6,8\} = \{2\}$$

$$\{2,3,5,7\} \cup \{2,4,6,8\} = \{2,3,4,5,6,7,8\}$$

Solve sets using the normal order of operations. Do what's in parentheses first, and then work from left to right.

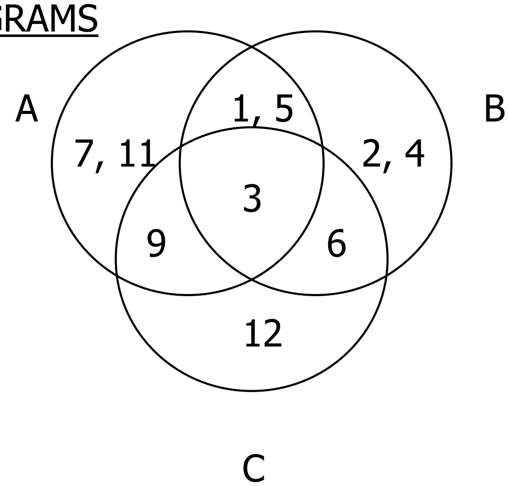
For example,

Prime Numbers $\cap$ (Numbers with 3 as a digit $\cap$ Numbers Less than 50)

First, you would find all the numbers that have 3 as a digit AND are less than 50.
{3, 13, 23, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 43}

Then, you would find all the numbers that are prime AND are in the set you just found. This would be your final answer.
{3, 13, 23, 31, 37, 43}

<u>VENN DIAGRAMS</u>



A Venn diagram includes two or more intersecting circles. The overlapping areas are the intersections of the sets.

To the left is where you place the following numbers if the elements of each set are as follows:

A: {1, 3, 5, 7, 9, 11}
B: {1, 2, 3, 4, 5, 6}
C: {3, 6, 9, 12}

Category 3
Number Theory
Meet #5 - April, 2019

Calculator Meet

1) Forty-three students on the track team run long-distance races while 37 run sprint races. There are 54 students on the team. Every student runs at least one race. How many students run both a long-distance race and a sprint race?

2) Set X = { multiples of 8 between 30 and 90 }
Set Y = {Factors of 240 }
$X \cap Y$ represents the intersection of set X and set Y, that is, the set of all elements (in this case, numbers) that belong to both set X and set Y. How many elements belong to $X \cap Y$?

3) There are 411 students at the Norwood Coakley Middle School.

*** 166 like Snickers,**

*** 201 like Reese's Peanut Butter Cups**

*** 178 like Butterfingers,**

*** 80 like Snickers and Reese's Peanut Butter Cups,**

*** 90 like Snickers and Butterfingers,**

*** 74 like Reese's Peanut Butter Cups and Butterfingers, and**

*** 51 like all three candies.**

How many students at the Norwood Coakley Middle School do not like any of the three candies?

Answers

1) _________

2) _________

3) _________

Solutions to Category 3
Number Theory
Meet #5 - April, 2019

Answers
1) 26
2) 3
3) 59

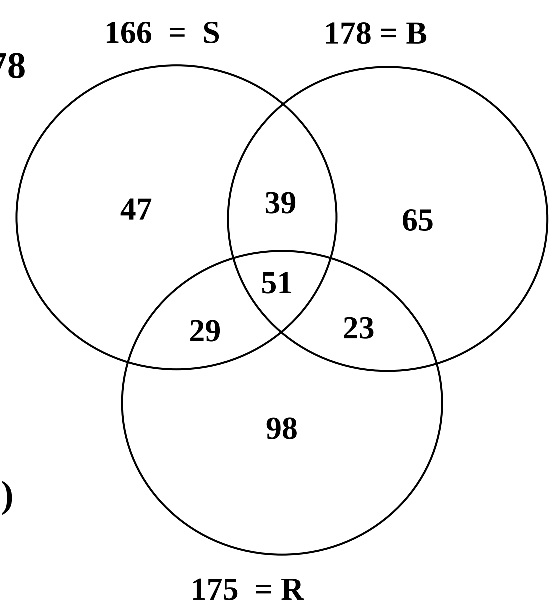
1) Since the number of distance runners plus the number of sprint runners exceeds the total number of students on the team, the overlap of those who run both types of races is X:

$$43 + 37 - X = 54$$
$$80 - X = 54$$
$$X = 26$$

Therefore, the number of students who run both the long-distance races and the sprint races is 26.

2) Set X = { multiples of 8 that are between 30 and 90 }
= { 32, 40, 48, 56, 64, 72, 80, 88 }
Set Y = { 1, 2, 3, 4, 5, 6, 8, 10, 12, 20, 24, 30, 40, 48, 60, 80, 120, 240 }
$X \cap Y$ = { 40, 48, 80 }
Therefore, there are three elements in the intersection of sets X and Y.

3) Since all three activities overlap, the following Venn diagram can help organize the given data. Let S = Snickers, B = Butterfingers, and R = Reese's:



Checking all conditions:
* Snickers = 47 + 39 + 51 + 29 = 166
* Butterfingers = 39 + 51 + 23 + 65 = 178
* Reese's = 29 + 51 + 23 + 98 = 175
* S and R = 51 + 29 = 80
* S and B = 39 + 51 = 90
* R and B = 51 + 23 = 74

The number of students who do not like any of the three candies is the difference between the total number of students and the sum of all the numbers in the Venn Diagram:
= 411 - (47 + 39 + 65 + 51 + 29 + 23 + 98)
= 411 - (352)
= 59.
Therefore, only 59 students do not like any of the three candies. On the next page, see one way to actually arrive at the solution.

Starting with the fact that 51 like all three, Subtract 51 from 80 to get 29 - the rest of those who like both snickers and Reese's. Subtract 51 from 74 to get 23 - the rest of those who like Reese's and Butterfingers. Subtract 51 from 90 to get 39 - the rest of those who like Butterfingers and Snickers. Subtract 29 + 51 + 39, or 119, from 166 to get 47 - the number who like just snickers. Subtract 39 + 51 + 23, or 113, from 178 to get 65 - the number who like just Butterfingers. Subtract 29 + 51 + 23, or 103, from 201 to get 98 - the number who like just Reese's. Finally, subtract the total of the numbers in the three intersecting circles, or 352, from 411 to get 59 - the number who do not like any of the three candies.

Category 3
Number Theory *Calculator Meet*
Meet #5 - April, 2017

1) Set A = {multiples of 4 between 1 and 170}
Set B = {multiples of 6 between 1 and 170}

How many numbers are in the intersection of sets A and B ?

2) A group of 74 people saw either the movie "Waterfront Nightmare" or the movie "The Secret Village" or both. Fifty-six people saw the first movie while 39 saw the second. How many people saw both ?

3) Set C = { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }
Set D = { 2, 3, 5, 7, 11, 13 }

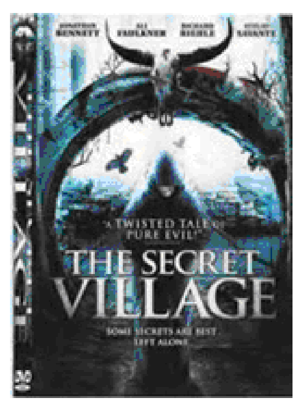
Set E = { common fractions $\frac{X}{Y}$ where X is an element of Set C and Y is an element of Set D }. How many elements are in Set E ? (Reminder: In a common fraction, the numerator and denominator do not share any common factors. The fraction may be proper or improper.

Answers

1) ________

2) ________

3) ________



Solutions to Category 3
Number Theory
Meet #5 - April, 2017

Answers
1) 14
2) 21
3) 52

1) The common multiple of 4 and 6 that lie between 1 and 170 are multiple of 12, or { 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168 }. There are 14 such multiples.

2) Let X = the number of people who saw both movies.

So, $(56 + 39) - X = 74$
$95 - X = 74$
$X = 21$

Therefore, 21 people saw both movies.

3) The common (simplified) fractions are the (11)(6) possible fraction, minus the numbers from Set C that are multiple of 2 (there are six), minus the remaining multiples of 3 (there are two), minus the remaining multiples of 5 (there is one), minus the remaining multiple of 7 (there is one), minus the remaining multiples of 11 (there is one).

$(11)(6) - (6 + 4 + 2 + 1 + 1)$
$= 66 - 14$
$= 52.$

Here is a full accounting of all the common fractions:

numerators of 2: 2/3, 2/5, 2/7, 2/11, 2/13 (five)
numerators of 3: 3/2, 3/5, 3/7, 3/11, 3/13 (five)
numerators of 4: 4/3, 4/5, 4/7, 4/11, 4/13 (five)
numerators of 5: 5/2, 5/3, 5/7, 5/11, 5/13 (five)
numerators of 6: 6/5, 6/7, 6/11, 6/13 (four)
numerators of 7: 7/2, 7/3, 7/5, 7/11, 7/13 (five)
numerators of 8: 8/3, 8/5, 8/7, 8/11, 8/13 (five)
numerators of 9: 9/2, 9/5, 9/7, 9/11, 9/13 (five)
numerators of 10: 10/3, 10/7, 10/11, 10/13 (four)
numerators of 11: 11/2, 11/3, 11/5, 11/7, 1/13 (five)
numerators of 12: 12/5, 12/7, 12/11, 12/13 (four)

There are 52 common fractions.

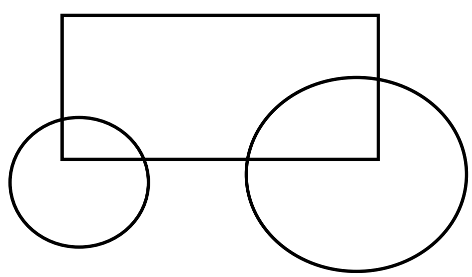
Category 3
Number Theory
Meet #5 - March, 2015
Calculator meet

1) Victoria has 83 coins in her pocket. Fifty-seven of the coins contain silver while 64 contain copper. The rest are silver-copper alloys, which are mixtures of the two metals. How many of the coins are a silver-copper alloy?

2) Set W = { multiples of 4 between 30 and 90 }
Set Y = { factors of 240 }

What is the sum of all the elements (members) in the set $W \cap Y$, that is, the intersection of sets W and Y ?

3) Santana arranged some of the toys that his school donated to Toys for Tots into roped areas as shown. All 824 trucks are in the rectangle. The 342 orange toys are in the smaller circle while the 487 red toys are in the larger circle. There are 72 red trucks and 119 orange trucks. If 2265 toys were donated in all, then how many of the toys were outside of the areas that Santana had roped off?



<u>Answers</u>

1) ______

2) ______

3) ______

Leonard Nimoy of Star Trek fame, born on March 26, 1931, lost his courageous battle with COPD and passed away peacefully on February 27, 2015.

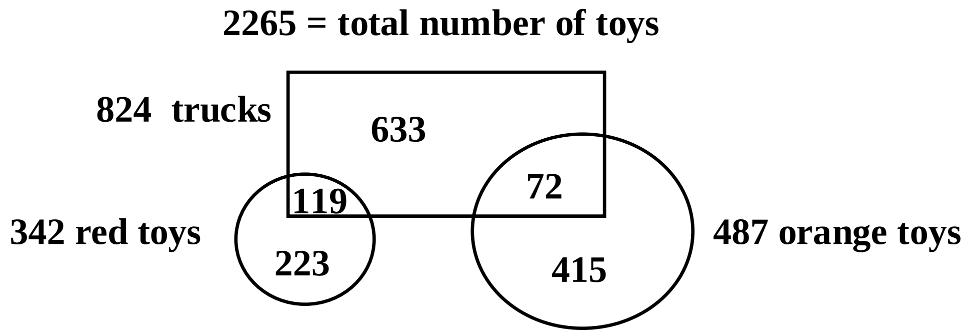
Solutions to Category 3
Number Theory
Meet #5 - March, 2015

Answers
1) 38
2) 228
3) 803

1) Subtract the number of coins from the sum of silver + copper: (57 + 64) - 83 = 121 - 83 = 38.

2) Set W = { 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88 }
The elements in the intersection = {40, 48, 60, 80 }.
Their sum is 40 + 48 + 60 + 80 = 228.

3) The numbers in this diagram reflect how many toys have the assigned characteristics:



The number of toys that lie outside the roped area is

2265 - (633 + 72 + 119 + 415 + 223)

= 2265 - 1462

= 803

Category 3
Number Theory
Meet #5, March/April 2013

You may use a calculator.

1. Set A is the students in Mr. Taylor's homeroom and set B is the students who ride bus #1. The mathematical statement $|A \cup B| = 49$ tells us that the union of sets A and B contains 49 students. The statement $|A \cap B| = 12$ tells us that the intersection of sets A and B contains 12 students. If there are 23 students in Mr. Taylor's homeroom, how many students ride bus #1?

2. Suppose our "universal set" *U* is all the positive integers less than or equal to 30. Suppose also that within this universal set, we have Set *P*, which is positive prime numbers, and set *E*, which is positive even numbers. Find the sum of all the elements in *U* that are not in $P \cup E$.

U = {1, 2, 3, … 30)
P = {2, 3, 5, … 29)
E = {2, 4, 6, … 30)

3. Kayla plays soccer on different teams in the Fall, the Winter and the Spring. There are six girls on Kayla's Fall soccer team (including Kayla) who are also on her indoor soccer team in the Winter. These six girls are one third of the Fall team and half the Winter team. There are four girls on the Winter team (including Kayla) who also play on Kayla's Spring team. There are five girls (including Kayla) who play on the Fall and Spring teams. These five girls are one third of the Spring team. If there are three girls who play on all three teams, how many girls play on exactly one of these teams?

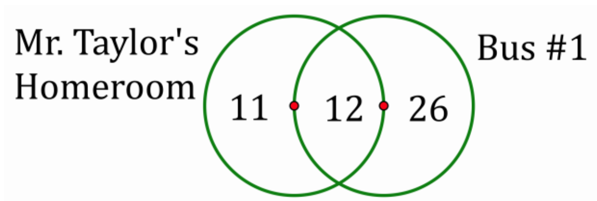
Answers

1. __________ students
2. ________________
3. ____________ girls

Solutions to Category 3
Number Theory
Meet #5, March/April 2013

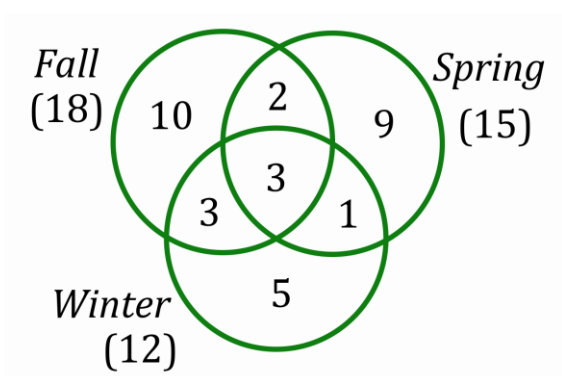
Answers
1. 38 students
2. 98
3. 24 girls

1. If we subtract the 23 students in Mr. Taylor's homeroom from the 49 students in the union, we get the 26 who ride bus #1 but are not in Mr. Taylor's homeroom. If we add the 12 students in the intersection, we get all **38 students** who ride bus #1.



2. The elements that are not in $P \cup E$ are the odd numbers less than 30 that are not prime. Their sum is 1 + 9 + 15 + 21 + 25 + 27 = **98**.

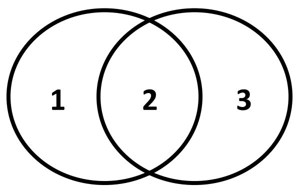
3. The Venn diagram below shows how many girls there are in each of the seven regions of the overlapping circles. We figured that there must be 6 × 3 = 18 girls on the Fall team, 6 × 2 = 12 girls on the winter team, and 5 × 3 = 15 girls on the Spring team. We placed a 3 in the very center for the 3 girls who play on all three teams. Once we had this 3 in the center, we can work out the number of girls who are on exactly two of the teams: 6 – 3 = 3 for Fall and Winter, 4 – 3 = 1 for Spring and Winter, 5 – 3 = 2 for Fall and Spring. Only then can we determine that there must be 10 girls who play only on the Fall team, 9 girls who play only on the Spring team, and 5 girls who play only on the Winter team. That's 10 + 9 + 5 = **24 girls** who play on exactly one of the teams.



Category 3 – Number Theory

1. Set A contains 10 elements. The union $A \cup B$ contains 20 elements, while the intersection $A \cap B$ contains 4 elements.
   How many elements are the in the set B?

2. In a general Venn diagram of 2 sets, we can have 3 different areas, as shown in this diagram:



   Where the common area in the middle represents the intersection of the two sets. How many different areas are possible in a Venn diagram of 4 sets?

3. How many subsets of the set $\{A, B, C, D, E, F\}$ contain at least one vowel?

Answers
1. ________________
2. ________________
3. ________________

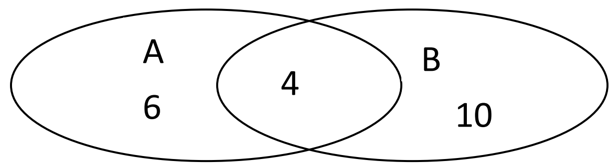
Solutions to Category 3 – Number Theory

Answers	
1.	14
2.	15
3.	48

1. The number of elements in a union of two sets is the sum of the number of elements in each one, minus the number of elements in their intersection. Denoting the number of elements in a set A as $|A|$, we can write this as:

   $$\left|A \bigcup B\right| = |A| + |B| - \left|A \bigcap B\right|$$

   Using the numbers in our case we get $|B| = 20 + 4 - 10 = 14$. We can see the number of elements in this Venn diagram:



2. There are 4 areas containing only one set each.

   There are 6 areas containing the intersections of exactly two sets each (AB, AC, AD, BC, BD, CD), 4 areas of intersection of exactly 3 sets (ABC, ABD, ACD, BCD), and one area of the intersection of $ABCD$.

   Overall 15 distinct areas.

   This is basically the number of subsets of $\{A, B, C, D\}$ without the empty set.

3. The set $\{A, B, C, D, E, F\}$ has 6 elements and therefore $2^6 = 64$ subsets.

   Removing the 2 vowels $\{A, E\}$ we're left with 4 elements that make $2^4 = 16$ subsets with no vowel in them. So the remaining 48 subsets must contain at least one vowel.

   Note we were careful to count the empty subset with the no-vowel group.

You may use a calculator today.

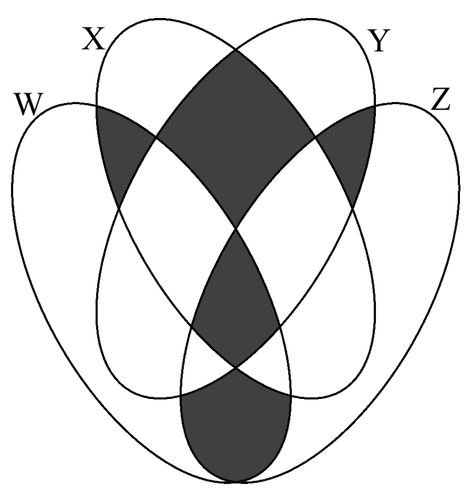
Category 3
Number Theory
Meet #5, March 2009

1. At the summer math camp there are 50 kids taking math classes. Thirty-five kids took the Number Theory class and 24 kids took the Probability class. If 7 kids took neither of the two classes, how many kids took both classes?

2. The set A contains only the vowels a, e, i, o, u and y. How many subsets of set A contain the letter y?

3. In the Venn diagram below, the four ovals represent 4 sets of numbers as described below. How many numbers fall in the shaded regions?

W = the set of positive even integers less than 50
X = the set of prime numbers less than 50
Y = the positive multiples of 3 less than 50
Z = the perfect squares less than 50



Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 3
Number Theory
Meet #5, March 2009

Answers

1. 16

2. 32

3. 5

1. 50 – 7 = 43 kids took at least one of the two classes. A nice way to think about this is that there are 35 Number Theory books being used and 24 Probability books being used for a total of 59 books. If each of the 43 kids had one book there would be 16 left over books. So 16 of the kids must take a second book and are taking both classes.

2. Subsets of set A can contain anywhere from none of the elements of A up to containing all of the elements of A. Since we are looking for the subsets that contain "y", we just need to determine whether or not the other letters are in a subset. There are 5 other letters and each can either be in the subset or not be in the subset resulting in 2 possibilities for each letter. That makes for $2 \times 2 \times 2 \times 2 \times 2 = 32$ possible subsets.

3. The five shaded areas represent the overlap between X and Y, between Y and Z, between Z and W, between X and W, and the region where all 4 overlap.

X and Y contain just 3
Y and Z contain just 9
Z and W contain 4 and 16
X and W contain just 2
The region where all circles overlap would be empty

That's a total of 5 numbers in the shaded regions.