Math League SCASD 2020

Meet #4

Number Theory

Self-study Packet

Problem Categories for this Meet (in addition to topics of earlier meets):

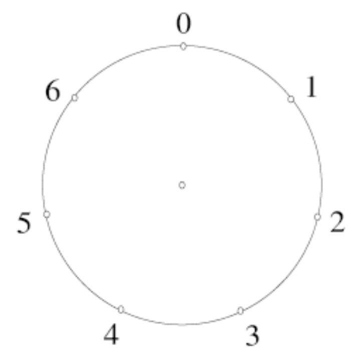
1. Mystery: Problem solving
2. Geometry: Properties of Circles
3. **Number Theory: Modular Arithmetic, Series and Sequences**
4. Arithmetic: Percent Applications
5. Algebra: Word Problems (linear, including direct proportions or systems)

Important Information you need to know about NUMBER THEORY:
Modular Arithmetic, Series and Sequences

Modular Arithmetic—Clocks are something common we use to describe in modular arithmetic. Our basic clock is in Mod 12, which means that it has the numbers 1-12 and then it repeats. Oftentimes, zero will be used instead of the Mod Number.

This clock is in Mod 7. 6 + 2 = 8. 8 ÷7 has a remainder of 1, so in Mod 7, 6 + 2 =1.

6•4 = 24. 24÷7 has a remainder of 3, so in Mod 7, 6•4 = 3



Series and Sequences

Arithmetic sequence: a sequence that adds or subtracts the same number each time

- To solve for a term in an arithmetic sequence, such as *What is the 100th term in the sequence 81, 84, 87, 90,…* first find the constant increase or decrease. In this case, +3. Multiply the constant rate by the number of times it would be added to the first term (in this case, 99, because the 100th term is 99 terms after term 1). Then add it to the first term. 99•3 + 81 = 378. The 100th term is 378.
- To find the sum of the terms in an arithmetic sequence, add the first and last terms and multiply the sum by the number of pairs (the number of terms divided by two). In the previous example, to find the sum of the first 100 terms, you would add 81 + 378 and get 459. You'd then multiply 459 by 50 because if there are 100 terms, there are 50 pairs that would all add up to 459. 50 •459 = 22,950

Category 3
Number Theory *Calculator Meet*
Meet #4 - February, 2019

1) What is the value of the 30th term (number) in the following arithmetic sequence?

17 20 23 26 29 . . .

2) Kenton's last birthday was 236 days ago. If today is Thursday, then on what day of the week was Kenton's last birthday?

3) The series below is the sum of consecutive integers whose first term is 26 and whose final term is 672 and contains all of the positive integers between 26 and 672 as well. What is that sum?

26 + 27 + 28 + 29 + . . . + 671 + 672

Answers

1) ________

2) ________

3) ________

Solutions to Category 3
Number Theory
Meet #4 - February, 2019

Answers	
1)	104
2)	Saturday
3)	225,803

1) Since there is a difference of 3 between any two consecutive terms, there is a connection between this sequence and the sequence of multiples of 3, following this formula: 3N + 14, where N is the number of the term. The 30th term is 3(30) + 14, or 104.

2) Divide 236 by 7 (the number of days in a week) to get 33 with remainder 5. Kenton was born 33 weeks ago . . . plus another 5 days in the past. Counting backwards 5 days from Thursday yields an answer of Saturday.

3) The sum of the given numbers can be considered as the difference between the sum of the integers from 1 through 672, inclusive, and the sum of the integers from 1 through 25, inclusive.

$$(1 + 2 + 3 + . . . + 672) - (1 + 2 + 3 + . . . + 25)$$

$$= [(n)(n + 1) / 2] - [(m)(m + 1) / 2]$$

$$= [(672)(673) / 2] - [(25)(26) / 2]$$

$$= 226{,}128 - 325$$

$$= 225{,}803$$

A common strategy is to add opposite ends of the series, working your way toward the middle:

$$(26 + 672) + (27 + 671) + (28 + 670) + . . .$$
$$= 698 + 698 + 698 + . . .$$

There are as many sums of 698 as half the number of terms in the series. That number of sums is (672 - 26 + 1) / 2, or 647 / 2, or 323.5
Then (323.5) (698) = 225,803, the same result as by using the formula above.

Category 3
Number Theory *Calculator Meet*
Meet #4 - February, 2017

1) The planet Mercury requires 88 days to orbit the sun. Savannah was born on Planet Earth on the 32nd day of Mercury's orbit. Her family is celebrating her 3rd birthday on Earth (when she is three years old). If Savannah has never experienced a leap year, then on which day of Mercury's orbit is her 3rd birthday? There are 365 days in an Earth year.

2) Find the value of the 438th term of the following arithmetic sequence:

63 70 77 84 91 . . .

3) Find the sum of the first 826 terms of the following arithmetic sequence:

78 87 96 105 114 . . .

Answers

1) ________

2) ________

3) ________

Solutions to Category 3
Number Theory
Meet #4 - March, 2017

Answers	
1)	71
2)	3122
3)	3,130,953

1) Savannah is 3(365) days old, or 1095. Add 1095 to 32 mod 88. First, divide 1095 by 88 to see how many Mercury orbits occur during the three years, and that is 12 with remainder 39. Add 39 to 32 to get 71 mod 88. So, Savannah's 3rd birthday falls on the 71st day of Mercury's orbit.

2)

Term #	Calculation of term value	Term value
1	63 + (0)(7)	63
2	63 + (1)(7)	70
3	63 + (2)(7)	77
4	63 + (3)(7)	84
N	63 + (N - 1)(7)	7N + 63
438	63 + (438 - 1)(7)	3122

3) Any two consecutive terms in this sequence have a difference of 9. First, find the value of the 826th term = 78 + (826 - 1)(9) = 7503. One way to find the sum of all 826 terms is to add the opposite ends (78 + 7503 = 7581) and the 2nd and 825th terms (87 + 7494 = 7581) and so on. There are 826 terms, but 826/2 pairs, or 413 pairs, that have a sum of 7581. So, the sum of all 826 terms is (413)(7581), or 3,130,953.

Category 3
Number Theory
Meet #4 - February, 2015
Calculator meet

1) If today is February 12 and Sabatino is exactly 633 months old, in what month was Sabatino born?

2) Below are seven terms of a geometric sequence. What is the value of A + B + C ? Express your answer as a decimal.

8 12 18 A B C 91.125

3) $\sum_{K=1}^{n} K^2 = \frac{(n)(n+1)(2n+1)}{6}$ **is the formula for adding consecutive square numbers in the following series:** $1^2 + 2^2 + 3^2 + \ldots + n^2$.

For example, $\sum_{K=1}^{4} K^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

We can achieve the same result by substituting 4 for n into the formula: $\frac{(n)(n+1)(2n+1)}{6} = \frac{(4)(4+1)(2(4)+1)}{6} = \frac{(4)(5)(9)}{6} = \frac{180}{6} =$ **30.**

*** * Important - this formula works for series that begin with K = 1.**

Find the value of the following sum: 64 + 81 + 100 + 121 + . . . + 3249.

Answers

1) ______

2) ______

3) ______

Bill Russell, shown above with his coach, Red Auerbach, was born on Feb 12, 1934 in West Monroe, Louisiana. Bill led the Boston Celtics professional basketball team to 11 world championships between 1956 and 1969. He was the guest of honor and keynote speaker at the 2005 Mathcounts National Competition held in Detroit, Michigan and was proud to sit and chat with the Massachusetts team, coached by Mr. Findell.

Solutions to Category 3
Number Theory
Meet #4 - February, 2015

Answers
1) May
2) 128.25
3) 63,225

1) Divide 633 by 12 . . . that will produce a quotient representing the number of complete years, as well as a remainder that represents the number of months prior to Sabatino's birthday month.
633 / 12 = 52 with remainder 9. Nine months prior to February (or, more easily, three months after February), is May.

2) The common ratio linking two consecutive terms is 12 / 8 = 1.5. So,
18 x 1.5 = 27 = A
27 x 1.5 = 40.5 = B
40.5 x 1.5 = 60.75 = C
Checking: 60.75 x 1.5 = 91.125, the final term.
So, A + B + C = 27 + 40.5 + 60.75 = 128.25 (must be expressed in decimal form).

3) To find the required sum, subtract the sum of the squares of the integers from 1 through 7, inclusive, from the sum of the squares of the integers from 1 through 57, inclusive.

$$\left(\sum_{K=1}^{57} K^2\right) - \left(\sum_{K=1}^{7} K^2\right) = \left(\frac{(57)(57+1)(2\cdot 57+1)}{6}\right) - \left(\frac{(7)(7+1)(2\cdot 7+1)}{6}\right)$$

$$= \left(\frac{(57)(58)(115)}{6}\right) - \left(\frac{(7)(8)(15)}{6}\right) = \left(\frac{380{,}190}{6}\right) - \left(\frac{840}{6}\right)$$

= 63,365 - 140 = 63,225.

Category 3
Number Theory
Meet #4, February 2013

You may use a calculator.

1. Find the 111th term in the arithmetic sequence below.

38, 44, 50,…

2. Find the sum of the first 8 terms in the geometric sequence below.

8, 24, 72, …

3. A mathematician grandmother arranged a Yankee gift swap among her 12 grandchildren in the following way. First she arranged their names in order from oldest to youngest and assigned them each a number from 1 to 12. She then determined that each grandchild *n* would buy a gift for grandchild 2^n (mod 13). For example, grandchild 4 would buy a gift for grandchild 3, since $2^4 = 16 = 3 \pmod{13}$. Find the number of the grandchild who would buy a gift for grandchild 5.

Answers

1. ___________________

2. ___________________

3. grandchild _______

Solutions to Category 3
Number Theory
Meet #4, February 2013

Answers
1. 698
2. 26,240
3. grandchild 9

1. Consecutive terms in an arithmetic sequence have a "common difference." The common difference between the terms in this sequence is 44 – 38 = 50 – 44 = 6.
If we want to get from the first term to the 111th term, we have to add six 110 times. The result is 38 + 110 × 6 = 38 + 660 = **698**. Some students may prefer to place a zeroth term of 32 before the first term 38. In that case, one would add 111 × 6 = 666 to 32 to get the same result of 698 for the 111th term.

2. Consecutive terms in a geometric sequence have a "common ratio." In this sequence, the common ratio is 24 ÷ 8 = 72 ÷ 24 = 3. To find the sum of the first 8 terms, we can actually calculate those 8 terms and keep a running sum, as shown in the table below.

Count	1	2	3	4	5	6	7	8
Term	8	24	72	216	648	1944	5832	17496
Sum	8	32	104	320	968	2912	8744	**26240**

Alternatively, we can use the formula below, with a = 8, r = 3, and n = 7.

$$S = a\left(\frac{1-r^{n+1}}{1-r}\right) = 8\left(\frac{1-3^8}{1-3}\right) = 8 \times \frac{-6560}{-2} = 26{,}240$$

3. We are trying to solve the equation $2^n = 5\left(\bmod 13\right)$ for n. We can just keep doubling, dividing by 13 whenever the result is greater than 13 and looking all the while for a remainder of 5. The table below shows the assignments for all 12 grandkids. We see that **grandchild 9** buys a present for grandchild 5.

n	1	2	3	4	5	6	7	8	**9**	10	11	12
2^n(mod 13)	2	4	8	3	6	12	11	9	5	10	7	1

Category 3 – Number Theory

1. What is the sum of the first 100 numbers in the series:
   $\{-93, -85, -77, -69, \dots\}$

2. A team of robots starts building a tower at $1pm$.
   It takes them one hour to construct the first floor, and each consecutive floor takes twice as long as the preceding one.
   What time is it when they're done building the 6^{th} floor?
   Be sure to note AM or PM in your answer!

3. When snow falls we measure its height on a clean surface, but when new snow falls then it weighs down the snow underneath it, so the height of a snow pile is actually less than the sum of accumulation.
   Let's assume that *each* new layer of $1''$ (1 inch) of snow compresses *all* the snow below it by a factor of 10%. *(So for example, if* $2''$ *of snow fall, then the bottom 1" will be compressed to* $0.9''$*, and the top* $1''$ *will still be* $1''$*).*
   If overall *12"* (twelve storms of $1''$ each) of snow fall , then how many inches are in the height of the snow pile?
   Round your answer to the nearest hundredth of an inch.

Answers

1. _______________

2. _______________

3. _______________

Solutions to Category 3 – Number Theory

Answers

1. 30,300
2. $4am$
3. 7.18

1. The N^{th} term in this series is given by $-93 + 8 \cdot (N - 1)$, and so the sum of the first 100 terms is:

   $S_{100} = (a_1 + a_{100}) * 50 = 50 * (-93 - 93 + 8 * 99) =$

   $= 50 * (606) = 30{,}300$

2. Since each floor takes twice as long as the preceding one, the total time required is: $1 + 2 + 4 + 8 + 16 + 32 = 63\ hours \equiv 15(mod\ 24)$

   15 hours past $1pm$ will be $4am$.

3. The top layer is still 1" thick, the second layer is $0.9 \cdot 1"$ thick, and so on, until the bottom layer which is $0.9^{11} \cdot 1"$ thick.

   The sum of these heights is a sum of a geometric series:

   $Sum = 1" \cdot (1 + 0.9 + 0.9^2 + \ldots + 0.9^{11}) = 1" \cdot \frac{1 - 0.9^{12}}{1 - 0.9} = 1" \cdot \frac{0.71757\ldots}{0.1} \cong 7.18"$

Category 3
Number Theory
Meet #4, February 2009

1. Mike's birthday is on Friday the 13th this year. Bob's next birthday is 180 days after Mike's. What day of the week is Bob's next birthday?

2. The 7th and 13th terms of an arithmetic sequence are -45 and 21, respectively. What is the 50th term in this sequence?

3. What is the greatest value of $M + N$ if M and N are single digit numbers, given the modular congruence below?

$$11M + 7N \equiv 23 \pmod{4}$$

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 3
Number Theory
Meet #4, February 2009

Answers

1. Wednesday

2. 428

3. 17

1. 180 days is equal to 25 full weeks and 5 more days. 25 full weeks after Mike's birthday would be still be a Friday, and 5 days after that would be a Wednesday.

2. The 13th term of the sequence is 6 terms after the 7th term. Since the sequence is arithmetic that means the same number has been added to the 7th term 6 times to get the 13th term. That means that $-45 + 6d = 21$ where d is the number being added each time. Solving that equation gives us $d = 11$. To get to the 50th term in the sequence, 11 will need to be added to the 13th term 37 more times. So the 50th term is $21 + 37(11) = 21 + 407 = 428$.

3.

$11M + 7N \equiv 23 \pmod 4$
$3M + 3N \equiv 3 \pmod 4$
$M + N = 1 \pmod 4$

So $M + N$ has a remainder of 1 when divided by 4. That means $M + N$ could be 1, 5, 9, 13, 17, 21, 25, ..., but the largest $M + N$ could possibly be if they are both single digit numbers is 18. Therefore the greatest value of $M + N$ that satisfies this modular congruence is 17.