Math League SCASD

Meet #3

Number Theory

2018 - Self-study Packet

Problem Categories for this Meet (in addition to topics of earlier meets):

1. Mystery: Problem solving
2. Geometry: Properties of Polygons, Pythagorean Theorem
3. **Number Theory: Bases, Scientific Notation**
4. Arithmetic: Integral Powers (positive, negative, and zero), roots up to the sixth
5. Algebra: Absolute Value, Inequalities in one variable including interpreting line graphs

Important information you need to know about NUMBER THEORY…
Bases, Scientific Notation

Number Bases

- We use the base-ten number system where each digit is a power of ten.
  $10^4 \quad 10^3 \quad 10^2 \quad 10^1 \quad 10^0$
  This means that every digit in the right column is worth 1, the 2nd (from the right) worth 10, the 3rd worth 100, the 4th worth 1000, the 5th worth 10,000 and so on.

- In any number base, n, the place values are always
  $n^4 \quad n^3 \quad n^2 \quad n^1 \quad n^0$ and continue on as our system does.

- To convert a number in another base into base 10, first find the value of each place. Add all the values together. Example: Convert $231_{\text{base 4}}$ to base 10. The far right's place value is 4^0 or 1. The middle's is 4^1 or 4. The left's is 4^2 or 16. There is 1 "1" (totaling 1), 3 "4's" (totaling 12) and 2 "16's" (totaling 32). If you add the individual totals, 1 + 12 + 32, you get 45. 45 is the base 10 equivalent of $231_{\text{base 4}}$.

- To convert a number from base 10 to another number system, first find out the value of each digit in another system. Example: Convert $37_{\text{base 10}}$ to base 5. Base 5's place values are worth $5^2 \quad 5^1 \quad 5^0$, or 25, 5, and 1. Starting from the left, there is one 25 in the base ten number 37 and there are 12 left over. There are two 5s in the base ten number 12 with 2 left over. There are 2 ones in the base ten number two. Therefore, $37_{\text{base 10}}$ is 122 in base 5 .

- In any number base, the largest digit is one less than the number base itself. For example, in base 10, the largest digit is 10-1 or 9. In base 6, the largest digit is 6-1, or 5.

- You can multiply, divide, add, or subtract in another number base as long as you remember to borrow or carry in that number base. You can also convert the number to base 10, perform the numerical operation, and then convert back to the original number base.

Scientific Notation

- The form for Scientific Notation is $a \cdot 10^n$ where *a* is a number between 1 and 10 ($1 \le a < 10$) multiplied by 10 to an integral power.

- When working with complex scientific notation problems, remember to cross-reduce and make the problem a lot simpler!

Category 3
Number Theory
Meet #3 - January, 2019

1) Write the base 2 numeral 11011 as a base 10 numeral.

2) There are approximately 100 trillion, that is 1×10^{14}, cells in a typical (average) human body. There are about as many atoms in a typical cell. Using these figures, about how many atoms are in a typical human body? If that number is written in scientific notation as $A \times 10^{B}$, then what is the value of A + B? Express your answer as a whole number.

3) Write the base 7 numeral 2436 as a base 3 numeral.

Answers

1) ________

2) ________

3) ________

Solutions to Category 3
Number Theory
Meet #3 - January, 2019

Answers
1) 27
2) 29
3) 1020200

1) The base 2 numeral 11011 has a base 10 value, from right to left:

1(1) + 1(2) + 0(4) + 1(8) + 1(16)
= 1 + 2 + 0 + 8 + 16
= 27

2) Multiply the number of atoms in a cell by the number of cells in a human body:

$$(1 \times 10^{14}) \times (1 \times 10^{14})$$
$$= 1 \times 10^{28}$$

So, A = 1 and B = 28, so A + B = 1 + 28 = 29.

3) Converting the base 7 numeral to a base 10 numeral, from right to left:

6(1) + 3(7) + 4(49) + 2(343)
= 6 + 21 + 196 + 686
= 909

Converting 909 base 10 to base 3:
First, divide 909 by the highest power of 3 that is less than 909, namely, 729. Then subtract 729 from 909, yielding 180. Now divide 180 by the highest power of 3 that is less than 180, namely 81, which is a power of 3. We get two 81s for a total of 162. Subtract 162 from 180 to get 18. With that, we get two 9s. Now, fill in zeroes for the powers of 3 for which we have none.
So we have 1(729) + 0(243) + 2(81) + 0(27) + 2(9) + 0(3) + 0(1), or 1020200.
Checking: 729 + 0 + 162 + 0 + 18 + 0 + 0 = 909. Check!

Category 3
Number Theory
Meet #3 - January, 2017

1) What is the base 10 value of the base 3 numeral 100110 ?

2) Using the symbol X as a sign of multiplication, what is the value of the following expression? Express your answer in scientific notation.

$$(6 \times 10^{8}) \times (7 \times 10^{-5}) \times (5 \times 10^{0}) \times (4 \times 10^{12})$$

3) Assume that a rocket can average a speed of 1.4×10^{3} miles per hour in space. How many days would it take for this rocket to reach the moon if the moon is 2.38×10^{5} miles from Earth on this date? Use the rounded figure that there are 24 hours in a day. Ignore fluctuations in distances due to the fact that the moon's orbit is not circular. Round your final answer to the nearest tenth of a day (as a decimal, but not in scientific notation).

Answers

1) ____________

2) ____________

3) ____________

Solutions to Category 3
Number Theory
Meet #3 - January, 2017

Answers	
1)	255
2)	8.4×10^{17}
3)	7.1

1) 100110 (base 3) = 243 + 9 + 3 = 255.

2) $(6 \times 10^8) \times (7 \times 10^{-5}) \times (5 \times 10^0) \times (4 \times 10^{12})$

$= (6 \times 7 \times 5 \times 4) \times (10^{8+(-5)+0+12})$

$= 840 \times 10^{15}$

$= 8.4 \times 10^{17}$ **(answer in scientific notation)**

3) Divide the distance to the moon in miles by the average speed in mph to get the number of hours it takes to get to the moon. Divide that answer by 24 to get the number of days.

$$\frac{2.38 \times 10^5}{1.4 \times 10^3 \times 24} = \frac{1.7 \times 10^2}{24} = \frac{170}{24} = 7.083... \approx 7.1$$

So, it would take about 7.1 days (rounded to the nearest tenth of a day) for the rocket to reach the moon.

Category 3
Number Theory
Meet #3 - January, 2015

1) Express the binary number 101001001 (base 2) as a base 10 numeral.

2) Compute. Express your answer in scientific notation:

$$\frac{200\times10^{7}}{0.04\times10^{6}}\div\frac{0.0005\times10^{-3}}{80\times10^{0}}$$

3) Convert the base 3 numeral 202110 to a base 5 numeral.

More than 517,000,000 of the famed Elvis Presley postage stamp have been sold since the stamp was first issued on January 8, 1993, more than any other single stamp ever issued by the United Sates Postal Service (USPS) since Benjamin Franklin became our first Postmaster General in 1775.

Answers

1) _______

2) _______

3) _______

Solutions to Category 3
Number Theory
Meet #3 - January, 2015

Answers

1) 329

2) 8×10^{12}

3) 4202

1) From right to left, 101001001 (base 2) =
1(1) + 0(2) + 0(4)+ 1(8) + 0(16) + 0(32) + 1(64) + 0(128) + 1(256)
= 1 + 0 + 0 + 8 + 0 + 0 + 64 + 0 + 256
= 329

2)

$$\frac{200\cdot10^7}{0.04\cdot10^6}\div\frac{0.0005\cdot10^{-3}}{80\cdot10^0}=\frac{200\cdot10^7}{0.04\cdot10^6}\cdot\frac{80\cdot10^0}{0.0005\cdot10^{-3}}$$

$$=\frac{16000\cdot10^7}{0.000020\cdot10^3}=\frac{16\cdot10^3\cdot10^7}{2\cdot10^{-5}\cdot10^3}=\frac{16\cdot10^{3+7}}{2\cdot10^{-5+3}}=\frac{16\cdot10^{10}}{2\cdot10^{-2}}=8\cdot10^{10-(-2)}=8\cdot10^{12}$$

3) First convert the base 3 numeral to a base 10 numeral and then to a base 5 numeral:
202110 (base 3) . . . (from right to left):
= 0(1) + 1(3) + 1(9) + 2(27) + 0(81) + 2(243)
= 0 + 3 + 9 + 54 + 0 + 486
= 552 (base 10)
To convert 552 (base 10) to base 5, divide 552 by subsequent powers of five until there is a remainder less than 5:
552 / 125 = 4 with remainder 52.
52 / 25 = 2 with remainder 2.
= 4(125) + 2(25) + 0(5) + 2(1)
= 4202 (base 5)

Category 3
Number Theory
Meet #3, January 2013

1. Complete the base-six addition problem below, giving your answer in base six.

$$\begin{array}{r} 3452_{six} \\ +\ 2405_{six} \\ \hline \end{array}$$

2. Simplify the expression below. Write your answer in scientific notation.

$$\frac{7.2\times10^{-3}}{4.8\times10^{7}}\times\left(3\times10^{12}\right)$$

3. Find the base-six value of the following expression.

$$111_2+222_3+333_4+444_5$$

Answers

1. __________ base six

2. ________________

3. __________ base six

Solutions to Category 3
Number Theory
Meet #3, January 2013

Answers
1. 10301 base six
2. 4.5×10^2
3. 1004 base six

1. When working in base 6, we have to remember to regroup whenever we get six in a place value. The number of groups of six will "carry" to the next place value, as shown below.

$$\begin{array}{r} 3452_{six} \\ +\ 2405_{six} \\ \hline 10301_{six} \end{array}$$

2. When evaluating this product, we can deal with the powers of ten separately. Also, we can "cancel" a common factor of 2.4 among the other factors, as shown below.

$$\frac{7.2 \times 10^{-3}}{4.8 \times 10^{7}} \times \left(3 \times 10^{12}\right) = \frac{7.2 \times 3}{4.8} \times \frac{10^{-3} \times 10^{12}}{10^{7}} = \frac{3 \times 3}{2} \times 10^{12-3-7} = 4.5 \times 10^{2}$$

3. It probably makes sense to convert all these numbers to base ten first. Recall that the place values in any base are powers of the base. We will multiply the digits in each place value by the base-ten equivalents of the place values as follows.

$$\begin{aligned} &111_2 + 222_3 + 333_4 + 444_5 = \\ &\left(1 \times 4 + 1 \times 2 + 1 \times 1\right) + \left(2 \times 9 + 2 \times 3 + 2 \times 1\right) + \\ &\left(3 \times 16 + 3 \times 4 + 3 \times 1\right) + \left(4 \times 25 + 4 \times 5 + 4 \times 1\right) = \\ &7 + 26 + 63 + 124 = 220_{ten} \end{aligned}$$

Notice that 7, 26, 63, and 124 are all one less than the next power of the base. In other words, $\left(8-1\right)+\left(27-1\right)+\left(64-1\right)+\left(125-1\right)=220_{ten}$.

Finally, we convert 220_{ten} to base six as follows:

$$220_{ten} = 216 + 4 = 1 \times 6^3 + 0 \times 6^2 + 0 \times 6^1 + 4 \times 6^0 = \mathbf{1004_6}$$

Category 3 – Number Theory

1. One gram of oxygen contains 1.88×10^{22} molecules.
   There are, of course, 1,000 grams in one kilogram.
   The Earth's atmosphere is estimated to weigh 5×10^{18} kilograms, of which 20% is oxygen. How many oxygen molecules are in the atmosphere?
   Express you answer in scientific notation.

2. Express 3.6×10^2 in Binary (Base 2).

3. Find the value of N that solves the equation below:

$$100_{Base\ 4} \times 100_{Base\ 5} = 620_{Base\ N}$$

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 3 – Number Theory

Answers

1. 1.88×10^{43}
2. 101,101,000 or 101101000 (commas optional)
3. 8

1. There are $20\% \times (5 \times 10^{18})$ kilograms of oxygen, each one containing $1{,}000 \times (1.88 \times 10^{22})$ molecules, so overall: $0.2 \times 5 \times 1.88 \times 10^{3+18+22} = 1.88 \times 10^{43}$ oxygen molecules. *(If you don't believe, count for yourself!).*

2. $3.6 \times 10^2 = 360 = 256 + 64 + 32 + 8 = 2^8 + 2^6 + 2^5 + 2^3 = 101{,}101{,}000_{Base\ 2}$

3. $100_{Base\ 4} \times 100_{Base\ 5} = 4^2 \times 5^2 = 400 = 620_{Base\ N} =$
   $= 6 \cdot N^2 + 2 \cdot N = 2 \cdot N \cdot (3 \cdot N + 1)$

   Trying a few values we can quickly realize that $N = 8$ is the solution.

   It is helpful to make these two observations:

   a. Since $400_{Base\ 10} = 620_{Base\ N}$ then we know $N < 10$
   b. We know that N is a factor of 400, since 620 ends in a '0'.

Category 3
Number Theory
Meet #3, January 2009

1. Express the base seven number 1234_7 as a base ten number.

$$1234_7 = \underline{\hspace{4cm}}_{10}$$

2. The diameter of the Earth at the equator is approximately 1.28×10^7 meters, while the average diameter of the smallest egg from a creature on Earth, the egg of the *Zenillia pullata*, is approximately 2×10^{-5} meters long. How many times as long as the diameter of the average *Zenillia pullata* egg is the diameter of the Earth at the equator? Express your answer in scientific notation.

3. Find the product of 101101_2 and 22_3 when written in base 2.

$$101101_2 \times 22_3 = \underline{\hspace{3cm}}_2$$

Answers
1. ________________
2. ________________
3. ________________ base 2

Solutions to Category 3
Number Theory
Meet #3, January 2009

Answers

1. 466

2. 6.4×10^{11}

3. 101101000_2

1. $1234_7 = 1(7^3) + 2(7^2) + 3(7^1) + 4(7^0) = 1(343) + 2(49) + 3(7) + 4(1) = 343 + 98 + 21 + 4 = 466$

2. To find how many times as long the Earth's diameter is compared to the egg's diameter we need to find the quotient of the two.

$$\frac{1.28\times10^{7}}{2\times10^{-5}} = \frac{1.28\times10^{7}\times10^{5}}{2} = \frac{1.28\times10^{12}}{2} = \mathbf{.64 \times 10^{12} = 6.4 \times 10^{11}}$$

3. Turning both numbers to base 10 first makes the problem $45 \times 8 = 360$. Turning 360 back to base 2 yields 101101000_2.

A more elegant way to do this is to notice that $22_3 = 8 = 1000_2$. Multiplying a base 2 number by 1000_2 has the same effect as multiplying a base 10 number by 1000. You just need to add 3 zeros to the end of the original number!