Math League SCASD

Meet #3

Geometry

2018 - Self-study Packet

Problem Categories for this Meet (in addition to topics of earlier meets):

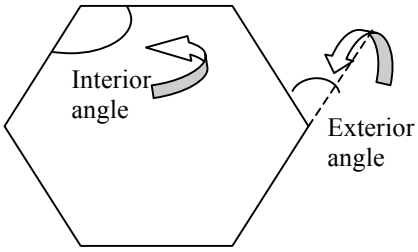
1. Mystery: Problem solving
2. **Geometry: Properties of Polygons, Pythagorean Theorem**
3. Number Theory: Bases, Scientific Notation
4. Arithmetic: Integral Powers (positive, negative, and zero), roots up to the sixth
5. Algebra: Absolute Value, Inequalities in one variable including interpreting line graphs

Important Information you need to know about GEOMETRY…

Properties of Polygons, Pythagorean Theorem

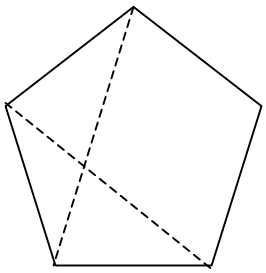
Formulas for Polygons where *n* means the number of sides:

- **Exterior Angle Measurement of a Regular Polygon:** 360÷n
- **Sum of Interior Angles:** 180(n – 2)
- **Interior Angle Measurement of a regular polygon:** $\frac{180(n-2)}{n}$
- **An interior angle and an exterior angle of a regular polygon always add up to 180°**



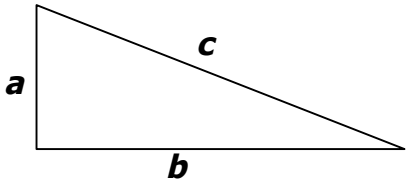
Diagonals of a Polygon where *n* stands for the number of vertices (which is equal to the number of sides):

- $\frac{n(n-3)}{2}$
- A diagonal is a segment that connects one vertex of a polygon to another vertex that is not directly next to it. The dashed lines represent *some* of the diagonals of this pentagon.



Pythagorean Theorem

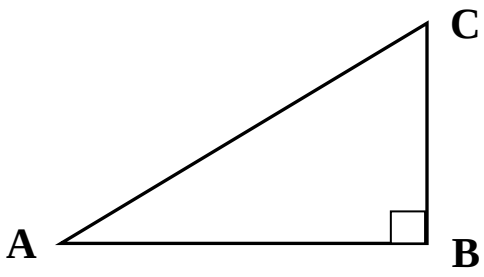
- $a^2 + b^2 = c^2$
- *a* and *b* are the legs of the triangle and *c* is the hypotenuse (the side opposite the right angle)



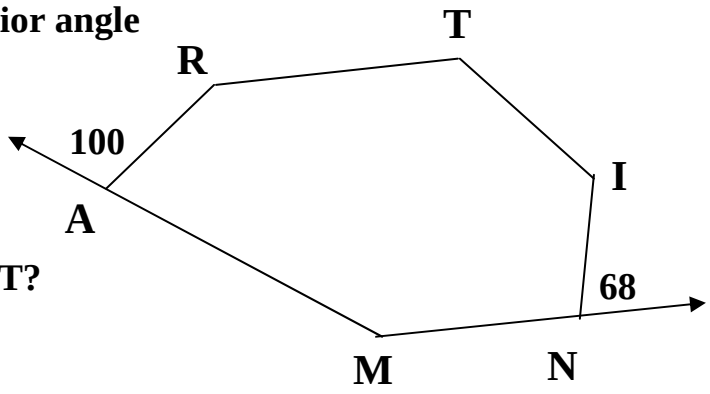
- Common Right triangles are ones with sides 3, 4, 5, with sides 5, 12, 13, with sides 7, 24, 25, and multiples thereof—Memorize these!

Category 2
Geometry
Meet #3 - January, 2019

1) In right triangle ABC, leg AB = 8 units and leg BC = 6 units. How many units long is the hypotenuse, AC?



2) Hexagon MARTIN has exterior angle measures as given, in degrees. Interior angles R, T, M, and I are congruent (and have the same measure). How many degrees are in the measure of interior angle T?



3) Jackson lives in Northborough. He walked 14 miles to the east, then 3 miles south, 22 miles east, and finally 12 miles south. Jackson walked an average speed of 3 miles per hour. If, instead, he had walked directly in a straight line from his starting point to his final destination, then how many hours fewer would it have taken him to do his walk?

Answers

1) _____ units

2) _____ degrees

3) _____ hours

Solutions to Category 2
Geometry
Meet #3 - January, 2019

Answers	
1)	10
2)	132
3)	4

1) Using the Pythagorean Theorem:

$$(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$$
$$(6)^2 + (8)^2 = (h)^2$$
$$36 + 64 = (h)^2$$
$$100 = (h)^2$$
$$10 = h$$

Therefore, AC = 10 units.

2) The sum of the interior angles of a convex polygon is 180 (n - 2), where n is the number of sides. Calculating the total number of degrees of the interior angles: 180 (6 - 2) = 180 (4) = 720 degrees. The given exterior angles of 100 and 68 degrees, respectively, have supplementary angles that are interior to the hexagon, and therefore measure 80 and 112 degrees, respectively. The total of the remaining four congruent interior angles, namely R, T, M, and I, is 720 - (80 + 112), or 720 - 192, or 528 degrees. Dividing 528 by 4 yields 132 degrees, the measure of each of the remaining interior angles. Therefore, angle T measures 132 degrees.

3) Adding the east distances yields 14 + 22 = 36 miles. Adding the south distances yields 3 + 12 = 15 miles. The distance from starting point to final destination is the hypotenuse of a right triangle with legs 36 and 15 miles. Use the Pythagorean Theorem to find the length of the hypotenuse:

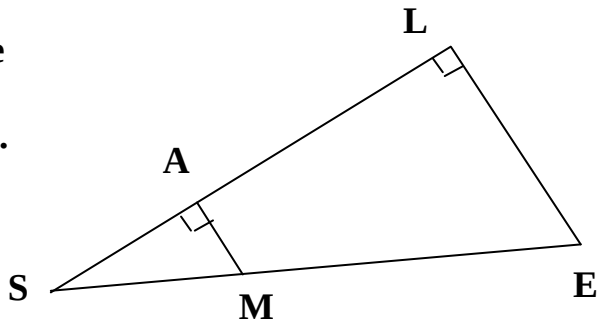
$$15^2 + 36^2 = d^2$$
$$225 + 1296 = d^2$$
$$1521 = d^2$$
$$39 = d$$

Oliver originally walked 15 + 36, or 51 miles. By taking the hypotenuse shortcut, he walked 12 miles fewer (51 - 39 = 12). At 3 miles per hour, he walked 4 hours fewer (12 / 3 = 4).

Category 2
Geometry
Meet #3 - January, 2017

1) How many inches long is a rectangle whose diagonal is 20 inches and whose width is 12 inches?

2) How many centimeters are in the perimeter of triangle SLE ? All measurements are in centimeters. AS = 4, SM = 5, and LE = 9.



3) How many more diagonals does a convex 15-sided polygon have than a 12-sided regular polygon?

Answers

1) _____

2) _____

3) _____

Solutions to Category 2
Geometry
Meet #3 - January, 2017

Answers	
1)	16
2)	36
3)	36

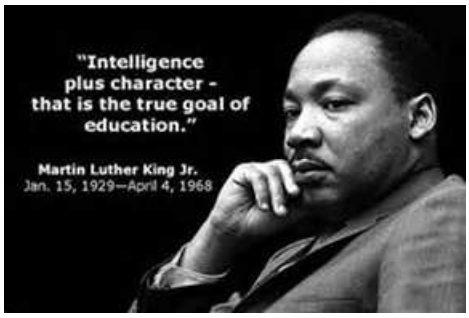
1) The diagonal of the rectangle forms the hypotenuse of a right triangle. The width and length of the rectangle form the legs of the right triangle. Use the Pythagorean Theorem to find the length:

$$(length)^2 + (width)^2 = (hypotenuse)^2$$
$$(L)^2 + (12)^2 = (20)^2$$
$$(L)^2 + 144 = 400$$
$$L^2 = 256$$
$$L = 16$$

2) Triangle ASM is similar to triangle LSE. To establish the scaling factor, we need to know the lengths of a pair of corresponding sides. We can find the length of AM because it is a leg of right triangle ASM. Using the Pythagorean Theorem, we find that AM = 3. The side of triangle LSE that corresponds to AM is LE = 9, so the scale is 9:3, or 3:1. Then LS = (3)(AS) = (3)(4) = 12 and SE = (3)(SM) = (3)(5) = 15. The perimeter of triangle SLE is 9 + 12 + 15 = 36 centimeters.

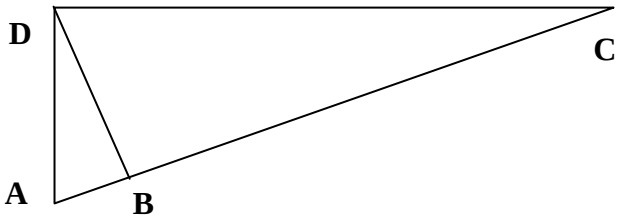
3) The number of diagonals in a convex polygon, where S = the # of sides, is given by the expression (S)(S - 3) / 2. So, for the convex 15-gon, the number of diagonals is (15)(12) / 2 = 90 and the number of diagonals in the 12-gon is (12)(9) / 2 = 54. The difference is 90 - 54, or 36.
Alternatively, at any vertex of the convex 15-gon, you can draw diagonals to all 15 vertices except itself and two neighbors, hence 15 - 3 = 12 diagonals per vertex. There are 15 vertices so 15 x 12 diagonals. But that counts all diagonals twice, at each end. So there are (0.5)(15 x 12) diagonals to the 15-gon. Same argument for the 12-gon.

Category 2
Geometry
Meet #3 - January, 2015

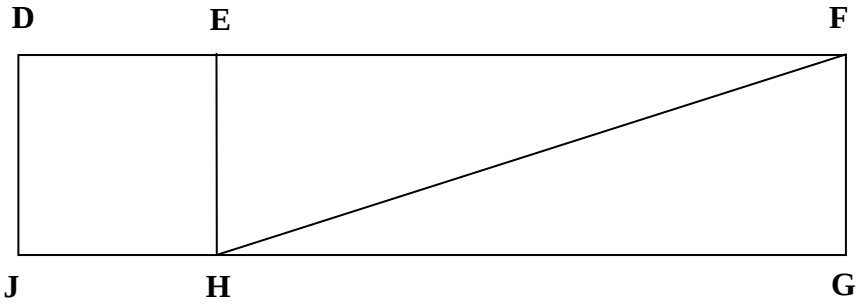


Figures are not necessarily drawn to scale.

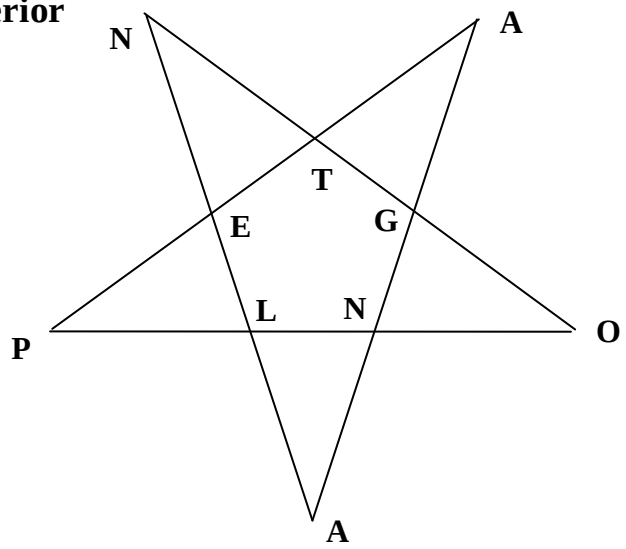
1) Angle ADC is a right angle. AB = 4 cm and BC = 9 cm. DB is perpendicular to AC. How many cm long is DB ?



2) DEHJ is a square with an area of 64 square meters. Diagonal HF = 17 meters. How many square meters are in rectangle DFGJ ?



3) Polygon PENTAGONAL is a pentagram (star) consisting of a regular pentagon with five isosceles triangles attached at its five edges. How many degrees are in one of the exterior angles (for example, angle PEN) ?



Answers

1) _____

2) _____

3) _____

Solutions to Category 2
Geometry
Meet #3 - January, 2015

Answers
1) 6
2) 184
3) 108

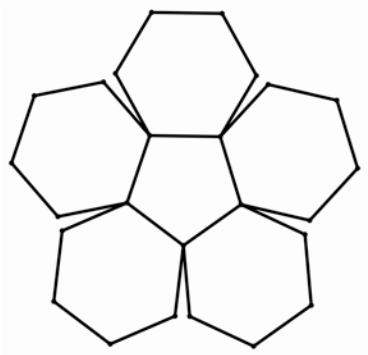
1) A student who knows the Pythagorean Theorem should also know that, at its foundation, is the notion of similar triangles. In this diagram are three similar triangles. Using triangles DAB and DBC, we can say that ratios of corresponding sides are proportional: $\frac{AB}{DB}=\frac{DB}{BC}$. **So,** $\frac{4}{DB}=\frac{DB}{9}$ **and cross products are equal, so** $(DB)^2=36$ **and DB = 6. A few students may recognize this diagram as representing this theorem: "The altitude to the hypotenuse of a right triangle is the geometric mean (or mean proportional) to the two segments of the hypotenuse into which it is divided."**

2) One side of square DEHJ is 8 meters because its area is 64 square meters. For one of the right triangles of rectangle EFGH, using the Pythagorean Theorem, $(EH)^2+(EF)^2=(HF)^2$. **So,** $(8)^2+(EF)^2=(17)^2$, **and** $64+(EF)^2=289$, **so** $(EF)^2=225$, **and EF = 15. So, rectangle DFGJ now measures 8 by (15 + 8), or 8 by 23, so its area is (8)(23), or 184 square meters.**

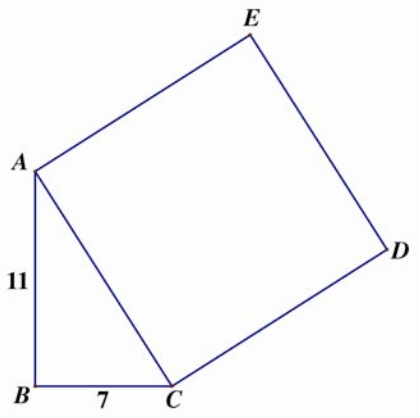
3) Each interior angle of the regular pentagon measures (3)(180)/5, or 108 degrees. Any one of the exterior angles of the pentagram is vertical to one of these interior 108 degree angles and, therefore, is equal to 108 degrees.

Category 2
Geometry
Meet #3, January 2013

1. Mia drew regular hexagons on each side of a pentagon. If she draws all the diagonals in all six shapes, how many diagonals will she have to draw? Note: A diagonal is a line segment that connects two vertices that are not already connected by a side.



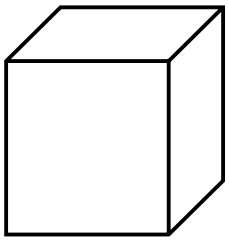
2. Right triangle ABC below has legs of length 7 units and 11 units. How many square units are there in the area of square ACDE which is constructed on the hypotenuse of this triangle?



3. A small rectangular box has sides of lengths 6 cm, 6 cm, and 7 cm. How many centimeters are there in the space diagonal of the box? Note: A space diagonal is a line that connect two opposite vertices of the box and goes through the interior space of the box.

Answers

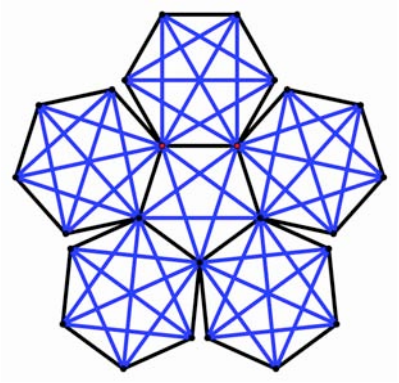
1. _____ diagonals
2. _____ sq. units
3. _____ cm



Solutions to Category 2
Geometry
Meet #3, January 2013

Answers
1. 50 diagonals
2. 170 sq. units
3. 11 cm

1. Three diagonals can be drawn from each of the six vertices on a hexagon, but this would count each diagonal at both ends. So there are $3 \times 6 \div 2 = 9$ diagonals in each hexagon. Similarly, there are $2 \times 5 \div 2 = 5$ diagonals in the pentagon. Mia will have to draw $5 \times 9 + 5 =$ **50 diagonals**.



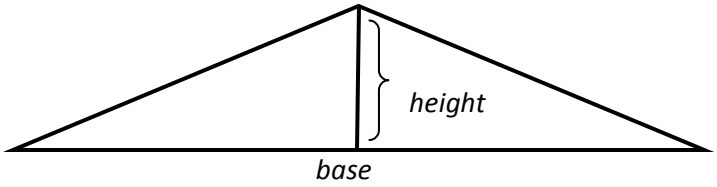
2. According to the Pythagorean theorem, the sum of the areas of the squares on the legs of a right triangle is equal to the area of the square on the hypotenuse. If we were to construct a square on leg AB, it would have an area of $11 \times 11 = 121$ square units. A square on leg BC would have an area of $7 \times 7 = 49$ square units. Their sum is $121 + 49 =$ **170 square units** and this is the area of square ACDE.

3. We can calculate the length of the space diagonal of the box by using the Pythagorean theorem twice. First we can find the length of the diagonal of the bottom face, which is $\sqrt{6^2+6^2} = \sqrt{72} = 6\sqrt{2}$ cm. Then we find the length of the space diagonal using this length and the height of the box as follows: $\sqrt{7^2+\left(6\sqrt{2}\right)^2} = \sqrt{49+72} = \sqrt{121} = \mathbf{11}$ cm.

Alternatively, we can use a 3-dimensional version of the Pythagorean theorem as follows: $\sqrt{6^2+6^2+7^2} = \sqrt{36+36+49} = \sqrt{121} = \mathbf{11}$ cm.

Category 2 – Geometry

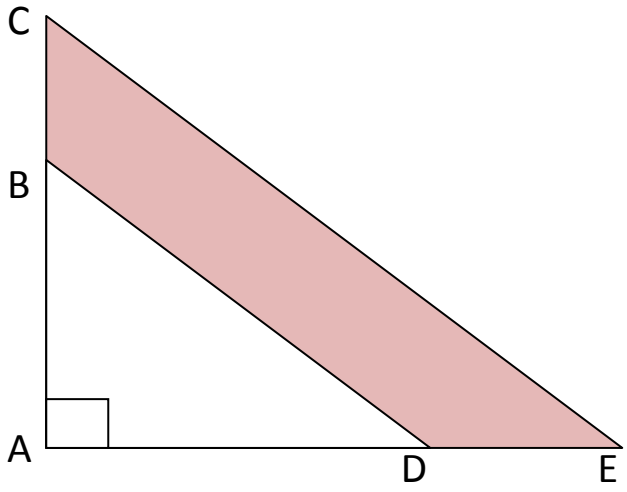
1. In the isosceles triangle below, the base measures 24 inches, and the height measures 5 inches. How many inches are in the triangle's perimeter?



2. Spongebob and Patrick each drew a regular polygon. Patrick's polygon had twice as many sides and six times as many diagonals as Spongebob's. How many sides to Patrick's polygon?

3. In the drawing below:
   - $\overline{BD}$ is parallel to $\overline{CE}$
   - $\overline{CB} = 3\ cm$, $\overline{DE} = 4\ cm$, and $\overline{BD} = 10\ cm$

How many centimeters squared are in the shaded trapezoid $BCED$?



Answers

1. _____

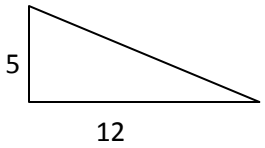
2. _____

3. _____

Solutions to Category 2 – Geometery

Answers	
1.	50
2.	12
3.	30

1. In each half of the triangle, we have $5^2 + 12^2 = 169 = Hypotenuse^2$ and therefore the Hypotenuse measures 13 inches, and the whole perimeter measures $13 + 13 + 24 = 50\ inches$.



2. If Spongebob's polygon has N sides, then it has $\frac{1}{2} \cdot N \cdot (N-3)$ diagonals. Patrick's polygon then has $2 \cdot N$ sides and $\frac{1}{2} \cdot (2 \cdot N) \cdot (2 \cdot N - 3)$ diagonals. So from the problem we know that: $(2 \cdot N) \cdot (2 \cdot N - 3) = \mathbf{6} \cdot N \cdot (N-3)$ which we can simplify to $(2 \cdot N - 3) = 3 \cdot (N-3)$. The solution is $N = 6$ and so Patrick's polygon has $2 \cdot N = 12$ sides.

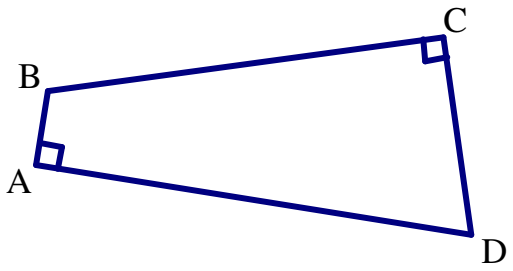
3. Since $\overline{BD}$ is parallel to $\overline{CE}$ then the triangles ABD and ACE are similar. Therefore $\frac{\overline{AB}}{\overline{AD}} = \frac{\overline{BC}}{\overline{DE}} = \frac{3}{4}$ and since in triangle ABD we have $\overline{AB}^2 + \overline{AD}^2 = 10^2$ we get that $\overline{AB} = 6cm$ and $\overline{AD} = 8cm$. Therefore the area of triangle ABD is $24\ cm^2$ and the area of triangle ACE is $54\ cm^2$, and the trapezoid is their difference, $30\ cm^2$.

Category 2
Geometry
Meet #3, January 2009

1. How many degrees are in the sum of the interior angles of a convex decagon?

2. Let the number of diagonals in a regular octagon be a, and the number of diagonals in a regular hexagon be b. What is the value of $a - b$?

3. Quadrilateral ABCD has right angles at A and C. The lengths of CD, BC, and AB are 7 cm, 11cm, and 1cm respectively. How many centimeters long is AD?



Answers

1. _____
2. _____
3. _____

Solutions to Category 2
Geometry
Meet #3, January 2009

Answers

1. 1440

2. 11

3. 13

1. For an n-sided convex polygon, the expression $180(n-2)$ will give the number of degrees in the sum of the interior angles of the polygon. So a decagon ($n = 10$) would have $180(10-2) = 180(8) = 1440$ degrees.

2. For an n-sided convex polygon, the expression $\frac{n(n-3)}{2}$ will give the number of diagonals in the polygon. So an octagon has $\frac{8(5)}{2} = 20$ diagonals and a hexagon has $\frac{6(3)}{2} = 9$ diagonals. So $a = 20$ and $b = 9$. Therefore $a - b = 20 - 9 = 11$.

3. First draw in segment BD. Since BD is the hypotenuse of triangle BCD, we can use the Pythagorean Theorem to find the length of drawn in segment BD. $BD^2 = 7^2 + 11^2 = 49 + 121 = 170$. However BD is also the hypotenuse of triangle ABD and therefore $BD^2 = 1^2 + AD^2 = 1 + AD^2 = 170$. So $AD^2 = 169$ and $AD = 13$.