Math League SCASD

Meet #2

Number Theory

Self-study Packet

Problem Categories for this Meet:

1. Mystery: Problem solving
2. Geometry: Angle measures in plane figures including supplements and complements
3. **Number Theory: Divisibility rules, factors, primes, composites**
4. Arithmetic: Order of operations; mean, median, mode; rounding; statistics
5. Algebra: Simplifying and evaluating expressions; solving equations with 1 unknown including identities

Meet #2 – Number Theory

Ideas you should know:

Prime Factorization: Every number 2,3,4,… can be written as a product of prime number factors. For example, 12 = 2x2x3, or $2^2\cdot3$. What is the prime factorization of 60? Answer: $2^2\cdot3\cdot5$, or $2^2\cdot3^1\cdot5^1$.

12	=	$2^2\cdot3^1$
24	=	$2^3\cdot3^1$
100	=	$2^2\cdot5^2$
210	=	$2^1\cdot3^1\cdot5^1\cdot7^1$

To the Zero Power: Any non-zero number to the zero power is one. $5^0=1$. This makes sense if you think that $5^3\div5=5^2$, and $5^2\div5=5^1$, so $5^1\div5=5^0$, which is the same as $5\div5=5^0=1$. (0^0 is undefined, not 1.)

$$5^0=1$$
$$5^1=1\text{x}5 \quad = 5$$
$$5^2=1\text{x}5\text{x}5=25$$

Least Common Multiple (LCM): LCM(a,b) is the least (smallest) number that is a multiple of both a and b. For example, LCM(6,10) = 30 because both 6 and 10 are factors of 30, and there is no smaller number than 30 that does this. One way of computing LCM(a,b) is by prime factoring a and b. Then, for each prime factor, take the *larger* of the two exponents. For example, $6=2^1\cdot3^1\cdot5^0$, and $10=2^1\cdot3^0\cdot5^1$. The larger exponents: $2^1\cdot3^1\cdot5^1=30$.

LCM(2, 3) = 6   LCM(2,4) = 4   LCM(10,15) = 30

LCM(6,10) = 30   LCM(8,12) = 24   LCM(6, ?) = 18 (?=9 or 18)

<u>Greatest Common Factor (GCF)</u>: GCF(a,b) is the greatest (largest) number that is a factor of both a and b. For example, GCF(6,10) = 2 because 2 is a factor of both 6 and 10, and there is no larger number that does this. One way of computing GCF(a,b) is by prime factoring a and b, and then, for each prime factor, take the *smaller* of the two exponents. For example, $6=2^1 \cdot 3^1 \cdot 5^0$, and $10=2^1 \cdot 3^0 \cdot 5^1$. The smaller exponents: $2^1 \cdot 3^0 \cdot 5^0 = 2$.

Useful trick: GCF(a,b) x LCM(a,b) = a x b.

GCF(2,3) = 1   GCF(4,6) = 2   GCF(12, 16) = 4

GCF(12,18)=6   GCF(17,24)=1   GCF(11, 22) = 11

GCF(10,?) x LCM(10,?) = 30   (? = 3, because GCFxLCM=3x10)

<u>Proper Factor</u>: A factor of a number other than the number itself or one. The proper factors of 6 are 2 and 3, but not 1 or 6. There are no proper factors of prime numbers.

Factors of 6: 1, 2, 3, 6   <u>Proper</u> Factors: 2, 3

Factors of 7: 1, 7   <u>Proper</u> Factors: None!

<u>Relatively Prime</u>: Two numbers are relatively prime if they have no common factors other than 1. Their GCF is 1. For example, 7 and 8 are relatively prime as they share no factors greater than 1. If the difference between two numbers is prime, then the numbers are relatively prime.

If GCF(A,B)=1, then A and B are relatively prime.

If A – B = prime, then GCF(A,B)=1

Are 21 and 91 relatively prime?   No, GCF(21,91)=7

Are 10 and 21 relatively prime?   Yes, 2x5 and 3x7 share no factors.

Number of Factors: 12 has 6 factors: 1,2,3,4,6,12 (we count 1 and 12). To count them, do prime factorization first. Take the exponents, add one to each, and multiply. For example, $12=2^2\cdot3^1$, so the exponents are 2 and 1. Add one to each: 3 and 2. Multiply: 2x3=6. So, 12 has 6 factors, same as we listed above. How many factors does 1000 have? $1000=2^3\cdot5^3$, so it has 4x4=16 factors.

Another way to count the number of factors is to list them in pairs and be sure you got them all! That's the hard part.

3+1=4 → *4x3=12 factors*

2+1=3

$$200=2^3\cdot5^2$$

$$\left.\begin{array}{l} 1\ x200 \\ 2\ x100 \\ 4\ x\ 50 \\ 5\ x\ 40 \\ 8\ x\ 25 \\ 10x20 \end{array}\right\} = 12\ factors$$

Perfect squares have an odd number of factors. 36=1x36, 2x18,3x12,4x9,6x6, or 9 distinct factors. $36=2^2x3^2$, (2+1)x(2+1)=9 factors.

Factors of 49: 1, 7, 49 = 3 factors – an odd number

Factors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36 = 9 factors (odd)

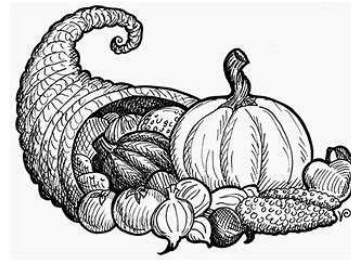
Euler's GCF Method (Fast way to compute Greatest Common Factor):
If A>B, then GCF(A,B) = GCF(A-B,B). This works because if A and B have a common factor, then A-B has the same factor. Repeat until obvious.

GCF(130,91) = GCF(130-91, 91) = GCF(39, 91) =

GCF(91, 39) = GCF(91-39, 39) = GCF(52,39) =

GCF(52-39, 39) = GCF(13,39) = 13

Category 3
Number Theory
Meet #2 - December, 2018



1) The GCF (greatest common factor) of A and B is the largest whole number that is a factor of both A and B.

X = the GCF of 12 and 20.
Y = the GCF of 15 and 35.
Z = the GCF of 8 and 21.

What is the sum of X + Y + Z ?

2) The prime factorization of 11,220 is 5 x A x B x C x 3 x 11. What is the sum of A + B + C ?

3) The GCF of F and G is 6. The LCM (lowest common multiple) of F and G is 72. F = 18. What is the value of G ?

Answers

1) __________

2) __________

3) __________

Dec. 8 is "Pretend to be a Time Traveler" Day.

Solutions to Category 3
Number Theory
Meet #2 - December, 2018

Answers	
1)	10
2)	21
3)	24

1) X = GCF (12, 20) = 4.
   Y = GCF (15, 35) = 5.
   Z = GCF (8, 21) = 1.
   X + Y + Z = 4 + 5 + 1 = 10.

2) The prime factorization can be accomplished by dividing 11,220 by increasing values of its prime factors.

   11,220 = 2 x 5,610
   = 2 x 2 x 2,805
   = 2 x 2 x 3 x 935
   = 2 x 2 x 3 x 5 x 187
   = 2 x 2 x 3 x 5 x 11 x 17.

   Although many students will opt for what is familiar, as above, the problem gives several of the factors. Dividing 11,220 by (5 x 3 x 11) gives 68. Prime factoring 68 will give the remaining prime factors: 68 = 2 x 2 x 17. Therefore, A + B + C = 2 + 2 + 17 = 21.

3) Students can guess and check their way to the answer. However, some may know this relationship: GCF (A, B) x LCM (A, B) = A x B.
   Substituting the given values gives (6)(72) = (18)(G)
   432 = 18G
   432 / 18 = G
   24 = G

Category 3
Number Theory
Meet #2 - December, 2016

1) Juan sees Thieu every 12 days and he sees Thrye every 28 days. Juan, Thieu, and Thrye are having lunch today. In how many days will they next see each other ?

2) 2520 = $2^3 \times 3^2 \times A \times B$ where A and B are prime numbers. What is the value of A + B ?

3) Find the smallest whole number that is a common multiple of 18 and 40 and is also a perfect square.

Answers
1) _______
2) _______
3) _______

Solutions to Category 3
Number Theory
Meet #2 - December, 2016

Answers
1) 84
2) 12
3) 3600

1) Find the LCM of 12 and 28. The numbers are small enough that the answer can be achieved by listing multiples of 12 and 28:
multiples of 12: 12 24 36 48 60 72 84
multiples of 28: 28 56 84

2) The prime factorization of 2520 = 2 x 2 x 2 x 3 x 3 x 5 x 7.
A + B = 5 + 7 = 12.

3) 18 = 2 x 3 x 3
40 = 2 x 2 x 2 x 5
Their LCM is 2 x 2 x 2 x 3 x 3 x 5 = 360.
The smallest multiple of this LCM must have an even number of like factors, so = 2 x 2 x 2 x 2 x 3 x 3 x 5 x 5 = 3600.

Category 3
Number Theory
Meet #2 - November, 2014

1) Simplify (reduce to lowest terms) the fraction $\frac{363}{594}$
if $363 = 3 \times 11^2$ and $594 = 2 \times 3^3 \times 11$.

2) Consider these factorizations:

$$C = 2 \times 3^2 \times N \quad \text{and} \quad D = 2^2 \times 5 \times N^2$$

If all of the lettered values are positive integers (whole numbers) and D = 1620, then what is the value of C ?

3) Jacqueline watches the TV sitcom "Big Bang Theory" every 12 days. She watches "Cosmos" every 18 days and "Scorpion" every 28 days. If she watched all three shows on a Tuesday, on what day of the week will she next watch all three shows ? Assume that all of the shows are aired every day of the week.

Answers
1) ______
2) ______
3) ______

Solutions to Category 3
Number Theory
Meet #2 - November, 2014

Answers

1) $\frac{11}{18}$

2) 162

3) Tuesday

1) $\frac{363}{594}=\frac{3\times11^2}{2\times3^3\times11}=\frac{11}{2\times3^2}=\frac{11}{18}$

2) If D=1620, then $1620=2^2\times5\times N^2$ and

$81=N^2$ and N = 9. Then substituting 9 for N in the first equation,

$C=2\times3^2\times9$ and C = 162.

3) First, find the LCM of 12, 18, and 28. Then divide the answer by 7 to see how many 7-day cycles there are. The remainder will be added onto Tuesday to determine the day of the week that all three shows will next be watched.

$12=2^2\times3$ and $18=2\times3^2$ and $28=2^2\times7$.

The LCM is the product of all the different factors, with each raised to the highest power that it appears in any of the factorizations. So,

the LCM is $2^2\times3^2\times7$ or 252.

Now 252 divided by 7 is 36 with a remainder of 0, so all three shows will next be watched on a Tuesday !

A shorter concluding step would be to acknowledge that the prime factorization of 252 contains the number 7 . . . thus making 252 divisible by 7 and there would be no remainder.

Category 3
Number Theory
Meet #2, November/December 2012

1. Some IMLEM Clusters will take Meet #2 on Thursday, 12/06. Find the prime factorization of 1206. You can use exponents or not, but you must write the prime factors in order from least to greatest.

2. Grace was supposed to find the greatest common factor (GCF) of 330 and 462, but she found the greatest common prime factor (GCPF) instead. What is the ratio of the GCF of 330 and 462 and the GCPF that Grace found?

3. At the Gadgets and Gizmos factory, workers complete a gadget every 588 minutes and a gizmo every 882 minutes. The factory is open 24 hours a day, seven days a week, and the workers always complete the gadgets and gizmos on schedule. If it happens that a gadget and a gizmo are completed at the same time at 3:00 PM on Thursday, November 29, how many additional times will a gadget and a gizmo be completed at the same time before 3:00 PM on Thursday, December 6?

Answers

1. ____________________

2. ____________________

3. ______________ times

Solutions to Category 3
Number Theory
Meet #2, November/December 2012

Answers

1. $\mathbf{2 \times 3^2 \times 67}$

or $\mathbf{2 \times 3 \times 3 \times 67}$

2. **6**

3. **5 times**

1. A systematic way to find the prime factorization of a number is to divide out primes in order of the primes as shown in the so-called ladder at right. The prime factorization of 1206 is $\mathbf{2 \times 3 \times 3 \times 67}$ or, using exponents, it's $\mathbf{2 \times 3^2 \times 67}$.

$$\begin{array}{r|l} 2 & \underline{1206} \\ 3 & \underline{603} \\ 3 & \underline{201} \\ 67 & \underline{67} \\ & 1 \end{array}$$

2. The prime factorization of 330 is $2 \times 3 \times 5 \times 11$ and that of 462 is $2 \times 3 \times 7 \times 11$. The GCF is $2 \times 3 \times 11 = 66$ and the GCPF is 11, so the desired ratio is 66/11 = **6**.
Editor note: The original problem asked "How many times greater is the actual GCF of 330 and 462 than the GCPF that Grace found?" This was changed here to remove any ambiguity.

3. The prime factorization of 588 is $2^2 \times 3 \times 7^2$, and that of 882 is $2 \times 3^2 \times 7^2$. The LCM of these two numbers is $2^2 \times 3^2 \times 7^2 = 1764$. This means that a gadget and a gizmo are completed simultaneously every 1764 minutes. There are $7 \times 24 \times 60 = 10{,}080$ minutes in a week. Five times 1764 = 8820 and $6 \times 1764 = 10584$, so gadgets and gizmos will be completed simultaneously an additional **5 times** in the next week.

Category 3 – Number Theory

1. What is the Least Common Multiple of 45 and 66?

2. The Greatest Common Factor (GCF) of two natural numbers A, B is 5, and their product $(A * B)$ is 1,000.
   What is the smallest possible sum of $A + B$?

3. In a far away galaxy, 3 comets are visible from planet 51:
   Comet Alpha is visible every 6 years, and was last seen in 2007.
   Comet Beta is visible every 7 years, and was last seen in 2009.
   Commet Gamma is visible every 8 years, and was last seen in 2009.
   In what year will all three comets be visible together again?

Answers
1. ________________
2. ________________
3. ________________

Answers	
1.	990
2.	65
3.	2,121

Solutions to Category 3 – Number Theory

1. $45 = 3^2 * 5$

   $66 = 2 * 3 * 11$

   The Least Common Multiple (LCM) is the product of all prime factors with their highest powers: $LCM(45,66) = 2 * 3^2 * 5 * 11 = 990$

2. The product of two numbers is equal to the product of their GCF and LCM: $A * B = GCF(A, B) * LCM(A, B)$, which in our case is 1,000, so we know that $LCM(A, B) = 1000 \div 5 = 200 = 2^3 \cdot 5^2$. Since $GCF(A, B) = 5$ then only one of the numbers can have 2 as a prime factor, not both of them. Both numbers have 5 as a factor, but only one of them has 5^2 as a factor (otherwise, the GCF would have been 5^2). So either $A = 2^3 \cdot 5^2 = 200$ and $B = 5$, or $A = 2^3 \cdot 5 = 40$ and $B = 5^2$. The second pair has the smaller sum, 65.

3. For convenience we can deduct 2000 from all years:

   Alpha is visible in years: $7, 13, 19, 25, \ldots$ (remainder 1 when divided by 6)

   Beta is visible in years: $9, 16, 23, 30, \ldots$ (remainder 2 when divided by 7)

   Gamma is visible in years: $9, 17, 25, 33, \ldots$ (remainder 1 when divided by 8)

   So to find the year when all appear, we need to find a number that fits all three conditions. Since (the years for) Alpha and Gamma both should leave a remainder of 1 when divided by 6 or 8, our year should leave a remainder of 1 when divided by any multiple of 6 and 8, and specifically by their LCM, 24. So our year can be one of the numbers $25, 49, 73, 97, 121$ ... but also needs to leave a remainder of 2 when divided by 7. The first number in this series to fit the bill is 121. So our year is 2,121.

Category 3
Number Theory
Meet #2, December 2008

1. What is the Greatest Common Factor of 216 and 504?

2. The prime factorization of X is $a^b \times c^d$. None of the values of $a, b, c,$ or d is equal to 2. Only one of the four values is 1. All four values of $a, b, c,$ and d are different. What is the smallest possible value of X?

3. The Least Common Multiple of positive integers a and b is 144. The Greatest Common Factor of a and b is 12. What is the value of $ab \div 27$?

Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 3
Number Theory
Meet #2, December 2008

Answers

1. 72

2. 405

3. 64

1. $216 = 2^3 \times 3^3$ and $504 = 2^3 \times 3^2 \times 7^1$
The Greatest Common Factor of 216 and 504 is equal to $2^3 \times 3^2 = 72$.

2. Since none of the values can be equal to 2, a and c should be 3 and 5 if we want X to be as small as possible (they can't be 1 since 1 isn't prime). Since we can use 1 just once, using it as the exponent of 5 will keep X smallest. That leaves the exponent for 3. The smallest value left to use is 4. So X = $3^4 \times 5^1 = 81 \times 5 = 405$.
Editor note: We can't use zero as no proper "prime factorization" has a zero as either a base or an exponent. Otherwise, there would be no unique prime factorization: 10 = $2^1 \cdot 5^1$ *or* $2^1 \cdot 5^1 \cdot 7^0$*, etc. See the Fundamental Theorem of Arithmetic.*

3. The LCM(a, b) times the GCF(a, b) will always be equal to the product of a and b. So $ab = 12 \times 144 = 1728$ and $ab \div 27 = 1728 \div 27 = 64$. A faster way to compute that would be $\frac{12\times144}{27} = \frac{4\times144}{9} = \frac{4\times16}{1} = 64$.