AP/ECE Calculus
Summer Work Packet 2024

This packet covers material you will need to know as you start your study of Calculus in the fall. The packet includes **<u>4</u> assignments**. It is important that you complete all of the assignments before the first day of school.

Each assignment includes:

- A Reading Section
  (This is intended to review things you have learned in previous math classes and teach you some new concepts.)

- Note Taking Instructions
  (This section gives you the terms and ideas you need to define in your Calculus Notebook. These concepts will be used throughout the year in Calculus.)

- A Problem Set
  *(You are expected to complete each problem set on a **<u>separate</u>** piece of paper, and have those assignments ready to hand in on the first day of school.)*

Due Dates:

- The Summer Work Assignments are due on the first day of school. ***(<u>Only</u> the problems for each assignment need to be turned in, and they should be done on separate paper, not written in this packet or included with your notes)***

- You will need to bring your Summer Work Packet <u>and</u> your Calculus Notebook with you the first day of school. We will be highlighting key concepts covered in the Summer Work Packet and looking at some applications of these concepts during the first 1 – 1 ½ weeks of school.

- At the end of the 2nd or 3rd week of school you will be tested on the first unit in Calculus, which will include material covered in the Summer Work Packet.

Questions?

As you work through the packet, you may have some questions regarding the new concepts. Since many of the ideas are new (or relatively new depending on how far you got in Precalculus), that is perfectly normal! To help with this, you can e-mail Mrs. Swanson at mswanson@cpsctg.org, even in the summer! I will generally get back to you within a few days of your e-mail.

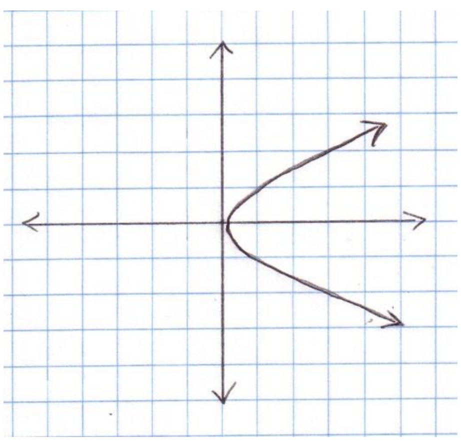
Assignment #1

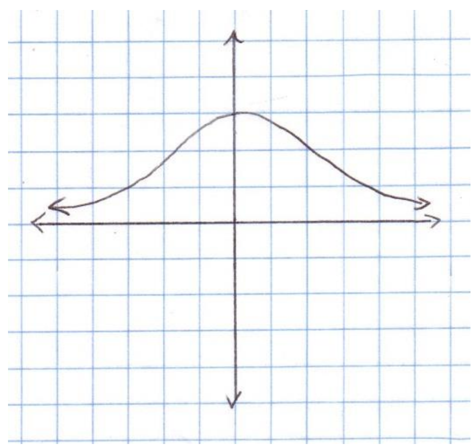
READING:

As we begin our study of Calculus, we need to review some important algebra concepts that will be used throughout our work. This first lesson will review the ideas of ***Function***, ***Domain*** and ***Range***, ***Symmetry***, ***Types of Functions*** and ***Inverse Functions***. You are expected to understand these concepts and be able to apply them throughout your Calculus work.

Long ago you learned about functions, and what makes an equation or a graph a function. To be sure we are all working with the same understanding, we have the following: A ***Function*** is an equation or graph for which each input (or independent variable value) has exactly one output (or dependent variable value). Algebraically we think of this definition this way: for each "x" value, we will get just one "y" value. It is possible for two different x-values to give the same y-value, but it is not possible to put an x-value into an equation and at different times get different y-values as answers. *FOR EXAMPLE:* Suppose our function equation is $y = x^2 + 2$. When x = 2 we get the ordered pair (2 , 6). When x = -2 we get the ordered pair (-2 , 6). However, since this is a function, at no time could we put x = 2 into the equation and get a different answer than y = 6. Similarly, at no time could we put x = -2 into the equation and get a different answer than y = 6. This, algebraically, is why our equation is a function.

Now, consider the graphs below. Are they functions? How do we know?

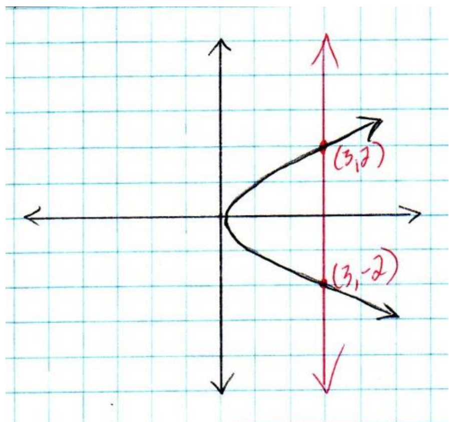
1. 

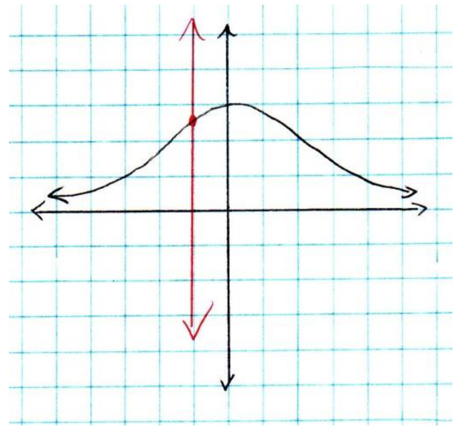
2. 

When we look at the two graphs, and the points on the graphs, we notice that graph #1 has places where the same x-value has two different points on the graph. (Look at x = 3 for example.) Because the points (3 , 2) and (3 , -2) are both on this graph, graph #1 is <u>not</u> a function. However, in considering graph #2 we do not find any x-values for which there is more than one point on the graph. So, graph #2 <u>is</u> a function.

The test we are using here is called the ***Vertical Line Test***. We know that every vertical line has the same x-value for every point on that line. So, the Vertical Line Test states: If at least one vertical line intersects a graph at more than one point, the graph is not a function. However, if every possible vertical line intersects the graph at just one point, the graph passes the Vertical Line Test and is a function.

In considering our graphs again with the Vertical Line Test:

1. 

2. 

We see that graph #1 fails the Vertical Line Test, where graph #2 passes the Vertical Line test. As a result, graph #1 is not a function, but graph #2 is a function.

Connected with the definition of function is the idea of ***domain*** and ***range*** of a function. The domain of a function is the set of all possible input values, or x-values, for the function. Similarly, the range of a function is the set of all possible output values, or y-values, for the function. Consider this example:

X	- 6	- 3	8	7	- 8	- 6	3	5	12
Y	- 8	- 2	20	18	- 12	- 8	10	14	28

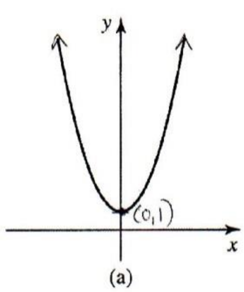
As we consider the set of ordered pairs, we see the data does represent a function because each input is paired with exactly one output. So, the domain and range of the function can be described as follows:

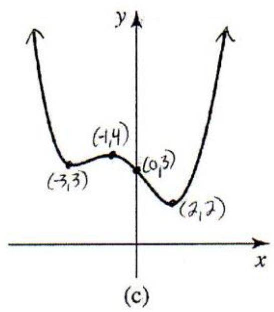
Domain {- 8, - 6, - 3, 3, 5, 7, 8, 12}

Range {- 12, - 8, - 2, 10, 14, 18, 20, 28}

Notice duplicate x and y values were not listed twice in the domain and range. Also notice the values for the domain and range were put in numerical order. While it is common practice to omit repeats, it is <u>not</u> necessary to put domain and range values in numerical order. While most textbooks will use numerical order, it is not something that will be required in your work.

Now consider the following graphs:

1. 

2. 

For graph #1, we see the graph is a function because it passes the Vertical Line Test. So, as we seek to describe the domain, we need to consider all possible values of x that would coincide with a point on this graph. Are there any values of x that would not have a point on this graph? The answer is no. So, we describe the domain of this function as: $-\infty < x < \infty$ or $(-\infty, \infty)$. Notice our description of the domain does not set x equal to positive or negative infinity. This is because infinity of any form is not a single value. Instead it is a description of extreme values in the positive or negative direction. As a result, we cannot say x is equal to an undefined value.

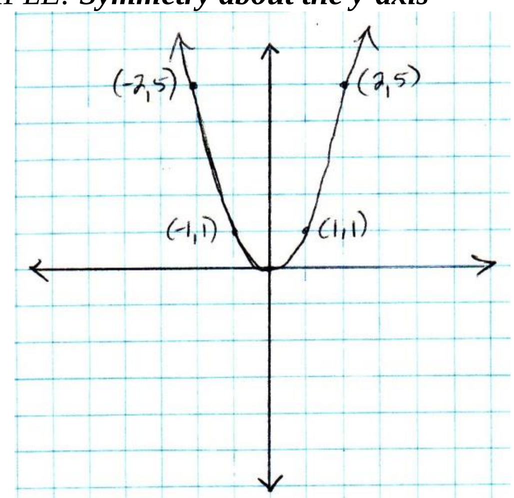
Notice also that the interval notation $(-\infty,\infty)$ uses parentheses. These parentheses also mean x is not equal to positive or negative infinity.

Now let's consider the range of graph #1. We notice that every point on the graph has a y-value of 1 or larger. As a result, we describe the range of this function as: $y \geq 1$ or $[1,\infty)$. Here again we have interval notation of $[1,\infty)$. The bracket before the one indicates that y could be equal to one, and the parenthesis after infinity indicates y cannot be equal to infinity. This ***interval notation*** is commonly used throughout Calculus. You should start trying to use this notation right away!

Now consider graph #2. Notice that this graph also passes the Vertical Line Test, and is therefore a function. As we compare the x-axis to the graph, we find the domain of graph #2 is $(-\infty,\infty)$. Also, as we compare the y-axis to graph #2, we find the range is $[2,\infty)$. If you are uncertain about how we determined these intervals, review the graph again, and bring your questions on the first day of school!

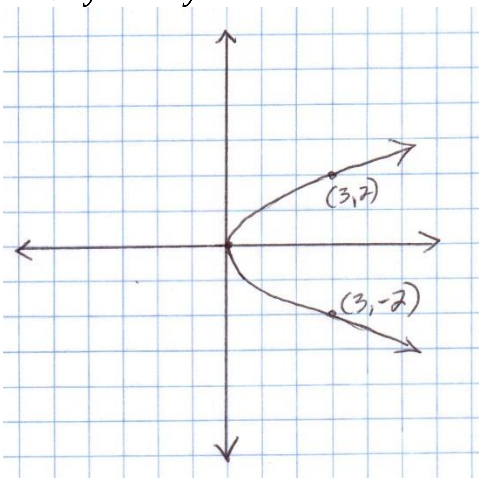
We will come back to the ideas of domain and range when we look at various types of functions, but for now let's move on to the idea of ***Symmetry***. There are three types of symmetry we need to be aware of as we study the behaviors of functions in Calculus. To demonstrate these symmetries and their definitions, consider the examples below:

EXAMPLE: ***Symmetry about the y-axis***



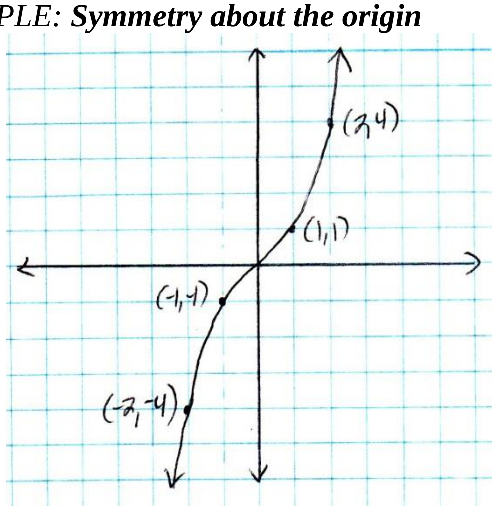
This graph demonstrates symmetry about the y-axis because, by definition, for every point (x , y) on the graph, the point (- x , y) is also on the graph.

EXAMPLE: ***Symmetry about the x-axis***



This graph, although not a function, demonstrates symmetry about the x-axis because, by definition, for every point (x , y) on the graph, the point (x , - y) is also on the graph.

EXAMPLE: ***Symmetry about the origin***



Finally, this graph demonstrates symmetry about the origin because, by definition, for every point (x , y) on the graph, the point (- x , - y) is also on the graph.

Symmetry is an idea that will become important to us as we develop ways to measure the area and volume of unusual shapes that do not have geometric equations. For now, a general understanding of the different types of symmetry are all the tools you will need to begin your study of Calculus.

Now we move to a quick overview if the different types of functions you should have already been exposed to in your prior math classes. In Calculus it is important that we understand the different types, and their proper names, because the tools we use to work with functions will vary depending on the type of function we are working with. So, the list that follows should be included in your notes:

<u>Polynomial Functions</u> – Functions formed by the sum and difference of terms that involve only positive integer exponents.

EXAMPLE: $y = x^3 + 7x^2 - 8x$

<u>Rational Functions</u> – Functions formed as quotients of polynomials.

EXAMPLE: $y = \dfrac{2x-8}{x^2+7}$

Root Functions – Functions involving fractional exponents.

EXAMPLE: $y = \sqrt[4]{x-3} = (x-3)^{\frac{1}{4}}$

Exponential Functions – Functions with a variable exponent.

EXAMPLES: $y = 8^x$ *or* $y = e^{(x+2)}$

Logarithmic Functions – Functions involving logarithms of any base.

EXAMPLES: $y = \log_5 x$ *or* $y = \frac{1}{3}\ln x$

Trigonometric Functions – Functions involving the six trigonometric functions.
(You should be very familiar with these from your study of Precalculus.)

Inverse Trigonometric Functions- Functions involving the inverses of the six trigonometric functions.
(Again you should be very familiar with these functions.)

Composite Functions – Functions with one or more functions nested inside each other. These functions take on the form $y = f(g(x))$.

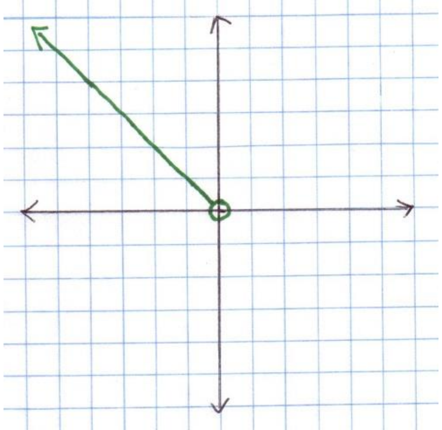
EXAMPLE: $y = \cos(\ln x)$

Piecewise Functions – Functions which have different equations for various parts of the domain.

EXAMPLE: $f(x) = \begin{cases} -x & \text{when } x < 0 \\ -2 & \text{when } x = 0 \\ x^2 - 3 & \text{when } x > 0 \end{cases}$

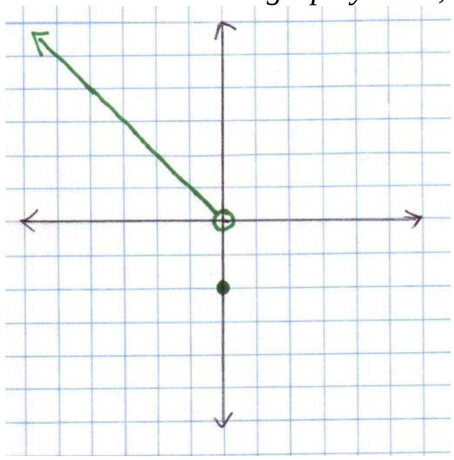
Let's work a bit with this piece wise function. As we study functional behavior in Calculus, piecewise functions and the graphs they create become a particular focus. So, let's graph this piecewise function:

First, we note that we want to graph $y = -x$ *, but only on the interval* $(-\infty, 0)$*:*

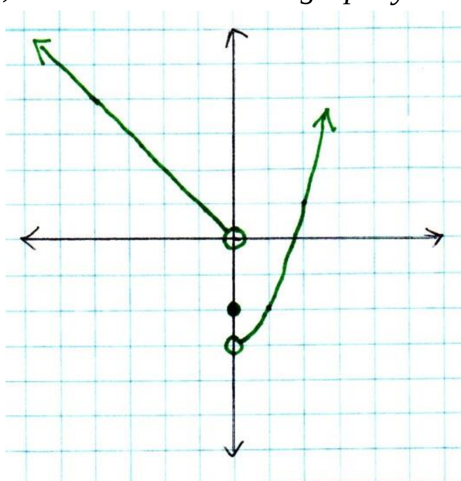


Notice the open circle at the point (0 , 0). This is because the first piece of our graph ($y = -x$) only applies to x < 0.

Next, we want to add to the graph y = - 2, but only when x = 0.



Finally, we want to add to the graph $y = x^2 - 3$, *but only on the interval* $(0, \infty)$.



This picture is the completed graph of the piecewise function. Notice that the graph <u>does</u> meet the definition of function since there are two open circles and just one filled in circle on the vertical line x = 0. Also notice that this graph has breaks in it, called jumps. These jumps are common in piecewise functions, and from a Calculus perspective form interesting areas of study. We will work more with piecewise functions in our study of Calculus, but for now you should have a general understanding of what they are, be able to graph them, and be able to find their domain and range.

Do you know the domain and range of our graphed function? Look at the graph and consider all possible x-values that have a point on the graph, and then consider all possible y-values that have a point on the graph. Be careful to notice that open circles are not points included in our function. With those ideas in mind, we have the following domain and range for our piecewise function: *Domain:* $(-\infty, \infty)$ *Range:* $(-3, \infty)$

Do you know why the range does not include the value – 3? Do you know why all real numbers are included in the domain? If not, review your graph again. Compare the x-axis to the graph to determine the domain, and compare the y-axis to the graph to determine the range.

Our final topic in this first lesson is the idea of ***inverse functions***. You have already become familiar with inverse trigonometric functions like $y = \sin^{-1} x$ or $y = \arccos x$. However, we want to expand this understanding to other functions, and learn to find the inverse of functions algebraically.

To begin, we need to be clear on a few points regarding inverse functions:

- The inverse of the function $y = f(x)$ is defined as $f^{-1}(y) = x$. This means the inputs and outputs are reversed. As a result, given an “output” from our original function, the inverse function can be used to determine which “input” created that output value.
- The inverse of a function must meet the definition of function in order for it to exist. That is, when working with the inverse it must still be true that each value you put into the function gives you one and only one answer.

Now, let’s look at the process of finding the inverse of the function $y = x^2 - 1$:

1) To find f^{-1} we begin by reversing x & y in the equation.

   So, $x = y^2 - 1$

2) Now, we algebraically solve for *y*:

   $x = y^2 - 1$

   $x + 1 = y^2$ *(add one to both sides)*

   $y = \sqrt{x+1}$ or $y = -\sqrt{x+1}$ *(take the square root of both sides)*

   So, in this case our function has two possible inverses:

   $f^{-1}(\text{x}) = \sqrt{x+1}$ or $f^{-1}(x) = -\sqrt{x+1}$

NOTE: Not every function will have two inverses. This generally occurs with even exponents and roots since even exponents do not preserve the sign of the value raised to that exponent.

EXAMPLE: $(-3)^4 = 81$ *and* $(3)^4 = 81$, *so* $\sqrt[4]{81} = 3$ *or* -3

Let’s try one more to be sure we understand the process of finding an inverse:

Consider $f(x) = \dfrac{4x-1}{2x+3}$

(re-write the function with y) $y = \dfrac{4x-1}{2x+3}$

(switch x and y) $x = \dfrac{4y-1}{2y+3}$

(multiply both sides by the denominator) $x(2y+3) = 4y-1$

(distribute) $2xy + 3x = 4y - 1$

(move all terms with y to one side, and all other terms to the other side)

$2xy - 4y = -3x - 1$

(factor out the y) $y(2x-4) = -3x-1$

(solve for y) $y = \dfrac{-3x-1}{2x-4}$

So, when $f(x) = \dfrac{4x-1}{2x+3}$, the inverse function is $f^{-1}(\text{x}) = \dfrac{-3x-1}{2x-4}$.

In this first lesson we have reviewed many of the algebra ideas you should already know. These concepts will be used in our study of Calculus, so it is important that you understand these ideas and can apply them. If you have questions on this lesson, make sure to bring them with you on the first day of school!

NOTE TAKING:

The following items from this reading must be included in your Calculus notebook:

- Definition of Function
- Definition of the Vertical Line Test
- Definition & Examples of Domain and Range of a Function
- Examples of Interval Notation
- Definition & Examples of Symmetry with respect to the x-axis, y-axis and origin
- Definition & Examples of Polynomials
- Definition & Examples of Rational Functions
- Definition & Examples of Root Functions
- Definition & Examples of Exponential Functions
- Definition & Examples of Logarithmic Functions
- Definition & Examples of Trigonometric Functions
- Definition & Examples of Inverse Trigonometric Functions
- Definition & Examples of Composite Functions
- Definition & Examples of Piecewise Functions
- Definition & Notation for Inverse Functions
- Examples of how to find the Inverse of a Function

PROBLEMS:

***** To be done on a separate piece of paper for handing in! *****

1) a) Graph the following: $f(x)=\begin{cases} x-3 & \text{when} \quad x<2 \\ x & \text{when} \quad 2\le x<4 \\ -2x+7 & \text{when} \quad x\ge 4 \end{cases}$

   b) Find the domain and range of the function.

2) Find the inverse of the function $f(x)=2x+6$. Be sure to show all of your work.

3) Find the inverse of the function $f(x)=x^3+2$. Be sure to show all of your work.

4) Find the inverse of the function $f(x)=\sqrt{10-3x}$. Be sure to show all of your work.

5) Explain how you could find the value of $\sin\left(\frac{\pi}{6}\right)$ <u>without</u> using your calculator. Show your work and explain your process. (*NOTE: This type of calculation is something you are expected to be able to solve on the <u>non-calculator</u> section of the AP Calculus Exam.*)

Assignment #2

READING:

In this lesson we are going to review the key ideas of ***Factoring***. Factoring is a crucial algebraic skill used throughout Calculus. It is expected that you have learned to factor before taking Calculus. So, we will use this lesson to review those ideas.

Before you learned to factor, you first learned the rules for using the ***Distributive Property*** along with the properties of exponents. In essence, once you learned to distribute, factoring is merely doing that process in reverse. So, let's briefly review distribution:

EXAMPLES: $x^2\left(3x-7\right)=3x^3-7x^2$

Here we simply distribute the x^2 *to each of the terms in the binomial.*

$\left(2x-7\right)\left(x+3\right)=2x^2+6x-7x-21=2x^2-x-21$

Here we must distribute both terms in the first binomial to both terms in the second binomial (hence the name Double Distribution). This method is also called the ***F.O.I.L.*** *Method because we multiply the First terms in each binomial (2x and x), then the Outside terms (2x and 3), then the Inside terms (-7 and x), and finally the Last terms in each binomial (-7 and 3). Once all of the multiplying is complete, we combine like terms to get our simplified answer.*

Once we have learned the tools of distribution, we then consider a few special products:

EXAMPLES: $(3x-5)(3x+5)=9x^2+15x-15x-25=9x^2-25$

This is the ***Product of the Sum and Difference of Identical Terms****. Since the two binomials have the same terms (3x and 5), but one includes addition and the other subtraction, the middle terms cancel out and our resulting product is a binomial. This phenomenon occurs every time we multiply the sum and difference of identical terms. Our answer is called a* ***Difference of Perfect Squares*** *because* $(3x)^2=9x^2$ *and* $5^2=25$.

$\left(y-8\right)^2=(y-8)(y-8)=y^2-16y+64$

This is an example of ***Squaring a Binomial****. Notice that we* <u>*do*</u> <u>*not*</u> *simply square the y and the 8 to get our product. Instead, we need to follow the process of Double Distribution (or F.O.I.L.) to get the correct product.*

These ideas of multiplying and distributing are certainly not new to you. So, from here we will move on to our review of factoring.

The first kind of factoring you learned was ***Factoring Out a Common Term***. For these problems we are looking for the <u>largest</u> <u>common</u> <u>factor</u> for all terms in the polynomial. When factoring out a common term, we want to factor out as much as possible, leaving the remaining polynomial as simple as possible. Consider these examples:

EXAMPLES: $35xy^3 + 7x^2y = 7xy\left(5y^2 + x\right)$

$-21gh^2 + 14gh + 7g = 7g\left(-3h^2 + 2h + 1\right)$

Notice for this example the last term becomes a one in the remaining trinomial. We do not simply eliminate the last term because, if we distributed the 7g back through, we would not get back to our original trinomial without the +1 term.

$-35a^2 - 15a - 20ab^2 = -5a(7a + 3 + 4b^2)$

Notice we factored out the negative along with the coefficient. Since our goal is to simplify the remaining trinomial as much as possible, we would like to factor out the negative whenever every term in the polynomial has a negative coefficient.

Once you learned how to factor out a common term, you learned how to factor trinomials into the product of two binomials. This factoring is essentially "undoing" the double distribution (or F.O.I.L.) process. However, in doing this there are several key ideas to keep in mind. Let's begin with trinomials with a coefficient of 1 on the x^2 term. Consider these examples:

Example 1: Factor $x^2 + 5x + 6$.

The first term in the trinomial is x^2. Since $x \cdot x = x^2$, the first term of each binomial is x.

$$x^2 + 5x + 6 = (x + \blacksquare)(x + \blacksquare)$$

To find the last terms, find a number pair whose product is 6 and whose sum is 5.

Product	Factors	Sum
6	1, 6	$1 + 6 = 7$
6	2, 3	$2 + 3 = 5$ ✓

Therefore, $x^2 + 5x + 6 = (x + 2)(x + 3)$.

Example 2: Factor $x^2 - 8x + 12$.

The first terms are both x. To find the last terms, find a number pair whose product is 12 and whose sum is -8.

Product	Factors	Sum
12	$-1, -12$	$-1 + (-12) = -13$
12	$-2, -6$	$-2 + (-6) = -8$ ✓
12	$-3, -4$	

Once the correct sum is found, it is not necessary to check any more factors. Therefore, $x^2 - 8x + 12 = (x - 2)(x - 6)$.

Example 3: Factor $x^2 - 2x - 15$.

The first terms are both x. To find the last terms, find a number pair whose product is -15 and whose sum is -2.

Product	Factors	Sum
-15	$1, -15$	$1 + (-15) = -14$
-15	$-1, 15$	$-1 + 15 = 14$
-15	$3, -5$	$3 + (-5) = -2$ ✓

Therefore, $x^2 - 2x - 15 = (x + 3)(x - 5)$.

After learning how to factor trinomials, you then learned how to factor when the x^2 term has a coefficient greater than one. For these problems, determining the exact factors is a bit more complex, but the process of how to get there is the same. Consider these examples:

Example 1: Factor $5x^2 + 37x + 14$.

The first term in the trinomial is $5x^2$. The only factors of 5 are 5 and 1, so the first terms of the binomials are $5x$ and x.

$$5x^2 + 37x + 14 = (5x + \blacksquare)(x + \blacksquare)$$

The last term in the trinomial is 14, which has two pairs of factors, 1 and 14, and 2 and 7. Try the factor pairs until you find the one that gives a middle term of $37x$.

First Terms	Last Terms	Binomial Pair	Middle Term	Trinomial
$5x, x$	$1, 14$	$(5x + 1)(x + 14)$	$x + 70x = 71x$	$5x^2 + 71x + 14$
$5x, x$	$14, 1$	$(5x + 14)(x + 1)$	$14x + 5x = 19x$	$5x^2 + 19x + 14$
$5x, x$	$2, 7$	$(5x + 2)(x + 7)$	$2x + 35x = 37x$	$5x^2 + 37x + 14$ ✓

Therefore, $5x^2 + 37x + 14 = (5x + 2)(x + 7)$.

Example 2: Factor $6x^2 - 23x + 7$.

There are two possible factor pairs of the first term, $2x$ and $3x$, and $6x$ and x. The last term is positive. The sum of the inside and outside terms is negative. So, the factors of 7 are -1 and -7. Try the factor pairs until you find the one that gives a middle term of $-23x$.

First Terms	Last Terms	Binomial Pair	Middle Term	Trinomial
$2x, 3x$	$-1, -7$	$(2x - 1)(3x - 7)$	$-3x - 14x = -17x$	$6x^2 - 17x + 7$
$3x, 2x$	$-1, -7$	$(3x - 1)(2x - 7)$	$-2x - 21x = -23x$	$6x^2 - 23x + 7$ ✓

Therefore, $6x^2 - 23x + 7 = (3x - 1)(2x - 7)$.

Notice for these trinomials we still begin by considering the first term, and then the last term before we determine which options would give us the middle term. This process is the same whenever we are factoring to “undo” the Double Distribution process.

Now that we recall how to factor trinomials, we need to review ***Factoring Special Products***. Here we are looking for ***Perfect Square Trinomials*** or the ***Difference of Two Perfect Squares***.

Consider these examples:

EXAMPLES: $x^2+10x+25=(x+5)(x+5)=(x+5)^2$

$x^2-2x+1=(x-1)(x-1)=(x-1)^2$

Notice for each of these examples, we got identical factors. This is why these trinomials are examples of ***Perfect Square Trinomials****.*

When factoring, we should note that all Perfect Square Trinomials take on one of two possible forms: $a^2+2ab+b^2=(a+b)(a+b)$

or

$a^2-2ab+b^2=(a-b)(a-b)$

MORE EXAMPLES: $x^2-49=(x+7)(x-7)$

$100g^2-36=(10g-6)(10g+6)$

Notice for each of these examples we were factoring a binomial, and obtained two binomials as factors. This occurs <u>only</u> when the binomial we are factoring is the ***Difference of Two Perfect Squares****.*

Finally, when factoring a problem, it is possible that we would need to factor out a common term, and then factor the remaining binomial or trinomial into the product of two binomials. When we do this, it is called ***Factoring Completely***. In general, every time we factor an expression our goal is to simplify the expression as much as possible. So, we always look to factor completely. Consider the following:

EXAMPLES: $10xw^2+15wx+5x=5x\left(2w^2+3x+1\right)=5x\left(2w+1\right)\left(w+1\right)$

$3r^4-24r^3+54r^2=3r^2\left(r^2-8r+16\right)=3r^2\left(r-4\right)\left(r-4\right)$

$6k^3-24k=6k\left(k^2-4\right)=6k\left(k-2\right)\left(k+2\right)$

So, now we have reviewed all of the factoring ideas you have learned. It's time to show what you know! Put the appropriate examples and terms in your notes, then complete the problem set!

NOTE TAKING:

The following items from this reading must be included in your Calculus notebook:

- Examples of The Distributive Property and Double Distribution
- Examples of Special Products
- Examples of Factoring Out a Common Term
- Examples of Factoring Trinomials
- Examples of Factoring Special Products
- Examples of Factoring Completely

PROBLEMS:

***** To be done on a separate piece of paper for handing in! *****

Factor all of the following completely.

1) $v^2 + 15v + 36$

2) $k^2 - 23k + 42$

3) $2m^2 - 18m - 44$

4) $5f^2 + 27f + 10$

5) $28y^2 - 40y + 12$

6) $4x^2y - 11xy - 3y$

7) $36h^2 - 12h + 1$

8) $4 - 25d^2$

9) $p^2q - 20pq + 100q$

10) $12t^2 - 27$

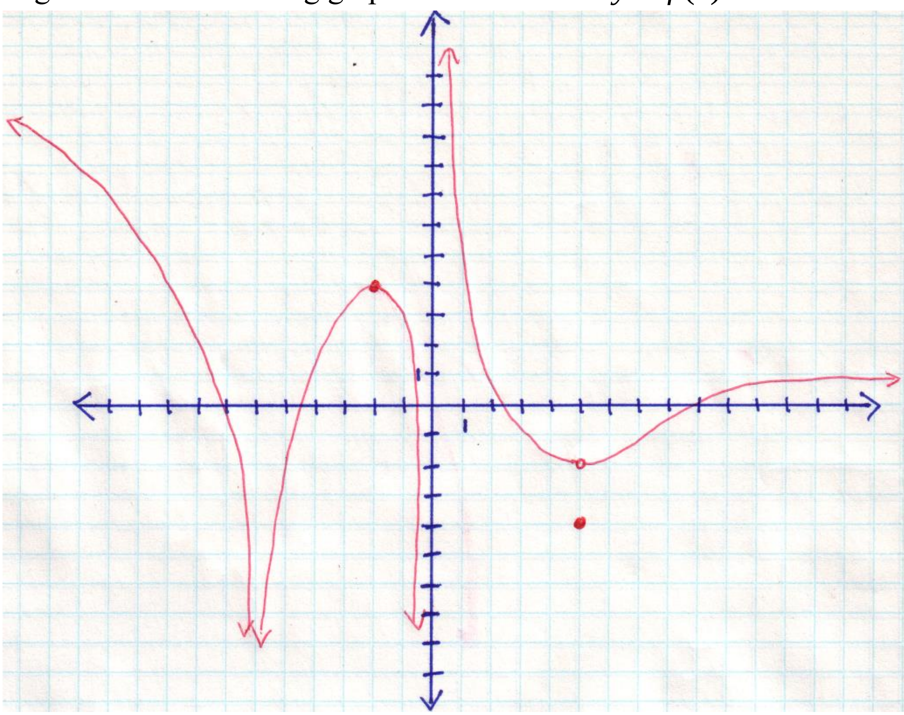
Assignment #3

READING:

In this lesson you will be introduced to the idea of ***Limit***, and the tools we will use as we begin our study of Calculus. You may have seen some of these limit ideas in Precalculus, but if not this reading should help you get started. If you have seen some limits, this should be a good review!

Up to this point in your study of mathematics, you have learned a variety of ways to write equations that model situations, solve those equations, and evaluate the reasonableness of your solutions. All of the equations you have used represent various types of functions. In Calculus, we seek to better understand the behavior of functions so we can use them to model and solve more complex problems. Along the way, we will apply the algebraic and geometric concepts you have learned.

Let's begin with the following graph of the function $y = f(x)$:



There are many unusual aspects of this function. So, we will begin to describe them one at a time.

- First we notice that there are many places where the functional value exists with no problems. One such place is at the point (- 2, 4). This point is on our function, and presents no problems since both sides of the curve $y = f(x)$ come at the point (- 2 , 4) as our x-values get closer and closer to x = - 2 from either side. As a result, we say *"The limit as x approaches -2 is 4"* since both sides of the curve are headed to a y-value of 4. Symbolically, we write this as:

$$\lim_{x \to -2} f(x) = 4$$

- There are many points on the graph like (- 2, 4) where the limit exists because the point is on the function. However, let's now turn our attention to the unusual places in the graph.

- We notice that the function $y = f(x)$ has a vertical asymptote at $x = -6$. It is also interesting to note that on both sides of the asymptote the graph is headed down toward negative y-values. Based on this behavior, we say *"The limit as x approaches -6 is negative infinity"* since both sides of the curve are headed to negative infinity around that asymptote. Symbolically, we write this as:

$$\lim_{x \to -6} f(x) = -\infty$$

- It is important to note that a limit being equal to either positive or negative infinity means the graph has a **vertical asymptote** at the x-value we are approaching.

- Now consider what the graph is doing at x = 0. Notice the function also has a vertical asymptote here. However, on the left side of the y-axis the graph is headed toward negative infinity, and on the right side of the y-axis the graph is headed toward positive infinity. As a result, we cannot say the graph is headed in the same direction or toward the same value on both sides of x = 0. In this case we say *"The limit as x approaches zero does not exist"* since the graph is not headed in the same direction as we approach zero from both sides. Symbolically, we write this as:
$$\lim_{x \to 0} f(x) = d.n.e.$$

- Notice that the limit does not exist at some vertical asymptotes, as well as at jumps in the graph. Jumps are created by piecewise functions. We will work more with jumps later.

- Finally, notice that when x = 5, the function gives a y-value of y = - 4 (because this is where the filled in point on the graph lies). However, the curve itself approaches the point (5 , - 2) as x approaches the value of 5. This place on the graph is called a ***Hole in the Graph*** because the functional value exists, $f(5) = -4$, but the curve approaches an open circle as we get close to x=5. In this case, ***it is possible for the functional value to be different from the limit value***. So, even though $f(5) = -4$, we say *"The limit as x approaches 5 is – 2"* since this is what the curve is headed towards on either side of x = 5. Symbolically, we write this as:
$$\lim_{x \to 5} f(x) = -2$$

- ***Notice that limit values describe the behavior of a curve on a small interval <u>around</u> the x-value we are approaching, not necessarily at the exact x-value itself.***

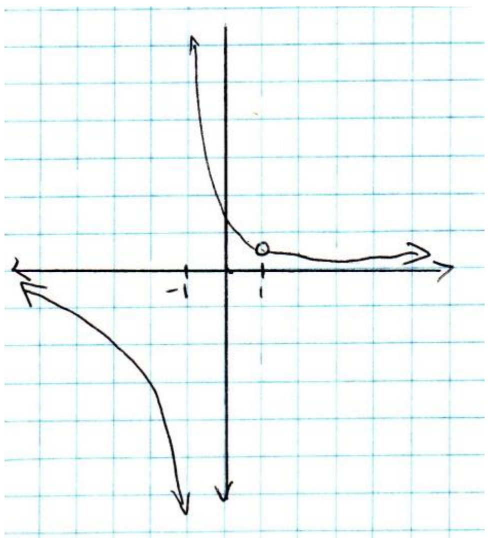
So, we have seen an example of a function with lots of unusual behavior, and ways that we describe that behavior using limits. Now we are ready for the definition of limit:

<u>Definition:</u> When the values of a function f(x) converge to some value L as we approach a given x-value (say x approaches a) from either side, we say the limit of the function f(x) as x approaches a equals L.
$$\lim_{x \to a} f(x) = L$$

Very often, as we investigate limits, we start by looking at the <u>graph</u> of the function. For this summer work, we will just look at graphs of functions to evaluate limits. However, as we work with limits further at the start of school, we will also develop other techniques for evaluating limits without always having to evaluate them graphically. For now, consider the following examples:

EXAMPLE: Find $\lim\limits_{x \to 1} \dfrac{x-1}{x^2-1}$.

We begin by looking at the graph of $f(x) = \dfrac{x-1}{x^2-1}$:

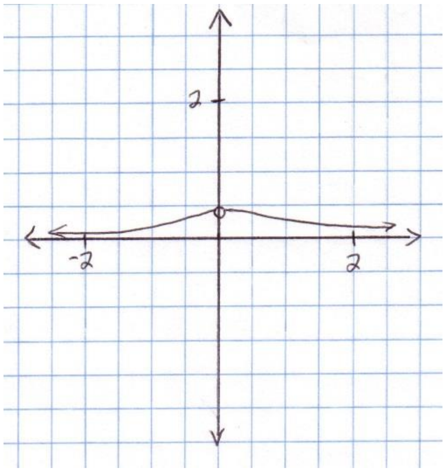


Notice there is a hole at x = 1, but both sides of the graph are headed toward the same value. If you graphed this on your graphing calculator and tried to trace over to x = 1, you would not get a y-value at this point since the functional value does not exist. However, by looking at the table and/or by tracing around on either side of x = 1, we find that:

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

EXAMPLE: Find $\lim\limits_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$

Again we begin by looking at the graph of $f(x)=\frac{\sqrt{x^2+9}-3}{x^2}$:

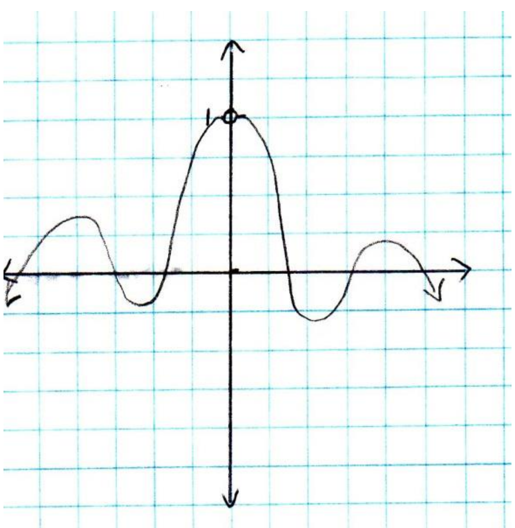


Notice there is a hole at x = 0, but both sides of the graph are headed toward the same value. If you graphed this on your graphing calculator and tried to trace over to x = 0, you would not get a y-value at this point since the functional value does not exist. However, by looking at the table and/or by tracing around on either side of x = 0, we find that:

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}=\frac{1}{6}$$

EXAMPLE: Find $\lim\limits_{t\to 0}\frac{\sin t}{t}$

Again we begin by looking at the graph of $f(t)=\frac{\sin t}{t}$:

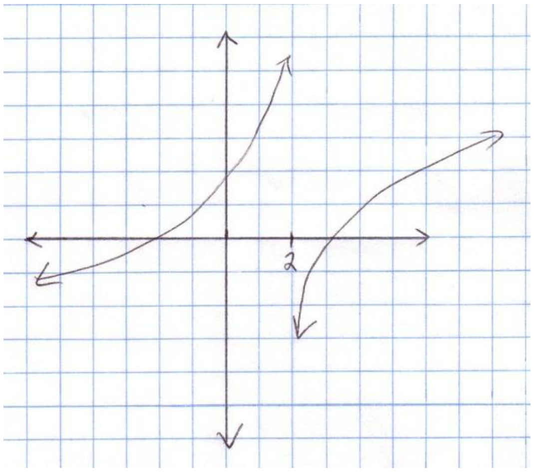


Notice there is a hole at x = 0, but both sides of the graph are headed toward the same value. If you graphed this on your graphing calculator and tried to trace over to x = 0, you would not get a y-value at this point since the functional value does not exist. However, by looking at the table and/or by tracing around on either side of x = 0, we find that:

$$\lim_{t\to 0}\frac{\sin t}{t}=1$$

Finally, consider the following: *Determine the value or values for which the limit does not exist for* $f(x) = \frac{x^2 - 5}{x - 2}$.

In this example, we may suspect that x = 2 could be a problem since that value makes the denominator zero, and it is undefined to divide by zero. However, to make sure we are correct, let's consider the graph of the function:



We see that, in fact, the function has a vertical asymptote at x = 2. Additionally, on the left side of the asymptote the function is going off to positive infinity, and on the right side of the asymptote the function is going off to negative infinity. As a result, $\lim_{x \to 2} \frac{x^2 - 5}{x - 2} = d.n.e.$. So, the limit of the function does not exist at x = 2.

NOTE TAKING:

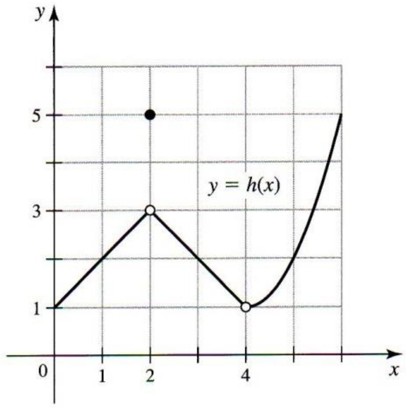
The following items from this reading must be included in your Calculus notebook:

- Examples of Limits and Limit Symbols
- Description of Possible Limit Values at a Vertical Asymptote
- Description of a Hole in the Graph and the Resulting Limit Value
- Description of What Limits Describe about a Functions' Behavior
- Definition of Limit
- Examples of Limits

PROBLEMS:

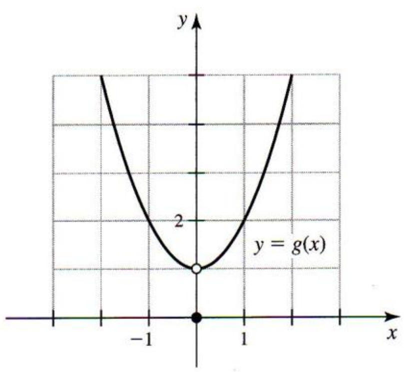
***** To be done on a separate piece of paper for handing in! *****

1) Use the graph below to find the following limits and values:



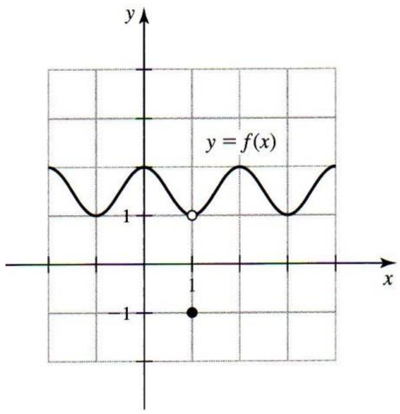
a) $\lim_{x \to 2} h(x) =$
b) h(2) =
c) $\lim_{x \to 4} h(x) =$
d) h(4) =
e) $\lim_{x \to 5} h(x) =$

2) Use the graph below to find the following limits and values:



a) g(0) =
b) $\lim_{x \to 0} g(x) =$
c) g(1) =
d) $\lim_{x \to 1} g(x) =$

3) Use the graph below to find the following limits and values:



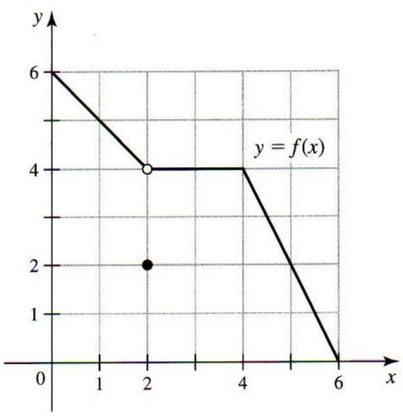
a) $\lim\limits_{x \to 1} f(x) =$

b) f(1) =

c) $\lim\limits_{x \to 0} f(x) =$

d) f(0) =

4) Use the graph below to find the following limits and values:



a) f(2) =

b) $\lim\limits_{x \to 2} f(x) =$

c) $\lim\limits_{x \to 4} f(x) =$

d) $\lim\limits_{x \to 5} f(x) =$

5) Determine the value or values for which the limit does not exist of the function $h(x) = \dfrac{x^2 - 5}{x - 3}$. Be sure to show your work, including your graph and use correct limit notation.

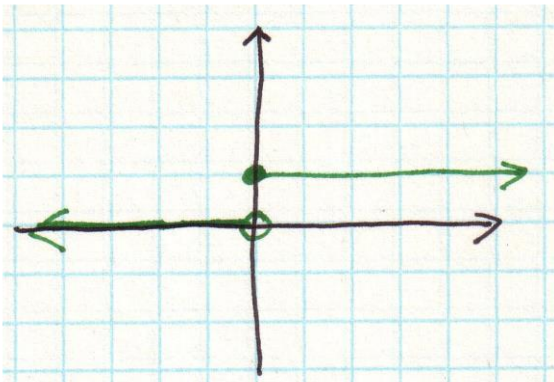
6) Find $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$. Be sure to show your work, including your graph.

Assignment #4

READING:

In this lesson we will expand on the idea of ***Limit***, and develop tools that can help us evaluate limits at asymptotes and jumps in a graph. Let's begin with an example of a piecewise function:

Consider: $H(t)=\begin{cases} 0 & if \quad t<0 \\ 1 & if \quad t\geq 0 \end{cases}$. If we graph this function, we get the following:



From this graph we can see that the limit would exist for any number *t* we want to approach, accept $t=0$. In fact, for any value $t<0$ we approach, the limit would be zero, and for any value $t>0$ we approach, the limit would be one.

It is at $t=0$ that we have a **Jump in the graph**. In many cases, the point of using limits to evaluate functional behavior is to help us describe unusual things that occur in the graph, like holes, or asymptotes, or jumps.

We can see that $\underset{t\to 0}{Lim}\,H(t)=d.n.e.$ because the two sides of the graph are not coming together to the same point as *t* approaches zero. However, if this was the only limit value we knew, and we did not have the graph available, we would not know if the limit does not exist because there is an asymptote with each side of the graph going in opposite directions, or if the limit does not exist because there is a jump on the graph. For this reason, we add the idea of **One Sided Limits**.

For our graph of $H(t)$, we want to demonstrate that there is a jump at $t=0$, and how large the jump is. To do that, we consider the limit as *t* approaches zero from <u>each</u> <u>side</u> of zero. First, imagine moving along the graph coming at zero from values larger than zero. As we move along heading toward zero, we see that *the limit as t approaches zero from the right is 1* since that is the point we reach as we get to zero. Symbolically we write this: $\underset{t\to 0^+}{Lim}\,H(t)=1$. Notice the 0^+ in our description of what *t* is approaching. This "superscript" of $+$ means we are approaching zero from the right (or values larger than zero). We read our limit as "the limit at *t* approaches zero from the right of *H(t)* is one".

Now, imagine moving along the graph coming at zero from values smaller than zero. As we move along heading toward zero, we see that *the limit as t approaches zero from the left is 0* since that is the point we reach as we get to zero. Symbolically we write this: $\underset{t\to 0^-}{Lim}\,H(t)=0$. Notice the 0^- in our description of that *t* is approaching. This "superscript" of – means we are approaching zero from the left (or values smaller than zero). We read our limit as "the limit as *t* approaches zero from the left of *H(t)* is zero".

This example demonstrates the definition of a one sided limit:

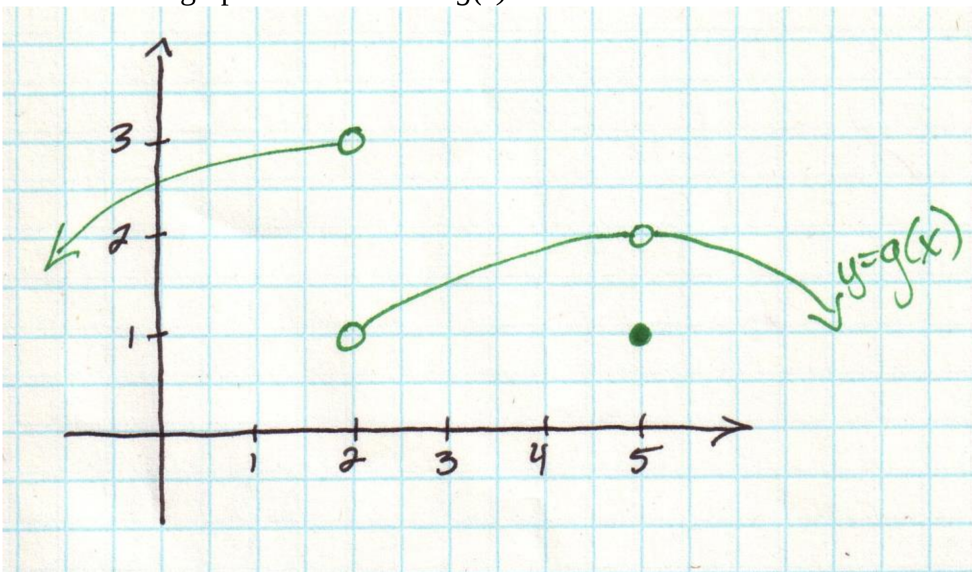
<u>**Definition:**</u> When the values of a function f(x) converge to some value L as we approach a given x-value (say x approaches a) from the right, we say the limit of the function f(x) as x approaches a from the right equals L.

$$\underset{x\to a^+}{Lim}\,f(x)=L$$

Similarly, when the values of a function f(x) converge to some value L as we approach a given x-value (say x approaches a) from the left, we say the limit of the function f(x) as x approaches a from the left equals L.

$$\lim_{x\to a^-} f(x) = L$$

Now consider the graph of the function *g(x)* below:



Based on the graph, we have the following limit values:

$\lim_{x\to 2^-} g(x) = 3$ since we are approaching 2 from values smaller than it, we arrive at the point (2,3).

$\lim_{x\to 2^+} g(x) = 1$ since we are approaching 2 from values larger than it, we arrive at the point (2,1).

$\lim_{x\to 2} g(x) = d.n.e.$ since approaching 2 from both sides does not bring us to the same point.

So, using limits we have demonstrated that there is a jump in the graph at $x = 2$. This leads us to the following:

<u>Definition:</u> The limit of a function f(x) as x approaches some value a exists only if the left and right sided limits as x approaches a are equal. That is,

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^+} f(x) = L \quad \text{and} \quad \lim_{x\to a^-} f(x) = L$$

Similarly, if the one sided limits are not equal, the limit of a function f(x) as x approaches a does not exist.

Also using our graph of *g(x)*:

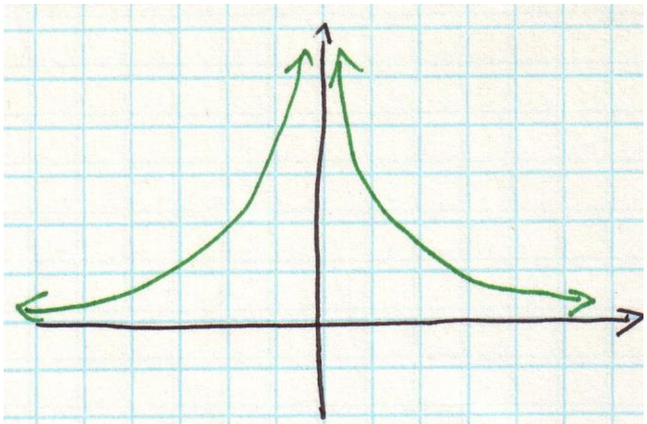
$\lim_{x\to 5^-} g(x) = 2$ since we are approaching 5 from values smaller than it, and the graph brings us to the point (5,2).

$\lim_{x\to 5^+} g(x) = 2$ since we are approaching 5 from values larger than it, and the graph also brings us to the point (5,2).

$\lim_{x\to 5} g(x) = 2$ since we are approaching 5 from both sides, and the graph brings us to the point (5,2).

It is important to notice in that last example that there is a whole in the graph at (5,2), and a filled in point at (5,1). This means that the actual functional value if we plugged in 5 into the equation of *g(x)* would give us the answer 1. However, all of the points around *x=5* are approaching the hole in the graph. So, the limit as we approach 5 is 2. This leads us to the very important idea that ***the limit value and the functional value <u>do</u> <u>not</u> always have to be the same value***.

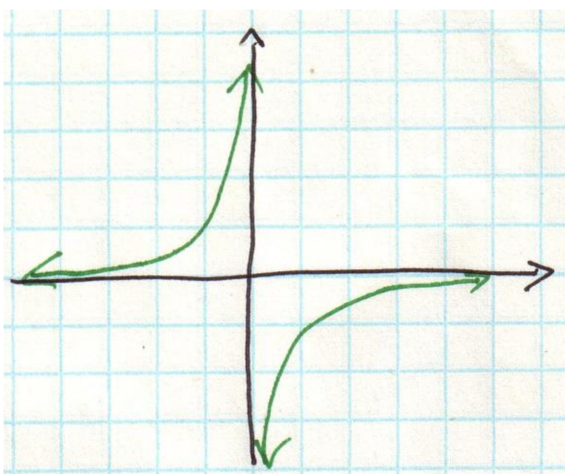
Now let's consider $\lim\limits_{x \to 0} \frac{1}{x^2}$. If we look at the graph of $\frac{1}{x^2}$:



Based on the graph, we have the following limit values:

$\lim\limits_{x \to 0^+} \frac{1}{x^2} = \infty$ and $\lim\limits_{x \to 0^-} \frac{1}{x^2} = \infty$, so $\lim\limits_{x \to 0} \frac{1}{x^2} = \infty$.

Now let's consider $\lim\limits_{x \to 0} \frac{-2}{x^3}$. If we look at the graph of $\frac{-2}{x^3}$:



Based on the graph, we have the following limit values:

$\lim\limits_{x \to 0^+} \frac{-2}{x^3} = -\infty$ and $\lim\limits_{x \to 0^-} \frac{-2}{x^3} = \infty$. Since the limits are not equal, $\lim\limits_{x \to 0} \frac{-2}{x^3} = d.n.e.$

Notice for these last two examples our graphs had vertical asymptotes. However, the answers we got for the limit as x approached the location of the asymptote (from both sides) were different. This leads us to the following definition of **vertical asymptote**:

Definition: The line $x = a$ is called a vertical asymptote of the function $y = f(x)$ if at least one of the following are true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

This means, if you take any limit of a function (one sided or two sided) and get an infinity answer, this means the function has a vertical asymptote at that *x*-value.

This has been an introduction to the idea of limits. When we begin our study of Calculus together in the fall, we will learn a variety of algebraic tools to evaluate limits, consider how limits apply to a variety of functions, consider what limits can tell us about vertical and horizontal asymptotes of a function, and use limits to solve problems related to functions, their behavior, and their rates of change. For now, you should have a general understanding of how to evaluate a limit from a graph, and the proper notation of limits.

NOTE TAKING:

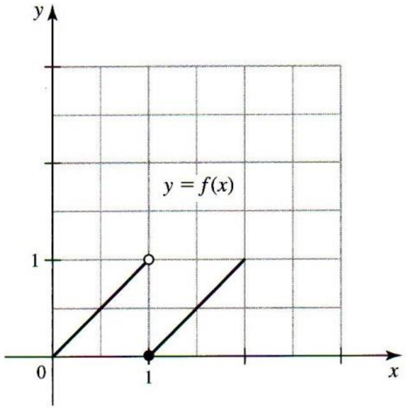
The following items from this reading must be included in your Calculus notebook:

- Definition of a Jump
- Examples of Jumps in graphs
- Definition of One Sided Limits
- Examples of One Sided Limits
- Notation for One Sided Limits
- Definition of the connection between One Sided and Two Sided Limits
- Examples of Limits around Vertical Asymptotes
- Definition of How to find Vertical Asymptotes using Limits

PROBLEMS:

***** To be done on a separate piece of paper for handing in! *****

1) Use the graph of the function *f(x)* below to evaluate each of the limits and functional values:



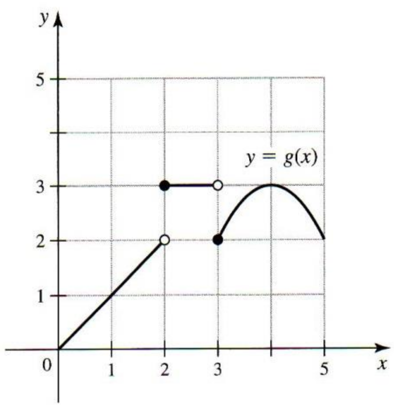
a) $f(1) =$

b) $\lim_{x \to 1^-} f(x) =$

c) $\lim_{x \to 1^+} f(x) =$

d) $\lim_{x \to 1} f(x) =$

2) Use the graph of the function *g(x)* below to evaluate each of the limits and functional values:



a) $g(2) =$

b) $\lim_{x \to 2^-} g(x) =$

c) $\lim_{x \to 2^+} g(x) =$

d) $\lim_{x \to 2} g(x) =$

e) $g(3) =$

f) $\lim_{x \to 3^-} g(x) =$

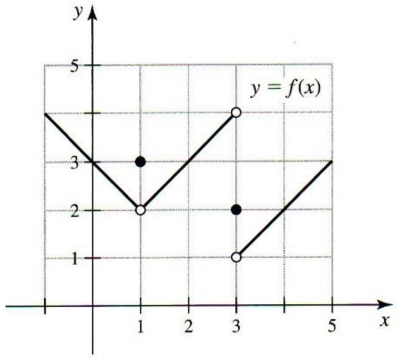
g) $\lim_{x \to 3^+} g(x) =$

h) $\lim_{x \to 3} g(x) =$

i) $g(4) =$

j) $\lim_{x \to 4} g(x) =$

3) Use the graph of the function *f(x)* below to evaluate each of the limits and functional values:



a) $f(1) =$

b) $\lim_{x\to 1^-} f(x) =$

c) $\lim_{x\to 1^+} f(x) =$

d) $\lim_{x\to 1} f(x) =$

e) $f(3) =$

f) $\lim_{x\to 3^-} f(x) =$

g) $\lim_{x\to 3^+} f(x) =$

h) $\lim_{x\to 3} f(x) =$

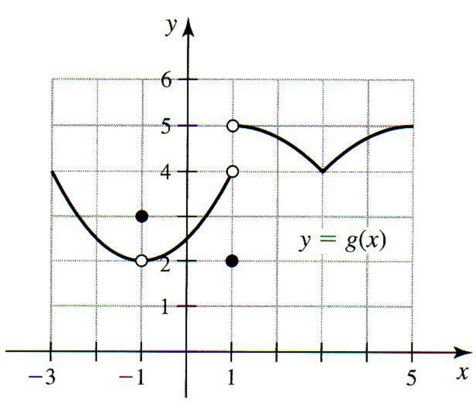
i) $f(2) =$

j) $\lim_{x\to 2^-} f(x) =$

k) $\lim_{x\to 2^+} f(x) =$

l) $\lim_{x\to 2} f(x) =$

4) Use the graph of the function *g(x)* below to evaluate each of the limits and functional values:



a) $g(-1) =$

b) $\lim_{x\to -1^-} g(x) =$

c) $\lim_{x\to -1^+} g(x) =$

d) $\lim_{x\to -1} g(x) =$

e) $g(1) =$

f) $\lim_{x\to 1} g(x) =$

g) $\lim_{x\to 3} g(x) =$

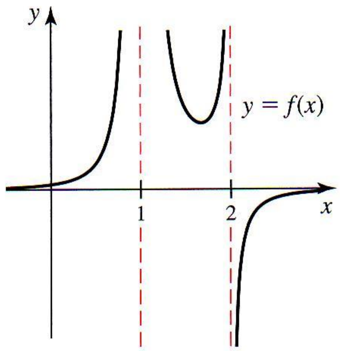
h) $g(5) =$

i) $\lim_{x\to 5^-} g(x) =$

5) Consider the graph of $f(x) = \dfrac{x}{(x^2 - 2x - 3)^2}$. Sketch the graph on your paper, then use the graph to evaluate the following:

a) $\lim\limits_{x \to -1} f(x) =$ b) $\lim\limits_{x \to 3} f(x) =$

6) Use the graph of the function *f(x)* below to evaluate each of the limits:

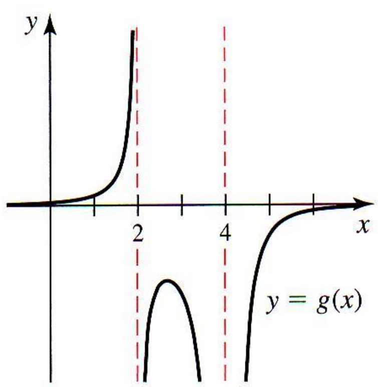


a) $\lim\limits_{x \to 1^-} f(\mathrm{x}) =$ b) $\lim\limits_{x \to 1^+} f(\mathrm{x}) =$

c) $\lim\limits_{x \to 1} f(\mathrm{x}) =$ d) $\lim\limits_{x \to 2^-} f(\mathrm{x}) =$

e) $\lim\limits_{x \to 2^+} f(\mathrm{x}) =$ f) $\lim\limits_{x \to 2} f(\mathrm{x}) =$

7) Use the graph of the function *g(x)* below to evaluate each of the limits:

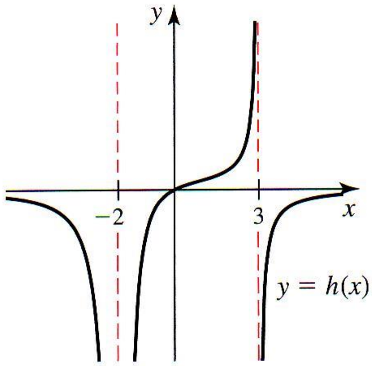


a) $\lim\limits_{x \to 2^-} g(x) =$ b) $\lim\limits_{x \to 2^+} g(x) =$

c) $\lim\limits_{x \to 2} g(x) =$ d) $\lim\limits_{x \to 4^-} g(x) =$

e) $\lim\limits_{x \to 4^+} g(x) =$ f) $\lim\limits_{x \to 4} g(x) =$

8) Use the graph of the function *h(x)* below to evaluate each of the limits:



a) $\lim_{x \to -2^-} h(x) =$

b) $\lim_{x \to -2^+} h(x) =$

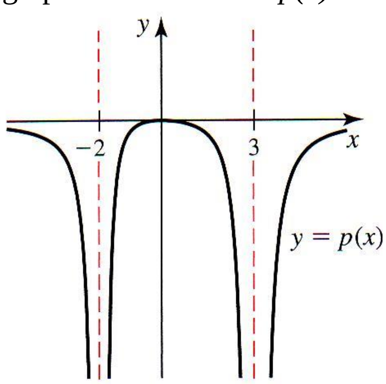
c) $\lim_{x \to -2} h(x) =$

d) $\lim_{x \to 3^-} h(x) =$

e) $\lim_{x \to 3^+} h(x) =$

f) $\lim_{x \to 3} h(x) =$

9) Use the graph of the function *p(x)* below to evaluate each of the limits:



a) $\lim_{x \to -2^-} p(x) =$

b) $\lim_{x \to -2^+} p(x) =$

c) $\lim_{x \to -2} p(x) =$

d) $\lim_{x \to 3^-} p(x) =$

e) $\lim_{x \to 3^+} p(x) =$

f) $\lim_{x \to 3} p(x) =$