

Lesson

5-9

Linear Programming

Vocabulary

linear-programming problem

► **BIG IDEA** A *linear-programming problem* is one in which you wish to find a solution to a system of inequalities that minimizes or maximizes a linear combination of the variables.

In Lesson 5-8, you graphed a system of linear inequalities to see the numbers of tables and chairs that could be made from a certain set of blocks. In this lesson, we consider combinations of actual tables and chairs that a furniture maker, Tim Burr, can build under the same constraints:

$$\begin{cases} 2T + C \leq 12 \\ 2T + 2C \leq 16 \\ T \geq 0 \\ C \geq 0 \end{cases}$$

Suppose that Tim earns \$900 for each table and \$600 for each chair he makes and sells. Under these constraints and assuming that Tim sells all the tables and chairs that he makes, how many of each should he produce to maximize revenue?

If T tables and C chairs are sold, the revenue R in dollars is given by the formula

$$900T + 600C = R.$$

For instance, suppose 3000 is substituted for R in the formula. The solutions (T, C) to $900T + 600C = 3000$ are ordered pairs that represent combinations of tables and chairs that will yield \$3000 in earnings. Two such solutions are $(2, 2)$ and $(0, 5)$. This means that Tim could make 2 tables and 2 chairs, or 0 tables and 5 chairs, and he would earn \$3000.

The feasible set for Tim's system of inequalities is the set of lattice points in the shaded region of the graph on the next page. The graph also includes the line for a \$3000 revenue and five other revenue lines that result from substituting different values of R into the revenue formula. Notice that all lines with equations of the form $900T + 600C = R$ are parallel because each has a slope of -1.5 .

Mental Math

Suppose a cleaning service charges a flat rate of \$25 for the first hour plus \$10 for each additional half hour or part of a half hour. What would the cleaning service charge if the time spent cleaning were

- 2 hours?
- 1 hour and 35 minutes?
- 3 hours and 15 minutes?
- 4 hours and 5 minutes?

Some of these lines intersect the feasible region and some do not. Lines such as L_3 and L_4 that do intersect the feasible region indicate possible revenues. The highest revenue line that intersects the feasible region represents the greatest possible revenue. This is the line L_3 , where the revenue line passes through vertex $(4, 4)$. So, to maximize revenue, Tim should make 4 tables and 4 chairs each week. The maximum revenue under these conditions is $900 \cdot 4 + 600 \cdot 4 = \6000 .

Problems such as this one, that involve wanting to maximize or minimize a quantity based on solutions to a system of linear inequalities, are called **linear-programming problems**. The word *programming* does not refer to a computer; it means that the solution gives a program, or course of action, to follow.

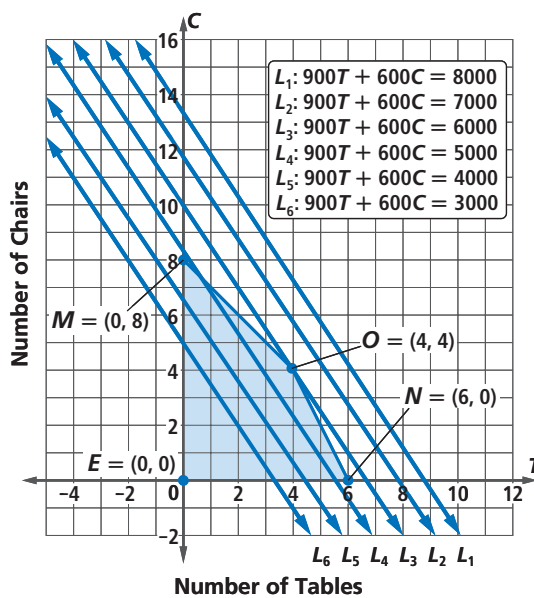
In 1826, the French mathematician Jean-Baptiste Joseph Fourier proved the following theorem. It tells you where to look for the greatest or least value of a linear-combination expression in a linear-programming situation, without having to draw any lines through the feasible region.

Linear-Programming Theorem

The feasible region of every linear-programming problem is convex, and the maximum or minimum quantity is determined at one of the vertices of this feasible region.

Linear programming is often used in industries in which all the competitors make the same product (such as gasoline, paper, appliances, clothing, and so on). Their efficiency in the use of labor and materials greatly affects their profits. These situations can involve as many as 5000 variables and 10,000 inequalities.

Testing vertices is a good solution method for problems involving a few constraints and only two variables. There are procedures for solving large linear-programming problems such as the *simplex algorithm* invented in 1947 by the mathematician George Dantzig, who worked on it with the econometrician Leonid Hurwicz, and the mathematician T.C. Koopmans, all from the United States. It is for this work, as well as other contributions, that Koopmans shared the Nobel Prize in 1975.



The Nobel Prize medal is made of 18-karat green gold plated with 24-karat gold.

Example

For a certain company, a bed sheet requires 2 pounds of cotton, 9 minutes of dyeing time, and 2 minutes of packaging time. A set of pillowcases requires 1 pound of cotton, 1.5 minutes of dyeing time, and 7.25 minutes of packaging time. Each day, the company has available 110 pounds of cotton, 405 minutes of dyeing time, and 435 minutes of packaging time.

- Let b = the number of bed sheets made per day and p = the number of sets of pillowcases made per day. Find the vertices of the region of pairs (b, p) the company can make.
- The company's daily revenue will be \$12 for each sheet and \$8 for each set of pillowcases. Assuming they sell everything they make, how many of each product should the company produce per day to maximize revenue?

Solution

- From the given information, write the constraints.

$$\text{Available cotton: } 2b + p \leq 110$$

$$\text{Dyeing time: } 9b + 1.5p \leq 405$$

$$\text{Packaging time: } 2b + 7.25p \leq 435$$

$$b \geq 0, p \geq 0$$

Graph the constraints. The graph at the right shows the feasible region. Because of the Linear-Programming Theorem, the revenue will be maximized at one of the vertices of this region.

There are 5 vertices to consider for the solution of this linear-programming problem. Each of these vertices is the intersection of two lines and is the solution to one of the five systems shown below.

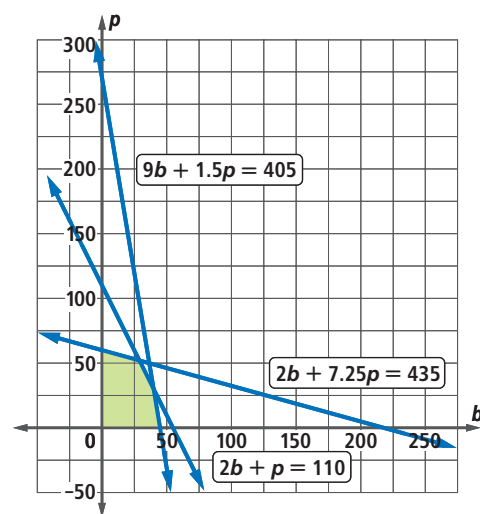
$$\begin{cases} p = 0 \\ 9b + 1.5p = 405 \end{cases} \quad \begin{cases} 2b + p = 110 \\ 9b + 1.5p = 405 \end{cases} \quad \begin{cases} 2b + p = 110 \\ 2b + 7.25p = 435 \end{cases}$$

$$\begin{cases} b = 0 \\ 2b + 7.25p = 435 \end{cases} \quad \begin{cases} b = 0 \\ p = 0 \end{cases}$$

Use solve to find each vertex (b, p) .

The vertices are $(45, 0)$, $(40, 30)$, $(29, 52)$, $(0, 60)$, and $(0, 0)$.

(continued on next page)



$$\text{solve}\{p=0 \text{ and } 9 \cdot b + 1.5 \cdot p = 405, \{b, p\}\}$$

$$b = 45 \text{ and } p = 0$$

- b. The company wants to maximize daily revenue R . Because the company sells bed sheets for \$12 and sets of pillowcases for \$8, a revenue formula is $R = 12b + 8p$. Use the Linear-Programming Theorem. The maximum value of R occurs at a vertex of the feasible region. Evaluate R at each vertex to see which combination of p and b gives the maximum revenue.

The maximum daily revenue of \$764 is obtained when $b = 29$ and $p = 52$, that is, when the company produces 29 bed sheets and 52 sets of pillowcases.

Define $f(b,p)=12\cdot b+8\cdot p$	Done
$f(45,0)$	540
$f(40,30)$	720
$f(29,52)$	764
$f(0,60)$	480
$f(0,0)$	0

Steps for Solving Linear-Programming Problems

The steps at the right are a good way to organize your work when solving a linear-programming problem. Use the steps in the following Activity.

Activity

MATERIALS CAS (optional)

A Prom Committee is responsible for prom decorations. There are two types of decorations needed; centerpieces for tables and topiaries for the dance floor. The committee decides to make its own decorations.

Each centerpiece requires 5 packages of artificial flowers and 2 twinkle lights; each topiary requires 3 packages of artificial flowers and 6 twinkle lights. A local florist helps out by donating 115 packages of artificial flowers and 130 twinkle lights.

Twenty prom committee members will meet for 6 hours to assemble the decorations, so they must complete the assembly job in 120 person-hours. Each centerpiece requires 2.5 hours, while each topiary requires 4 hours of labor. At least 10 centerpieces and at least 1 topiary will be needed.

After the prom, the committee plans on selling the decorations. Centerpieces will sell for \$20 each and topiaries will sell for \$15 each.

Step 1 Identify the variables and write a system of constraints.

Let x = number of centerpieces and y = number of topiaries.

Complete a table like the one at the top of the next page to organize the information. Then write three constraints that summarize the table, and two more that represent the minimum values for x and y .

LINEAR-PROGRAMMING PROBLEMS

1. Identify the variables and write a system of constraints.
2. Determine which intersections of lines define the vertices of the feasible region. (A sketch of the inequalities found in Step 1 may help.)
3. Find the vertices of the feasible region.
4. Write a formula or an expression to be maximized or minimized.
5. Apply the Linear-Programming Theorem.
6. Interpret the results.

	Number of Centerpieces (x)	Number of Topiaries (y)	Constraint Value
No. of packages of flowers	?	?	115
No. of packages of twinkle lights	?	?	?
No. of hours to make	?	?	?

Step 2 Determine which intersections of lines define the vertices of the feasible region. Make a sketch of the feasible region. There are four vertices of the feasible region, defined by the solutions of four systems. Fill in the blanks below to record the systems. Which constraint, if any, does not help define the feasible region?

$$\begin{cases} x = 10 \\ y = 1 \end{cases} \quad \begin{cases} x = 10 \\ ? \end{cases} \quad \begin{cases} y = 1 \\ ? \end{cases} \quad \begin{cases} ? \\ ? \end{cases}$$

Step 3 Find the vertices of the feasible region. Use a CAS to solve the four systems from Step 2.

The four vertices of the region are $?$, $?$, $?$, and $?$.

$$\text{solve}(x=10 \text{ and } 2 \cdot x + 6 \cdot y = 130, x, y)$$

$$x=10. \text{ and } y=18.3333$$

Step 4 Write a formula or an expression to be maximized or minimized. Write an equation for the revenue R if all decorations made are sold.

Step 5 Apply the Linear-Programming Theorem. Using your results from Steps 3 and 4, find the revenue for each vertex. Round vertex coordinates down because the committee cannot sell partial centerpieces or topiaries.

Step 6 Interpret the results. To maximize profits, how many centerpieces and topiaries should be made? What would be the maximum revenue from the sales?

Questions

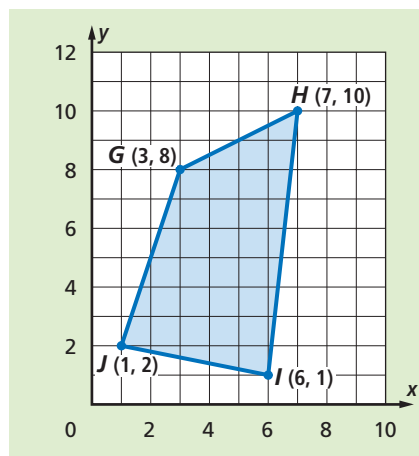
COVERING THE IDEAS

- Refer to the discussion of Tim Burr at the beginning of this lesson.
 - What is Tim trying to maximize? How is it represented?
 - Find the revenue for making 5 tables and 1 chair.
 - What do the coordinates of points with integer coordinates inside the feasible region represent?
 - True or False** The line $900T + 600C = 950$ intersects the feasible region.

2. Refer to the Example.
 - a. What does the linear combination R represent?
 - b. Why is the vertex $(52, 29)$ a solution to the problem?
 - c. Why is $p \geq 0$ and $b \geq 0$?
 - d. Assume that the available cotton has changed from 110 pounds to 120 pounds. Write a new constraint for cotton.
3. What does Jean-Baptiste Fourier have to do with the content of this lesson?
4. Why do industries test vertices of the feasible region of a system of inequalities?

In 5 and 6, refer to the Activity.

5. Write the coordinates of the vertex that maximizes revenue. What does this vertex mean for the committee?
6. Why should you not round the vertex values up to the next integer value?
7. What is the simplex algorithm?
8. Who developed the simplex algorithm, and when?
9. Use the feasible set graphed at the right.
 - a. Which vertex maximizes the profit equation $P = 30x + 18y$?
 - b. Which vertex minimizes the cost equation $C = 25x + 13y$?



APPLYING THE MATHEMATICS

10. Apples and oranges can meet certain nutritional needs. Suppose a person needs to consume at least 1000 mg of calcium and at least 1000 IU (international units) of vitamin A each day. An apple has about 73 IU of vitamin A and 10 mg calcium and an orange has about 20 IU of vitamin A and 60 mg calcium. Suppose an apple costs 30¢ and an orange costs 45¢. You want to minimize costs, yet meet the nutritional needs.
 - a. Identify the variables and constraints. Translate them into a system of inequalities.
 - b. Use a CAS and find the vertices of the feasible region.
 - c. Write the expression to be minimized.
 - d. Find the number of apples and oranges that will minimize the cost.

11. A muffin recipe uses $\frac{9}{16}$ pound of flour, $\frac{3}{4}$ pound of sugar, and $\frac{3}{4}$ pound of nuts to make 1 tray of muffins. A nut-bread recipe uses $\frac{1}{2}$ pound of flour, $\frac{3}{4}$ pound of sugar, and $\frac{1}{6}$ pound of nuts to yield 1 loaf of nut bread. Able Baker Carl has 50 pounds of flour, 40 pounds of sugar, and 15 pounds of nuts available.
- Identify the variables and constraints. Translate them into a system of inequalities.
 - Sketch a graph of the feasible region. Which constraint from Part a is not a boundary of the feasible region?
 - How many of each item should Carl make to maximize his revenue, if each tray of muffins sells for \$8.00 and each loaf of nut bread sells for \$12.00?
12. Landscaping contractor Pete Moss uses a combination of two brands of fertilizers, each containing different amounts of phosphates and nitrates as shown in the table. A certain lawn requires a mixture of at least 24 lb of phosphates and at least 16 lb of nitrates. If Pete uses a packages of Brand A and b packages of Brand B, then the constraints of the problem are given by the system of inequalities at the right.
- Graph the feasible region for this situation.
 - If a package of Brand A costs \$6.99 and a package of Brand B costs \$17.99, which acceptable combination of packages (a, b) will cost Pete the least?

	Brand A	Brand B
Phosphates (lb)	4	6
Nitrates (lb)	2	5

$$\begin{cases} a \geq 0 \\ b \geq 0 \\ 4a + 6b \geq 24 \\ 2a + 5b \geq 16 \end{cases}$$

REVIEW

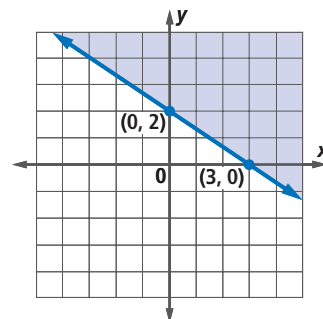
In 13 and 14, graph the solution set to the system. (Lesson 5-8)

13. $\begin{cases} y > 3 - x \\ y < x - 3 \end{cases}$ 14. $\begin{cases} x + y \geq 10 \\ x - y \leq 10 \end{cases}$

15. Write an inequality to describe the shaded region at the right. (Lesson 5-7)

In 16–18, consider the system $\begin{cases} A = s^2 \\ A + 2s = 8 \end{cases}$.

- Solve the system by substitution. (Lesson 5-3)
- Solve the system using a CAS. (Lesson 5-2)
- Solve the system by graphing. (Lesson 5-1)



EXPLORATION

19. Suppose $a, b, c, d, e,$ and f are positive integers and $a + b + c + d + e + f = 1000$. What integer values of a, b, c, d, e and f will make the product $abcdef$
- as large as possible?
 - as small as possible?