

Lesson

5-8

Systems of Linear Inequalities

Vocabulary

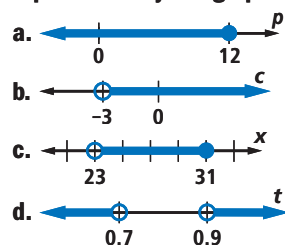
feasible set, feasible region

BIG IDEA The solution to a system of linear inequalities in two variables is either the empty set, the interior of a polygon, or a region bounded by line segments and rays.

A linear inequality in two or more variables may be used to model a situation in which one or more resources limit, or *constrain*, the values of the variables. In the previous lesson, Aleta's money limited the number of pencils and erasers she could buy. In this lesson we show how systems of linear inequalities can model even more complicated situations.

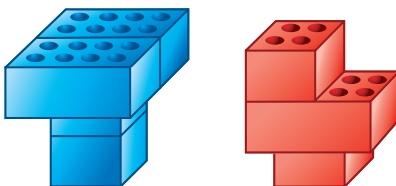
Mental Math

What inequality is represented by the graph?



Activity

Suppose you have a collection of 16 square (2×2) and 12 long (2×4) interlocking blocks to form into tables and chairs. It takes 2 long blocks and 2 square blocks to make a table. It takes 1 long block and 2 square blocks to make a chair. What combinations of tables and chairs can you make with your collection?

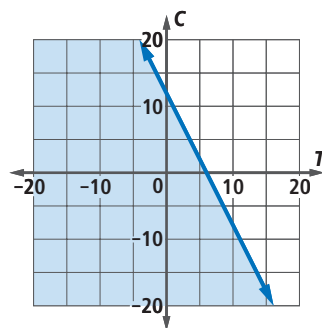


| | Table | Chair |
|----------------------|-------|-------|
| Long Blocks Needed | 2 | 1 |
| Square Blocks Needed | 2 | 2 |

Step 1 Let T = the number of tables and C = the number of chairs. Notice that T and C are nonnegative integers. Write an inequality using T and C that relates the total number of long blocks needed to the number of long blocks available. Graph this inequality.

Step 2 Write an inequality using T and C that relates the total number of square blocks needed to the number of square blocks available. Graph this inequality on the same axes as the inequality from Step 1.

Step 3 Determine whether each ordered pair (T, C) in the table on the next page satisfies the inequalities from Steps 1 and 2. Then plot the four ordered pairs on your graph.

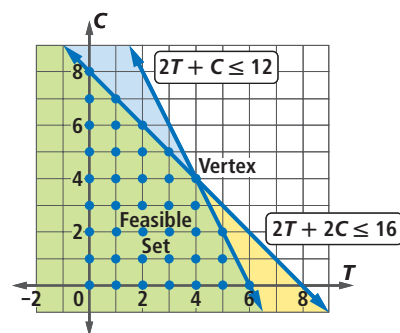


| (T, C) | Satisfies Long Block Inequality | Satisfies Square Block Inequality |
|----------|---------------------------------|-----------------------------------|
| (2, 4) | ? | ? |
| (6, 2) | ? | ? |
| (2, 7) | ? | ? |
| (5, 4) | ? | ? |

Step 4 Which point(s) from Step 3 satisfy both the long-block inequality and the square-block inequality? Where do these points appear on the graph?

In the Activity, the limited numbers of square and long blocks are constraints to solving the problem. These constraints are represented by the inequalities $2T + C \leq 12$, $2T + 2C \leq 16$, $T \geq 0$, and $C \geq 0$. The set of possible combinations of tables and chairs is the intersection of the solution sets of all these inequalities, as shown in the graph at the right.

Because the graph of a linear inequality in two variables is a half-plane, the graph of the solution to a system of linear inequalities is the intersection of half-planes. Points in this intersection are often called the **feasible set** or **feasible region** for the system.



STOP QY

A feasible region is always bounded by either segments or rays. The intersections of boundary segments or rays are *vertices* of the feasible region. One vertex of the feasible region from the Activity is labeled in the graph. Because the boundary may not be included in the solution set of an inequality, the vertices and boundary segments or rays may not be part of the feasible region.

GUIDED

Example 1

An electronics firm makes two kinds of televisions: plasma and projection. The firm assumes it will sell all the sets it makes. It profits \$2000 on each plasma set it sells and \$1500 on each projection TV. If it wants a profit of over \$100,000, how many plasma sets and how many projection TVs should it make?

Solution Let L = the number of plasma sets made. Then the profit the firm makes on these sets is $\underline{\quad? \quad} \frac{\text{dollars}}{\text{set}} \cdot \underline{\quad? \quad} \text{sets} = \underline{\quad? \quad}$ dollars. Similarly, let R = the number of projection sets made. The profit on these sets is $\underline{\quad? \quad}$ dollars.

(continued on next page)

QY

Multiple Choice

Which of these combinations of tables and chairs are in the feasible region from the Activity?

- A (5, 2) B (6, 1)
C (0, 12) D (0, 0)
E (4, 4)

So the total profit is $\underline{\quad} + \underline{\quad}$ dollars, and the pair (L, R) of numbers of televisions the firm should make needs to satisfy

$$\underline{\quad} + \underline{\quad} > 100,000.$$

Divide both sides by 100 to make the numbers easier to manage.

$$\underline{\quad} + \underline{\quad} > 1000$$

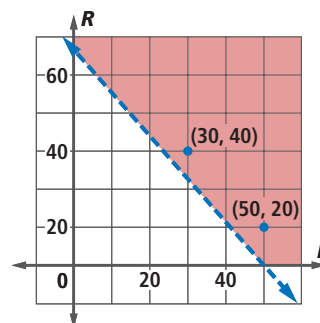
Because L and R are numbers of televisions, they are nonnegative integers. So $L \geq 0$ and $R \geq 0$. These three inequalities form the boundary of the feasible region.

Graph the three boundary lines. Shade the feasible region as shown at the right.

The firm can make any pair (L, R) of numbers of televisions that is a lattice point in the feasible region.

In this situation, there are too many lattice points to plot distinctly, so the feasible region is shaded and two discrete solutions, $(\underline{\quad}, \underline{\quad})$ and $(\underline{\quad}, \underline{\quad})$, are noted on the graph.

These points mean that the firm could sell $\underline{\quad}$ plasma sets and $\underline{\quad}$ projection sets or $\underline{\quad}$ plasma sets and $\underline{\quad}$ projection sets and meet their goal.



In real life, there may be other constraints on the situation in Example 1. It takes time to make each set. It takes different amounts of various kinds of metals, plastics, and electronics to make each set. Each constraint may add a line to the boundary and make the problem a little more complicated.

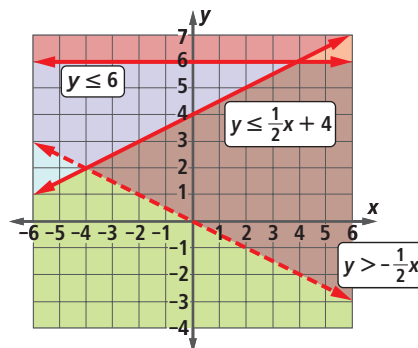
Example 2

Consider the system
$$\begin{cases} y > -\frac{1}{2}x \\ y \leq \frac{1}{2}x + 4 \\ y \leq 6 \end{cases}$$

- Graph the solution to the system by hand and check your solution.
- Find the vertices of the feasible region.

Solution

- The boundary lines for all the inequalities are easily sketched because the inequalities are in slope-intercept form. The first inequality has a dotted boundary, and the half-plane above is shaded. The second has a solid boundary, and the half-plane below it is shaded. The third inequality has a solid boundary, and the half-plane below it is shaded. The three inequalities are graphed on the same grid shown here.



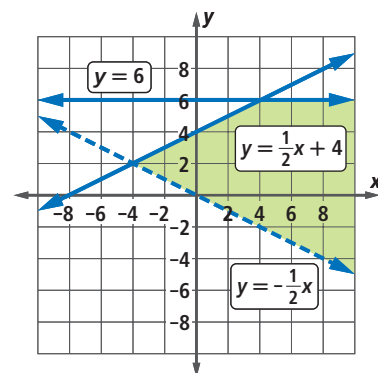
The solution to the system is all the points in the intersection of the three half-planes. A graph of the feasible region is shown at the right. The region extends forever to the right, below and on $y = 6$ and $y = -\frac{1}{2}x + 4$ and above $y = -\frac{1}{2}x$.

Choose an ordered pair in the feasible region to see if it satisfies all three inequalities. We choose $(1, 0)$.

$$\text{Is } 0 > -\frac{1}{2}(1)? \quad \text{Yes.}$$

$$\text{Is } 0 \leq \frac{1}{2}(1) + 4? \quad \text{Yes.}$$

$$\text{Is } 0 \leq 6? \quad \text{Yes, so the solution checks.}$$



- b. The coordinates of each vertex of the feasible region can be found by reading the graph or by solving pairs of equations using the substitution method.

$$\begin{cases} y = \frac{1}{2}x + 4 \\ y = 6 \end{cases}$$

Substitute 6 for y in $y = \frac{1}{2}x + 4$.

$$6 = \frac{1}{2}x + 4$$

$$x = 4$$

$(4, 6)$ is one vertex.

$$\begin{cases} y = \frac{1}{2}x + 4 \\ y = -\frac{1}{2}x \end{cases}$$

Substitute $-\frac{1}{2}x$ for y in $y = \frac{1}{2}x + 4$.

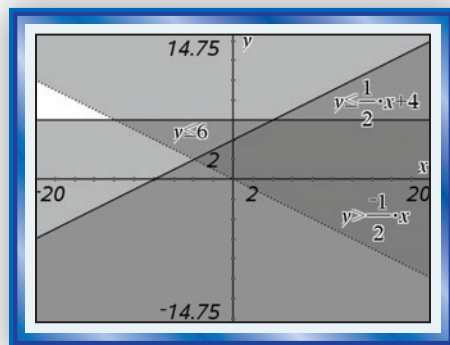
$$-\frac{1}{2}x = \frac{1}{2}x + 4$$

$$x = -4$$

Substitute -4 for x in either equation to find that $y = 2$.

$(-4, 2)$ is the other vertex.

A graphing utility can graph systems of inequalities. However, the feasible region may not be clear on all machines. A graph of the system from Example 2 on one grapher is shown below. The darkest region is the feasible region.



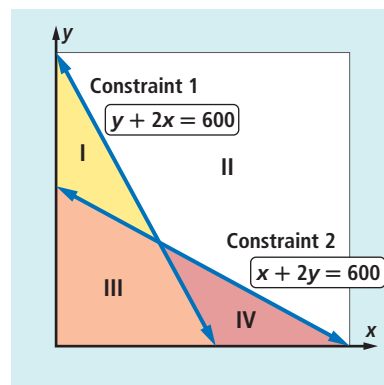
Questions

COVERING THE IDEAS

- Fill in the Blank** The solution to a system of linear inequalities can be represented by the ___?___ of half-planes.
- The solution set to a system of linear inequalities is often called the ___?___.

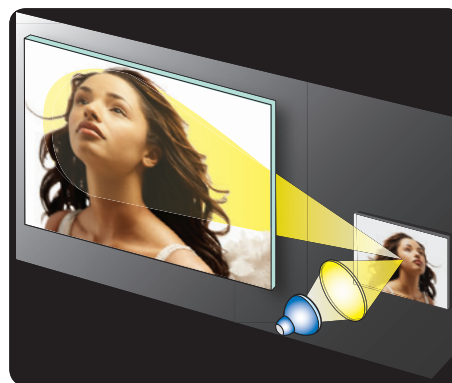
In 3–5, refer to the Activity.

- Use your graph to determine whether each combination of tables and chairs is possible. If it is not, describe which block, long or square, is in short supply.
 - 7 tables and 2 chairs
 - 5 tables and 2 chairs
 - 2 tables and 5 chairs
 - 2 tables and 7 chairs
- The point (3, 2.5) satisfies all the constraints, but it is not a solution in this situation. Explain why not.
- What system of equations can you solve to find the vertex of the feasible region located in the first quadrant?
 - Solve the system from Part a to find the coordinates of the vertex.
- The graph at the right represents a system of inequalities. In what region(s) are the solutions to each system, with $x > 0$ and $y > 0$?
 - $y + 2x < 600$ and $x + 2y < 600$
 - $y + 2x > 600$ and $x + 2y < 600$
 - $y + 2x < 600$ and $x + 2y > 600$
 - $y + 2x > 600$ and $x + 2y > 600$



In 7 and 8, refer to Example 1.

- How much profit will the company make if it sells
 - 50 plasma sets and 40 projection TVs?
 - 20 plasma sets and 40 projection TVs?
 - 50 plasma sets and 5 projection TVs?
- Will the firm meet its goal of surpassing a \$100,000 profit if it sells 11 plasma sets and 45 projection TVs?



Most projection TVs are rear-projection systems; the image is displayed on the back of the screen and the projector is contained in the TV.

In 9 and 10, a system of inequalities is given.

- Graph the solution set.
- Find the coordinates of each vertex of the feasible region.

$$9. \begin{cases} y \leq 2x + 3 \\ y > -3x - 4 \end{cases} \qquad 10. \begin{cases} x + 2y \leq 16 \\ 3x + y < 18 \\ y \leq 7 \\ x \geq 0 \end{cases}$$

APPLYING THE MATHEMATICS

11. A school's film club shows movies after school and sells popcorn in small and large sizes. The bags cost \$0.15 for a small and \$0.20 for a large. A small bag holds 1.25 ounces of popcorn; a large bag holds 2.5 ounces of popcorn. The club has 400 ounces of popcorn and a budget of \$40 for popcorn bags.

- Fill in the Blanks** Let S be the number of small bags and L be the number of large bags. Complete the translation of this situation into a system of inequalities.

$$\begin{cases} S \geq \frac{?}{?} \\ L \geq \frac{?}{?} \\ 0.15S + 0.2L \leq \frac{?}{?} \\ \frac{?}{?} \leq 400 \end{cases}$$

- Graph the feasible region of points (S, L) for this system and label the vertices.
12. Graph the solutions to the system $\begin{cases} y < 2x + 4 \\ 4x - 2y \leq 6 \end{cases}$.
13. In his shop, Hammond Wrye makes two kinds of sandwiches: plain turkey and turkey-and-cheese. Each sandwich uses 2 pieces of bread. The plain turkey sandwiches use 3.5 ounces of turkey, while the turkey-and-cheese sandwiches use 2.5 ounces of turkey and 1 slice of cheese. Hammond has 100 slices of bread, 40 slices of cheese, and 150 ounces of turkey. Let x be the number of turkey-and-cheese sandwiches and y be the number of plain turkey sandwiches he makes.
- Translate the situation into a system of inequalities.
 - Graph the feasible set for this system, and label the vertices.
14. Refer to Example 1. Suppose it takes 30 worker-hours to make a plasma TV and 25 worker-hours to make a projection TV. If the company has 15,000 worker-hours available, can it make its desired profit of \$100,000? Explain your answer.



Americans consume an average of 17.3 billion quarts of popcorn each year.

REVIEW

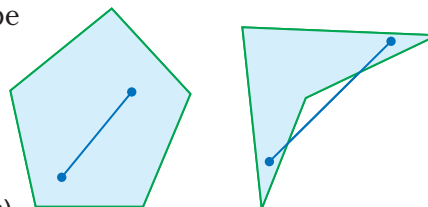
In 15 and 16, graph the inequality. (Lesson 5-7)

15. $y \leq -4x + 3$

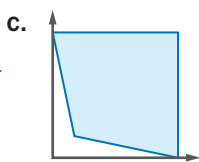
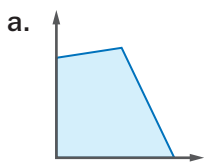
16. $3x - 2y > 14$

17. **True or False** If the coefficient matrix of a system has determinant zero, then the system has no solutions. (Lesson 5-6)

18. Recall from geometry that a region of the plane is said to be *convex* if and only if any two points of the region can be connected by a line segment which is itself entirely within the region. The pentagonal region at the right is convex, but the quadrilateral region is not.



Tell whether the shaded region is convex. (Previous Course)



19. Cesar runs for x to $x + 5$ minutes every Tuesday. Every year has either 365 or 366 days in it (depending on whether or not it is a leap year). What are the minimum and maximum amounts of time Cesar could spend on his weekly runs in any given year? (Previous Course)

EXPLORATION

20. Consider the system
$$\begin{cases} x > 0 \\ y > 0 \\ y < mx + b \end{cases}$$
.

- For what values of m and b is the graph of the solution set to this system the interior of a triangle?
- Find the area of the triangle in terms of m and b .

QY ANSWER

A, D, and E