Lesson 5-6 Solving Systems Using Matrices

Vocabulary

matrix form of a system
coefficient matrix
constant matrix

BIG IDEA A system of linear equations in standard form can be written as a matrix equation that can often be solved by multiplying both sides of the equation by the inverse of the *coefficient matrix.*

The matrix equation $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$ is called the **matrix form of the system** $\begin{cases} 3x - y = 14 \\ 2x + 4y = 0 \end{cases}$. The matrix $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ is called the **coefficient matrix** because it contains the coefficients of the variables in the system. The matrix $\begin{bmatrix} 14 \\ 0 \end{bmatrix}$ is called the **constant matrix** because it contains the constants on the right side of the equations in the system.

STOP QY1

How do you solve a system in matrix form? Think of how you solve the equation $3x = 6$. You might divide both sides of the equation by 3, or multiply both sides of the equation by $\frac{1}{3}$, the multiplicative inverse of 3. This idea is employed to solve linear systems in matrix form.

Mental Math

Calculate the determinant of the matrix.

a. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

c. $\begin{bmatrix} -1 & 5 \\ -2 & -1 \end{bmatrix}$

d. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

QY1

Expand $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$ to verify that it represents $\begin{cases} 3x - y = 14 \\ 2x + 4y = 0 \end{cases}$.

GUIDED

Example 1

Solve the system $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$.

Solution First, find the inverse of the coefficient matrix.

$$\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{?}{?} & \frac{?}{?} \\ \frac{?}{?} & \frac{?}{?} \end{bmatrix}$$

Next, multiply both sides of the equation on the left by the inverse.

$$\begin{bmatrix} \frac{?}{?} & \frac{?}{?} \\ \frac{?}{?} & \frac{?}{?} \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{?}{?} & \frac{?}{?} \\ \frac{?}{?} & \frac{?}{?} \end{bmatrix} \cdot \begin{bmatrix} 14 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{?}{?} & \frac{?}{?} \\ \frac{?}{?} & \frac{?}{?} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{?}{?} \\ \frac{?}{?} \end{bmatrix}$$

Because you multiplied by the inverse, the left side of the equation should now be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \underline{?} \\ \underline{?} \end{bmatrix}$, so $x = \underline{\ ?\ }$ and $y = \underline{\ ?\ }$.

Check Check your answer by substituting your values for x and y into the original matrix equation.

$\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$ It checks.

This method for solving a system of linear equations with matrices was developed in the middle of the nineteenth century by the British mathematician Arthur Cayley and can be generalized. For any invertible coefficient matrix A and constant matrix B, with $X = \begin{bmatrix} x \\ y \end{bmatrix}$, the solution to the matrix equation $AX = B$ is $X = A^{-1}B$. Example 2 shows how this method works on systems with three linear equations in three variables.

Example 2

Use matrices to solve $\begin{cases} 4x + 2y - 2z = 2 \\ 2x + 4z = 28 \\ 3y - 2z = -16 \end{cases}$.

Solution Write the matrix form of the system.

$$\underset{A}{\begin{bmatrix} 4 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 3 & -2 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 2 \\ 28 \\ -16 \end{bmatrix}}$$

For each side of the equation use a CAS to multiply on the left by the inverse of the coefficient matrix.

$$A^{-1} \cdot A \cdot X \quad = \quad A^{-1} \cdot B$$

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 3 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 4 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 3 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ 28 \\ -16 \end{bmatrix} \quad \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$

(continued on next page)

Write an equation with the results from the multiplications.

$$X = A^{-1} \cdot B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$. So, $x = 4$, $y = -2$, and $z = 5$.

Check Substitute for x, y, and z in each of the three given equations.

Does $4(4) + 2(-2) - 2(5) = 2$? Yes, $16 + -4 - 10 = 2$.

Does $2(4) + 4(5) = 28$? Yes, $8 + 20 = 28$.

Does $3(-2) + -2(5) = -16$? Yes, $-6 + -10 = -16$.

The Number of Solutions to Linear Systems

Matrices can be used to determine the number of solutions to a linear system. For example, consider the system $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$.

If the determinant $ad - bc$ of the coefficient matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not 0, the matrix has an inverse and the system has exactly one solution. When $ad - bc = 0$, the coefficient matrix has no inverse. In that case, the system has either infinitely many solutions or none at all.

System-Determinant Theorem

An $n \times n$ system of linear equations has exactly one solution if and only if the determinant of the coefficient matrix is *not* zero.

In the previous lesson you calculated the determinant of a 2×2 matrix. While the calculation of a determinant of a larger square matrix is tedious by hand, a calculator can find the determinant automatically.

STOP **QY2**

When the determinant of the coefficient matrix is zero, to determine whether there are no solutions or infinitely many solutions, find a solution to one of the equations and test it in the other equations. If it satisfies all the other equations, there are infinitely many solutions to the system. If the solution does not satisfy all the other equations, there are no solutions to the system.

▶ QY2

Use a calculator to find the determinant of the coefficient matrix in Example 2. Does its value support the solution to that Example?

Example 3

Consider the system $\begin{cases} 3x + 2y = 14 \\ 6x + 4y = 10 \end{cases}$.

a. Show that this system does not have exactly one solution.

b. Determine how many solutions this system has.

Solution 1

a. Let A be the coefficient matrix $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$.

Then $\det A = (3)(4) - (2)(6) = 0$.

So, by the System-Determinant Theorem, the system does not have exactly one solution.

b. Either the system has infinitely many solutions or no solutions. To decide, find an ordered pair that satisfies one equation.

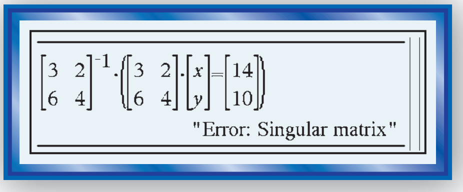
(4, 1) satisfies the first equation.

Next, does it satisfy the other equation?

Does $6(4) + 4(1) = 10$? No. Thus, because (4, 1) does not satisfy all the equations, the system has no solutions.

Solution 2

a. Use a calculator to solve the system with matrices. One calculator gives the result at the right. The error message means .that the coefficient matrix does not have an inverse.



b. Proceed as in Part b of Solution 1.

STOP QY3

QY3

Is the system in Example 3 consistent or inconsistent?

Questions

COVERING THE IDEAS

1. a. Write the matrix form of the system $\begin{cases} 0.5x - y = 1.75 \\ 3x + 8y = 5 \end{cases}$.

   b. Solve the system.

2. a. Write a system of equations whose matrix form is

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 5 & 2 \\ 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 10 \end{bmatrix}.$$

   b. Which matrix in Part a is the coefficient matrix? Which is the constant matrix?

   c. Use technology to solve this system.

In 3–5, determine how many solutions the system has. Justify your answer.

3. $\begin{cases} 8x + 12y = 40 \\ 4x + 6y = 25 \end{cases}$

4. $\begin{cases} 8x + 12y = 40 \\ 6x + 20y = 52 \end{cases}$

5. $\begin{cases} 8x + 12y = 40 \\ 6x + 9y = 30 \end{cases}$

In 6 and 7, solve each system using matrices.

6. $\begin{cases} 18 = 2d - 3h \\ 7 = 5d + 2h \end{cases}$

7. $\begin{cases} 4x - 3y + z = 4 \\ 2x = -2 \\ 3y + 5z = 40 \end{cases}$

8. Solve the system in Example 3 using the linear-combination method. Explain how that method shows there is no solution.

APPLYING THE MATHEMATICS

9. Set up a system and solve it using matrices to answer this question: Two types of tickets are available for Lincoln High School plays. Student tickets cost \$3 each, and nonstudent tickets cost \$5 each. On opening night of the play *Proof* by David Auburn, 937 total tickets were sold for \$3943. How many of each type of ticket were sold?

10. Forensic scientists and anthropologists can estimate the height h of a person based on the length b of the person's femur using the following equations. All lengths are measured in inches.

Male: $h = 32.010 + 1.880b$

Female: $h = 28.679 + 1.945b$

a. Even though the equation for male height has a larger h-intercept, there is a point where males and females are expected to have the same height and length of femur. How do you know this from the equations?

b. Use matrices to find the point of intersection of the lines represented by the equations. (Hint: Remember to put the system in standard form before setting up each matrix.)

c. **Fill in the Blanks** According to these linear models, a man and a woman with the same height and femur length are __?__ inches tall and have a femur length of __?__ inches.

11. A hotel has standard, special, and deluxe rooms. A meeting planner needs 15 rooms. If 6 standard, 6 special, and 3 deluxe rooms are booked, the cost will be \$2835 a night. Is this enough information to determine the cost of each kind of room? If so, determine the costs. If not, explain why not.

12. **a.** Write the matrix form of the system $\begin{cases} 4x = 12 \\ 8y = 16 \end{cases}$ and solve.

b. Give a geometric interpretation of the solution to the system.

In 13 and 14, determine all values of *n* that satisfy the condition.

13. $\begin{cases} 5x + 3y = 4 \\ 20x - ny = 16 \end{cases}$ has infinitely many solutions.

14. $\begin{cases} 2x + 7y = 1 \\ 4x + 14y = n \end{cases}$ has no solution.

15. Solve the system $\begin{cases} 2w - x - y + z = -1 \\ 3.2w + 1.4z = 2.4 \\ 1.8x - 3y + 7z = 23.8 \\ 5w - 3x + 7y + z = 10 \end{cases}$.

REVIEW

16. A 2 × 2 *diagonal matrix* is of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Show that the inverse (if it exists) of a diagonal matrix is another diagonal matrix. **(Lesson 5-5)**

17. **a.** Find the inverse of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

b. Check your answer using matrix multiplication. **(Lessons 5-5, 4-3)**

18. Solve and check: $\begin{cases} A = 19 - 3D \\ 2A - 4D = -12 \end{cases}$. **(Lesson 5-3)**

19. Find the equation in slope-intercept form for the line through the origin and perpendicular to the line with equation $8x - 5y = 20$. **(Lesson 4-9)**

20. A sequence L has the explicit formula $L_n = 4 - 2.5n$. Write a recursive formula for the sequence. **(Lesson 3-8)**

EXPLORATION

21. From a reference, find a formula for the determinant and inverse of a 3 × 3 matrix. Check the formula for the inverse for the matrix $A = \begin{bmatrix} 2 & -2 & 5 \\ 4 & 6 & 12 \\ 3 & 1 & 7 \end{bmatrix}$.

QY ANSWERS

1. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - y \\ 2x + 4y \end{bmatrix}$, so, $\begin{bmatrix} 3x - y \\ 2x + 4y \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$, which represents $\begin{cases} 3x - y = 14 \\ 2x + 4y = 0 \end{cases}$.

2. –52; yes; it does.

3. inconsistent