Lesson 5-4
Solving Systems Using Linear Combinations

▶ **BIG IDEA** Some systems of equations can be solved by creating a new equation that is a linear combination of the original equations.

In the last lesson, you solved systems using the Substitution Property, which may be faster than using a CAS in cases when one equation is already solved for one variable in terms of the other variables. When linear equations are written in standard form, it may be more efficient to solve the system using the Addition and Multiplication Properties of Equality.

Recall that an expression of the form $Am + Bn$ is called a *linear combination* of m and n. In this lesson we use what we call the **linear-combination method** of solving systems because it involves adding multiples of the given equations.

Mental Math

Write a linear combination to answer the question. How much did I spend if I bought

a. p pairs of shoes at $39 each and s pairs of socks at $4 each?

b. 3 pairs of shoes at d_1 each and 6 pairs of socks at d_2 each?

c. x pairs of shoes at d_1 each and y pairs of socks at d_2 each?

Example 1

Solve the system $\begin{cases} x + 2y = 11 \\ 3x - 9y = 23 \end{cases}$ by the linear-combination method.

Solution The equations in this system represent the lines graphed at the right. The solution to the system is the point of intersection of the lines.

To solve the system using linear combinations, multiply the first equation by –3.

$-3(x + 2y) = -3 \cdot 11$ Multiplication Property of Equality

$-3x - 6y = -33$ Distributive Property, arithmetic

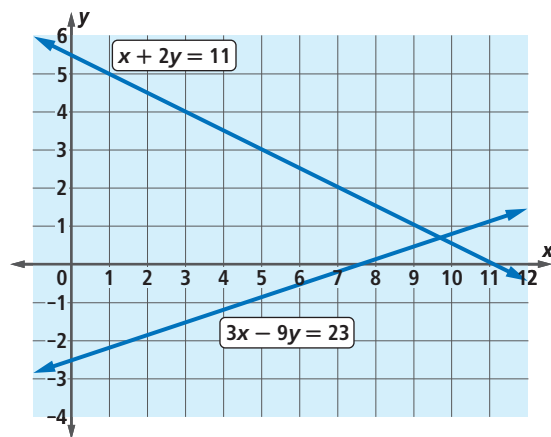

This equation is equivalent to the first equation of the original system. So, rewrite the system using this equation.

$\begin{cases} -3x - 6y = -33 \\ 3x - 9y = 23 \end{cases}$

Add these two equations. The result has no term in x.

$-15y = -10$ Addition Property of Equality

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Solve this equation for y.

$$y = \frac{-10}{-15} = \frac{2}{3}$$

Thus, $\frac{2}{3}$ is the y-coordinate of the point of intersection.

Substitute $\frac{2}{3}$ for y in any equation above to find x. We choose the first equation.

$$x + 2\left(\frac{2}{3}\right) = 11$$

$$x = 11 - \frac{4}{3} = \frac{29}{3}$$

So the solution to the system is $\begin{cases} x = \frac{29}{3} \\ y = \frac{2}{3} \end{cases}$.

You can write $(x, y) = \left(\frac{29}{3}, \frac{2}{3}\right)$.

Check 1 Substitute $\frac{29}{3}$ for x and $\frac{2}{3}$ for y in the second equation. Does $3 \cdot \frac{29}{3} - 9 \cdot \frac{2}{3} = 23$? Yes. It checks.

Check 2 The graphs of the lines with the equations $x = \frac{29}{3}$ and $y = \frac{2}{3}$ intersect at the same point as the lines in the other systems. The last system is equivalent to the other two systems on the previous page.
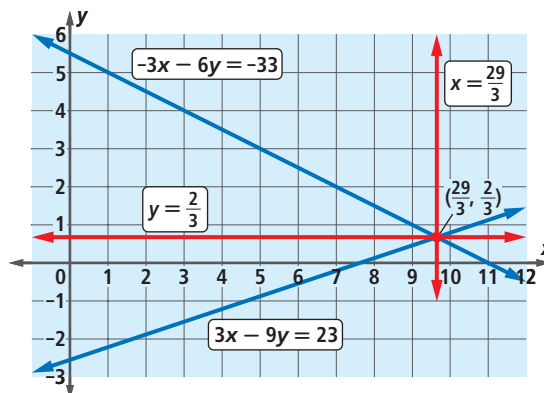
Systems arise in important practical endeavors. For example, we all need to eat foods that meet minimum requirements of protein, vitamins, minerals, and calories.

Activity

MATERIALS CAS (optional)

Long-distance runner Mary Thawn wants to get 800 calories and 50 grams of protein from a dinner course of chicken and pasta with sauce. Protein and calorie counts for each item are given below. How many ounces of each type of food does Mary need to eat? Work with a partner to solve this problem in two different ways using the linear-combination method.

Scientific studies suggest that marathoners eat 0.8–1.7 grams of protein per kilogram of body weight per day.

Step 1 Organize the given information into a table.

	Chicken	Pasta with Sauce	Total Needed
Protein	9 g/oz	2 g/oz	50 g
Calories	65 cal/oz	45 cal/oz	800 cal

Step 2 Together, write a system to model this situation if c = the number of ounces of chicken Mary should eat, and p = the number of ounces of pasta with sauce that Mary should eat.

Step 3 In the linear-combination method, you want to eliminate a variable by making the coefficients for that variable into opposites. Have one partner work to eliminate c and the other to eliminate p. By what numbers can you multiply the equations in the system to make the coefficients of your variable opposites?

Step 4 Carry out your multiplications from Step 3 and add to eliminate your variable. You can do this on a CAS or by hand. If you use a CAS, you might want to assign the two original equations to variables such as $eq1$ and $eq2$. Then you can multiply $eq1$ and $eq2$ by constants and add the results without retyping whole equations.

$9{\cdot}c{+}2{\cdot}p{=}50 \rightarrow eq1$	$9{\cdot}c{+}2{\cdot}p{=}50$
$65{\cdot}c{+}45{\cdot}p{=}800 \rightarrow eq2$	$65{\cdot}c{+}45{\cdot}p{=}800$
$-65{\cdot}eq1{+}9{\cdot}eq2$	$275{\cdot}p{=}3950$

Step 5 Solve for the remaining variable and substitute into one of the original equations to find the value of the other variable. Then answer the question in the problem.

Step 6 Compare your results with those of your partner. Did you get the same solution? Was eliminating p easier than eliminating c, or vice versa? Was using a CAS easier or harder than doing it by hand?

STOP QY

Linear Combinations with Systems of Three Equations

The linear-combination method can also be used to solve a system with three linear equations. First, use a linear combination of any pair of equations to eliminate one of the variables. Then eliminate the same variable from another pair of equations by using another linear combination. The result is a system of two equations with two variables. Then you can solve the simpler system using the methods of Example 1 and the Activity.

▶ **QY**

If m is one of your equations in the Activity, and n is the other equation, then $Am + Bn$ is the linear combination you used to eliminate your variable. What were your A and B? What were your partner's A and B?

GUIDED

Example 2

Solve the system $\begin{cases} 2x + y + 4z = 20 \\ 3x - 3y + 2z = 27 \\ 4x + 5y - 2z = 4 \end{cases}$.

Solution We choose to eliminate z first because its coefficients in the last two equations are already opposites. Add the last two equations to get an equation in terms of x and y.

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$$3x - 3y + 2z = 27$$
$$\underline{4x + 5y - 2z = 4}$$
$$\underline{}\ \underline{?}\ +\ \underline{?}\ = 31 \quad \text{Add.}$$

Now consider the first two equations. The coefficients of z are 4 and 2, so you can multiply the second equation by -2, then add the result to the first equation to get an equation in terms of x and y.

$$2x + y + 4z = 20 \rightarrow \qquad 2x + y + 4z = 20$$
$$3x - 3y + 2z = 27 \rightarrow \underline{\ ?\ } + \underline{\ ?\ } - 4z = \underline{\ ?\ } \qquad \text{Multiply by } -2.$$
$$\underline{\ ?\ } + \underline{\ ?\ } = -34 \qquad \text{Add.}$$

The result is the system below.

$$\begin{cases} -4x + 7y = -34 \\ \underline{\ ?\ } + \underline{\ ?\ } = \underline{\ ?\ } \end{cases}$$

Continue using the linear-combination method on this system of two equations in two variables. We choose to eliminate x.

$$-4x + 7y = -34 \rightarrow \qquad -28x + \underline{\ ?\ } = \underline{\ ?\ } \qquad \text{Multiply by 7.}$$
$$7x + 2y = 31 \rightarrow \qquad \underline{\ ?\ } + 8y = \underline{\ ?\ } \qquad \text{Multiply by 4.}$$
$$\underline{\ ?\ } = -114 \qquad \text{Add.}$$
$$y = \underline{\ ?\ } \qquad \text{Solve for } y.$$

Substitute $y = -2$ into one of the two equations involving just two variables. We substitute into $-4x + 7y = -34$.

$$-4x + 7(-2) = -34$$
$$-4x = \underline{\ ?\ }$$
$$x = \underline{\ ?\ }$$

Now substitute $x = 5$ and $y = -2$ into one of the original equations of the system. We use $2x + y + 4z = 20$.
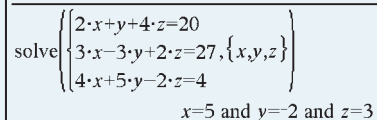
$$2(5) + (-2) + 4z = 20$$
$$4z = \underline{\ ?\ }$$
$$z = \underline{\ ?\ }$$

So the solution is $x = \underline{\ ?\ }$, $y = \underline{\ ?\ }$, and $z = \underline{\ ?\ }$. This can be written as the ordered triple $(x, y, z) = (\underline{\ ?\ }, \underline{\ ?\ }, \underline{\ ?\ })$.

Check 1 Substitute the values for x, y, and z in each of the original equations to make sure they check.

Check 2 Solve using a CAS. One CAS solution is shown at the right.

$$\text{solve}\begin{pmatrix}\begin{cases} 2 \cdot x + y + 4 \cdot z = 20 \\ 3 \cdot x - 3 \cdot y + 2 \cdot z = 27, \{x, y, z\} \\ 4 \cdot x + 5 \cdot y - 2 \cdot z = 4 \end{cases}\end{pmatrix}$$
$$x = 5 \text{ and } y = -2 \text{ and } z = 3$$

The systems in Examples 1 and 2 are consistent, and each has a unique solution. However, suppose you try to solve systems such as

$$\begin{cases} x + 2y = 11 \\ x + 2y = 12 \end{cases} \quad \text{or} \quad \begin{cases} x + 2y = 11 \\ 2x + 4y = 22 \end{cases}.$$
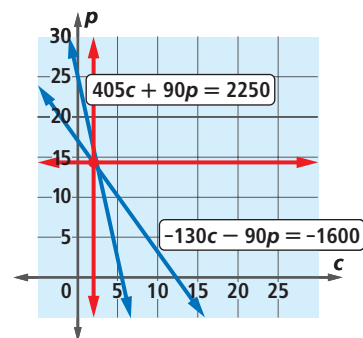
If you use the linear-combination method with the first system, the result is a false statement, such as $0 = 1$, so the system is inconsistent and has no solutions. However, with the second system, the linear-combination method yields a result that is always true, such as $0 = 0$. This means the system is consistent, and there are infinitely many solutions.

As Example 2 shows, the linear-combination method takes some work with a 3×3 system. This method is more complicated and impractical when there are large numbers of variables and equations in the system. In Lesson 5-6, you will see how matrices provide an efficient way to solve large systems.

Questions

COVERING THE IDEAS

1. Refer to Example 1. What properties are used to obtain the equation $-15y = -10$ from the two original equations?

2. What equation results when $5x - 4y = 12$ is multiplied by -2? What property is being applied?

3. Refer to the Activity. One student graphed the equations she obtained in Steps 4 and 5. Her graph is at the right.
   a. Which variable was eliminated in Step 4?
   b. What is an equation for the horizontal line? What is the equation for the vertical line?
   c. What is the solution to the system?

4. Refer to the system involving only x and y in Example 2. Multiply the first equation by $\frac{7}{4}$ and solve the resulting system.

5. The table at the right gives the number of grams of protein and the number of calories in one ounce of each of two foods. Mary Thawn from the Activity still wants to get 800 calories from her meal and obtain 50 grams of protein.
   a. Let h be the number of ounces of chicken strips and r be the number of ounces of french fries she eats. Write a system of equations that describes these conditions.
   b. How many ounces of each food should Mary eat?

	Chicken Strips	French Fries
Protein	5 g/oz	1 g/oz
Calories	80 cal/oz	95 cal/oz

In 6–9, solve the system using the linear-combination method.

6. $\begin{cases} a + b = \frac{1}{2} \\ a - b = \frac{1}{3} \end{cases}$

7. $\begin{cases} 4u - 2v = 24 \\ 5u + 6v = 13 \end{cases}$

8. $\begin{cases} 10g + 15h = 60 \\ 0.02g + 0.12h = 0.3 \end{cases}$

9. $\begin{cases} 2x - y + 2z = 4 \\ 5x + 2y - 3z = 43 \\ x + y - z = 11 \end{cases}$

10. In the process of solving a system using the linear-combination method, a student obtained the result $0 = 0$. How should the result be interpreted?

In 11 and 12, use the linear-combination method to determine whether the system is inconsistent or consistent.

11. $\begin{cases} 5x + y = 16 \\ 10x + 2y = 20 \end{cases}$

12. $\begin{cases} 4a + 10b = 12 \\ 6a + 15b = 18 \end{cases}$

APPLYING THE MATHEMATICS

13. Suppose 3 turkey wraps and 2 juices cost $26.50, while 2 turkey wraps and 3 juices cost $24.75. What is the cost of one juice?

14. N liters of a 1.3 moles per liter (mol/L) nitric acid solution are mixed with A liters of a 6.8 mol/L nitric acid solution. The result is 2 liters of a solution that is 3.5 mol/L.

    a. Write an equation relating N, A, and the total number of liters.

    b. The amount of nitric acid in the resulting solution is 2 liters · 3.5 mol/L, or 7 mols. Write an equation relating the amount of nitric acid in the two initial solutions and the resulting solution.

    c. Solve the system represented by your answers in Parts a and b. How many liters of each solution are needed?

Nitric acid is sometimes used in etching as a *mordant*, a solution that eats away unwanted metal.

15. Refer to Example 2.

    a. Suppose you use the first and second equations to eliminate x. What can you multiply each equation by so that when you add the results, you get an equation in terms of y and z only?

    b. Suppose you use the first and third equations to eliminate x. What can you multiply each equation by so that when you add the results, you get an equation in terms of y and z only?

    c. Do the multiplications in Parts a and b and solve the resulting system of 2 equations in y and z. Then substitute to find x.

    d. **True or False** The method used in Parts a–c gives the same solution as in Example 2.

16. Solve the system $\begin{cases} 3x^2 - 2y^2 = 40 \\ 2x^2 + 4y^2 = 48 \end{cases}$ using the linear-combination method.

REVIEW

17. Solve by substitution and check: $\begin{cases} x - 3y + 2z = 18 \\ y + z = -9 \\ z = 1 \end{cases}$. **(Lesson 5-3)**

18. Consider the scale change transformations with matrices $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. **(Lessons 4-7, 4-5, 4-4)**

    a. What is the matrix for the composite of these two transformations?

    b. What type of transformation is represented by the matrix you found in Part a?

In 19 and 20, find the product. (Lesson 4-3)

19. $\begin{bmatrix} 4 & 3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

20. $\begin{bmatrix} 2 & -4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 \\ -0.05 & 0.1 \end{bmatrix}$

21. a. Rewrite the following using scalar multiples.

$$\begin{bmatrix} 4 & -6 & 5.5 \\ 0 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & -6 & 5.5 \\ 0 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & -6 & 5.5 \\ 0 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 7 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$- \begin{bmatrix} 7 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

    b. Find the sum in Part a. **(Lesson 4-2)**

EXPLORATION

22. In a talk to students, mathematician Raymond Smullyan gave this problem as an example of one that can be solved without algebra:

> A family has 12 pets, a combination of cats and dogs. Every night, they give each cat two treats and each dog three treats. If they give 27 treats a night, how many of each kind of pet do they have?

    a. Show how to solve this problem without any algebra and without using trial-and-error.

    b. Show how to answer Question 14 without algebra or trial-and-error.

QY ANSWER

Answers vary. Sample: one partner, $A = 65$, $B = -9$; the other partner, $A = -45$, $B = 2$