Lesson 5-3 Solving Systems Using Substitution

Vocabulary

substitution method

consistent system

inconsistent system

BIG IDEA Some systems of equations in two (or more) variables can be solved by solving one equation for one variable, substituting the expression for that variable into the other equation, and solving the resulting equation.

Mental Math

Find the perimeter of the polygon.

a. a square with side length $x + 1$

b. a right triangle with leg lengths 3 and 5

c. a parallelogram with one side of length 5 and one side of length L

d. a regular n-gon with side length $\frac{\sqrt{3}}{2}$

Tables and graphs can be used to solve systems, but they do not always give exact solutions. You can find exact solutions with paper and pencil or on a CAS by using the **Substitution Property of Equality**: if $a = b$, then a may be substituted for b in any arithmetic or algebraic expression. For instance, if $H = 4V$, you can substitute $4V$ for H in any other expression. The following Activity illustrates how to use the Substitution Property of Equality as part of a **substitution method** to solve a system of two equations.

Activity

MATERIALS CAS

The Policeville Bandits sports stadium seats 60,000 people. Suppose 600 tickets are reserved for the two teams. The home team gets 4 times as many tickets as the visiting team. Let H be the number of tickets for the Bandits and V be the number of tickets for the visiting team. How many tickets does each team receive?

Step 1 Work with a partner. Write a system of equations describing this situation.

Step 2 Have one partner solve one equation in the system for H and use the Substitution Property of Equality to rewrite the other equation in terms of V only. Have the other partner solve one equation in the system for V and then rewrite the other equation in terms of H only.

Step 3 Solve your last equation in Step 2 for the remaining variable. Then substitute your solution in one of the equations in the original system and solve for the other variable. How many tickets does each team receive?

Step 4 Check your answers from Step 3 by comparing them to your partner's answers. Was one substitution easier than the other? Why or why not?

Step 5 A CAS applies the Substitution Property of Equality when the `such that` command is used. Enter the equations into a CAS using the `such that` command, and explain the result.

$h+v=600|h=4\cdot v$

Step 6 Solve the result of Step 5 for *V*.

Step 7 Use the second equation in the system and the such that command to find *H* for your value of *V* from Step 6.

Solving Systems with Three or More Linear Equations

You can also use the substitution method when there are more than two variables and two equations.

GUIDED

Example 1

Able Baker makes a total of 160 dozen regular muffins, mini-muffins, and jumbo muffins for his stores each day. He makes twice as many regular muffins as mini-muffins and 5 times as many jumbo muffins as mini-muffins. How many of each does he make?

Solution Solve by hand using substitution.

Let R = the number of dozens of regular muffins,

M = the number of dozens of mini-muffins, and

J = the number of dozens of jumbo muffins.

Then the system to be solved is $\begin{cases} R + M + J = 160 \\ R = \underline{\ ?\ } \\ J = \underline{\ ?\ } \end{cases}$.

Substitute your expressions for *R* and *J* in the last two equations into the first equation and solve for *M*.

$$\underline{\quad ?\quad} + M + \underline{\quad ?\quad} = 160$$

$$8M = 160$$

$$M = \underline{\ ?\ } \text{ dozen}$$

Substitute to find *R*. $R = \underline{\ ?\ }$ dozen

Substitute to find *J*. $J = \underline{\ ?\ }$ dozen

Carl makes __?__ dozen regular muffins, __?__ dozen mini-muffins, and __?__ dozen jumbo muffins.

Check Does $R = 2M$? __?__ Does $J = 5M$? __?__ Does $R + M + J = 160$? __?__ If the answers are yes, then your solution to the system is correct.

Solving Nonlinear Systems

You can also use the substitution method to solve some systems with nonlinear equations. Write one equation in terms of a single variable, and substitute the expression into the other equation.

Example 2

Solve the system $\begin{cases} y = 4x \\ xy = 36 \end{cases}$.

Solution Substitute $4x$ for y in the second equation and solve for x.

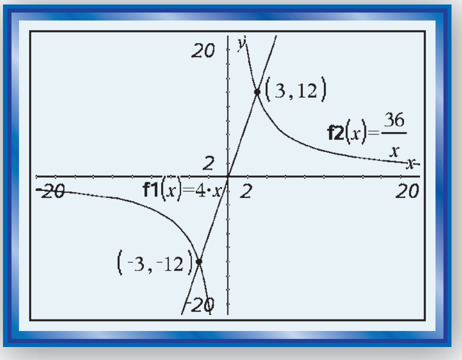
$x(4x) = 36$	Substitute.
$4x^2 = 36$	Simplify.
$x^2 = 9$	Divide both sides by 4.
$x = 3$ or $x = -3$	Take the square root of both sides.

The word *or* means that the solution set is the union of all possible answers. So substitute each value of x into either of the original equations to get two corresponding values of y. We substitute into $y = 4x$.

If $x = 3$, then $y = 4(3) = 12$. If $x = -3$, then $y = 4(-3) = -12$.

The solution set is $\{(3, 12), (-3, -12)\}$.

Check Graph the equations. This calculator display shows the two solutions (3, 12) and (–3, –12). It checks.



Consistent and Inconsistent Systems

A system that has *one or more solutions* is called a **consistent system**. Examples 1 and 2 involve consistent systems. A system that has *no solutions* is called an **inconsistent system**. Example 3 illustrates an inconsistent system.

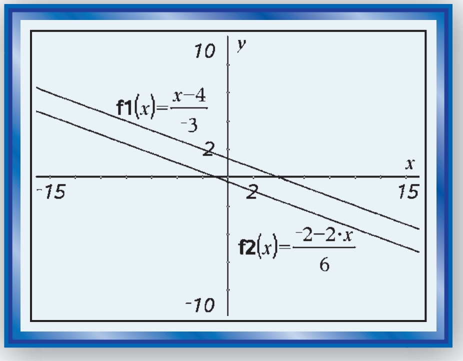
Example 3

Solve the system $\begin{cases} x = 4 - 3y \\ 2x + 6y = -2 \end{cases}$.

Solution 1 Substitute $4 - 3y$ for x in the second equation.

$2(4 - 3y) + 6y = -2$	
$8 - 6y + 6y = -2$	Use the Distributive Property.
$8 = -2$	Add like terms.

We came up with the statement $8 = -2$, which is never true! This false conclusion indicates that what we started with is impossible. That is, **the system has no solutions. In other words, the solution is ∅, the empty set.** A graph of the system shows parallel lines, which supports this conclusion.



Solution 2 Solving the system on a CAS returns `false`, as shown below. This indicates that **the system is inconsistent and has no solution.**

```
solve(x=4−3·y and 2·x+6·y=−2,{x,y})
                                  false
```

Example 4 illustrates a consistent system with infinitely many solutions.

Example 4

Solve the system $\begin{cases} y = -3x + 2 \\ 15x + 5y = 10 \end{cases}$.

Solution 1 Substitute $-3x + 2$ for y in the second equation.

$15x + 5(-3x+2) = 10$ Substitute.

$15x - 15x + 10 = 10$ Use the Distributive Property.

$10 = 10$ Add like terms.

The statement $10 = 10$ is *always* true. **The solutions to the system are all ordered pairs satisfying either equation: $\{(x, y) | y = -3x + 2\}$.** The graph of each equation is the same line. So, this system has an *infinite number of solutions.*

Solution 2 Substituting the first equation into the second on a CAS returns `true`. This means that every solution to the first equation is a solution to the second equation.

```
15·x+5·y=10|y=−3·x+2          true
```

STOP QY

▶ QY

Show that the two equations in Example 4 are equations describing the same line by putting the second one in slope-intercept form.

Questions

COVERING THE IDEAS

1. State the Substitution Property of Equality.

2. Solve $\begin{cases} x + y = 20 \\ x = 2y \end{cases}$ using the `such that` comand on a CAS.

In 3 and 4, refer to Example 1.

3. After the expressions in the second and third equations are substituted into the first equation, how many variables are in this new equation?

4. For a holiday when business was slow, Able Baker cut back the total number of muffins he made to 120 dozen. How many jumbo muffins did he make?

5. Verify that (-3, -12) is a solution to the system of Example 2.

In 6 and 7, solve the system by hand using the substitution method.

6. $\begin{cases} y = 4 - 2x \\ 3x + 4y = 11 \end{cases}$

7. $\begin{cases} x - 2y + 4z = 9 \\ x = 2z + 2 \\ y = -4z \end{cases}$

In 8–10, a system is given.

a. Solve the system using substitution.

b. Tell whether the system is consistent or inconsistent.

c. Graph the system to verify your answers to Parts a and b.

8. $\begin{cases} y = 5x \\ xy = 500 \end{cases}$

9. $\begin{cases} y = \frac{2}{3}x + 3 \\ x = \frac{3}{2}y - 4 \end{cases}$

10. $\begin{cases} y = 2.75x - 1 \\ 11x - 4y = 4 \end{cases}$

11. Two lines in a plane are either the same line, parallel non-intersecting lines, or lines that intersect in a single point. What does this tell you about the possible number of points in the solution set of a system of linear equations in two variables?

APPLYING THE MATHEMATICS

12. Cheap Airways offers to fly the Underdog team to the playoffs for \$250 per person. Comfort Flights offers to take the team for \$1500 plus \$100 per person. Let x = the number of team members on the flight and y = the total cost of the flight. The Underdog coach, who is also a math teacher, wants to analyze the rates to get the best deal.
    a. Write a system of equations to describe the relationship between x and y for the two airlines.
    b. For what number of passengers will the cost be the same for both airlines?
    c. What is this cost?
    d. The Underdog team and staff consists of 23 people. Which airline will cost the least? How much is saved over the other airline?

13. Consider the system $\begin{cases} y = x^2 \\ y = x - 3 \end{cases}$.
    a. Solve the system by substitution.
    b. Tell whether the system is consistent or inconsistent.
    c. Graph the system to verify your answer to Parts a and b.

14. For the system $\begin{cases} xy = 4 \\ y = x \end{cases}$, a student wrote "The solution is $x = -2$ or 2 and $y = -2$ or 2." Explain why this is *not* a correct answer.

15. Sand, gravel, and cement are mixed with water to produce concrete. One mixture has these three components in the extended ratio 2:4:1. For a total of 50 cubic yards of concrete, how much sand, gravel, and cement should be mixed?
    a. Write a system of three equations to describe this situation.
    b. Solve the system to determine how much of each ingredient should be used.

16. Suppose the circumference C of a circle is 1 meter longer than the diameter d of the circle.
    a. Write a system of equations relating C and d.
    b. Solve the system.
    c. Estimate the radius of the circle to the nearest millimeter.

To test the consistency of concrete, inspectors perform a slump test by forming a cone of concrete and measuring how much it slumps due to gravity.

REVIEW

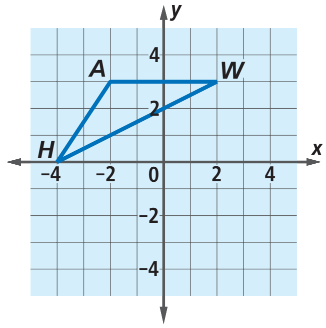
17. The Outdoors Club at Larkchester High School budgeted \$20 for the year to print color flyers for their trips. Printing costs 9 cents per page. **(Lesson 5-1)**

a. Write an inequality relating the number x of flyers the Outdoors Club can print this year.

b. Solve your inequality from Part a.

c. How many flyers can the club print?

18. a. Write a matrix for $\triangle HAW$ at the right.



b. Use matrix multiplication to determine the coordinates of $\triangle H'A'W'$, the reflection image of $\triangle HAW$ over the line $y = x$. **(Lessons 4-6, 4-1)**

19. Give the domain and range of each variation function. **(Lessons 2-6, 2-5, 2-4)**

a. $d = \frac{10}{t}$

b. $y = -2.5x^2$

c. $I = \frac{300}{d^2}$

d. $C = 10.09n$

20. Tell whether each relation is a function. Justify your answer. **(Lesson 1-2)**

a. $g: x \rightarrow 3x^2$

b. $\{(-2, 0), (1, 4), (2, 14), (1, 3), (-2, -5)\}$

c.

s	-1	-0.5	0	0.5	1	1.5	2
r	-12	-6	0	-6	-12	-18	-24

(Treat s as the independent variable.)

d. The relation that maps Blake's age onto his height at that age.

EXPLORATION

21. Consider the system $\begin{cases} y = x^2 \\ y = x + k \end{cases}$. Explore these equations and find the values of k for which this system has

a. two solutions.

b. exactly one solution.

c. no solution.

QY ANSWER

$$15x + 5y = 10$$
$$5y = 10 - 15x$$
$$y = \frac{1}{5}(10 - 15x)$$
$$y = -3x + 2$$