Lesson 5-1

Inequalities and Compound Sentences

Vocabulary

compound sentence
intersection of two sets
double inequality
union of two sets

BIG IDEA The graph of a linear inequality in one variable is a ray either with or without its endpoint.

Compound Sentences

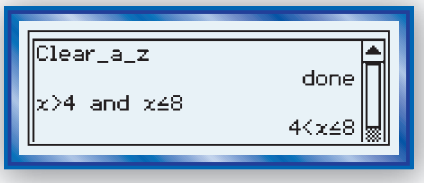
In English, you can use the words *and* and *or* as conjunctions to join two or more clauses. In mathematics, these two words are used in a similar way. A sentence in which two clauses are connected by the word *and* or by the word *or* is called a **compound sentence**. For example, the requirement that a soccer ball's circumference C satisfy the single sentence $68 \text{ cm} \leq C \leq 70 \text{ cm}$ is mathematical shorthand for the compound sentence "$68 \text{ cm} \leq C$ *and* $C \leq 70$ cm."

You can enter compound sentences on a CAS.

Activity

MATERIALS CAS

Step 1 Make a table like the one below. Clear all variables in your CAS memory, then enter each compound sentence into your CAS and record the output.



Compound Sentence Entry	$x > 4$ and $x \leq 8$	$x > 4$ or $x \leq 8$	$a < 5$ and $a < 20$	$a < 5$ or $a < 20$	$n < 0$ and $n > 5$	$n < 0$ or $n > 5$
CAS Output	?	?	?	?	?	?

Step 2 In some cases, the output is only one word, either true or false. What do you notice about situations that lead to these results?

Step 3 Examine each pair of compound sentences that only differ by the connecting word *and* versus *or* (such as "$x > 4$ and $x \leq 8$" and "$x > 4$ or $x \leq 8$"). In each pair, which output includes more values in its solution? Explain why you think this is the case.

Mental Math

Find the union or intersection.

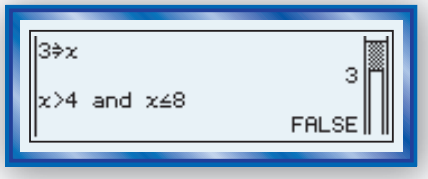
a. $\{1, 4, 7, 11\} \cap \{3, 5, 7, 9, 11\}$

b. the set of all odd integers $\cup$ the set of all even integers

c. the set of all odd integers $\cap$ the set of all even integers

d. the set of all real numbers $\cup$ the set of all integers

Step 4 Store values for variables x, a, and n, as indicated in the table below. Then enter each sentence and record the output.



Stored Variable	$3 \to x$	$3 \to x$	$1 \to a$	$1 \to a$	$2 \to n$	$2 \to n$
Compound Sentence Entry	$x > 4$ and $x \leq 8$	$x > 4$ or $x \leq 8$	$a < 5$ and $a < 20$	$a < 5$ or $a < 20$	$n < 0$ and $n > 5$	$n < 0$ or $n > 5$
CAS Output	?	?	?	?	?	?

Step 5 Summarize the results of Step 4. In general, when are outputs of compound sentences with assigned variables true and when are they false?

STOP **QY1**

▶ QY1

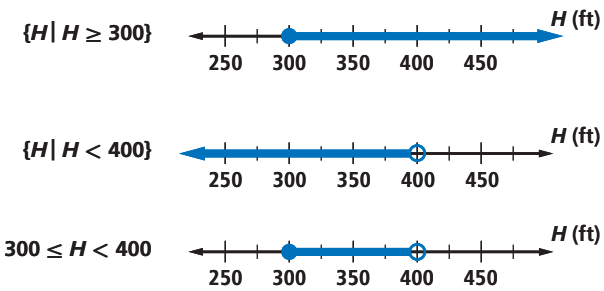
If 8 is stored as x, what output do you expect from your CAS if you enter "$x = 8$ and $x = 5$"? If you enter "$x = 8$ or $x = 5$"?

Compound Sentences Using the Word *and*

The solution set for a compound sentence using *and* consists of the *intersection* of the solution sets of the individual sentences. Recall that the **intersection of two sets** is the set consisting of those values common to both sets. The graph of the intersection consists of the points common to the graphs of the individual sets.

For instance, roller coasters are classified according to maximum height achieved during the ride. A gigacoaster is a roller coaster that reaches a height of at least 300 feet, but less than 400 feet. In symbols, $H \geq 300$ and $H < 400$. The solution sets of the individual sentences, $\{H \mid H \geq 300\}$ and $\{H \mid H < 400\}$, are *intervals*.

The graph of the interval $\{H \mid H \geq 300\}$ is a ray. The graph of $\{H \mid H < 400\}$ is a ray without its endpoint.



The intersection of these two graphs is the graph of required height. The intersection can be described by the single compound sentence $300 \leq H < 400$, which is called a **double inequality** because it has two inequality symbols.

Recall that the symbol used for intersection is $\cap$. So the intersection of sets A and B is written $A \cap B$. In set notation,

$$\{H \mid 300 \leq H < 400\} = \{H \mid H \geq 300\} \cap \{H \mid H < 400\}.$$

This can be read "the set of numbers from 300 up to but not including 400 equals the intersection of the set of numbers greater than or equal to 300 and the set of numbers less than 400."

When describing an interval in English, it is sometimes difficult to know whether endpoints are included. In this book, we use the following language:

"x is *from* 3 to 4" means $3 \le x \le 4$. The endpoints are included.

"x is *between* 3 and 4" means $3 < x < 4$. The endpoints are not included.

This is consistent with the use of the word "between" in geometry. When just one endpoint is included, as in "$3 \le x < 4$," we say "x is 3 or between 3 and 4," or "x is at least 3 and less than 4."

STOP **QY2**

▶ **QY2**

Graph
$\{x \mid 4 < x \text{ and } x \le 8\}$.

Compound Sentences Using the Word *or*

Recall that the **union of two sets** is the set consisting of those values in *either one or both* sets. This meaning of *or* is somewhat different from the everyday meaning *either, but not both*. The symbol often used for union is $\cup$. The union of sets A and B is written $A \cup B$.

The solution set for a compound sentence using *or* consists of the union of the solution sets to the individual sentences. For example, suppose you are the quality-control examiner in a factory that produces cell-phone batteries. For one type of battery, the specification states that battery length L must be 3 cm $\pm$ 0.06 cm. (Recall that the symbol $\pm$ means *plus or minus*.) Batteries produced with a length outside the acceptable range are rejected. L must lie in the interval

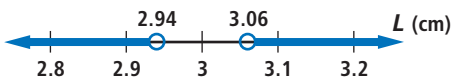
$$3 \text{ cm} - 0.06 \text{ cm} \le L \le 3 \text{ cm} + 0.06 \text{ cm}, \text{ or}$$

$$2.94 \text{ cm} \le L \le 3.06 \text{ cm}.$$

So, the battery will be rejected if $L < 2.94$ or $L > 3.06$.

In set notation, $\{L \mid L < 2.94 \text{ or } L > 3.06\} = \{L \mid L < 2.94\} \cup \{L \mid L > 3.06\}$.

These sets are graphed below.



STOP **QY3**

▶ **QY3**

Graph
$\{A \mid A < 12 \text{ or } A > 65\}$.

Solving Linear Inequalities

Solving a linear inequality is very much like solving a linear equation. The only difference is that when you multiply or divide each side of an inequality by a negative number, you must *reverse* the inequality sign.

Properties of Inequality

For all real numbers a, b, and c:

Addition Property of Inequality
If $a < b$, then $a + c < b + c$.

Multiplication Property of Inequality
If $a < b$ and $c > 0$, then $ac < bc$.
If $a < b$ and $c < 0$, then $ac > bc$.

This Example illustrates an application of linear inequalities.

Example

Penny Nichols has $500 to buy stock options at $17.50 per option. She wants to stop buying options as soon as she has less than $100. (Stock options from a company give an employee the right to buy shares of the company's stock at a designated price.)

a. How many options should she buy before stopping?

b. How much money will she have left?

Solution

a. Let s = the number of options bought. After buying s options Penny will have $500 - 17.50s$ dollars left. She will stop buying stock as soon as $500 - 17.50s < 100$. Solve the inequality.

$$500 - 17.50s < 100$$

$$-17.50s < -400 \quad \text{Subtract 500 from both sides.}$$

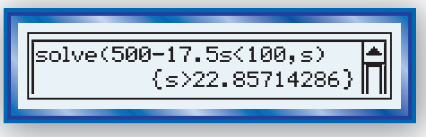
$$s > \frac{-400}{-17.5} \quad \text{Divide both sides by } -17.5. \text{ (Notice that the inequality sign is reversed in this step.)}$$

$$s > 22.86$$

So, Penny will have less than $100 left when $s > 22.86$. Penny should purchase 23 options before stopping.

b. To find how much money Penny has left, we substitute 23 into the expression in Part a. Penny has $\$500 - \$17.50(23) = \$97.50$ left.

Check To check the solution, use the `solve` command on a CAS. It gives the same result.

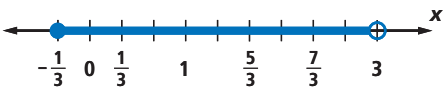


Questions

COVERING THE IDEAS

In 1–3, translate the statement into a mathematical inequality and graph the inequality.

1. To ride the T-Bar in Blizzard Beach at Disney World, you must be under 4' tall.
2. In order to fit snugly into the box, the paperweight has to be between 7.5 and 7.6 cm in diameter.
3. In Louisville, Kentucky, the number of hours of daylight in a day ranges over a year from about 14 hr 48 min to about 9 hr 29 min.
4. A new soccer ball was manufactured for the 2006 World Cup Soccer Tournament. The ball had a circumference between 69 and 69.25 cm, a weight between 441 and 444 g, and a possible weight gain due to water absorption less than 0.1%.
   a. Represent circumference C, weight w, and percent weight gain g due to water absorption as inequalities.
   b. Refer to the first page of the chapter. Does this ball meet the FIFA requirements for circumference and weight?
5. **Multiple Choice** Which inequality represents the statement "The weight L of airplane carry-on luggage may be no more than 25 pounds"?
   A $25 < L$ B $L \leq 25$
   C $25 > L$ D $L \geq 25$
6. Write an inequality for the set of numbers graphed at the right.



7. a. The solution set for a compound sentence using *and* is the ___?___ of the solution sets to the individual sentences.
   b. The solution set for a compound sentence using *or* is the ___?___ of solution sets to the individual sentences.
8. Assume all variables are cleared in a CAS memory.
   a. **Matching** Match each CAS entry at the left with its output at the right.
      i. $x > 1$ and $x < 4$ A `x < 1 or x > 4`
      ii. $x > 1$ or $x < 4$ B `false`
      iii. $x < 1$ or $x > 4$ C `1 < x < 4`
      iv. $x < 1$ and $x > 4$ D `true`
   b. Graph the solution set for each compound sentence.

In 9 and 10, graph the solution set on a number line.

9. $x \leq -7$ or $x > -2$
10. $t \leq -7$ and $t > -2$

11. Three roller coasters and their maximum heights are listed below. Which ones are gigacoasters?

Rollercoaster	Location	Maximum Height (ft)
Kingda Ka	Jackson, NJ	456
Steel Dragon 2000	Nagashima, Japan	318
Superman El Último Escape	Mexico City, Mexico	220

Source: Ultimate Rollercoaster

12. In the Example, suppose Penny had \$750 to spend and stock options cost \$19.00 per share. If she stops buying stock when she first has \$50 or less left, how many shares of stock will she buy?

13. **Fill in the Blanks** When you ___?___ or ___?___ both sides of an inequality by a negative number, you must ___?___ the inequality sign.

In 14 and 15, solve by hand or with a CAS. Graph all solutions.

14. $\frac{2}{3}x \leq \frac{1}{2}$

15. $-5m - 0.4 > 1$

APPLYING THE MATHEMATICS

16. Suppose a small plane weighs 1615 pounds and an average passenger weighs 146 pounds.
    a. Write an expression representing the total weight of the airplane with p people on board.
    b. How many people can the airplane hold if the total weight limit is 2300 pounds?

17. *From The Mixed-up Files of Mrs. Basil E. Frankweiler* (Konigsburg, 1967) tells the story of Claudia and Jamie, who run away from home and live in the Metropolitan Museum of Art in New York City for a short while. Short on money, Jamie argues with Claudia about spending money to ride the bus. He wants to save their \$24.43 for food and other necessities.
    a. In 1967, it cost 20 cents per person to ride the bus in New York City. If Jamie and Claudia both ride the bus, how many rides r could they take together? Write your answer as a double inequality that includes all possible values.
    b. In 2008, it cost \$2.00 to ride the bus in New York City. Write a double inequality that includes all possible values for r in 2008.

Six New York City bus routes have stops for the Metropolitan Museum of Art.

18. Omar solved $x^2 = 5$ and wrote "$x = \sqrt{5}$ and $x = -\sqrt{5}$." What is wrong with Omar's answer?

19. Solve the compound sentence $7m + 2 < 23$ and $8 - 3m \leq 9$ for m. Write the final answer using set-builder notation and provide a graph of the solution set.

REVIEW

In 20 and 21, line m has equation $y = -2x + 8$. Line m' is the image of line m under a translation. Suppose line m' contains the point (4, 8).

20. Find an equation for m'. **(Lessons 4-10, 3-4)**

21. Find a translation that maps m onto m'. **(Lesson 4-10)**

22. Find an equation for the line that passes through the origin and is perpendicular to the line with equation $2x - 3y = -6$. **(Lessons 4-9, 3-4)**

23. **Multiple Choice** Which of the equations below is equivalent to the equation $4x - 3y = 12$? **(Lesson 3-3)**

A $4x + 3y = 12$ **B** $-3x + 4y = 12$

C $8x - 6y = 24$ **D** $-8x + 6y = 2$

24. A *duathlon* is a sporting event involving running and cycling. While training for a duathlon, Gustavo ran for R hours at 10 kilometers per hour and cycled for C hours at 28 kilometers per hour. He went a total of 50 kilometers. **(Lessons 3-3, 3-2)**

a. Write a linear combination equation describing this situation.

b. Graph your equation from Part a with R as the independent variable.

c. What does the point (1.5, 1.25) represent on the graph?

EXPLORATION

25. Data on average daily temperature highs and lows are collected for many cities by month, and published on the Internet. Find January temperature data for three different cities. For each city, write the temperature range of average daily lows and highs as a compound inequality.

26. In ordinary usage, replacing *or* by *and* can dramatically change the meaning of a sentence. For instance, "Give me liberty and give me death" differs from Patrick Henry's famous saying only by that one word. Find examples of other sayings that have a change in meaning when "and" is replaced with "or," or vice versa.

QY ANSWERS

1. false; true

2.

3.