Lesson 7-8
Negative Rational Exponents

▶ **BIG IDEA** The expressions $x^{\frac{m}{n}}$ and $x^{-\frac{m}{n}}$ are reciprocals.

For mammals, a typical relationship between body weight w in kilograms and resting heart rate r in $\frac{\text{beats}}{\text{minute}}$ is modeled by $r = 241w^{-\frac{1}{4}}$. So far in this chapter you have deduced theorems from the postulates of powers that provide meaning for all positive rational exponents. Now you will use the properties of powers to determine the meaning of negative rational exponents, enabling you to work with equations such as this resting heart rate formula.

Mental Math

Calculate.

a. $16 \cdot 256^{\frac{1}{4}}$

b. $16^{\frac{1}{4}} \cdot 256^{\frac{1}{4}}$

c. $(16 \cdot 256)^{\frac{1}{4}}$

Evaluating Powers with Negative Rational Exponents

Consider the power $x^{-\frac{m}{n}}$. Because $-\frac{m}{n} = -1 \cdot m \cdot \frac{1}{n}$, and the factors can be multiplied in any order, you have the choice of taking the reciprocal, the mth power, or the nth root first.

GUIDED

Example 1

Evaluate $\left(\frac{16}{625}\right)^{-\frac{3}{4}}$ in three different ways.

Solution

Method 1: First find the reciprocal of $\frac{16}{625}$, then take the fourth root, and then calculate the third power.

$$\left(\frac{16}{625}\right)^{-\frac{3}{4}} = \left(\left(\left(\frac{16}{625}\right)^{-1\frac{1}{4}}\right)\right)^3 = \left(\left(\underline{\quad?\quad}\right)^{\frac{1}{4}}\right)^3 = \left(\underline{\quad?\quad}\right)^3 = \underline{\quad?\quad}$$

Method 2: First take the fourth root of $\frac{16}{625}$, then calculate the third power, and then find the reciprocal.

$$\left(\frac{16}{625}\right)^{-\frac{3}{4}} = \left(\left(\left(\frac{16}{625}\right)^{\frac{1}{4}}\right)^3\right)^{-1} = \left(\left(\underline{\quad?\quad}\right)^3\right)^{-1} = \left(\underline{\quad?\quad}\right)^{-1} = \underline{\quad?\quad}$$

Method 3: First calculate the third power of $\frac{16}{625}$, then take the fourth root, and then find the reciprocal.

$$\left(\frac{16}{625}\right)^{-\frac{3}{4}} = \left(\left(\left(\frac{16}{625}\right)^3\right)^{\frac{1}{4}}\right)^{-1} = \left(\left(\underline{\quad?\quad}\right)^{\frac{1}{4}}\right)^{-1} = \left(\underline{\quad?\quad}\right)^{-1} = \underline{\quad?\quad}$$

Example 2

Use the formula $r = 241w^{-\frac{1}{4}}$ to estimate a 180-lb person's resting heart rate in beats per minute.

**Solution** In the formula, w is in kilograms, so convert the weight in pounds to kilograms.

$$180 \text{ pounds} \cdot \frac{1 \text{ kilogram}}{2.2 \text{ pounds}} \approx 81.8 \text{ kilograms}$$

Substitute $w = 81.8$ into the formula and evaluate using a calculator.

$$r = 241(81.8)^{-\frac{1}{4}}$$

$$r \approx 80$$

So, a normal resting heart rate for a 180-lb person is about 80 beats per minute. Note, however, that resting heart rate is affected by many factors and that there is a wide range of variability in human heart rates. Any rate in the range $50 \le r \le 100$ might be considered normal for an individual.
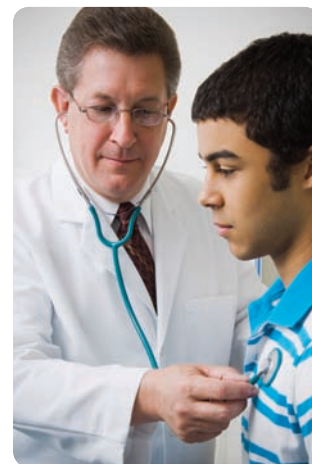
Diet, fitness, and other health-related issues can all affect a person's resting heart rate.

Solving Equations Involving Negative Rational Exponents

The ideas used in Lesson 7-7 to solve equations with positive rational exponents can be used with negative rational exponents as well. For example, if you have a savings goal of G dollars and you already have saved S dollars, then a formula relating S and G to the annual percentage yield r needed over time t in years to reach the savings goal is

$$S = G(1 + r)^{-t}.$$

Example 3

Suppose Jett has $50,000 of the $150,000 he hopes to have for his child's college education in $12\frac{1}{2}$ years. Find the interest rate Jett needs in order to meet his savings goal.

Solution Substitute $G = 150{,}000$, $S = 50{,}000$, and $t = 12\frac{1}{2} = \frac{25}{2}$ into the formula above.

$$150{,}000(1 + r)^{-\frac{25}{2}} = 50{,}000$$

$$(1 + r)^{-\frac{25}{2}} = \frac{1}{3}$$

(continued on next page)

The reciprocal of $-\frac{25}{2}$ is $-\frac{2}{25}$, so raise each side to the $-\frac{2}{25}$ power.

$$\left((1+r)^{-\frac{25}{2}}\right)^{-\frac{2}{25}} = \left(\frac{1}{3}\right)^{-\frac{2}{25}}$$

$$(1+r) = \left(\frac{1}{3}\right)^{-\frac{2}{25}} \approx 1.092$$

$$r \approx 0.092$$

Jett needs to find an investment with an APY of about 9.2% to meet his savings goal.

Check Check by substitution.

Is $150,000(1+r)^{-\frac{25}{2}} \approx 50,000$? $150,000(1.092)^{-\frac{25}{2}} \approx$ 49,924. This is close enough to 50,000 given the estimate, so it checks.

Questions

COVERING THE IDEAS

In 1–3, evaluate without using a calculator.

1. $1000^{-\frac{1}{3}}$

2. $16^{-\frac{3}{2}}$

3. $\left(\frac{625}{81}\right)^{-\frac{3}{4}}$

4. Tell whether or not the expression equals $a^{-\frac{2}{3}}$ for $a > 0$.

   a. $\left(-a^{\frac{1}{3}}\right)^2$

   b. $\dfrac{1}{\left(a^2\right)^{\frac{1}{3}}}$

   c. $\left(\left(a^{-1}\right)^{-2}\right)^{-\frac{1}{3}}$

   d. $-a^{\frac{2}{3}}$

   e. $\left(a^{\frac{1}{3}}\right)^{-2}$

   f. $\dfrac{1}{a^{\frac{2}{3}}}$

In 5–7, approximate to the nearest thousandth.

5. $75^{-\frac{1}{2}}$

6. $20 \cdot 4.61^{-\frac{3}{5}}$

7. $8^{-0.056}$

In 8 and 9, refer to Example 2.

8. An adult mouse weighs about 20 grams. Estimate its heart rate in beats per minute.

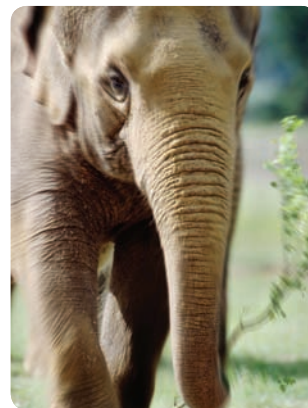9. If the resting heart rate of an Asian elephant is about 30 beats per minute, estimate its weight.

In 10–12, solve without using a calculator.

10. $r^{-\frac{1}{3}} = 3$

11. $w^{-\frac{2}{5}} = 4$

12. $4z^{-\frac{3}{2}} = \frac{1}{16}$

In 2008, the average in-state public 4-year college tuition was over $6000 per year. For private institutions it was over $23,000.

Asian elephants may live as long as 60 years.

13. Solve $\frac{x^{-\frac{2}{3}}}{2} - 11 = 0$ to the nearest thousandth.

In 14 and 15, refer to Example 3.

14. **a.** Rewrite the formula for G in terms of S.

    b. Jett adopts another child and now needs to save $300,000 in 14.5 years. What APY is needed to meet this goal?

15. A $2.00 hamburger in 2007 cost as little as 5¢ in 1923. What yearly percentage growth in the price of the hamburger does this represent?

APPLYING THE MATHEMATICS

In 16–18, True or False. Justify your answer.

16. The value of an expression with a negative exponent is always less than zero.

17. If $y = x^{-\frac{1}{5}}$ and $z = -5$, then $y^z = x$.

18. $\left(125^{-\frac{1}{3}}\right)^0 = 0$

19. Find t if $\left(\frac{99}{100}\right)^{-\frac{1}{2}} = \left(\frac{100}{99}\right)^t$.

20. The amount F of food in grams that a mouse with body mass m must eat daily to maintain its mass is estimated by $F = km^{\frac{2}{3}}$. For a 25-gram mouse, suppose $k = 11.7$. Find the amount of food in grams that this mouse must eat daily to maintain its weight.

In 21–24, rewrite each expression in the form ax^n. Check your answer by substituting a value for x in both the original and the rewritten expression.

21. $\left(x^{-4}\right)^{-\frac{1}{4}}$

22. $\left(36x^{-2}\right)^{-\frac{3}{2}}$

23. $\frac{x}{6x^{-\frac{4}{3}}}\left(3x^{\frac{1}{3}}\right)$

24. $\frac{-\frac{5}{6}x^{-\frac{5}{6}}}{\frac{1}{6}x^{\frac{1}{6}}}$

Baby mice raised in captivity without their mother may be fed milk or formula.

REVIEW

In 25–27, simplify without a calculator. (Lesson 7-7)

25. $100{,}000{,}000{,}000^{\frac{5}{11}}$

26. $12 \cdot 81^{\frac{3}{4}}$

27. $0.064^{\frac{2}{3}}$

28. German astronomer Johannes Kepler (1571–1630) made observations on planetary orbits. His results became known as Kepler's laws of planetary motion. Kepler's third law states that the ratio of the squares of the periods of any two planets equals the ratio of the cubes of their mean distances from the Sun. (The period of a planet is the length of time it takes the planet to go around the Sun.) If the periods of any two planets are t and T, and their mean distances from the sun are d and D, respectively, then $\frac{T^2}{t^2} = \frac{D^3}{d^3}$. Find the ratio of the periods, $\frac{T}{t}$. (**Lesson 7-7**)

29. Use a graph to explain why negative real numbers have no real nth roots when n is even. (**Lessons 7-6, 7-1**)

Johannes Kepler

30. Cassandra is playing a board game where she rolls two 6-sided dice each turn, and then she adds the two rolls to find how many squares she should move forward. In 50 turns, Cassandra moved forward 2 squares five times. Cassandra thinks the dice might be unfair.

    a. What was her relative frequency of rolling two 1s with these dice?

    b. What seems to be the probability of rolling two 1s with these dice? Is Cassandra's concern valid? (**Lesson 7-6**)

31. Simplify $\left(\frac{a}{b}\right)^3 \cdot \frac{a^2 b}{(2ab^2)^2}$. (**Lesson 7-2**)

32. Expand $(2x + 3y)^2$. (**Lesson 6-1**)

EXPLORATION

33. Research the average adult weight and heart rate of a mouse and of an Asian elephant. Compare these data to your answers for Questions 8 and 9. How accurate does the model $r = 241w^{-\frac{1}{4}}$ seem to be for predicting the heart rates of these two mammals? Does the weight of the mammal seem to affect the accuracy of the model?