Lesson 7-7 Positive Rational Exponents

BIG IDEA The expression $x^{\frac{m}{n}}$ is only defined when x is nonnegative and stands for the positive nth root of the mth power of x, or, equivalently, the mth power of the positive nth root of x.

The Meaning of Positive Rational Powers

In Lesson 7-6, you learned that $x^{\frac{1}{n}}$ stands for the positive nth root of x. For instance, $64^{\frac{1}{6}}$ is the positive 6th root of 64. In this lesson, we ask what $x^{\frac{m}{n}}$ means when m and n are positive integers. For instance, what does $64^{\frac{5}{6}}$ mean? The answer can be found by rewriting the fraction $\frac{5}{6}$ and using the Power of a Power Postulate.

$$64^{\frac{5}{6}} = 64^{\left(\frac{1}{6}\cdot 5\right)} \qquad \frac{5}{6} = \frac{1}{6}\cdot 5$$

$$= \left(64^{\frac{1}{6}}\right)^5 \qquad \text{Power of a Power Postulate}$$

Thus, $64^{\frac{5}{6}}$ is the 5th power of the positive 6th root of 64. With this interpretation, $64^{\frac{5}{6}} = \left(64^{\frac{1}{6}}\right)^5 = 2^5 = 32$.

Notice also that $64^{\frac{5}{6}} = 64^{\left(5\cdot\frac{1}{6}\right)} = \left(64^5\right)^{\frac{1}{6}}$.

So $64^{\frac{5}{6}}$ is also the positive 6th root of the 5th power of 64. With this interpretation, $64^{\frac{5}{6}}$ can be simplified as follows:

$$64^{\frac{5}{6}} = \left(64^5\right)^{\frac{1}{6}} = (1{,}073{,}741{,}824)^{\frac{1}{6}} = 32.$$

In general, when an exponent is a simple fraction, the numerator is the power and the denominator is the root. The proof is a generalization of the argument above.

Rational Exponent Theorem

For any nonnegative real number x and positive integers m and n,
$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$, the mth power of the positive nth root of x, and
$x^{\frac{m}{n}} = \left(x^m\right)^{\frac{1}{n}}$, the positive nth root of the mth power of x.

Mental Math

Tyra is painting her bedroom. The room is 8 feet by 12 feet and has 10-foot ceilings. One quart of paint covers about 90 square feet. How many quarts of paint does Tyra need if she wants to paint

a. only the largest wall?

b. all four walls, not including a 24-square-foot window and a 28-square-foot doorway?

Proof $x^{\frac{m}{n}} = x^{\frac{1}{n} \cdot m}$ $\quad \frac{m}{n} = \frac{1}{n} \cdot m$

$= \left(x^{\frac{1}{n}}\right)^m$ $\quad$ Power of a Power Postulate

Also, $x^{\frac{m}{n}} = x^{m \cdot \frac{1}{n}}$ $\quad \frac{m}{n} = m \cdot \frac{1}{n}$

$= (x^m)^{\frac{1}{n}}.$ $\quad$ Power of a Power Postulate

To simplify an expression with a rational exponent you can find powers first or roots first. By hand, it is usually easier to find the root first because you end up working with smaller numbers and fewer digits. With a calculator, you can work more directly.

STOP **QY1**

Properties of Positive Rational Exponents

The exponent gives information about the value of each power relative to the base.

Activity 1

Step 1 Complete the following chart for powers of 8.

Rational Power	8^0	$8^{\frac{1}{3}}$	$8^{\frac{1}{2}}$	$8^{\frac{3}{4}}$	8^1	$8^{\frac{5}{4}}$	$8^{\frac{3}{2}}$	$8^{\frac{5}{3}}$	8^2
Decimal Power	8^0	$8^{0.\overline{3}}$	?	$8^{0.75}$	8^1	?	?	?	8^2
Value	1	?	?	?	8	?	?	?	64

Step 2 When the exponent n is between 0 and 1, between what values is 8^n?

Step 3 When the exponent n is between 1 and 2, between what values is 8^n?

Step 4 As the exponent n gets larger, what happens to the value of 8^n?

Step 5 Without calculating, predict between what values $8^{\frac{7}{4}}$ will be. Explain your answer.

In general, if the base is greater than 1, then the greater the exponent, the greater the value. Thus, even without calculating, you can conclude that $8^{\frac{7}{5}}$ is between 8 and 64 because $\frac{7}{5} = 1.4$ is between 1 and 2. In Question 19, you are asked to examine what happens if the base is less than 1 but greater than zero.

STOP **QY2**

▸ QY1

a. Estimate $7^{\frac{2}{5}}$ by using the key sequence 7^(2÷5).

b. Is $7^{\frac{2}{5}}$ greater than, equal to, or less than $7^{0.4}$?

▸ QY2

Which is smaller, $47^{\frac{3}{4}}$ or $47^{0.78}$?

The properties of powers in Lesson 7-2 hold for all positive exponents.

Example 1

Suppose $y \geq 0$. Simplify $\left(64y^9\right)^{\frac{4}{3}}$.

Solution

$$\left(64y^9\right)^{\frac{4}{3}} = (64)^{\frac{4}{3}}\left(y^9\right)^{\frac{4}{3}} \qquad \text{Power of a Product Postulate}$$

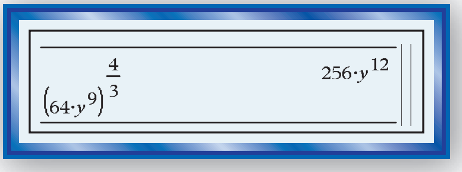
$$= 64^{\frac{4}{3}} \cdot y^{\frac{36}{3}} \qquad \text{Power of a Power Postulate}$$

$$= \left(64^{\frac{1}{3}}\right)^4 \cdot y^{12} \qquad \text{Rational Exponent Theorem}$$

$$= 4^4 \cdot y^{12} \qquad \tfrac{1}{n}\text{ Exponent Theorem}$$

$$= 256y^{12} \qquad \text{Arithmetic}$$

Check This CAS display shows that $\left(64y^9\right)^{\frac{4}{3}} = 256y^{12}$.



Solving Equations with Positive Rational Exponents

Properties of powers can be used to solve equations with positive rational exponents. To solve an equation of the form $x^{\frac{m}{n}} = k$, raise each side of the equation to the $\frac{n}{m}$ power. This can be done because in general, if $a = b$, then $a^n = b^n$.

Example 2

Solve $x^{\frac{3}{4}} = 7.2$ for x.

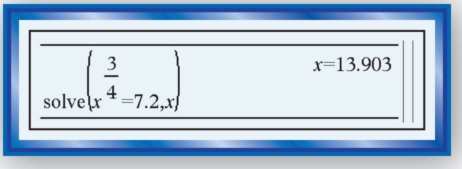
Solution

$$\left(x^{\frac{3}{4}}\right)^{\frac{4}{3}} = (7.2)^{\frac{4}{3}} \qquad \text{Raise both sides to the } \tfrac{4}{3} \text{ power.}$$

$$x^1 = (7.2)^{\frac{4}{3}} \qquad \text{Power of a Power Postulate}$$

$$x \approx 13.9 \qquad \text{Arithmetic}$$

Check Solve on a CAS as shown at the right.



STOP **QY3**

QY3

Find a solution to the nearest thousandth for $x^{\frac{8}{5}} = 50$.

Applications of Rational Exponents

Rational exponents have many applications, including growth situations, investments, and radioactive decay.

Example 3

The base price of a convertible sports car was $4037 in 1963 and $51,390 in 2006. What was the average annual percent increase in price over these years?

A 1963 Austin-Healey Model 3000

Solution Let p_t be the price of the sports car where $t = 1$ represents 1963 and $t = 44$ represents 2006. Then $p_1 = 4037$ and $p_{44} = 51{,}390$. Use a geometric sequence P with $p_t = p_1 r^{t-1}$ to model this situation.

$p_{44} = p_1 r^{44-1}$	Geometric sequence model
$51{,}390 = 4037r^{43}$	Substitution
$12.730 \approx r^{43}$	Divide both sides by 4037.
$(12.730)^{\frac{1}{43}} \approx r$	Raise each side to the $\frac{1}{43}$ power.
$1.0609 \approx r$	Arithmetic

The constant ratio r represents the previous year's price plus the increase of $0.0609 = 6.09\%$. So, the price of the car increased by about 6.09% per year from 1963 to 2006.

Why Don't We Use Rational Exponents with Negative Bases?

Difficulties arise when working with noninteger rational exponents when the base is negative.

Activity 2

MATERIALS CAS (optional)

Set a CAS or graphing calculator to real-number mode and use the standard window.

Step 1 a. Do you think the graph of $y = \left(x^{\frac{1}{6}}\right)^2$ will be the same as the graph of $y = x^{\frac{1}{3}}$? Why or why not?

b. Graph $y = x^{\frac{1}{3}}$.

c. Clear the window and graph $y = \left(x^{\frac{1}{6}}\right)^2$. Was your prediction correct?

Step 2 **a.** Repeat Step 1 with $y = (x^{10})^{\frac{1}{2}}$ and $y = x^5$.

b. Predict what the graph of $y = \left(x^{\frac{1}{2}}\right)^{10}$ will look like. Clear the graph screen and check your prediction by graphing $y = \left(x^{\frac{1}{2}}\right)^{10}$.

Step 3 Predict what the graph of $y = x^{0.5} \cdot x^{0.5}$ will look like. Then clear the screen and graph the equation to check your prediction.

Step 4 Based on these examples, what properties of powers do not hold for noninteger rational exponents with negative bases? Compare your results with the others in your class and discuss any differences.

The results of Activity 2 illustrate that to keep the properties of powers valid, the Rational Exponent Theorem requires that the base be a nonnegative real number.

Questions

COVERING THE IDEAS

In 1 and 2, write as a power of *x*.

1. the 3rd power of the 7th root of x
2. the fifth root of the square of x
3. a. Rewrite $10{,}000{,}000^{\frac{4}{7}}$ in two ways as a power of a power of 10,000,000.
   b. Which way is easier to calculate mentally?
   c. Calculate $10{,}000{,}000^{\frac{4}{7}}$.

In 4–6, simplify without a calculator.

4. $16^{\frac{3}{4}}$
5. $8^{\frac{5}{3}}$
6. $49^{\frac{3}{2}}$

In 7–9, evaluate with a calculator.

7. $243^{\frac{3}{5}}$
8. $(1.331)^{\frac{4}{3}}$
9. $169^{1.50}$

In 10–12, suppose that the value of each variable is nonnegative. Simplify.

10. $(27x^9)^{\frac{2}{3}}$
11. $C^{\frac{3}{4}} \cdot C^{\frac{8}{6}}$
12. $\frac{7}{8}y^{\frac{7}{8}} \cdot \frac{8}{7}y^{\frac{8}{7}}$

In 13–15, solve.

13. $R^{\frac{2}{5}} = 100$
14. $j^{\frac{4}{3}} = 3^4$
15. $s^{\frac{4}{7}} - 10 = 0$

16. **True or False** For all real numbers x, $x^{\frac{4}{14}} = x^{\frac{2}{7}}$. Explain your answer.

Fill in the Blank In 17 and 18, complete with >, <, or =.

17. When $y > 1$, $y^{\frac{3}{4}}$ ___?___ $y^{\frac{4}{3}}$.
18. When $0 < z < 1$, $z^{\frac{5}{6}}$ ___?___ $z^{\frac{3}{4}}$.

19. Refer to Activity 1.

a. Complete the table at the right.

b. As the exponent n gets larger, what happens to the value of $\left(\frac{1}{4}\right)^n$?

c. Without calculating, predict values that $\left(\frac{1}{4}\right)^{\frac{5}{3}}$ will be between. Explain your answer.

d. **Fill in the Blank** Complete the following statement: If the base is smaller than 1 (but greater than zero), as the exponent gets larger, the value gets __?__.

Rational Power	Decimal Power	Value
$\left(\frac{1}{4}\right)^0$	$\left(\frac{1}{4}\right)^0$	1
$\left(\frac{1}{4}\right)^{\frac{1}{4}}$	?	?
$\left(\frac{1}{4}\right)^{\frac{1}{2}}$	?	?
$\left(\frac{1}{4}\right)^{\frac{2}{3}}$	?	?
$\left(\frac{1}{4}\right)^1$	$\left(\frac{1}{4}\right)^1$	0.25
$\left(\frac{1}{4}\right)^{\frac{4}{3}}$	?	?
$\left(\frac{1}{4}\right)^{\frac{3}{2}}$	?	?
$\left(\frac{1}{4}\right)^{\frac{7}{4}}$	?	?
$\left(\frac{1}{4}\right)^2$	$\left(\frac{1}{4}\right)^2$	0.063

20. Refer to Example 3. Estimate what the base price of the sports car was in 2007.

APPLYING THE MATHEMATICS

21. Suppose n is a positive integer. Write $32^{\frac{n}{5}}$ as an integer power of an integer.

22. **Fill in the Blanks** This question gives one reason why rational exponents are used only with positive bases.

a. If $(-27)^{\frac{1}{3}}$ were to equal the cube root of -27, then $(-27)^{\frac{1}{3}} =$ __?__.

b. If $(-27)^{\frac{2}{6}}$ follows the Rational Exponent Theorem, then $(-27)^{\frac{2}{6}} = \left((-27)^2\right)^{\frac{1}{6}} =$ __?__; and $(-27)^{\frac{2}{6}} = \left(-27^{\frac{1}{6}}\right)^2 =$ __?__.

c. In this question, does $(-27)^{\frac{1}{3}} = (-27)^{\frac{2}{6}}$?

d. Check the answers to this question on a CAS. Is there a difference?

e. Reread the Rational Exponent Theorem's hypothesis and explain why the base is restricted to nonnegative numbers.

In 23–25, apply the Power of a Quotient Theorem, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, to simplify.

23. $\left(\frac{16}{81}\right)^{\frac{3}{4}}$ 24. $\left(\frac{125}{27}\right)^{\frac{2}{3}}$ 25. $(0.00032)^{\frac{3}{5}}$

26. The diameter D of the base of a tree of a given species roughly varies directly with the $\frac{3}{2}$ power of its height h.

a. Suppose a young sequoia 6 meters tall has a base diameter of 20 centimeters. Find the constant of variation.

b. The most massive living tree is a California sequoia called General Sherman. Its base diameter is about 11.1 meters. According to the variation in Part a, about how tall is General Sherman?

General Sherman

27. Recall from Lesson 2-3 the ancient rodent *Phoberomys*. There you used the direct variation equation $w = kd^3$ to estimate the weight of the rodent given its femur diameter d. This equation actually gives an overestimate of *Phoberomys*'s weight. Scientists have concluded that a better model is $w = kd^{2.5}$. Use this newer model to estimate how the weight of the rodent compares to the weight of a modern guinea pig if the *Phoberomys's* femur diameter is 18 times that of the guinea pig. How does this compare to your estimate in Lesson 2-3?

REVIEW

28. In Byzantine music theory, an octave is divided into 72 notes. What is the ratio of the frequency of a note to the frequency of the note below it in this system? Assume that the ratios between consecutive notes are equal. **(Lesson 7-6)**

29. Verify that $-1 - \sqrt{3}\,i$ is a third root of 8. **(Lessons 7-6, 6-9)**

30. **Fill in the Blanks** Write <, =, or > in each blank. Suppose the geometric sequence with formula $g_n = ar^{n-1}$ has $g_1 < 0$ and $g_2 > 0$. Then a __?__ 0 and r __?__ 0. **(Lesson 7-5)**

31. If you graph the function $y = x^n$ for $x \le 0$, what is the range when n is

 a. even?
 b. odd? **(Lesson 7-1)**

32. Suppose $f(x) = 3x^2 + 5$. Write an expression for $f(8 - 2x)$ in the standard form of a quadratic. **(Lessons 6-1, 1-3)**

33. Find a transformation T such that $T \circ (S_3 \circ S_{0.5,4}) = I$. **(Lessons 4-7, 4-5, 4-4)**

EXPLORATION

34. Sandwiching irrational powers, such as $x^{\sqrt{2}}$ or x^{π}, between two close rational powers is one way to give a meaning to irrational powers of nonnegative numbers. Set a CAS to approximate mode so your answers will be decimals.

 a. Evaluate several irrational powers such as $5^{\sqrt{2}}$, $7^{\sqrt{6}}$, and $\sqrt{2}^{\sqrt{3}}$ and some others of your choosing.

 b. Recall that $1.41 < \sqrt{2} < 1.42$. Show that $5^{1.41} < 5^{\sqrt{2}} < 5^{1.42}$.

 c. Given that $\sqrt{6} \approx 2.4494$, is it true that $7^{\sqrt{6}}$ is between $7^{2.449}$ and $7^{2.450}$?

QY ANSWERS

1. **a.** ≈ 2.1779
 b. equal

2. $47^{\frac{3}{4}}$

3. 11.531