Lesson

7-6 *n*th Roots

Vocabulary

cube root

*n*th root

BIG IDEA If y is the nth power of x, then x is an nth root of y. Real numbers may have 0, 1, or 2 real nth roots.

Mental Math

Determine whether the discriminant of the quadratic equation is positive, negative, or zero. Then, tell how many real roots the equation has.

a. $5x^2 + 2x - 1 = 0$

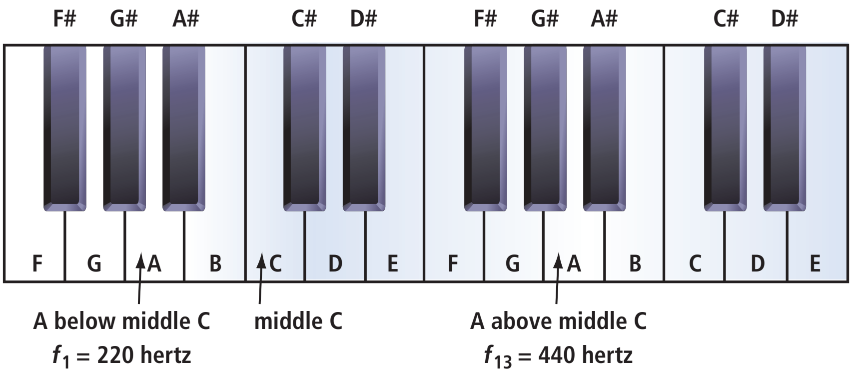
b. $-t^2 - 4t + 1 = 0$

c. $2h^2 + 3h + 2 = 0$

d. $s^2 - 6s + 9 = 0$

Geometric Sequences in Music

A piano tuner adjusts the tension on the strings of a piano so the notes are at the proper pitch. Pitch depends on frequency, measured in hertz or cycles per second. It is common today to tune the A above middle C to 440 hertz. Pythagoras and his followers discovered that a note has exactly half the frequency of the note one octave higher. Thus, the A below middle C is tuned to a frequency of 220 hertz. In most music today, an octave is divided into twelve notes as shown below.



In order for a musical piece to sound much the same in any key, notes in the scale are tuned so that ratios of the frequencies of consecutive notes are equal. To find these frequencies, let f_1 = the frequency of the A below middle C and f_n = the frequency of the nth note in the scale. Then f_{n+1} is the frequency of the next higher note. The frequency of A *above* middle C is f_{13} because there are 12 notes in each octave.

Let r = the ratio of the frequencies of consecutive notes. Then for all integers $n \geq 1$,

$$\frac{f_{n+1}}{f_n} = r.$$

Multiply both sides of this equation by f_n.

$$f_{n+1} = rf_n, \text{ for all integers } n \geq 1$$

Together with the known value $f_1 = 220$, this is a recursive formula for a geometric sequence. It indicates that the frequencies of consecutive notes on a piano (when in tune) are the elements of a geometric sequence. The first term is $f_1 = 220$ and the 13th term of this sequence is $f_{13} = 440$.

From the explicit formula for the nth term of a geometric sequence, substituting 13 for n,

$$f_{13} = 220r^{13-1} = 220r^{12}.$$

To find r, substitute 440 for f_{13}. $\quad 440 = 220r^{12}$

Divide both sides by 220. $\quad 2 = r^{12}$

The ratio of the frequencies of consecutive in-tune keys on a piano is called a *12th root* of 2.

A piano being tuned

What Is an *n*th Root?

Recall that x is a square root of t if and only if $x^2 = t$. Similarly, x is a **cube root** of t if and only if $x^3 = t$. For instance, 4 is a cube root of 64 because $4^3 = 64$. Square roots and cube roots are special cases of the following more general idea.

Definition of *n*th Root

Let n be an integer greater than 1. Then b is an ***n*th root** of x if and only if $b^n = x$.

For example, $-\frac{1}{3}$ is a 5th root of $-\frac{1}{243}$ because $\left(-\frac{1}{3}\right)^5 = -\frac{1}{243}$. There are no special names for nth roots other than *square roots* (when $n = 2$) and *cube roots* (when $n = 3$). Other nth roots are called *fourth roots*, *fifth roots*, and so on. In the piano-tuning example above, $r^{12} = 2$. This is why r is a 12th root of 2. And because $(-r)^{12} = r^{12}$ for all real numbers r, there is a negative number that is also a 12th root of 2.

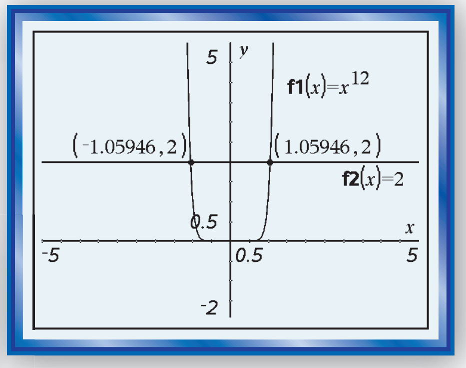
Example 1

Approximate the real 12th roots of 2 to find the ratio r of the frequencies of consecutive in-tune notes, to the nearest hundred-thousandth.

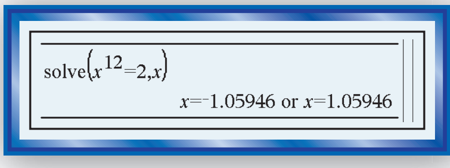
Solution The real 12th roots of 2 are the real solutions to $x^{12} = 2$. So they are the x-coordinates of the points of intersection of $y = x^{12}$ and $y = 2$. Graph these functions using a graphing utility.

(continued on next page)

From the graph at the right, you can see that that there are two real 12th roots of 2. This calculator shows that the real 12th roots of 2 are approximately –1.05946 and 1.05946. Only the positive root has meaning in this context. So, the ratio r of the frequencies of consecutive in-tune keys is about 1.05946.



Check Use a CAS in real-number mode to solve $x^{12} = 2$.

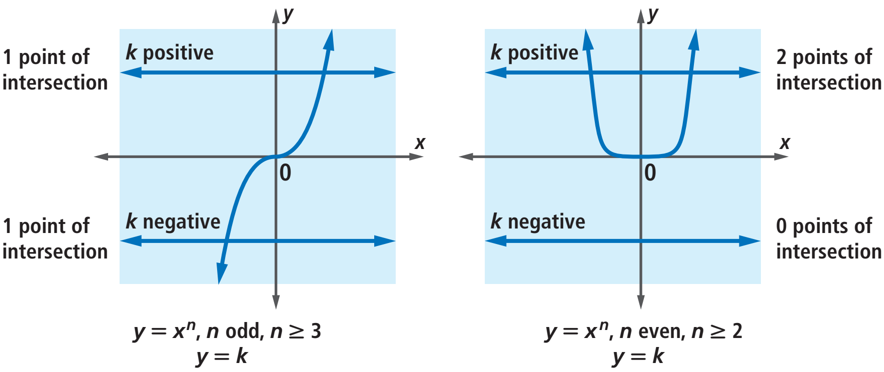


The positive value $x = 1.05946$ checks.

STOP **QY1**

How Many Real *n*th Roots Does a Real Number Have?

The number of real *n*th roots of a real number *k* is the number of points of intersection of the line $y = k$ with the power function $y = x^n$. The number of intersections is determined by whether the value of *n* is odd or even, and whether the real number *k* is positive or negative, as illustrated in the graphs below.



Each intersection determines an *n*th root of *k*, suggesting the following theorem.

Number of Real Roots Theorem

Every positive real number has:	2 real *n*th roots, when *n* is even. 1 real *n*th root, when *n* is odd.
Every negative real number has:	0 real *n*th roots, when *n* is even. 1 real *n*th root, when *n* is odd.
Zero has:	1 real *n*th root.

QY1

If the A below middle C is tuned to 220 cycles per second, find the frequency of the D above middle C, the 5th note above this A.

For instance, –4 has no real square roots, 4th roots, or 6th roots. It has one real cube root, one real 5th root, and one real 7th root.

Roots and Powers

One reason that powers are important is that the positive *n*th root of a positive number *x* is a power of *x*. The power is directly related to the root: the square root is the $\frac{1}{2}$ power; the cube root is the $\frac{1}{3}$ power; and so on. The general property is stated in the following theorem and can be proved using properties of powers you already know.

$\frac{1}{n}$ Exponent Theorem

When $x \geq 0$ and n is an integer greater than 1, $x^{\frac{1}{n}}$ is an nth root of x.

Proof By the definition of nth root, b is an nth root of x if and only if $b^n = x$.

Suppose $b = x^{\frac{1}{n}}$.

Then $b^n = \left(x^{\frac{1}{n}}\right)^n$ Raise both sides to the nth power.

$= x^{\left(\frac{1}{n} \cdot n\right)}$ Power of a Power Postulate

$= x^1$

$= x.$

Thus, $x^{\frac{1}{n}}$ is an nth root of x.

Mathematicians could decide to let the symbol $x^{\frac{1}{n}}$ be any of the nth roots of x. However, to ensure that $x^{\frac{1}{n}}$ has exactly one value, we restrict the base x to be a nonnegative real number and let $x^{\frac{1}{n}}$ stand for the ***unique nonnegative*** nth root. For example, $x^{\frac{1}{2}}$ is the positive square root of x, and $2^{\frac{1}{4}}$ is the positive fourth root of 2.

Pay close attention to parentheses when applying the $\frac{1}{n}$ Exponent Theorem. Do not consider negative bases with these exponents because there are properties of powers that do not apply to them. You may use your calculator to find the nth roots of a nonnegative number b by entering `b^(1÷n)`, as in the next Example. You will read more about how your calculator interprets numbers like $(-8)^{\frac{1}{3}}$ in Lesson 7-7.

GUIDED

Example 2

Approximate all real solutions to $x^5 = 77$ to the nearest thousandth.

(continued on next page)

Solution By the definition of *n*th root, the real solutions of $x^5 = 77$ are the real __?__ roots of __?__. So, one solution is $x = 77^{\underline{?}}$.

Enter 77^(1÷5) into a calculator. The result, to the nearest thousandth is __?__. So, $77^{\frac{1}{5}} \approx$ __?__. By the Number of Real Roots Theorem, $x \approx$ __?__ is the only real solution because 5 is a(n) __?__ number.

STOP QY2

> **QY2**
>
> Approximate the real solution to $4n^3 = -100$ to the nearest hundredth.

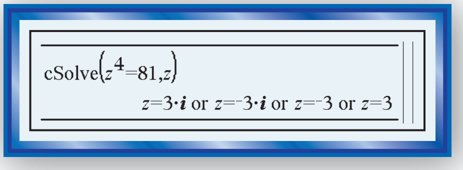
Nonreal *n*th Roots

Some of the *n*th roots of a real number are not real.

Example 3

Use the cSolve or Solve command on a CAS to find all 4th roots of 81.

Solution Set a CAS to complex-number mode. z is a 4th root of 81 if and only if $z^4 = 81$. So solve $z^4 = 81$. One CAS shows four solutions: $z = 3i, -3i, -3$, or 3.



Check Verify that each solution satisfies $z^4 = 81$.

$$\begin{aligned} 3^4 &= 3 \cdot 3 \cdot 3 \cdot 3 = 81 \\ (-3)^4 &= (-3)(-3)(-3)(-3) = 81 \\ (3i)^4 &= 3^4 \cdot i^4 = 81 \cdot 1 = 81 \\ (-3i)^4 &= (-3)^4 \cdot i^4 = 81 \cdot 1 = 81 \end{aligned}$$

So, 3, −3, 3i, and −3i are fourth roots of 81.

It can be proved that every nonzero real number has *n* distinct *n*th roots. In a later mathematics course you will learn how to find them.

Questions

COVERING THE IDEAS

In 1–3, refer to the discussion of the musical scale at the start of this lesson.

1. What is the exact ratio of the frequencies of consecutive notes in this scale?
2. What is the frequency of E above the middle C, to the nearest hundredth?
3. In 1879, the Steinway & Sons piano manufacturing company tuned the note A above middle C to 457.2 hertz. Given this frequency, what is the frequency of A below middle C?

4. **Fill in the Blank** Let n be an integer greater than 1. Then x is an nth root of d if and only if ___?___.

In 5–7, find the positive real root without a calculator.

5. fourth root of 81
6. cube root of 64
7. fifth root of 32

In 8–10, suppose the graphs of $y = x^n$ and $y = k$ are drawn on the same set of axes.

8. How are the points of intersection related to the nth roots of k?
9. If n is odd and $k > 0$, at how many points do the graphs intersect?
10. If $k < 0$ and the graphs do not intersect, is n even or odd?
11. Approximate all real solutions of $m^7 = 5$ to the nearest hundredth.

In 12 and 13, use a CAS to find all solutions of the equation.

12. $x^3 = 1331$
13. $x^4 = 20{,}736$
14. a. Find two solutions to $x^4 = \frac{81}{625}$ in your head.
    b. Your answers to Part a are nth roots of $\frac{81}{625}$. What is n?
15. **True or False** $x^{\frac{1}{n}}$ is defined for $x \geq 0$ and any real number n.
16. Explain why $(-7)^4 = 2401$, but $2401^{\frac{1}{4}} \neq -7$.
17. **Multiple Choice** Which of the following is *not* a 4th root of 6561? Justify your answer.
    A $6561^{\frac{1}{4}}$ B -9 C $9i$ D $6561^{-\frac{1}{4}}$

APPLYING THE MATHEMATICS

18. In the nautilus shell discussed on the first page of the chapter, the eleventh chamber is about twice as long as the smallest chamber. Assuming that the ratio r of the lengths of two adjacent chambers is constant, estimate r to the nearest hundredth.

In 19 and 20, use the Compound Interest Formula and nth roots to determine a rate of growth.

19. In an old animated show, the main character travels to the future and finds out he is a billionaire. Assume that he started with \$563 in his account and after 248 years the account held \$1,000,000,000. If the interest rate is constant for the entire time period, what is the annual percentage yield?
20. Suppose a house was purchased in 1990 for \$100,000.
    a. If its value is increasing by r% each year, what is its value n years after 1990?
    b. If its value in 2005 was \$180,000, estimate r to the nearest tenth.
21. Give a value of x that makes the inequality $x^{\frac{1}{2}} > x$ true.

22. a. Verify that $i\sqrt{3}$ is a 4th root of 9.
b. Why is $9^{\frac{1}{4}} \neq i\sqrt{3}$?

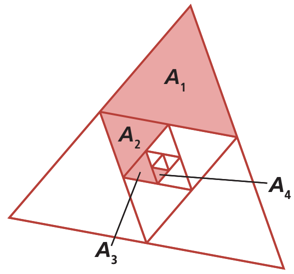
Fill in the Blank In 23 and 24, which symbol, <, =, or >, makes it a true statement?

23. $(16.1)^{\frac{1}{4}}$ ___?___ 2

24. $0.25^{\frac{1}{2}}$ ___?___ $0.25^{\frac{1}{3}}$

REVIEW

25. A Baravelle spiral is shown at the right. The ratios of the areas of consecutive shaded regions are equal. Suppose the area of region $A_1 = 32$ square units and the area of region $A_4 = \frac{1}{2}$ square unit. What is the ratio of consecutive areas? **(Lesson 7-5)**



In 26–28, determine whether the sequence is geometric. If it is, write an explicit formula for its *n*th term. (Lesson 7-5)

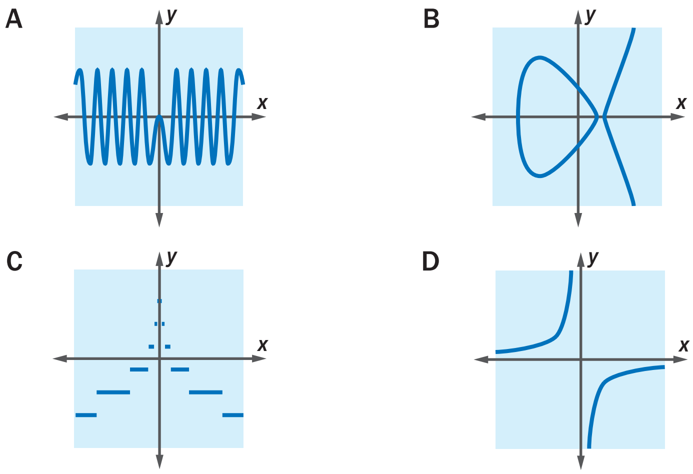
26. 4, 9, 16, 25, …

27. 5, 10, 20, 40, …

28. $6, -2, \frac{2}{3}, -\frac{2}{9}, \ldots$

29. **True or False** If two savings accounts have the same published interest rate, then the one with more compoundings per year will always have a higher annual percentage yield. **(Lesson 7-4)**

30. **Multiple Choice** Which of the following graphs show y as a function of x? There may be more than one correct answer. **(Lesson 1-4)**



EXPLORATION

31. Use a CAS to find all the complex solutions to $x^8 = 1$.
a. Choose any two solutions and multiply them. What do you notice?
b. Add all of the solutions. What do you notice?
c. Repeat Parts *a* and *b* for the solutions for $x^9 = 1$. Make two conjectures about your findings.

QY ANSWERS

1. about 293.660 Hz
2. $n \approx -2.92$