Lesson 7-5 Geometric Sequences

Vocabulary

geometric sequence, exponential sequence

constant multiplier

constant ratio

BIG IDEA Geometric sequences are generated by multiplying each term by a constant to get the next term, just as arithmetic sequences are generated by adding a constant to each term to get the next.

Mental Math

Tell whether the sequence could be arithmetic. If it could be, give the constant difference.

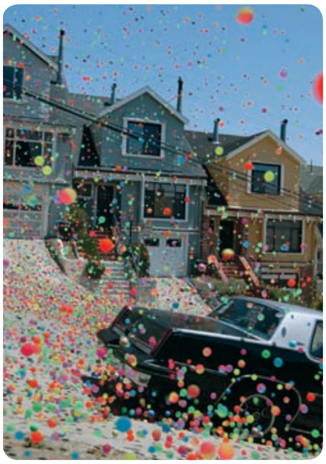
a. $1\frac{5}{6}, \frac{5}{6}, -\frac{5}{6}, -1\frac{5}{6}, -2\frac{5}{6}, \ldots$

b. 17, 28, 39, 50, ...

c. 1.017, 0.917, 0.817, 0.717, ...

d. 2, −4, 6, −8, ...

In 2005, a major television manufacturer filmed an advertisement featuring 250,000 balls bouncing down the hills of San Francisco. The maximum height of a ball after each bounce can be modeled by a *geometric sequence*.



Recursive Formulas for Geometric Sequences

Recall that, in an arithmetic (linear) sequence, each term after the first is found by adding a constant difference to the previous term. If, instead, each term after the first is found by *multiplying* the previous term by a constant, then a **geometric** (or **exponential**) **sequence** is formed.

The constant in a geometric sequence is called a **constant multiplier**. For instance, the geometric sequence with first term 30 and constant multiplier 2 is

$$30, 60, 120, 240, 480, 960, \ldots .$$

Replace 30 by g_1 and 2 by r and you have the general form for a geometric sequence.

Recursive Formula for a Geometric Sequence

Let r be a nonzero constant. The sequence g defined by the recursive formula

$$\begin{cases} g_1 = x \\ g_n = rg_{n-1}, \text{ for integers } n \geq 2 \end{cases}$$

is the geometric, or exponential, sequence with first term x and constant multiplier r.

Solving the sentence $g_n = rg_{n-1}$ for r yields $\frac{g_n}{g_{n-1}} = r$. This indicates that in a geometric sequence, the ratio of successive terms is constant. For this reason, the constant multiplier r is also called the **constant ratio**.

Alternatively, you can write a recursive formula for a geometric sequence g using the $(n + 1)$st term as

$$\begin{cases} g_1 = x \\ g_{n+1} = rg_n, \text{ for integers } n \geq 1. \end{cases}$$

Example 1

Give the first six terms and the constant multiplier of the geometric sequence g where

$$\begin{cases} g_1 = 6 \\ g_n = 3g_{n-1}, \text{ for integers } n \geq 2. \end{cases}$$

Solution 1 The value $g_1 = 6$ is given. The rule for g_n tells you that each term after the first is found by multiplying the previous term by 3.

The constant multiplier is 3.

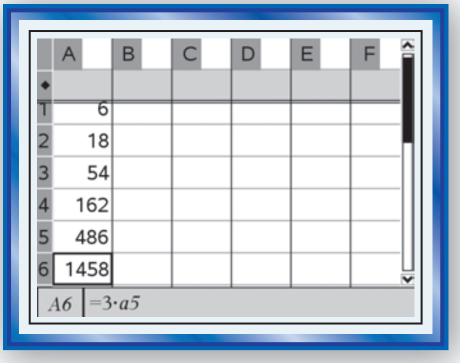
$$\begin{aligned} g_2 &= 3g_1 = 3 \cdot 6 = 18 \\ g_3 &= 3g_2 = 3 \cdot 18 = 54 \\ g_4 &= 3g_3 = 3 \cdot 54 = 162 \\ g_5 &= 3g_4 = 3 \cdot 162 = 486 \\ g_6 &= 3g_5 = 3 \cdot 486 = 1458 \end{aligned}$$

The first six terms of the sequence are 6, 18, 54, 162, 486, 1458.

Solution 2 Use a spreadsheet.

Enter 6 in cell A1. Then enter = 3*A1 in cell A2.

Copy and paste cell A2 into cells A3–A6.



STOP QY

> **QY**
>
> Write a recursive formula for the geometric sequence in Example 1 using g_{n+1}.

Explicit Formulas for Geometric Sequences

You may have received a letter or e-mail that promises good luck as long as you send the letter to five friends asking each to forward it to five of their friends, and so on. Such chain letters are illegal in the U.S. if the mailer asks for money.

Part of the appeal of chain letters is that a very large number of people can receive them quickly. This is because the number of letters sent by each *generation* of mailers forms a geometric sequence. The data at the right represent the first five generations of a chain letter in which a person sends an e-mail to 12 people and asks each person receiving the e-mail to forward it to 4 other friends, and no person receives two letters.

Generation	Number of Letters Sent
1	12
2	$12 \cdot 4 = 48$
3	$12 \cdot 4^2 = 192$
4	$12 \cdot 4^3 = 768$
5	$12 \cdot 4^4 = 3072$

In this sequence, the constant multiplier is $r = 4$. If $g_1 = 12$, the number of letters the first person sends out, then g_2 is the number of letters sent out by everyone in generation 2, and $g_n = 12(4)^{n-1}$ is the number of letters sent out in generation n. This pattern can be generalized to find an explicit formula for the nth term of any geometric sequence.

Explicit Formula for a Geometric Sequence

In the geometric sequence g with first term g_1 and constant ratio r,
$g_n = g_1(r)^{n-1}$, for integers $n \geq 1$.

Notice that in the explicit formula, the exponent of the nth term is $n - 1$. When you substitute 1 for n to find the first term, the constant multiplier has an exponent of zero.

$$\begin{aligned} g_1 &= g_1(r)^{1-1} \\ &= g_1 r^0 \end{aligned}$$

This is consistent with the Zero Exponent Theorem which states that for all $r \neq 0$, $r^0 = 1$.

Constant multipliers in a geometric sequence can be negative. Then the terms of the sequence alternate between positive and negative values.

GUIDED

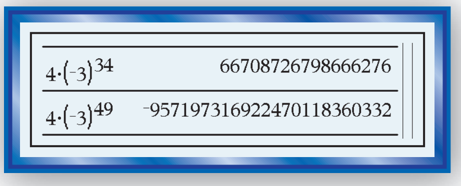
Example 2

Write the 1st, 5th, 10th, 35th, and 50th terms of the sequence a defined by $a_n = 4(-3)^{n-1}$.

Solution Substitute $n = 1, 5, 10, 35$, and 50 into the formula for the sequence.

$a_1 = 4(-3)^{1-1} = 4 \cdot \underline{\ ?\ } = \underline{\ ?\ }$
$a_5 = 4(-3)^{\underline{\ ?\ }} = 4 \cdot \underline{\ ?\ } = \underline{\ ?\ }$
$a_{10} = \underline{\ ?\ } = \underline{\ ?\ } = -78{,}732$

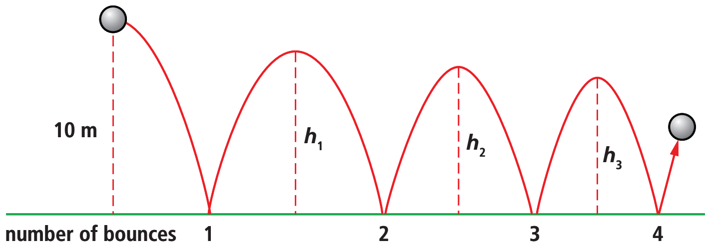
A calculator display of a_{35} and a_{50} is shown at the right.



Constant multipliers in a geometric sequence can also be between 0 and 1. This is the case for the sequence that models the height of a bouncing ball.

Example 3

Suppose a ball is dropped from a height of 10 meters, and it bounces up to 80% of its previous height after each bounce. (A bounce is counted when the ball hits the ground.) Let h_n be the maximum height of the ball after the nth bounce.



a. Find an explicit formula for h_n.

b. Find the maximum height of the ball after the seventh bounce.

Solution

a. Because each term is 0.8 times the previous term, the sequence is geometric. On the first bounce, the ball bounces up to 10(0.8) meters, or 8 meters. So, $h_1 = 8$. Also, $r = 0.8$. So,

$$h_n = 8(0.8)^{n-1}.$$

b. On the seventh bounce, $n = 7$, so

$$h_7 = 8(0.8)^{7-1} \approx 2.10.$$

So, after the seventh bounce, the ball will rise to a height of about 2.10 meters.

Look back at the sequences generated in Examples 1 to 3. Notice that in Example 1, $r > 1$ and g_n increases as n increases. In Example 3, $0 < r < 1$ and g_n decreases as n increases. In Example 2, $r < 0$ and as n increases, g_n alternates between positive and negative values. These properties are true for all geometric sequences.

Questions

COVERING THE IDEAS

1. **Fill in the Blanks** In an arithmetic sequence, each term after the first is found by __?__ a constant to the previous term. In a geometric sequence, each term after the first is found by __?__ the previous term by a constant.

In 2–4, state whether the numbers can be consecutive terms of a geometric sequence. If they can, find the constant ratio and write an explicit formula.

2. 12, 48, 192, ...

3. 5, 10, 20, 25, ...

4. $12, \frac{4}{3}, \frac{1}{6}, \frac{1}{48}$ ...

5. Let g be a sequence with $g_n = 200 \cdot (0.10)^{n-1}$.
   a. Find terms 1, 5, 20, and 50.
   b. Use a spreadsheet to generate the first 15 terms of g.

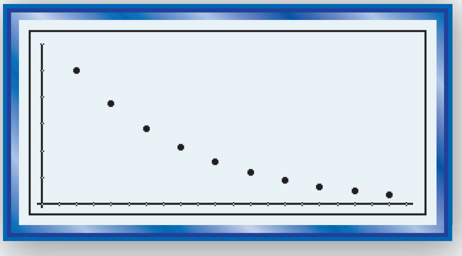
6. Find the first five terms of the sequence
$$\begin{cases} t_1 = 1.2 \\ t_n = 5 \cdot t_{n-1}, \text{ for integers } n \geq 2. \end{cases}$$

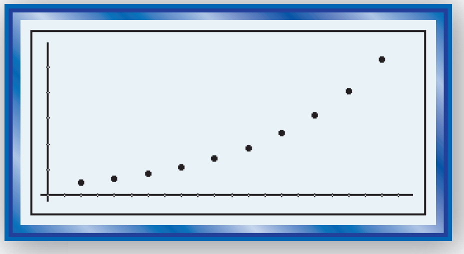
7. Consider $\begin{cases} g_1 = x \\ g_{n+1} = rg_n, \text{ for integers } n \geq 1 \end{cases}$.
   a. Write r in terms of g_{n+1} and g_n.
   b. Write an explicit formula for g_{n+1}.

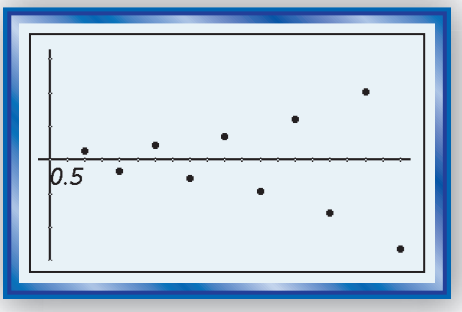
8. a. Write the first six terms of the geometric sequence whose first term is -3 and whose constant ratio is -2.
   b. Give a recursive formula for the sequence in Part a.
   c. Give an explicit formula for the sequence in Part a.

9. Suppose $g_n = 1.85 \cdot 0.38^{n-1}$ for integers $n \geq 1$.
   a. Find each ratio.
      i. $\frac{g_2}{g_1}$ ii. $\frac{g_3}{g_2}$ iii. $\frac{g_4}{g_3}$ iv. $\frac{g_{20}}{g_{19}}$
   b. What is true about the values in Part a?

10. **Matching** Each graph below is a graph of a geometric sequence. Match each graph with its range of possible common ratios r.
   i. $r > 1$ ii. $r < 0$ iii. $0 < r < 1$

a.


b.


c.


11. Suppose a ball dropped from a height of 11 feet bounces up to 60% of its previous height after each bounce.

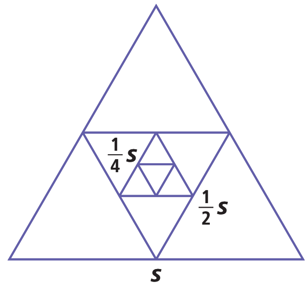
a. Find an explicit formula for the maximum height of the ball after the nth bounce.

b. Find the height of the ball, to the nearest inch, after the tenth bounce.

12. In the figure at the right, the midpoints of the sides of the largest equilateral triangle have been connected to create the next smaller equilateral triangle, and this process has been continued. If the side length of the largest triangle is s, then the side length of the next smaller triangle is $\frac{1}{2}s$. The sides of the consecutively smaller triangles form a geometric sequence.

a. What is the first term of the sequence?

b. What is the constant multiplier?



APPLYING THE MATHEMATICS

13. Willie Savit invested $5000 in an account at 2.25% interest compounded annually after t years.

a. Write an explicit formula for how much money Willie will have in his account after t years.

b. How much money will Willie have after 20 years?

c. If Willie had received simple interest instead of compound interest, the value of his investment over the first five years would be as shown at the right. Do the amounts in his account under simple interest form a geometric sequence? Why or why not?

Year	Amount ($)
0	$5000.00
1	$5112.50
2	$5225.00
3	$5337.50
4	$5450.00

14. The fourth term of a geometric sequence is 1. The constant multiplier is $\frac{1}{5}$.

a. What is the seventh term? **b.** What is the first term?

In 15 and 16, the first few terms of a geometric sequence are given. Find the next two terms. Then, find an explicit formula for the *n*th term of the geometric sequence.

15. 14, 28, 56, 112, ... **16.** 9, 3, 1, ...

17. a. Write a recursive formula for the geometric sequence whose first four terms are 23, –23, 23, –23, ...

b. Write an explicit formula for the sequence in Part a.

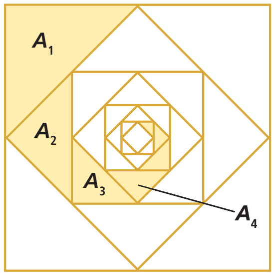
18. y, xy^2, x^2y^3, x^3y^4, and x^4y^5 are the first five terms of a sequence s.

a. Find each ratio.

i. $\frac{s_2}{s_1}$ ii. $\frac{s_3}{s_2}$ iii. $\frac{s_4}{s_3}$ iv. $\frac{s_5}{s_4}$

b. Could s be a geometric sequence? Explain why or why not.

19. In the figure at the right, the midpoints of the sides of each square are connected to form the next smaller square. The ratios of the areas of consecutive shaded regions are equal. This is called a Baravelle Spiral. Assume that the side of the largest square has length 1 unit.



a. Find the area of region A_1.

b. The areas of the consecutively smaller and smaller regions $A_2, A_3, A_4, \ldots$ form a geometric sequence A. Find an explicit formula for A_n.

REVIEW

20. A bank account has an annual interest rate of 3.8% compounded daily. **(Lesson 7-4)**

a. If $10,000 is placed in this account, calculate the value of the account 5 years from now.

b. What is the annual percentage yield of this account?

c. If $10,000 is placed into an account that compounds interest annually at a rate equal to the annual percentage yield in Part b, what will the value of this new account be in 5 years? How does this amount compare to your answer to Part a?

21. Suppose N varies directly as a and inversely as the square of b and the third power of c. Using negative exponents, write a formula for N that does not contain a fraction. **(Lessons 7-3, 2-8)**

22. Rewrite $y = x^2 - 46x + 497$ in vertex form. **(Lesson 6-5)**

23. A circle and a rectangle have the same area. The circumference of the circle is 20 cm and one side of the rectangle equals the diameter of the circle. Determine whether the perimeter of the rectangle is longer or shorter than the circle's circumference. **(Lesson 6-2)**

EXPLORATION

24. a. Refer to the chain letter example from the lesson. The number of e-mails actually sent after the first generation seldom matches the numbers in the table. Why is that?

b. Suppose only 2 of 4 people who receive an e-mail pass it along to 4 others. What, then, is the minimum number of generations it will take for the e-mail to reach 100 million people?

QY ANSWER

$$\begin{cases} g_1 = 6 \\ g_{n+1} = 3g_n, \text{ for } n \geq 1 \end{cases}$$