

Lesson

7-4

Compound Interest

► **BIG IDEA** If money grows at a constant interest rate r in a single time period, then after n time periods the value of the original investment has been multiplied by $(1 + r)^n$.

Interest Compounded Annually

Penny Wise, a high school junior, works part-time during the school year and full-time during the summer. Suppose that Penny decides to deposit \$3000 this year in a 3-year certificate of deposit (CD).

The investment is guaranteed to earn interest at a yearly rate of 4.5%. The interest is added to the account at the end of each year. If no money is added or withdrawn, then after one year the CD will have the original amount invested, plus 4.5% interest.

$$\begin{aligned} \text{amount after 1 year: } 3000 + 0.045(3000) &= 3000(1 + 0.045)^1 \\ &= 3000(1.045) \\ &= 3135 \end{aligned}$$

Penny's CD is worth \$3135 after one year.

Notice that to find the amount after 1 year, you do not have to add the interest separately; you can just multiply the original amount by 1.045. Similarly, at the end of the second year, there will be 1.045 times the *balance* (the ending amount in the account) from the first year.

$$\begin{aligned} \text{amount after 2 years: } 3000(1.045)(1.045) &= 3000(1.045)^2 \\ &\approx 3276.075 \end{aligned}$$

Since banks round down, Penny's CD is worth \$3276.07 after two years.

$$\begin{aligned} \text{amount after 3 years: } 3000(1.045)^2(1.045) &= 3000(1.045)^3 \\ &\approx 3423.49 \end{aligned}$$

The value of Penny's CD has grown to \$3423.49 after three years. Notice the general pattern.

$$\text{amount after } t \text{ years: } 3000(1.045)^t$$

Vocabulary

annual compound interest
principal
semi-annually
compounding daily
annual percentage yield, APY

Mental Math

Suppose a quiz has two sections.

- The first section has 4 multiple-choice questions each with 3 possible answers. What is the probability of guessing all answers in this section correctly?
- The second section has 3 true/false questions. What is the probability of guessing all answers in this section correctly?
- What is the probability of guessing correctly on all the quiz questions?

When interest is earned at the end of each year, it is called *annual compound interest*. To find a more general formula for interest, replace 4.5% with r , the annual interest rate, and 3000 with P , the **principal** or original amount invested.

Annual Compound Interest Formula

Let P be the amount of money invested at an annual interest rate r compounded annually. Let A be the total amount after t years. Then

$$A = P(1 + r)^t.$$

In the Annual Compound Interest Formula, notice that A varies directly as P . For example, doubling the principal doubles the amount at the end. However, A does not vary directly as r ; doubling the rate does not necessarily double the amount earned.



QY

Interest Compounded More Than Once a Year

In most savings accounts, interest is compounded more than once a year. If money is compounded **semi-annually**, the interest rate at each compounding is *half of the annual interest rate* but there are *two compoundings each year* instead of just one. So if your account pays 4.5% compounded semi-annually, you earn 2.25% on the balance every six months. At the end of t years, interest paid semi-annually will have been paid $2t$ times. Therefore, the compound interest formula becomes

$$A = P\left(1 + \frac{r}{2}\right)^{2t}.$$

If money is compounded *quarterly*, the compound interest formula becomes

$$A = P\left(1 + \frac{r}{4}\right)^{4t}.$$

This pattern leads to a general compound interest formula.

General Compound Interest Formula

Let P be the amount invested at an annual interest rate r compounded n times per year. Let A be the amount after t years. Then

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

The number of times that the interest is compounded makes a difference in the amount of interest earned.

QY

How much more money would Penny earn if she were able to earn 9% rather than the 4.5% for 3 years?

Example 1

Suppose \$10,000 is placed into an account that pays interest at a rate of 5%. How much will be earned in the account in the first year if the interest is compounded as indicated?

- a. annually b. semi-annually c. quarterly

Solution

a. Since interest is compounded only once, the interest is simply $0.05 \cdot \$10,000 = \500 . The account will earn \$500.

b. and c. Substitute into the General Compound Interest Formula to determine the account's value.

For Part b, $P = \$10,000$;
 $r = 5\%$, $n = 2$, and $t = 1$ year.

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ &= \$10,000\left(1 + \frac{0.05}{2}\right)^{2 \cdot 1} \\ &= \$10,000(1.025)^2 \\ &= \$10,000(1.050625) \\ &= \$10,506.25 \end{aligned}$$

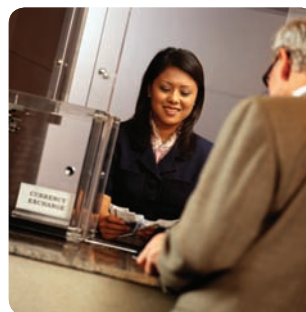
For Part c, $P = \$10,000$;
 $r = 5\%$, $n = 4$, and $t = 1$ year.

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ &= \$10,000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 1} \\ &= \$10,000(1.0125)^4 \\ &\approx \$10,000(1.050945) \\ &\approx \$10,509.45 \end{aligned}$$

Now subtract the \$10,000 principal to find the amount of interest that was earned.

The account will earn
\$506.25.

The account will earn
\$509.45.



In Example 1, the difference after one year between compounding semi-annually and compounding quarterly is only \$3.20. However, if you withdraw your money before a year is up, you may have received interest in the account that pays quarterly while you may not have received interest in the account that pays semi-annually. For instance, if interest is compounded quarterly and you withdraw your money after 10 months, you will have received 3 of the 4 quarterly compound interest payments and have a total of $10,000\left(1 + \frac{0.05}{4}\right)^{3 \cdot 1} \approx \$10,379.70$.

However, if interest is compounded semi-annually, then after 10 months you will have received 1 of 2 semi-annual compound interest payments and have only $10,000\left(1 + \frac{0.05}{2}\right)^{1 \cdot 1} = \$10,250$, a difference of over \$125!

To avoid angering their customers, most savings institutions guarantee that accounts will earn interest “from the date of deposit until the date of withdrawal.” They can do this by **compounding daily**. Daily compounding uses either 360 or 365 as the number of days in a year.

Annual Percentage Yield

Because of the many different ways of calculating interest, savings institutions are required by federal law to disclose the **annual percentage yield**, or **APY**, of an account after all the compoundings for a year have taken place. This allows consumers to compare savings plans. For instance, to determine the APY of an account paying 5% compounded quarterly (as in Example 1), find the interest \$1 would earn in the account in one year.

$$1 \cdot \left(1 + \frac{0.05}{4}\right)^{4 \cdot 1} \approx 1.0509$$

So the interest earned is $\$1.0509 - \$1 = \$0.0509$. This means that the APY on an account paying 5% compounded quarterly is 5.09%.

Product	Rate%	Yield%
THE PREFERRED ACCOUNT Checking with high-rate savings		
Savings	4.26	4.35
24-MONTH CD	5.12	5.25
60-MONTH CD	6.30	6.50

GUIDED

Example 2

What is the APY for a 5.5% interest rate compounded

- quarterly?
- daily, for 365 days per year?

Solution To find the APY on an account, use \$1 as the principal amount to keep the computations simple.

a. $1 \cdot \left(1 + \frac{?}{4}\right)^{4 \cdot 1} \approx ?$

So, the interest earned is \$? - \$1 = \$?.

This is an APY of ?%.

b. $1 \cdot \left(1 + \frac{?}{?}\right)^{?} \approx ?$

So, the interest earned is \$? - \$1 = \$?.

This is an APY of ?%.

Going Back in Time

In both compound interest formulas, you can think of P either as the principal or as the *present amount*. In each of the previous examples, A is an amount that is determined after compounding. Then, because A comes after P , the time t is represented by a positive number. But it is also possible to think of A as an amount some years ago that was compounded to get the present amount P . Then the time t is represented by a negative number.

Example 3

Zero-coupon bonds do not pay interest during their lifetime, typically 20 or 30 years. They are bought for much less than their final value and earn a fixed rate of interest over their life. When a bond *matures*, its value is equal to the initial investment plus all the interest earned over its lifetime.

Suppose a 30-year zero-coupon bond has a value at maturity of \$20,000 and is offered at 5.5% interest compounded semi-annually. How much do you need to invest to buy this bond?



Series I savings bonds are sold at face value and earn interest based on a fixed interest rate and inflation. The interest earned is tax free when the bond is used to pay for higher education expenses. Albert Einstein appears on the \$1000 Series I savings bond.

Solution 1 Think of how much you would need to have invested 30 years ago to have \$20,000 now. Use the General Compound Interest Formula with a present value of $P = 20,000$, $r = 0.055$, $n = 2$, and $t = -30$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{General Compound Interest Formula}$$

$$A = 20,000\left(1 + \frac{0.055}{2}\right)^{2 \cdot -30} \quad \text{Substitution}$$

$$A = \frac{20,000}{1.0275^{60}} \quad \text{Arithmetic and Negative Exponent Theorem}$$

$$A \approx 3927.54 \quad \text{Arithmetic}$$

You need to invest \$3927.54 to buy this bond.

Solution 2 Use the General Compound Interest Formula. You know $A = 20,000$, $r = 0.055$, $n = 2$, and $t = 30$. Solve for P .

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{General Compound Interest Formula}$$

$$20,000 = P\left(1 + \frac{0.055}{2}\right)^{2 \cdot 30} \quad \text{Substitution}$$

$$20,000 \approx P(1.0275)^{60} \quad \text{Arithmetic}$$

$$P \approx 3927.54 \quad \text{Divide both sides by } 1.0275^{60}$$

You need to invest \$3927.54 to buy this bond.

Questions**COVERING THE IDEAS**

- Suppose Penny Wise buys \$2500 worth of government bonds that pay 3.7% interest compounded quarterly. If no money is added or withdrawn, find out how much the bonds will be worth after 1, 2, 3, 4, and 5 years.
- Find the interest earned in the fourth year for Penny's 4.5% CD described in this lesson.

3. Write the compound interest formula for an account that earns interest compounded
 - a. monthly.
 - b. daily in a leap year.
4. To what amount will \$8000 grow if it is invested for 12 years at 6% compounded quarterly?
5. Find the APY of a savings account earning 4% interest compounded daily. Use 360 for the number of days in a year.
6. Suppose a zero-coupon bond matures and pays the owner \$30,000 after 10 years, paying 4.5% interest annually. How much was invested 10 years ago?
7. **True or False** An account earning 8% compounded annually earns exactly twice as much interest in 6 years as an account earning 8% compounded annually earns in 3 years. Explain your answer.
8. **True or False** Justify your answer. In the General Compound Interest Formula,
 - a. A varies directly as t .
 - b. A varies directly as P .
 - c. A varies inversely as n .
 - d. A varies directly as r .

APPLYING THE MATHEMATICS

9. Refer to Penny's certificate of deposit.
 - a. Calculate the value of Penny's CD at the end of 3 years using the **simple interest formula** $I = Prt$, where I is the amount of interest earned, P is the principal, r is the annual percentage rate, and t is time in years.
 - b. How much money would Penny have if she earned annual compound interest over the same 3 years?
 - c. Should Penny prefer simple interest to annual compound interest? Explain why or why not.
10. On a CAS, define a function $\text{gencompint}(p, r, n, t)$ to calculate the value of an investment using the General Compound Interest Formula. Use the function to verify the answers to Example 1.
11. Rich takes a \$2000 cash advance against his credit card to fund an investment opportunity he saw on the Internet. The credit card charges an annual rate of 18% and compounds the interest monthly on all cash advances. How much interest does Rich owe if he does not make any payments for 3 months?

Define $\text{gencompint}(p, r, n, t) = p \cdot \left(1 + \frac{r}{n}\right)^{nt}$
Done

12. Stores often advertise a “90-day same as cash” method for making purchases. This means that interest is compounded starting the day of purchase (the interest *accrues*), but no interest is charged if the bill is paid in full before 90 days go by. However, on the 91st day, the accrued interest is added to the purchase price. Suppose Manny purchased a sofa for \$3000 under this plan with an annual interest rate of 20.5% compounded monthly. If Manny forgets to pay the purchase in full, how much will he owe on the 91st day?



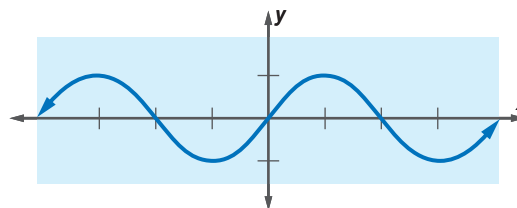
Many stores offer 90 days same as cash during sales.

REVIEW

13. Rewrite $\left(\frac{1}{3^{-3}}\right)^{-2}$ without exponents. (Lesson 7-3)
14. The Stefan-Boltzmann constant in physics is $\frac{2\pi^5k^4}{15h^3c^2}$. Rewrite this expression in the form $\frac{2}{15}M$, where M is an expression that does not involve fractions. (Lesson 7-3)
15. **True or False** For all positive integers a , b , and c , $(a^b)^c = a^{(b^c)}$. Justify your answer. (Lesson 7-2)
16. Solve $x^2 - 3x + 7 = 2x^2 + 5x - 9$ by
 a. graphing.
 b. using the Quadratic Formula. (Lessons 6-7, 6-4)
17. What translation maps the graph of $y = \sqrt{x - 3} + 5$ onto the graph of $y = \sqrt{x} - 3$? (Lesson 6-3)
18. The graph at the right appears to have a certain symmetry. Give a matrix describing a transformation that would map this graph onto itself. (Lesson 4-8)
19. Consider the sequence t_n defined recursively as follows. (Lessons 3-6, 1-8)

$$\begin{cases} t_1 = 4 \\ t_n = 3t_{n-1} \text{ for } n \geq 2 \end{cases}$$

- a. Find the first five terms of the sequence.
 b. Is t an arithmetic sequence? Why or why not?



EXPLORATION

20. Find an interest rate and an APY for a 3-year CD at a savings institution in your area. Show how to calculate the APY from the interest rate.

QY ANSWER

\$461.59