Lesson

7-3

Negative Integer Exponents

▶ **BIG IDEA** The powers x^m and x^{-m} represent reciprocals when they are both defined.

Because of the Quotient of Powers Theorem, for any positive base b and real exponents m and n, or any nonzero base b and integer exponents m and n,

$$\frac{b^m}{b^n} = b^{m-n}.$$

Let us look at this theorem in terms of the values of m and n. If $m > n$, then $m - n$ is positive and b^{m-n} is the product of $(m - n)$ b's. If $m = n$, then $m - n = 0$, and $b^{m-n} = b^0 = 1$. What happens when $m < n$? Then $m - n < 0$, and b^{m-n} has a negative exponent. You have seen negative exponents with base 10 before when writing numbers in scientific notation. For example,

$$10^6 = 1,000,000 = \text{one million}$$
$$10^{-6} = 0.000001 = \text{one millionth}.$$

Note that $1,000,000 \cdot 0.000001 = 1$. This means 10^6 and 10^{-6} are reciprocals. The same kind of relationship holds for bases other than 10.

Activity
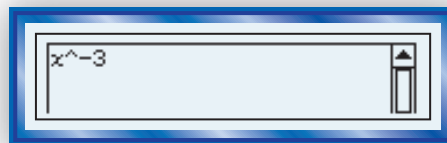
MATERIALS CAS

Step 1 What does a CAS display when you enter the following powers of x?

    a. x^{-3} **b.** x^{-8} **c.** x^{-11}

Step 2 Based on the results of Step 1, make a conjecture: $x^{-n} = $ ___?___. Test your conjecture by entering x^{-n} into a CAS. If your calculator does not rewrite x^{-n}, use `expand(x^-n)` to create an equivalent expression with only positive exponents.

The results of the Activity should be instances of the theorem at the top of the next page.

Negative Exponent Theorem

For any positive base b and real exponent n, or any nonzero base b and integer exponent n, $b^{-n} = \frac{1}{b^n}$.

Proof

1. Suppose $b^{-n} = x$. We want to determine x.

2. $b^n \cdot b^{-n} = b^n \cdot x$ Multiplication Property of Equality (Multiply both sides by b^n.)

3. $b^0 = b^n \cdot x$ Product of Powers Postulate

4. $1 = b^n \cdot x$ Zero Exponent Theorem

5. $\frac{1}{b^n} = x$ Divide both sides by b^n (which can always be done because $b \neq 0$).

6. Thus $b^{-n} = \frac{1}{b^n}$. Transitive Property of Equality (Steps 1 and 5)

It helps to think of the Negative Exponent Theorem as stating that b^n and b^{-n} are reciprocals. In particular, $b^{-1} = \frac{1}{b}$, so b^{-1} is the reciprocal of b.

GUIDED

Example 1

Write 2^{-5} as a decimal.

Solution By the Negative Exponent Theorem, $2^{-5} = \frac{1}{?}$ as a fraction and __?__ as a decimal.

The Negative Exponent Theorem allows expressions with fractions to be rewritten without them.

Example 2

Critical buckling load is the minimum weight that would cause a column to buckle. Rewrite Euler's formula for critical buckling load, $P = \frac{\pi^2 EI}{L^2}$, using negative exponents and no fractions. (In this formula, E is a constant related to the material used to construct the column, I is the moment of inertia of a cross-section of the column, and L is the length of the column.)

Solution $P = \frac{\pi^2 EI}{L^2}$

$P = \pi^2 EI \frac{1}{L^2}$ Algebraic definition of division

$P = \pi^2 EI L^{-2}$ Negative Exponent Theorem

▶ **READING MATH**

In many math-related fields the term *critical* refers to a point or measurement at which some quality undergoes a drastic change. In the case of critical buckling load, this change is from stable to collapsed.

Properties of Negative Integer Exponents

All the postulates and theorems involving powers stated in Lesson 7-2 hold when the exponents are negative integers.

GUIDED

Example 3

Rewrite each expression as a single power or a single number in scientific notation.

a. $\dfrac{r^5}{r^{12}}$ (Assume $r > 0$.)   b. $4^{-3} \cdot 4^5$ c. $(2 \cdot 10^3)^2 \cdot (1.5 \cdot 10^{-2})$

Solution

a. Use the Quotient of Powers Theorem.
$$\frac{r^5}{r^{12}} = r^{\underline{\ ?\ } - \underline{\ ?\ }} = r^{\underline{\ ?\ }}$$

b. Use the Product of Powers Postulate.
$$4^{-3} \cdot 4^5 = 4^{\underline{\ ?\ } + \underline{\ ?\ }} = 4^{\underline{\ ?\ }}$$

c. Use the Power of a Product and Power of a Power Postulates.
$$(2 \cdot 10^3)^2 \cdot (1.5 \cdot 10^{-2}) = (\underline{\ ?\ } \cdot 10^{\underline{\ ?\ }}) \cdot (1.5 \cdot 10^{-2})$$

Use the Commutative Property of Multiplication and the Product of Powers Postulate.
$$\underline{\ ?\ } \cdot 10^{\underline{\ ?\ }}(10^{-2}) = \underline{\ ?\ } \cdot 10^{\underline{\ ?\ }}$$

Check Use a CAS to check whether the original expression and your answer are equivalent.

$\dfrac{r^5}{r^{12}}$ $\qquad$ $\dfrac{1}{r^7}$

Caution: A negative number in an exponent does not make the value of an expression negative. All powers of positive numbers are positive.

Example 4

Rewrite $(3c)^{-3}(2c)^4$ as a fraction with the variable c appearing only once.

Solution 1

$$\begin{aligned}
(3c)^{-3}(2c)^4 &= 3^{-3} \cdot c^{-3} \cdot 2^4 \cdot c^4 &&\text{Power of a Product Postulate}\\
&= 3^{-3} \cdot 2^4 \cdot c^{-3} \cdot c^4 &&\text{Commutative Property of Multiplication}\\
&= 3^{-3} \cdot 2^4 \cdot c &&\text{Product of Powers Postulate}\\
&= \frac{1}{3^3} \cdot 2^4 \cdot c &&\text{Negative Exponent Theorem}\\
&= \frac{16c}{27}
\end{aligned}$$

Solution 2

$$(3c)^{-3}(2c)^4 = \frac{1}{(3c)^3} \cdot (2c)^4 \qquad \text{Negative Exponent Theorem}$$

$$= \frac{(2c)^4}{(3c)^3} \qquad \text{Multiply.}$$

$$= \frac{2^4 \cdot c^4}{3^3 \cdot c^3} \qquad \text{Power of a Product Postulate}$$

$$= \frac{2^4 c}{3^3} \qquad \text{Quotient of Powers Postulate}$$

$$= \frac{16c}{27}$$

Questions

COVERING THE IDEAS

In 1 and 2, write as a single power.

1. $\dfrac{t^3}{t^{10}}$

2. $\dfrac{6^6}{6^7}$

In 3 and 4, write an equivalent expression without negative exponents.

3. x^{-4}

4. 3^{-7}

5. Solve for n: $8^{-7} = \dfrac{1}{8^n}$.

6. If $p^{-1} = \dfrac{4}{13}$, find p.

7. Write as a whole number or a fraction without an exponent.

   a. 1.8^0 b. 1.8^{-1} c. 1.8^{-2}

8. Write as a fraction without an exponent.

   a. 9^{-1} b. 9^{-3} c. 9^{-5}

9. Write 8^{-3}

   a. as a decimal.

   b. in scientific notation.

10. The time t it takes you to read a book is inversely proportional to the number p of pages in the book. Let k be the constant of variation.

    a. Write an inverse variation equation to represent this situation using positive exponents.

    b. Rewrite your inverse variation equation using negative exponents.

In 11–13, write an equivalent expression without negative exponents.

11. $\dfrac{10y}{15y^{-2}}$

12. $(3e^2)^{-3}(5e)^4$

13. $\dfrac{2a^{-3}}{b^4}$

In 14–16, write an equivalent expression without an exponent.

14. $10^4 \cdot 10^{-5}$

15. $1.982^0 \cdot 1^{47}$

16. $3^{-3} \cdot \left(\dfrac{1}{3}\right)^{-1}$

17. **Multiple Choice** If $b > 0$, for what integer values of n is $b^n < 0$?

    A $n < 0$ B $0 < n < 1$

    C   all values of n D   no values of n

18. If $y^{-6} = 9$, what is the value of y^6?

APPLYING THE MATHEMATICS

In 19 and 20, rewrite the right side of the formula using negative exponents and no fractions.

19. $A = \dfrac{kh}{g^5}$

20. $T = \dfrac{k}{m^2 n^5}$

21. Write an expression equivalent to $\left(\dfrac{3x^{-3}y}{2z^{-5}}\right)^{-2}\left(\dfrac{x^4 y^{-3}}{z^3}\right)^8$ without any negative exponents.

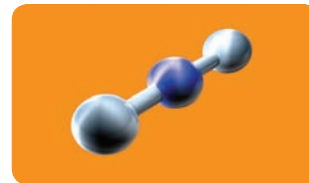22. Write each expression in standard scientific notation $x \cdot 10^n$, where $1 \le x < 10$ and n is an integer.

    a. $(3 \cdot 10^{-4})^2$

    b. $(6.2 \cdot 10^{-6})(4.6 \cdot 10^9)$

23. Carbon dioxide is a colorless and odorless gas that absorbs radiation from the sun and contributes to global warming. Before the industrial age, carbon dioxide made up about 280 parts per million (ppm) of the atmosphere. Carbon dioxide is released when coal, gasoline, and other fossil fuels are burned, so the level of the gas in the atmosphere has been increasing. Some scientific models have predicted that by 2100, the concentration will range from 490 ppm to 1260 ppm.

    a. Write all three rates as fractions using positive powers of 10.

    b. Write each rate in scientific notation.

The molecular structure of carbon dioxide shows that one carbon atom in the center is flanked by two oxygen atoms.

24. Benjamin Franklin was one of the most famous scientists of his day. In one experiment he noticed that oil dropped on the surface of a lake would not spread out beyond a certain area. In modern units, he found that 0.1 cm^3 of oil spread to cover about 40 m^2 of the lake. About how thick is such a layer of oil? Express your answer in scientific notation. (We now know that the layer of oil stops spreading when it is one molecule thick. Although in Franklin's time no one knew about molecules, Franklin's experiment resulted in the first estimate of a molecule's size.)

In 25 and 26, write an equivalent expression without a fraction.

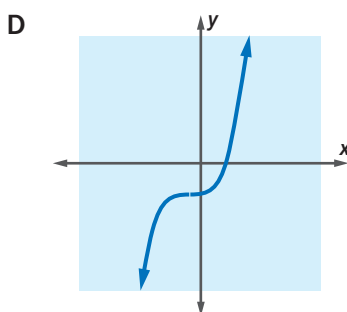25. $\dfrac{4x^2}{y^3}$

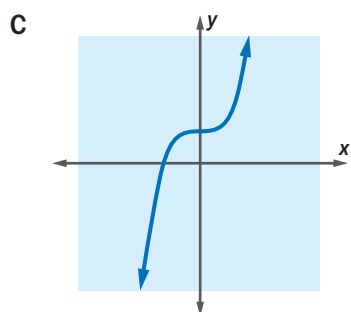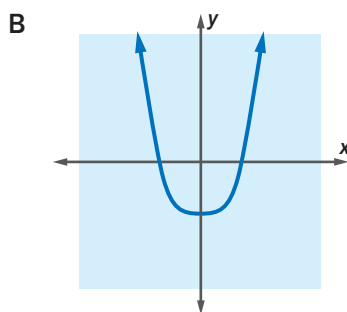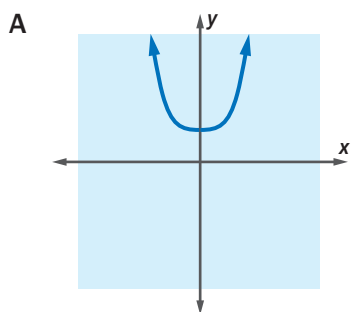26. $\dfrac{12a^4}{19b^5 c^2}$

Benjamin Franklin

REVIEW

In 27 and 28, write without a fraction. (Lesson 7-2)

27. $\left(\dfrac{a^3}{a^2}\right)^3 \cdot 20a$

28. $\dfrac{(xy)^m y^n}{x^2}$

29. **Multiple Choice** Which of the following could be the graph of
$y = x^4 + 1$? Justify your answer. (**Lessons 7-1, 6-3**)

A

B

C

D

30. A bowler needs 12 strikes in a row to bowl a perfect game of 300.
   a. If the probability that Khadijah gets a strike is $\frac{1}{8}$ and strikes
   are independent of each other, what is the probability that
   Khadijah's next game will be a perfect game?
   b. Is your answer to Part a greater than or less than one
   billionth? (**Lesson 7-1**)

31. a. What are the domain and range of the function with equation
   $y = -\frac{4}{x^2}$?
   b. What are the domain and range of the function with equation
   $y = -\frac{4}{x}$? (**Lesson 2-6**)

EXPLORATION

32. a. Examine the table at the right closely. Describe two
   patterns relating the powers of 5 on the left to the powers
   of 2 on the right.
   b. Make a chart similar to the one in Part a using the powers
   of 4 and of 2.5.
   c. Describe how the patterns in the chart from Part b are
   similar to the patterns in Part a.
   d. Find another pair of numbers with the same properties.

Powers of 5	Powers of 2
$5^6 = 15{,}625$	$2^6 = 64$
$5^5 = 3125$	$2^5 = 32$
$5^4 = 625$	$2^4 = 16$
$5^3 = 125$	$2^3 = 8$
$5^2 = 25$	$2^2 = 4$
$5^1 = 5$	$2^1 = 2$
$5^0 = 1$	$2^0 = 1$
$5^{-1} = 0.2$	$2^{-1} = 0.5$
$5^{-2} = 0.04$	$2^{-2} = 0.25$
$5^{-3} = 0.008$	$2^{-3} = 0.125$
$5^{-4} = 0.0016$	$2^{-4} = 0.0625$
$5^{-5} = 0.00032$	$2^{-5} = 0.03125$
$5^{-6} = 0.000064$	$2^{-6} = 0.015625$