Lesson 7-2 Properties of Powers

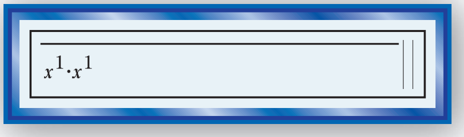
BIG IDEA There are single powers equivalent to $x^m \cdot x^n$, $\frac{x^m}{x^n}$, $x^m \cdot y^m$, $\frac{x^m}{y^m}$, and $(x^m)^n$ for all x and y for which these powers are defined.

This lesson reviews the properties of powers for positive integer exponents that you have learned in previous courses. In Lessons 7-3 and 7-7, you will see that these properties also apply to exponents that are negative integers or nonzero rational numbers.

Activity 1

Set a CAS to real-number mode and clear all variables.

Step 1 What does the CAS display when you enter each product of powers?



a. x^1x^1 **b.** x^6x^4 **c.** x^3x^{12}

Step 2 Based on your results in Step 1, write the general property: For all positive integers m and n, $x^mx^n = \underline{\ \ ?\ \ }$.

Step 3 What does the CAS display when you enter each power of products?

a. $(x \cdot y)^5$ **b.** $(x \cdot y)^2$ **c.** $(x \cdot y)^{15}$

Step 4 Based on your results in Step 3, make a conjecture: $(xy)^m = \underline{\ \ ?\ \ }$.

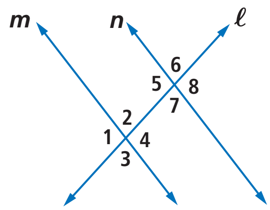
Products and Quotients of Powers with the Same Base

The general pattern in Step 2 of Activity 1 is summarized in the following postulate.

Product of Powers Postulate

For any nonnegative base b and nonzero real exponents m and n, or any nonzero base b and integer exponents m and n, $b^m \cdot b^n = b^{m+n}$.

Mental Math



Refer to the diagram above. Assume *m* and *n* are parallel. True or false?

a. $\angle 1$ and $\angle 7$ form a linear pair.

b. $\angle 1$ and $\angle 7$ are supplementary.

c. $m\angle 3 = m\angle 6$

d. $m\angle 1 + m\angle 8 = m\angle 5 + m\angle 4$

Consider $10^4 \cdot 10^8 = 10^{12}$. The Product of Powers Postulate shows a correspondence between a product of powers and a sum of exponents.

$$10^4 \cdot 10^8 = 10^{12} \quad \text{and} \quad 4 + 8 = 12.$$

Related facts yield two other correspondences, between quotients of powers and differences of exponents.

$$\frac{10^{12}}{10^4} = 10^8 \quad \text{and} \quad 12 - 4 = 8;$$

$$\frac{10^{12}}{10^8} = 10^4 \quad \text{and} \quad 12 - 8 = 4.$$

In general, from the Product of Powers Postulate, we can prove a theorem about quotients of powers.

Quotient of Powers Theorem

For any positive base b and real exponents m and n, or any nonzero base b and integer exponents m and n, $\frac{b^m}{b^n} = b^{m-n}$.

GUIDED

Example 1

Solve $14^3 \cdot 14^x = 14^{10}$.

Solution Write a correspondence like the one above:

$14^3 \cdot 14^x = 14^{10}$ and $\underline{\ ?\ } + x = \underline{\ ?\ }$.

Now solve for x: $x = \underline{\ ?\ }$.

Power of a Product or a Quotient

The general pattern in Step 4 of Activity 1 is summarized in the following postulate.

Power of a Product Postulate

For any nonnegative bases a and b and nonzero real exponent m, or any nonzero bases a and b and integer exponent m, $(ab)^m = a^m b^m$.

You can think about the Power of a Product Postulate as meaning that *powering distributes over multiplication*.

STOP **QY1**

▸ QY1

Give an example to show that powering does not distribute over addition.

Example 2

How many zeros are at the end of the number $N = 2^6 \cdot 5^6 \cdot 17$ when written in base 10?

Solution Use the Power of a Product Postulate.

$2^6 \cdot 5^6 = (2 \cdot 5)^6 = 10^6$. So, $N = 17 \cdot 10^6$, which is 17 followed by six zeros. There are six zeros at the end of the number.

Check Multiply: $2^6 \cdot 5^6 \cdot 17 = 17{,}000{,}000$. It checks.

STOP QY2

From the Power of a Product Postulate, a theorem about the power of a quotient can be deduced.

Power of a Quotient Theorem

For any positive bases a and b and real exponent n, or any nonzero bases a and b and integer exponent n, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Zero as an Exponent

Suppose you divide any power by itself. For instance, consider $\frac{3^7}{3^7}$. By the Quotient of Powers Theorem, $\frac{3^7}{3^7} = 3^{7-7} = 3^0$. But it is also true that $\frac{3^7}{3^7} = 1$. These statements and the Transitive Property of Equality prove that $3^0 = 1$.

In general, for any nonzero real number b,

	$\frac{b^n}{b^n} = b^{n-n}$	Quotient of Powers Theorem
	$= b^0$	Arithmetic
Also,	$\frac{b^n}{b^n} = 1$	A number divided by itself is 1.
So,	$b^0 = 1$	Transitive Property of Equality

This proves the following theorem.

Zero Exponent Theorem

If b is a nonzero real number, $b^0 = 1$.

Notice that the argument above does not work when $b = 0$. 0^0 is not defined because it would involve dividing by 0.

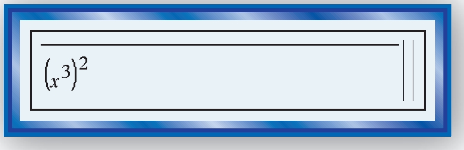
QY2

How many zeros are at the end of $5^3 \cdot 13 \cdot 2^3$ when written in base 10?

Powers of Powers

Activity 2

Set a CAS to real-number mode and clear all variables.



Step 1 What does the CAS display when you enter each power of powers?

a. $(x^3)^2$ **b.** $(y^5)^4$ **c.** $(z^6)^3$ **d.** $(m^0)^5$

Step 2 Based on your results in Step 1, make a conjecture: $(x^m)^n = \underline{\ ?\ }$.

The general pattern in Activity 2 is the last assumed property of powers.

Power of a Power Postulate

For any nonnegative base b and nonzero real exponents m and n, or any nonzero base b and integer exponents m and n, $(b^m)^n = b^{mn}$.

For example, using the Power of a Power Postulate,

$$(a^7)^4 = a^{7 \cdot 4} = a^{28}.$$

You can check this with the Product of Powers Postulate.

$$(a^7)^4 = a^7 \cdot a^7 \cdot a^7 \cdot a^7 = a^{7+7+7+7} = a^{28}$$

Example 3

A multiple-choice test is 3 pages long. Each page has 5 questions. Each question has c possible choices. How many different ways can you complete the test if you leave no answer blank?

Solution 1 If each of the 5 questions on one page has c possible choices, there are c^5 ways to answer the questions on that page. Because there are 3 pages, there are $(c^5)^3 = c^{5 \cdot 3} = c^{15}$ ways to complete the test.

Solution 2 The whole test has $5 \cdot 3 = 15$ questions, each with c choices. So, there are c^{15} ways to answer the questions on the test.

Using the Properties of Powers

Properties of powers are often used when working with numbers expressed in scientific notation.

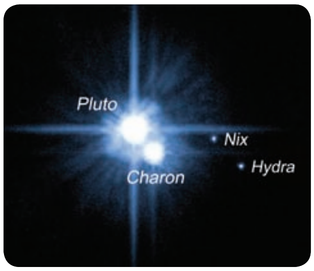
Example 4

In 2005, astronomers using the Hubble Space Telescope discovered two tiny moons, named Hydra and Nix, orbiting Pluto. Hydra's mass is believed to be between 1×10^{17} kg and 9×10^{18} kg. Assume that Hydra's mass is actually 2.4×10^{18} kg. The mass of Earth's moon is 7.35×10^{22} kg. About how many times as massive is Earth's moon as Hydra?

Solution Divide the two numbers:

$$\frac{\text{mass of Earth's moon}}{\text{mass of Hydra}} \approx \frac{7.35 \cdot 10^{22} \text{ kg}}{2.4 \cdot 10^{18} \text{ kg}} = \frac{7.35}{2.4} \cdot \frac{10^{22}}{10^{18}}$$
$$\approx 3 \cdot 10^{22-18} = 3 \cdot 10^4$$

Earth's moon is about 30,000 times as massive as Hydra.



All four of these names have their base in Greek mythology. Nix was named for Nyx, the mother of Charon, and Hydra was named for the nine-headed serpent that guarded Pluto's realm.

Properties of powers can also be used to simplify quotients or products of algebraic expressions containing exponents.

GUIDED

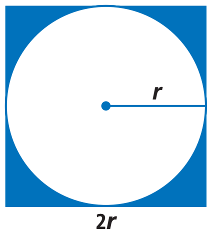
Example 5

Farmers often use circular irrigators on square plots of land, leaving the regions at the corners unirrigated. How does the percentage of irrigated land depend on the radius of the circle?

Solution The diameter of the circle and the side of the square are each $2r$. The area of the circle is πr^2 and the area of the square is $(\underline{\ ?\ })^2 = \underline{\ ?\ }$ by the Power of a Product Postulate. To find the percentage of irrigated land, divide:

$$\frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi r^2}{?} = \frac{\pi}{?} \approx \underline{\ ?\ }.$$

So, regardless of the radius of the circle, $\underline{\ ?\ }$% of the land is irrigated.



Questions

COVERING THE IDEAS

In 1–5, give an example to illustrate each postulate or theorem.

1. Product of Powers Postulate
2. Power of a Product Postulate
3. Power of a Power Postulate
4. Quotient of Powers Theorem
5. Power of a Quotient Theorem

In 6–8, write the expression as a single power using a postulate or theorem from this lesson. Then, check your answer using the repeated multiplication model of powering.

6. $17^3 \cdot 17^4$
7. $(5^2)^3$
8. $\frac{10^5}{10^2}$

9. What postulate or theorem justifies $\left(\frac{2}{y}\right)^4 = \frac{16}{y^4}$?

10. Solve $6^{2x} \cdot 6^3 = 6^7$ for x.

In 11–16, simplify by hand and check using a CAS.

11. $\frac{n^{77}}{n^7}$

12. $60m^6 \cdot \frac{m^3}{6}$

13. $(3x^4)^3$

14. $\frac{w^8}{(w^4)^2}$

15. $v^2 \cdot v^{a-2}$

16. $j^{289} \cdot j^0$

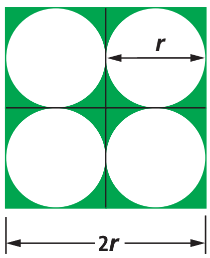
17. How many zeros are at the end of $2^8 \cdot 3^{10} \cdot 5^6$ when written in base 10?

18. A multiple-choice test has 6 pages and 5 questions on each page. If each question can be answered in a different ways, how many ways are there to complete the test?

19. Refer to Example 4. Pluto itself has a mass of approximately 1.3×10^{22} kg. How many times as massive is Pluto than its moon Hydra?

APPLYING THE MATHEMATICS

20. Refer to Example 5. Suppose the farmer decides to split the field into four smaller squares of side length r and irrigate each one separately.



a. Compute the total area of the four circles.

b. Compute the percent of the field that is irrigated in the new arrangement.

Many farmers irrigate their crops using a center pivot irrigation system.

21. a. Graph $y = (x^3)^2$.

b. Find a number n such that the graph of $y = x^n$ coincides with the graph in Part a.

22. a. Check that $\left(\frac{x^{12}}{x^{10}}\right)\left(\frac{x}{2}\right)^2 = \frac{x^4}{4}$ by graphing the functions $f(x) = \left(\frac{x^{12}}{x^{10}}\right)\left(\frac{x}{2}\right)^2$ and $g(x) = \frac{x^4}{4}$ separately.

b. Is this check more or less reliable than testing whether the equation is true for a single value of x?

23. The prime factorization of 875 is $5^3 \cdot 7$. The prime factorization of 8575 is $5^2 \cdot 7^3$. Use this information to find the prime factorization of $875 \cdot 8575$.

24. Waneta computed $(40)^3$ and obtained the answer 640. Does her answer end with enough zeros to be correct? How do you know? What is the correct answer?

25. x^3 and x^4 are powers of x whose product is x^7. Find three more pairs of powers of x whose product is x^7.

26. Below is a proof of the Quotient of Powers Theorem. Suppose b is a nonzero real number, and n and m are integers. Fill in each justification.

$$\begin{aligned} b^{m-n} \cdot b^n &= b^{(m-n)+n} && \text{a. } \underline{\ ?\ } \\ &= b^{m+(-n+n)} && \text{b. } \underline{\ ?\ } \\ &= b^{m+0} && \text{c. } \underline{\ ?\ } \\ b^{m-n} \cdot b^n &= b^m && \text{d. } \underline{\ ?\ } \\ b^{m-n} &= \frac{b^m}{b^n} && \text{e. } \underline{\ ?\ } \end{aligned}$$

In 27–29, write an equivalent expression in which each variable appears once.

27. $\dfrac{18x^4}{(-3x^2)^3}$ **28.** $\dfrac{8a^2b^6c^8}{(4a)^2b^6c^6}$ **29.** $\left(\dfrac{8}{w^3}\right)^2\left(\dfrac{w^2}{4}\right)^3$

REVIEW

30. Fill in the Blank The graph of $f(x) = x^n$ has rotation symmetry whenever n is __?__. **(Lesson 7-1)**

31. Fill in the Blank The graph of $f(x) = x^n$ has reflection symmetry whenever n is __?__. **(Lesson 7-1)**

32. How many real roots does the quadratic equation $2p^2 - 4p + 2 = 0$ have? **(Lesson 6-10)**

33. Calculate $\sqrt{-3} \cdot \sqrt{-5} \cdot \sqrt{-15}$. **(Lesson 6-8)**

EXPLORATION

34. a. Copy and fill in the table at the right. What do you notice about the last digits of your answers?

n	1	2	3	4	5	6	7	8	9	10
n^5	?	?	?	?	?	?	?	?	?	?

b. Copy and fill in the table at the right using scientific notation. Round the constant to the nearest tenth.

n	10	20	30	40	50	60	70	80	90	100
n^5	?	?	?	?	?	?	?	?	?	?

c. An old magic trick has a member of the audience pick a number n between 1 and 99, compute n^5, and tell the magician the result. The magician then instantly tells the audience member the original number by using the observation in Part a to determine the number's last digit, and by comparing the size of the number to the values in the second table (which the magician memorizes) to determine the number's first digit. For example, the audience member might say the number 1,419,857. Use the tables above (but no calculator) to determine the original number.

QY ANSWERS

1. Answers vary. Sample: $(a + b)^3 \neq a^3 + b^3$. Let $a = 1$, $b = 1$. The left side is equal to 8, the right side is equal to 2.

2. 3 zeros