Lesson

7-1 Power Functions

BIG IDEA When n is a positive integer greater than 1, x^n can be interpreted as repeated multiplication.

Recall that the expression x^n, read "x to the nth power" or "the nth power of x", is the result of an operation called **powering** or **exponentiation**. The variable x is called the **base**, n is called the **exponent**, and the expression x^n is called a **power**.

Defining b^n When n Is a Positive Integer

When n is a positive integer and b is any real number, one way to think of b^n is as the nth term of the sequence

$$b^1, b^2, b^3, \dots b^n, \dots .$$

This sequence can be defined recursively as

$$\begin{cases} b^1 = b \\ b^n = b \cdot b^{n-1}, \text{ for } n > 1 . \end{cases}$$

For example, $14^1 = 14$, $14^2 = 14 \cdot 14^1 = 196$, $14^3 = 14 \cdot 14^2 = 14 \cdot 196 = 2744$, and so on.

STOP QY1

As a result of the recursive definition of b^n for any real number b, when n is a positive integer ≥ 2,

$$b^n = \underbrace{b \cdot b \cdot b \cdot \dots \cdot b}_{n \text{ factors}} .$$

This is the *repeated multiplication* definition of a power. For instance, $x^6 = x \cdot x \cdot x \cdot x \cdot x \cdot x$. The definition enables you to use multiplication to calculate positive integer powers of any number without having to calculate each preceding power. For example,

$$-\left(\frac{2}{3}\right)^5 = -\frac{2}{3} \cdot -\frac{2}{3} \cdot -\frac{2}{3} \cdot -\frac{2}{3} \cdot -\frac{2}{3}$$
$$= -\frac{32}{243}.$$

Vocabulary

powering, exponentiation
base
exponent
power
nth-power function
identity function
squaring function
cubing function

Mental Math

Give a number that fits the description or tell if no such number exists.

a. an integer that is not a natural number

b. an irrational number that is not a real number

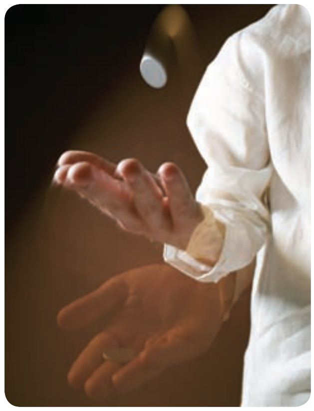
c. a complex number that is not a pure imaginary number

d. a real number that is not a complex number

QY1

Calculate 14^4 without a calculator given that $14^3 = 2744$.

An Example of a Power Function



Powers often arise in counting and probability situations. For example, in the play *Rosencrantz & Guildenstern are Dead,* by Tom Stoppard, the character Rosencrantz spends a lot of time flipping a fair coin. It always lands heads up! This, of course, is very unlikely. The probability that a fair coin lands heads up is $\frac{1}{2}$. Because each flip is independent, the probability of as few as four heads in a row is very small:

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^4 = 0.0625.$$

Had Rosencrantz been tossing a 6-sided die, his probability of tossing the same number four times in a row would have been even smaller:

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \left(\frac{1}{6}\right)^4 \approx 0.0007716.$$

Example 1 generalizes Rosencrantz's situation.

Example 1

Suppose that the probability of an event happening is p. Let A be the probability of the event happening four times in a row.

a. Find a formula for A in terms of p.

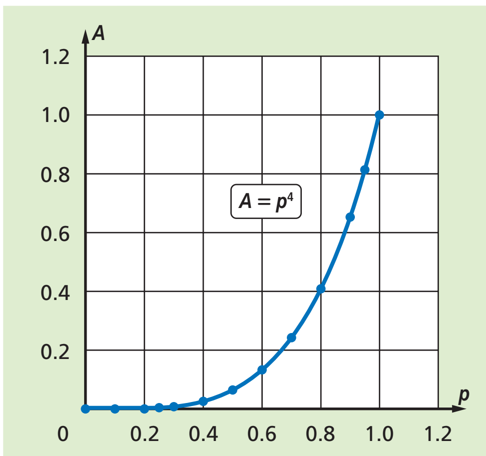
b. Make a table of values and a graph for typical values of p.

Solution

a. The probability of the event happening four times in a row is $p \cdot p \cdot p \cdot p$, so $A = p^4$.

b. The probability p must be a number from 0 to 1. A table and graph are shown below.

p	$A = p^4$
0	0
0.1	0.0001
0.2	0.0016
0.25	0.0039
0.3	0.0081
0.4	0.0256
0.5	0.0625
0.6	0.1296
0.7	0.2401
0.8	0.4096
0.9	0.6561
0.95	0.8145
1	1



Notice from Example 1 that an event with as large as a 90% chance of happening has only a $(0.90)^4 \approx 0.6561$ probability of happening four times in a row. That's less than a 66% chance.

The general formula is a simple application of nth powers.

Probability of Repeated Independent Events

If an event has probability p, and if each occurrence of the event is independent of all other occurrences, then the probability that the event occurs n times in a row is p^n.

Caution: The events must be independent. For instance, if there is a 40% probability of rain on a typical day in a particular location, there is not a $(40\%)^7$ chance that there will be rain all 7 days because rain on one day can affect the chances of rain on the next day.

STOP QY2

QY2

If the probability of an independent event happening is 0.6, what is the probability that the event will occur n times in a row?

Some Simple Power Functions

In general, the function f defined by $f(x) = x^n$, where n is a positive integer, is called the ***n*th-power function**. The function with equation $y = x^5$ is the 5th-power function. The graph in Example 1 is a part of the graph of the 4th-power function.

The simplest power function has the equation $f(x) = x^1$ and is called the **identity function** because the output is identical to the input. The quadratic function f with $f(x) = x^2$ is the 2nd-power, or **squaring function**.

Any real number can be raised to the first or second power. So the domain of each of these positive integer power functions is the set of all real numbers. The range of the identity function is also the set of real numbers. However, because the result of squaring a real number is always nonnegative, the range of the squaring function is the set of all nonnegative real numbers.

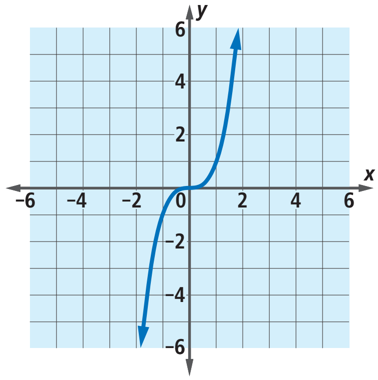
The function with equation $f(x) = x^3$ is called the **cubing function**. The nth-power functions where $n > 3$ do not have special names.

Example 2

Draw a graph and state the domain and range for the cubing function.

Solution Make a table of values and plot points. Connect the points with a smooth curve.

x	$f(x) = x^3$
–5	$(-5)^3 = -125$
–4	$(-4)^3 = -64$
–3	$(-3)^3 = -27$
–2	$(-2)^3 = -8$
–1	$(-1)^3 = -1$
0	$0^3 = 0$
1	$1^3 = 1$
2	$2^3 = 8$
3	$3^3 = 27$
4	$4^3 = 64$
5	$5^3 = 125$



Because x can be any real number, the domain is the set of real numbers.

Because x^3 can be as large or as small as we want, positive or negative, the range is the set of real numbers.

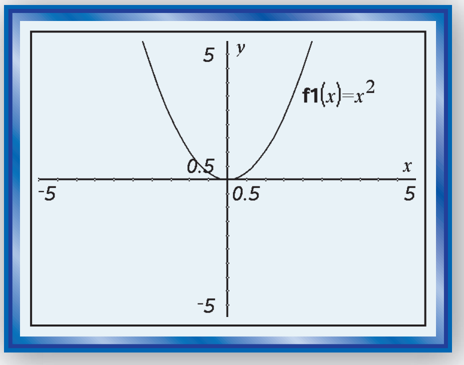
Properties of Power Functions

Activity

MATERIALS CAS or graphing calculator

Work with a partner.

Step 1 Graph several functions of the form $y = x^n$ in a window where $\{x| -5 \le x \le 5\}$ and $\{y| -5 \le y \le 5\}$. One of you should graph functions for even values of n and the other should graph functions for odd values of n. Graph at least four functions each in the same window. Sketch your results on a separate sheet of paper and label each function.



Step 2 State the domain and range for each of the functions you graphed.

Step 3 Compare and contrast your graphs, then discuss the results with your partner. What other properties do these functions seem to have?

The table on the next page summarizes some properties of the nth power functions f with $f(x) = x^n$, where n is a positive integer. As in the Activity, these properties depend on whether n is even or odd.

Properties of Power Functions with Equations $y = x^n$, for integers $n > 0$		
	n is even.	n is odd.
Some ordered pairs on the graph	(0, 0) (1, 1) (–1, 1)	(0, 0) (1, 1) (–1, –1)
Domain	set of all real numbers	set of all real numbers
Range	$\{y \mid y \geq 0\}$	set of all real numbers
Quadrants	I and II	I and III
Symmetry	Reflection symmetry over the y-axis For all x, $f(x) = f(-x)$.	Rotation symmetry of 180° about the origin For all x, $f(-x) = -f(x)$.

STOP QY3

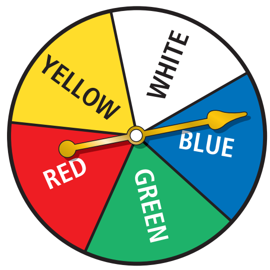
QY3

What type of symmetry does the graph of $y = x^{19}$ have?

Questions

COVERING THE IDEAS

1. Given that $2^{10} = 1024$, calculate 2^{11} and 2^{12} without a calculator.
2. What is the first term of the sequence s with $s_n = (0.9)^n$ in which $s_n < 0.01$?
3. Refer to Example 1. What is the meaning of the number 0.0081 in the table?
4. A spinner has 5 congruent sectors, as pictured at the right. If the spinner is fair, what is the probability of landing in the blue sector 6 times in 6 spins?

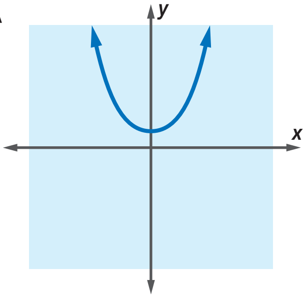


5. Alvin forgot to study for a multiple-choice test with 10 independent questions. Find the probability of getting all questions correct if the probability of getting a single question correct is
   a. 0.5. **b.** 0.25. **c.** p.
6. What is the function f with equation $f(x) = x^3$ called?

In 7 and 8, an equation for a power function f is given.
   a. Sketch a graph without plotting points or doing any calculations.
   b. State the domain and range of the function.
   c. Describe any symmetry the graph may have.

7. $f(x) = x^5$
8. $f(x) = x^{10}$
9. **Fill in the Blanks** If n is even, the range of $y = x^n$ is __?__ and the graph is in Quadrants __?__ and __?__.
10. **Fill in the Blanks** If n is odd, the range of $y = x^n$ is __?__ and the graph is in Quadrants __?__ and __?__.

11. **Multiple Choice** Which of the following graphs could represent the function with equation $y = x^8$? Justify your answer.

A


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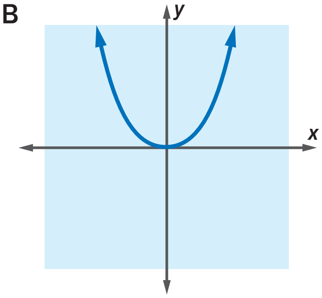

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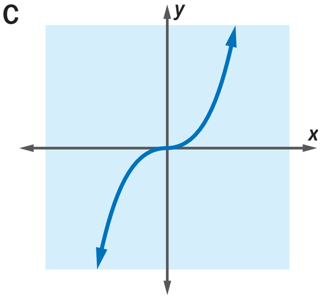


APPLYING THE MATHEMATICS

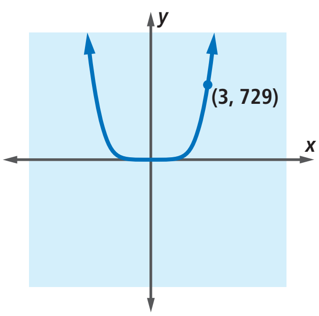
12. Refer to Example 1. Suppose Rosencrantz gets a new fair coin to flip. This coin always seems to alternate heads and tails with each flip: H T H T H T H T H … . Is the pattern H T H T more likely, less likely, or equally as likely as four heads in a row? Justify your answer.

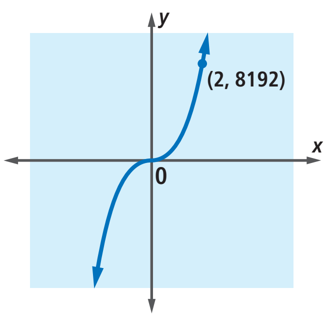
13. Izzie Wright forgets to study for a history quiz and makes random guesses at each answer. The quiz has 5 true/false questions and 10 multiple-choice questions with 4 choices each.
    a. What is the probability Izzie will guess the correct answer on a true/false question?
    b. What is the probability Izzie will guess the correct answer on a multiple-choice question?
    c. Calculate the probability that Izzie will answer all 15 questions correctly.

14. **True or False** The graphs of the odd power functions have no minimum or maximum values.

15. Consider the graph of each function. For what values of x is the graph above the x-axis, and for what values of x is the graph below the x-axis?
    a. $f : x \to x^{33}$
    b. $g : x \to x^n$, where n is even

In 16 and 17, write an equation for the *n*th-power function that is graphed.

16.


17.


18. The point (-7, 2401) is on the graph of an even nth power function. What point (other than the origin, (1, 1), and (-1, 1)) must also be on the graph of this function? Why?

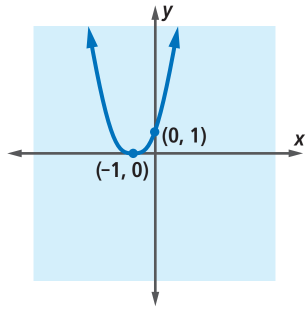
19. a. Graph $f(x) = x^2$ and $g(x) = x^6$ on the same set of axes.
   b. For what value(s) of x is $f(x) = g(x)$?
   c. For what values of x is $f(x) > g(x)$?
   d. As x increases from 0 to 1, what happens to the difference between $f(x)$ and $g(x)$?

REVIEW

20. **Fill in the Blanks** The graph of the quadratic function Q where $Q(x) = x^2 + 5x + 2$ intersects the x-axis in __?__ point(s) and the y-axis in __?__ point(s). **(Lesson 6-10)**

21. a. Write $(1 + i)(1 - i)$ in $a + bi$ form.
   b. Calculate $(1 + i)(2 + 3i)(1 - i)$. **(Lesson 6-9)**

22. Give an equation for the quadratic function graphed at the right. **(Lesson 6-3)**



23. Solve $\begin{cases} x + 2y - 4z = -8 \\ 2x + z = 2 \\ 4x - 4y - z = 16 \end{cases}$. **(Lesson 5-6)**

24. In 2006, the St. Louis Cardinals defeated the Detroit Tigers in baseball's World Series. The win-loss records for the two teams in the three regular seasons leading up to this World Series are given in the matrices below. Find a matrix for the total win-loss record of each team over all three seasons. **(Lesson 4-2)**

2004

	Wins	Losses
Detroit	72	90
St. Louis	105	57

2005

	W	L
D	71	91
SL	100	62

2006

	W	L
D	95	67
SL	83	78

25. **Fill in the Blank** $2x^{15} \cdot$ __?__ $= 3x^{18}$ **(Previous Course)**

EXPLORATION

26. Consider nth power functions with equations of the form $f(x) = ax^n$, where a is any real number.
   a. What are the domain and range of f if n is even and a is negative?
   b. What are the domain and range of f if n is odd and a is negative?
   c. How do the graphs of $f(x) = ax^n$ and $f(x) = -ax^n$ compare?

QY ANSWERS

1. $14^4 = 14 \cdot 14^3 = 14 \cdot 2744 = 38{,}416$

2. $(0.6)^n$

3. rotation symmetry about the origin