# Lesson 7-7 Comparing Linear Increase and Exponential Growth

**BIG IDEA** In the long run, exponential growth always overtakes linear (constant) increase.

In the patterns that are constant increase/decrease situations, a number is repeatedly *added*. In exponential growth/decay situations, a number is repeatedly *multiplied*. In this lesson, we compare what happens as a result.

### Mental Math

**What is the date of the *x*th day of the year in a nonleap year when**

**a.** $x = 100$?

**b.** $x = 200$?

**c.** $x = 300$?

**GUIDED**

## Example

**Suppose you have \$10. For two weeks, your rich uncle agrees to do one of the following.**

**Option 1: Increase what you had the previous day by \$50.**

**Option 2: Increase what you had the previous day by 50%.**

**Which option will give you more money?**

**Solution** Make a table to compare the two options for the first week. Use the Now/Next method to fill in the table. The exponential growth factor is 1.50.

Start = \$10
Next = Now + \$50

| Day | Option 1: Add \$50. |
|---|---|
| 0 | \$10 |
| 1 | ? |
| 2 | ? |

(+ \$50 from each day to the next)

Start = \$10
Next = Now • 1.50

| Day | Option 2: Multiply by 1.50. |
|---|---|
| 0 | \$10 |
| 1 | ? |
| 2 | ? |

(• 1.50 from each day to the next)

Continue the table until day 14. You should find that at first, you get more money from Option 1. But the table shows that starting on day __?__, Option 2 gives more money. In the long run, Option 2, increasing by 50% each day, is the better choice.

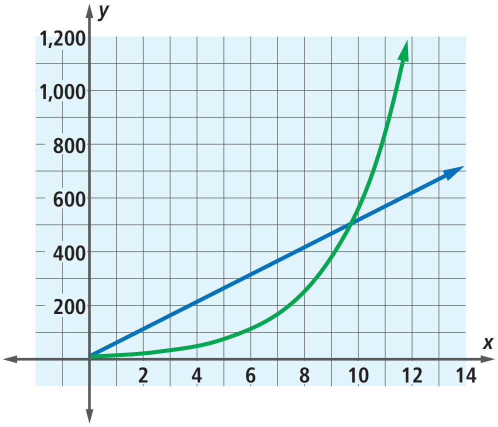
Above, the two options were described by telling how the amounts changed each day. In that situation, the Now/Next method works well. But to graph the situation on your calculator, you need equations for these functions.

## Comparing Using a Graph

To find the equations for the functions in the Guided Example, you can make a table to compare the two options during the first week. The exponential growth factor is 1.50. Let $L(x)$ = the amount given to you under Option 1, and let $E(x)$ = the amount given to you under Option 2.

| Day | Option 1: Add \$50. | Option 2: Multiply by 1.50. |
|---|---|---|
| 0 | $L(0) = 10 = \$10.00$ | $E(0) = 10 = \$10.00$ |
| 1 | $L(1) = 10 + 50 \cdot 1 = \$60.00$ | $E(1) = 10 \cdot 1.50^1 = \$15.00$ |
| 2 | $L(2) = 10 + 50 \cdot 2 = \$110.00$ | $E(2) = 10 \cdot 1.50^2 = \$22.50$ |
| 3 | $L(3) = 10 + 50 \cdot 3 = \$160.00$ | $E(3) = 10 \cdot 1.50^3 = \$33.75$ |
| 4 | $L(4) = 10 + 50 \cdot 4 = \$210.00$ | $E(4) = 10 \cdot 1.50^4 = \$50.63$ |
| 5 | $L(5) = 10 + 50 \cdot 5 = \$260.00$ | $E(5) = 10 \cdot 1.50^5 = \$75.94$ |
| $x$ | $L(x) = 10 + 50x$ | $E(x) = 10 \cdot 1.50^x$ |

We see how the values compare by graphing the two functions $E$ and $L$. The graphs of $L(x) = 10 + 50x$ and $E(x) = 10 \cdot 1.5^x$ are shown at the right.




The line $L(x) = 10 + 50x$ has a constant rate of change. The graph of $y = 10 \cdot 1.5^x$ is a curve that gets steeper and steeper as you move to the right. Notice that at first the exponential curve is below the line. But toward the middle of the graph, it intersects the line and passes above it. On later days, the graph of the curve rises farther and farther above the line.

The longer your uncle gives you money, the better Option 2 is compared to Option 1.

## Comparing Using Spreadsheets

Activity 1 shows how to use a spreadsheet to confirm the results of Guided Example 1.

### Activity 1

**Step 1** Create a spreadsheet similar to the one at the right. Be sure to have titles in row 1. In cells A2 through A16 enter the numbers 0 to 14.

| ◇ | A | B | C |
|---|---|---|---|
| 1 | Day $x$ | Option 1 | Option 2 |
| 2 | 0 | 10 | 10 |
| 3 | 1 | =B2+50 | |

**Step 2** Type =B2+50 in cell B3. Press ENTER. What appears in cell B3?

**Step 3** Type the formula for Option 2 into cell C3. (*Hint*: What is the Now/Next formula for Option 2?)

**An advantage of spreadsheets is that you don't have to type a formula into each cell. When you type the formula =B2+50 into cell B3, the spreadsheet remembers this as: "Into this cell put 50 plus the number that is in cell B2 above." For example, if you copy cell B3 to cell D5, the formula copied will change to =D4+50 because one cell above D5 is D4. This way of copying in spreadsheets is called *replication.***

**Step 4** Replicate the formula in cell B3 into cells B4 through B16.

**Step 5** Replicate the formula in cell C3 into cells C4 through C16.

**Step 6** Compare your spreadsheet to the table on page 440. Experiment by changing the starting amount of $10 to other values. Then go back to the original starting amount of $10 before doing the next step.

**Step 7** Add two more columns to your spreadsheet.

| ◇ | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | Day $x$ | Option 1 | Option 2 | $L(x) = 10 + 50x$ | $E(x) = 10 \bullet 1.5^x$ |
| 2 | 0 | 10 | 10 | | |
| 3 | 1 | =B2+50 | | | |

**Step 8** Type =10+50*A2 in cell D2. This puts into D2 the value of the function *L* for the domain value in cell A2.

**Step 9** Replicate the formula in cell D2 into cells D3 through D16.

**Step 10** Compare the values in columns B and D. If they are the same, then you know that you have done the previous steps correctly.

**Step 11** Type =10*1.5^A2 into cell E2. This puts into E2 the value of the function *E* for the domain value in cell A2.

**Step 12** Replicate the formula in cell E2 into cells E3 through E16. What should happen? Have you done the previous steps correctly?

## Activity 2

In 1993, Florida introduced 19 mountain lions into its northern region. With animals in the wild, there are two scenarios. If there are no limiting factors, the population of animals tends to grow exponentially. However, if limiting factors are established, the population growth tends to be linear. Limiting factors can be things such as climate, availability of food, predators, and hunting.

Suppose the scientists who introduced the mountain lions into northern Florida used one of the following options to model the population growth.

Option 1: There are limiting factors so that 2 more mountain lions appear each year.

Option 2: There are no limiting factors so that the population grows by 6% each year.

A typical male mountain lion patrols 50 to 300 square miles, depending on how plentiful food is.

Source: *USAToday*

**Step 1** First, create a spreadsheet similar to the one below.

| ◇ | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | Year | Option 1 | Option 2 | $L(x)$ | $E(x)$ |
| 2 | 1993 | 19 | 19 | | |
| 3 | 1994 | | | | |

**Step 2** Enter formulas into B3 and C3 to calculate the population using the Now/Next method.

**Step 3** Copy and paste B3 into cells B4 and lower. Also copy and paste C3 into cells C4 and lower. Be sure to gather enough data and compare the populations in column B to those in column C.

**Step 4** Enter formulas for $L(x)$ and $E(x)$ in columns D and E and copy these for as many rows as you used in Step 3.

**Step 5** Graph each option's population equation on the same axes. Let the $x$-coordinates be the number of years since 1993. (Let $x = 0$ be 1993.) You can use the chart feature of the spreadsheet to create the graphs.

**Step 6** Answer the following questions using the collected data and the graphs.

1. In 2010, which option would provide a larger population of mountain lions?
2. In which year (if ever), would Option 2 create a larger population of mountain lions?

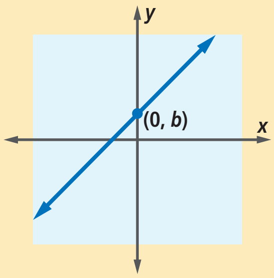
## A Summary of Constant Increase and Exponential Growth

In this lesson, you have seen that differences between linear and exponential models can be seen in the equations that describe them, the tables that list ordered pairs, and the graphs that picture them. You have seen that if the growth factor is greater than 1, exponential growth always overtakes constant increase. Here is a summary of their behavior.

### Constant Increase

- Begin with an amount $b$.
- Add $m$ (the slope) in each of the $x$ time periods.
- After $x$ time periods, the amount is given by the function $L(x) = mx + b$.

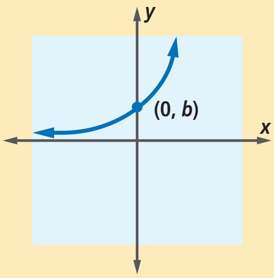
$L(x) = mx + b, m > 0$




### Exponential Growth

- Begin with an amount $b$.
- Multiply by $g$ (the growth factor) in each of the $x$ time periods.
- After $x$ time periods, the amount is given by the function $E(x) = b \cdot g^x$.

$E(x) = b \cdot g^x,\ g > 1$




## Questions

### COVERING THE IDEAS

1. What is the difference between a constant increase situation and an exponential growth situation?

**In 2–5, let $L(x) = 20 + 3x$ and $E(x) = 20(1.03)^x$.**

2. Calculate $L(5)$ and $E(5)$.
3. Sketch a graph of both functions on the same axes.
4. Give an example of a value of $x$ for which $E(x) > L(x)$.
5. What kind of situation could have led to these equations?

6. Two friends found $100 and split it equally between them. Alexis put her half in a piggy bank and added $7 to it each year. Lynn put her half in a bank with an annual yield of 7%.
   a. Make a spreadsheet to illustrate how much money each friend has at the end of each year for the next 25 years. Have one column represent Alexis and one column represent Lynn.
   b. Sketch a graph to represent the amount of money each friend has over the next 25 years.

7. Rochelle started to make the following spreadsheet. She replicated the formula in cell A2 into cells A3 and A4.

| ◇ | A | B |
|---|---|---|
| 1 | 28 | 6 |
| 2 | =A1+13 | =1.2*B1 |
| 3 | | |
| 4 | | |
| 5 | | |

   a. Give the formulas that will occur in cells A3 and A4.
   b. What numbers result from the formulas in A3 and A4?
   c. How will the values in A3 and A4 change if Rochelle changes the start value in A1 to –5?
   d. Does column A illustrate constant increase or exponential growth? Explain.

8. Repeat Question 7 for column B.

9. The number of deer in the state of Massachusetts is a problem. In 1998, the deer population was estimated to be about 85,000. The Massachusetts Division of Fisheries and Wildlife had to decide whether to allow hunting (a limiting factor) or to ban hunting (no limiting factor). If hunting is allowed, they predict the deer population to increase at a constant rate of about 270 deer a year. If hunting is not allowed, the prediction is the deer population would grow exponentially by 15% each year.
   a. Write a Now/Next formula for the deer population if hunting is allowed.
   b. Write a Now/Next formula for the deer population if hunting is banned.
   c. Let $L(x)$ = the number of deer $x$ years after 1998 if hunting is allowed. Find a formula for $L(x)$.
   d. Let $E(x)$ = the number of deer $x$ years after 1998 if hunting is not allowed. Find a formula for $E(x)$.
   e. The state allowed hunting. The 2006 deer population was estimated between 85,000 and 95,000. Was the prediction correct?

**In 1906, the U.S. deer population was a sparse 500,000. Today, experts estimate that 20 million deer roam the nation.**

Source: Tufts University

## APPLYING THE MATHEMATICS

**10.** Refer to the spreadsheet at the right.

**a.** What Now/Next formula could be used to generate the numbers in column A?

**b.** Let $f(x)$ be the value in column A at time $x$. What is a formula for $f(x)$?

| ◇ | A |
|---|---|
| 1 | |
| 2 | 15 |
| 3 | 165 |
| 4 | 1815 |
| 5 | 19965 |
| 6 | 219615 |
| 7 | 2415765 |
| 8 | 26573415 |
| 9 | 292307565 |
| 10 | 3215383215 |
| 11 | 35369215365 |

**11.** Suppose you are reading a 900-page novel at the rate of 25 pages per hour. You are currently at page 67.

**a.** Is the number of pages you read in the book an example of constant increase or exponential growth? Explain.

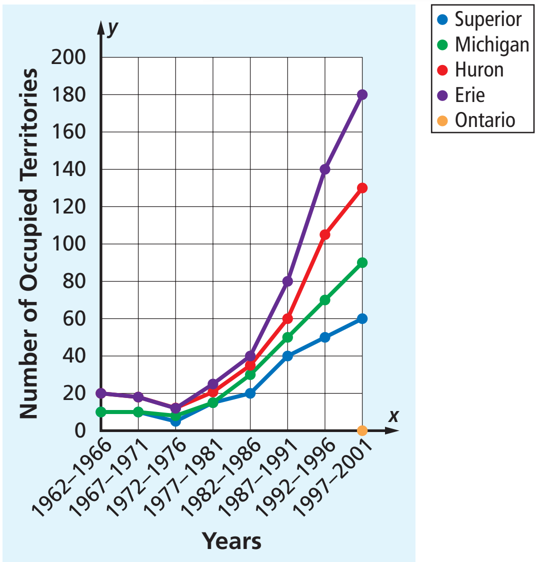
**b.** Write an equation to describe the pages finished $x$ hours from now.

**c.** How many hours will it take you to finish the book?

**12.** The graph at the right shows the number of territories in which bald eagles nest around the five Great Lakes.

**a.** Would you describe the graphs as constant increase or exponential growth? Explain your answer.

**b.** The graph of which lake can be represented by $y = 14.5 \cdot 1.053^x$, where $x$ is the years since 1962? Explain your answer.

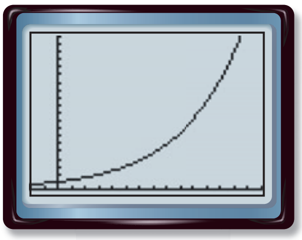



**MATCHING** In 13–16, each graph is drawn on the window $-2 \le x \le 15$, $0 \le y \le 2{,}000$. Match the graph with its equation.

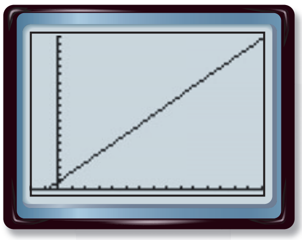
**a.** $f(x) = 100 \cdot 1.25^x$ **b.** $g(x) = 100 + 125x$

**c.** $h(x) = 100 + 60x$ **d.** $j(x) = 100 \cdot 1.1^x$

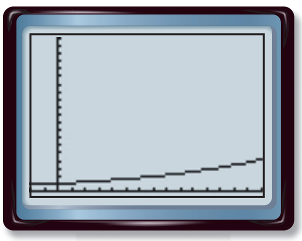
**13.**



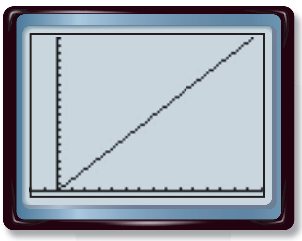
**14.**



**15.**



**16.**



17. The principal of a high school is making long-range budget plans. The number of students dropped from 2,410 students to 2,270 in one year. Student enrollment is dropping, as shown in rows 2 and 3 of the spreadsheet. The situation may be modeled by a linear function or an exponential function.

a. Make a spreadsheet similar to the one at the right. Show the future enrollments in the two possible situations.

| ◇ | A | B | C |
|---|---|---|---|
| 1 | | Constant Decrease | Exponential Decrease |
| 2 | 0 | 2410 | 2410 |
| 3 | 1 | 2270 | 2270 |

b. What are the predicted enrollments 5 years from the year shown in row 3? By how much do they differ?

c. What are the predicted enrollments 15 years from year shown in row 3? By how much do they differ?


**The safest way to transport children to and from school and school-related activities is in a school bus.**

Source: National Association of State Directors of Pupil Transportation Services

## REVIEW

18. Let $M(x) = 13x - 18$. Find the value of $x$ for which $M(x) = -1.2$. **(Lesson 7-6)**

19. **True or False** If $a$ is any real number, then $a$ is in the range of the function with equation $y = 3x$. **(Lesson 7-5)**

20. Write an exponential expression for the number, which, written in base 10, is 7 followed by $n$ zeroes (For example when $n = 3$, this number is 7,000.) **(Lesson 7-2)**

21. Write the equation $\frac{y - 3x + 2}{15} - \frac{1}{3} = \frac{7x + 2y}{5}$ in standard form. **(Lessons 6-8, 3-8, 2-2)**

22. Do the points (–1, 3), (5, 4), and (0, –8) lie on a line? How can you tell? **(Lesson 6-6)**

23. If you ask a random person the date of his or her birth, what is the probability that it will be one of the first ten days of the month? **(Lesson 5-6)**

24. Calculate $5(-5)^5$. **(Lesson 2-4)**

## EXPLORATION

25. The statement, "If the growth factor $g$ is greater than 1, exponential growth always overtakes constant increase," was made at the start of this lesson. Write a similar statement that could describe the relationship between exponential decay and constant decrease.