Lesson 7-3 Exponential Decay

Vocabulary

exponential decay

half-life

▶ **BIG IDEA** Decay at a constant percentage rate can be described by an expression of the form bg^x, where $0 < g < 1$ and the variable is in the exponent.

The Growth Factor and the Type of Exponential Change

Three children were arguing. Tammy said, "If you multiply 5 by a positive number, the answer is always greater than 5." Nancy said, "No, you're wrong! I can multiply 5 by something and get an answer less than 5." Leon said, "I can multiply 5 by something and get 5 for an answer."

In the children's arguments, what matters is how the multiplier compares to 1. If Nancy chooses a multiplier between 0 and 1, multiplying by 5 gives a result that is less than 5. For example, $5 \cdot \frac{1}{10} = \frac{1}{2}$. Of course, Leon can do $5 \cdot 1$ and get 5 for an answer.

This relates to exponential equations of the form $y = b \cdot g^x$ because the growth factor g can be greater than, equal to, or less than 1. In the last lesson, you saw only situations in which the growth factor was greater than 1, so there was an increase over time in each case. While a growth factor always has to be positive, it can be less than 1. When this is true, there is a decrease over time. This happens in situations of **exponential decay.**

Mental Math

Find the number.

a. 7 less than 4 times the number is 13.

b. –10 times the number, plus 80, is 10.

c. 14 minus 3 times the number is 121.

Examples of Exponential Decay

Psychologists use exponential decay models to describe learning and memory loss. In Example 1, the growth factor is less than 1 so the amount remembered decreases.

Example 1

Assume that each day after cramming, a student forgets 20% of the vocabulary words learned the day before. A student crams for a French test on Friday by learning 100 vocabulary words Thursday night. But the test is delayed from Friday to Monday. If the student does not study over the weekend, how many words is he or she likely to remember on Monday?

(continued on next page)

Solution If 20% of the words are forgotten each day, 80% are remembered.

Day	Day Number	Number of Words Remembered
Thursday	0	100
Friday	1	100(0.80) = 80
Saturday	2	$100(0.80)(0.80) = 100(0.80)^2 = 64$
Sunday	3	$100(0.80)(0.80)(0.80) = 100(0.80)^3 = 51.2 \approx 51$
Monday	4	$100(0.80)(0.80)(0.80)(0.80) = 100(0.80)^4 = 40.96 \approx 41$

On Monday the student is likely to remember about 41 vocabulary words.

STOP QY1

As they get old, cars and other manufactured items often wear out. Therefore, their value decreases over time. This decrease, called *depreciation,* is often described by giving the percent of the value that is lost each year. If the item is worth r percent less each year, then it keeps $(1 - r)$ percent of its previous value. This is the growth factor. (The word "growth" is used even though the value is shrinking.)

QY1

If the student does not study between now (Thursday) and the test, how many words will be remembered x days from Thursday?

Example 2

In 1998, a new car cost $21,000. Suppose its value depreciates 15% each year.

a. Find an equation that gives the car's value y when it is x years old.

b. What was the predicted value of the car in 2005? How close is this to the actual price of a 1998 car, which was $7,050 in 2005?

c. Graph the car's value for the interval $0 \leq x \leq 8$.

About 35% of the cost of owning and operating a car comes from depreciation.

Source: Federal Highway Administration

Solutions

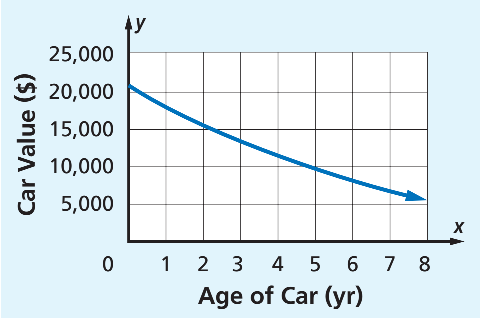
a. If the car loses 15% of its value each year, it keeps 85%, so the growth factor is $1 - 0.15 = 0.85$. In the exponential growth equation $y = b \cdot g^x$, b is 21,000, the new car's price, and g is 0.85. An equation that gives the value of the car is $y = 21{,}000 \cdot (0.85)^x$.

b. The year 2005 is 7 years after 1998, so $x = 7$.
$21{,}000(0.85)^7 \approx 6{,}732.12$. The predicted value was $6,732.12. The deviation between the actual price and the predicted price was $\$7{,}050 - \$6{,}732.12 = \$317.88$, so the model gives a fairly accurate prediction.

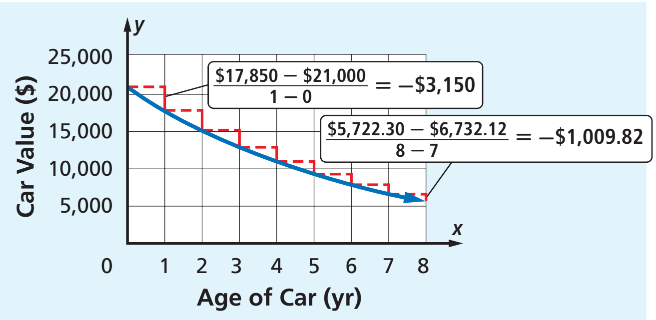
c. The table below shows the value of the car each year. The initial value, \$21,000, is the *y*-intercept of the graph. In the table, to move from one *y* value to the next, you can multiply by 0.85. The decrease is seen in the graph by the fact that the curve goes downward as you go to the right.

x	y (dollars)
0	21,000.00
1	17,850.00
2	15,172.50
3	12,896.63
4	10,962.13
5	9,317.81
6	7,920.14
7	6,732.12
8	5,722.30

· 0.85
· 0.85



As with graphs of exponential growth, the points of an exponential decay relationship lie on a curve rather than a straight line. Another graph of the car value is shown below, with segments showing the rate of change between points. Notice that as you go from one year to the next, the amount of decrease is decreasing. For example, in going from new ($x = 0$) to 1 year old ($x = 1$), the value of the car dropped \$3,150. But between $x = 7$ and $x = 8$, the value lost was only about \$1,000.



In each case, 15% of the value is lost. But as time passes, this is 15% of a smaller number.

STOP **QY2**

In Example 3 on the next page, the "population" is the amount of medication in a person's body.

QY2

A new boat costs \$32,000. Its value depreciates by 8% each year. Give an equation for *y*, the value of the boat, when it is *x* years old.

GUIDED

Example 3

A common medicine for people with diabetes is insulin. Insulin breaks down in the bloodstream quickly, with the rate varying for different types of the medication. Suppose that initially there are 10 units of insulin in a person's bloodstream and that the amount decreases by 3% each minute.

a. Write an equation to describe *y*, the amount of insulin in the bloodstream, after *x* minutes have passed.

b. Make a calculator table for the equation from Part a. Use the table to find when 5 units of insulin remain in the bloodstream. (This is half of the initial amount. The amount of time it takes half the quantity to decay is called the half-life.)

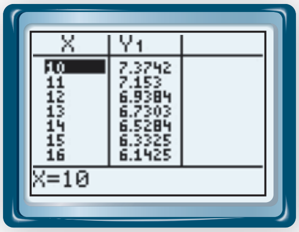
c. How much insulin remains after 4 hours?

d. According to the equation, when will the amount of insulin in the body be zero?

Solutions

a. The amount of insulin starts at $\underline{\ ?\ }$ units. Because 3% of the insulin is lost each minute, the growth factor is $\underline{\ ?\ }$. The exponential equation is $y = \underline{\ ?\ } \cdot \underline{\ ?\ }^{x}$.

b. The screen at the right shows the starting value of 10 units, with x increasing by 1 for each row. Scroll down the table to find where y is close to 5 units. This happens at $x = \underline{\ ?\ }$. So 5 units of insulin remain after about $\underline{\ ?\ }$ minutes.



c. First change 4 hours into $\underline{\ ?\ }$ minutes. When $x = \underline{\ ?\ }$ minutes, then $y = \underline{\ ?\ } \cdot \underline{\ ?\ }^{\underline{?}} \approx 0.007$ unit of insulin remain. So after 4 hours, there is almost no insulin left.

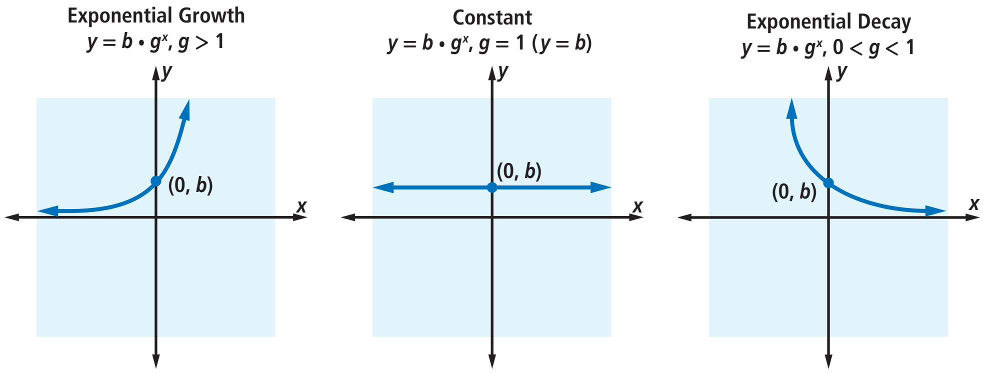
d. Using this equation, the amount of insulin in the body will never be zero. No matter how great the value of x becomes, y will always be greater than zero.

The above example illustrates that a quantity decaying exponentially will theoretically never reach zero. Because this is true, there will never be an ordered pair with a y-coordinate of exactly 0. So the graph of an exponential equation $y = b \cdot g^x$ does not intersect the x-axis.

Graphs and Growth Factors

Exponential growth and exponential decay are both described by an equation of the same form, $y = b \cdot g^x$.

The value of g determines whether the equation describes growth or decay. A third situation happens when there is no change. The three possibilities are graphed below.



The growth factor can be given in words like "double," "triple," or "half," which indicate a factor of 2, 3, or $\frac{1}{2}$, respectively. But it is also common to describe the growth factor by using percent. If the quantity is growing by a percent r, the value of growth factor g is greater than 1 and the exponential equation is $y = b(1 + r)^x$. If the quantity is shrinking, g is less than 1 and the equation is $y = b(1 - r)^x$. Recall that the graph of equation $y = k$ is a horizontal line, like that shown in the middle graph above. In this situation, in which the original quantity remains constant, there is neither growth nor decay. It can be described by an exponential equation where g is 1. For example, the horizontal line $y = 5$ is also the graph of $y = 5 \cdot 1^x$, since $1^x = 1$.

Questions

COVERING THE IDEAS

In 1–3, give the growth factor for a quantity with the given characteristic.

1. decreases exponentially by 17%
2. increases exponentially by 2.5%
3. does not change
4. Many teachers have policies about late work that lower a student's grade for each day that it is late. Suppose a teacher lowers the grade of a 50-point assignment by 20% for every day late.
   a. Let x = the number of days late an assignment is and y = the number of points the assignment would earn. Make a table of values using $x = 0, 1, 2, 3, 4$, and 5.
   b. Write an equation to describe the relationship.

5. Suppose a new car costs \$32,000 in 2006. Find the value of the car in one year if the following is true.
   a. The car is worth 85% of its purchase price.
   b. The car depreciated 20% of its value.
   c. The value of the car depreciated d%.

6. A new piece of industrial machinery costs \$2,470,000 and depreciates at a rate of 12% per year.
   a. Find the value of the machine after 15 years.
   b. Find the value of the machine after t years.

7. A person with diabetes requires a dose of 15 units of insulin. Assume that 3% of the insulin is lost from the bloodstream each minute. How much insulin remains in the bloodstream after 30 minutes? After x minutes?

8. a. Complete the table for the exponential decay situation at the right.
   b. Write an equation to describe the relationship.

x	y
0	160
1	?
2	?
3	?

$\cdot\ 0.75$
$\cdot\ 0.75$
$\cdot\ 0.75$

In 9–11, classify the pattern in the table as exponential growth, exponential decay, or constant.

9.

x	y
0	35
1	?
2	?

$\cdot\ 1$
$\cdot\ 1$

10.

x	y
0	0.26
1	?
2	?

$\cdot\ 1.13$
$\cdot\ 1.13$

11.

x	y
0	458
1	?
2	?

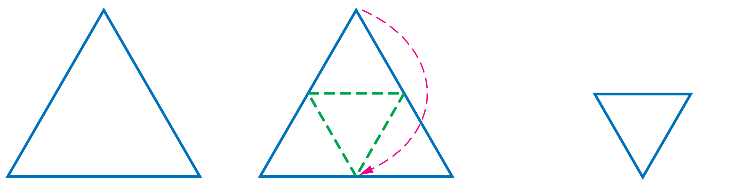
$\cdot\ 0.32$
$\cdot\ 0.32$

12. **Fill in the Blank** Fill in each blank with “decay” or “growth.”
   a. $\frac{2}{3}$ can be the growth factor in an exponential ___?___ situation.
   b. $\frac{3}{2}$ can be the growth factor in an exponential ___?___ situation.

APPLYING THE MATHEMATICS

13. Suppose a school has 2,500 students and the number of students is decreasing by 2% each year.
   a. If this rate continues, write an equation for the number of students after x years.
   b. If this rate continues, how many students will the school have 10 years from now?

14. Imagine that you begin with a cutout of an equilateral triangle. If you fold on the dotted lines as shown below, each vertex will touch the midpoint of the opposite side. Four regions will be formed.



If you repeat this process, you will get a sequence of successively smaller equilateral triangles. Suppose the original triangle has an area of 100 square centimeters.

a. Complete the chart below.

Set of folds	0	1	2	3
Regions	1	4	?	?
Area of a Region (cm^2)	100	?	?	?

b. Write an equation to describe the number of regions y after x sets of folds.

c. Write an equation to describe the area of each region y after x sets of folds.

15. Suppose one plate of tinted glass allows only 60% of light to pass through. The amount of light y that will pass through x panes of glass can then be described by the exponential decay equation $y = 0.6^x$.

a. Plot and label 5 points on a graph of this equation.

b. If enough panes of glass are put together, will the amount of light passing through the panes ever be zero according to this model?

16. The amount of a radioactive substance decreases over time. The *half-life* of a substance is the amount of time it takes half of the material to decay. Strontium-90 has a half-life of 29 years. This means that in each 29-year period, one half of the strontium-90 decays and one half remains. Suppose you have 2,000 grams of strontium-90.

a. How much strontium-90 will remain after 5 half-life periods?

b. How much strontium-90 will remain after 10 half-life periods?

c. How many years equal 10 half-life periods of strontium-90?

Colored light patterns reflect on the floor of the Old Louisiana State Capitol in Baton Rouge during the building's restoration in the early 1990s.

17. Consider the equation $y = \left(\frac{1}{2}\right)^x$.

a. Make a table of values giving y as both a fraction and a decimal when $x = 0, 1, 2, 3, 10$, and 20.

b. Find all solutions to the equation $\left(\frac{1}{2}\right)^x = 0$.

REVIEW

18. The rule of 72 is a simple finance method that can be used to estimate the number of periods that it will take an investment to double in value. The method is to divide 72 by the growth rate expressed as a percent, and the result will be the approximate number of periods. For example, if \$50 is invested at 6% per year, then the rule of 72 says that after $\frac{72}{6} = 12$ years, a \$50 investment will be worth $50 \cdot 2 = \$100$. Calculation shows that after 12 years, this investment is worth \$100.61. **(Lessons 7-2, 7-1)**

a. Use the rule of 72 to estimate the time it will take for \$250 invested at 9% to double in value.

b. Use the compound interest formula to give an exact value of a \$250 investment after the number of periods found in Part a.

19. Refer to the table at the right that shows the average consumption of bottled water per person in the United States. **(Lesson 6-7, 6-4)**

a. Draw a scatterplot with *year* on the x-axis and *gallons per person* on the y-axis.

b. Find an equation of an eyeballed line to the data.

c. Write the slope-intercept form of your equation for the line of fit from Part b.

d. Use your equation to predict the consumption of bottled water in the United States in 2007.

Year	Bottled Water (gal/person)
2000	17.3
2001	18.8
2002	20.9
2003	22.4
2004	24.0
2005	25.7

Source: www.beveragemarketing.com

20. Solve $4.8q + 9.1 < 12.3q - 7.4$. **(Lessons 4-5, 3-8)**

EXPLORATION

21. Archaeologists use radioactivity to determine the age of ancient objects. Carbon-14 is a radioactive element that is often used to date fossils. Find the half-life of carbon-14 and describe how it is used to help date fossils.

QY ANSWERS

1. $100(0.80)^x$

2. $y = 32{,}000 \cdot (0.92)^x$