Lesson 7-2

Exponential Growth

Vocabulary

exponential growth

growth factor

exponential growth equation

BIG IDEA Growth at a constant percentage rate can be described by an expression of the form bg^x, where $g > 1$ and the variable is in the exponent.

Powering and Population Growth

An important application of powers is in population growth situations. As an example, consider rabbit populations, which can grow quickly. In 1859, 24 rabbits were imported to Australia from Europe as a new source of food. Rabbits are not native to Australia, but conditions there were ideal for rabbits and so they flourished. Soon, there were so many rabbits that they damaged grazing land. By 1887, the government was offering a reward for a way to control the rabbit population. How many rabbits might there have been in 1887? Example 1 provides an estimate.

Mental Math

Solve each inequality.

a. $2x < 5$

b. $-4m + 3 > 14$

c. $9 + 3b \leq 3 - b$

Example 1

Twenty-six rabbits are introduced to another area. Assume that the rabbit population doubles every year. How many rabbits would there be after 28 years?

Solution Since the population doubles every year, in 28 years it will double 28 times. The number of rabbits will be

$$26 \cdot \underbrace{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}_{28 \text{ factors}}.$$

To evaluate this expression on a calculator, rewrite it as $26 \cdot 2^{28}$. Use the y^x or ^ key. There would be 6,979,321,856, or about 7 billion rabbits after 28 years.

A pet rabbit's diet should be made up of good quality pellets, fresh hay (alfalfa, timothy, or oat), water, and fresh vegetables.

Source: House Rabbit Society

What Is Exponential Growth?

The rabbit population in Example 1 is said to grow exponentially. In **exponential growth,** the original amount is repeatedly *multiplied* by a positive number called the **growth factor.**

Growth Model for Powering

If a quantity is multiplied by a positive number g (the growth factor) in each of x time periods, then, after the x periods, the quantity will be multiplied by g^x.

In Example 1, the population doubles (is multiplied by 2) every year, so $g = 2$. There are 28 time periods, so $x = 28$. The original number of rabbits, 26, is multiplied by g^x, or 2^{28}. So the population y of rabbits after x years is given by the formula $y = 26 \cdot 2^x$.

In general, if the amount at the beginning of the growth period is b, the growth factor is g, and y is the amount after x time periods, then $y = b \cdot g^x$. We call this the **exponential growth equation.**

The compound interest formula $A = P(1 + r)^t$ is another example of an exponential growth equation. Suppose \$5,000 is invested at an annual yield of 4%. Using the variable names of the exponential growth equation, $b = 5{,}000$ and $g = 1.04$. So y, the value of the investment after x years, is given by $y = 5{,}000 \cdot 1.04^x$.

If the money is kept invested for 11 years, then $x = 11$, and the total amount will be $5{,}000 \cdot 1.04^{11}$, which is \$7,697.27.

What Happens If the Exponent Is Zero?

In the exponential growth equation $y = b \cdot g^x$, x can be any real number. Consider the situation when $x = 0$. In 0 time periods no time has elapsed. The starting amount b has not grown at all and so it remains the same. It can remain the same only if it is multiplied by 1. This means that $g^0 = 1$, regardless of the value of the growth factor g. This property applies also when g is a negative number.

Zero Exponent Property

If x is any nonzero real number, then $x^0 = 1$.

In words, the zero power of any nonzero number equals 1. For example, $4^0 = 1$, $(-2)^0 = 1$, and $\left(\frac{5}{7}\right)^0 = 1$. The zero power of 0, which would be written 0^0, is undefined.

What Does a Graph of Exponential Growth Look Like?

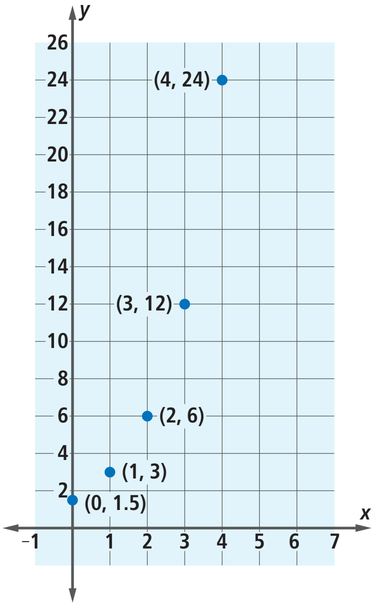
An equation of the form $y = b \cdot g^x$, where g is a number greater than 1, can describe exponential growth. Graphs of such equations are not lines. They are *exponential growth curves.*

Example 2

Graph the equation $y = 1.5 \cdot 2^x$, when x is 0, 1, 2, 3, and 4.

Solution Substitute $x = 0, 1, 2, 3$, and 4 into the formula $y = 1.5 \cdot 2^x$. Below we show the computation and the results listed as (x, y) pairs.

Computation	(x, y)
$1.5 \cdot 2^0 = 1.5 \cdot 1 = 1.5$	(0, 1.5)
$1.5 \cdot 2^1 = 1.5 \cdot 2 = 3$	(1, 3)
$1.5 \cdot 2^2 = 1.5 \cdot 4 = 6$	(2, 6)
$1.5 \cdot 2^3 = 1.5 \cdot 8 = 12$	(3, 12)
$1.5 \cdot 2^4 = 1.5 \cdot 16 = 24$	(4, 24)

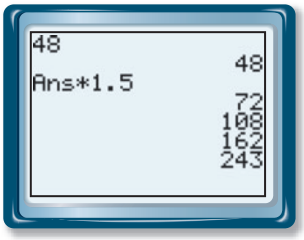


Notice that the y-intercept of the graph in Example 2 is 1.5. Also, the graph does not have a constant rate of change. When something grows exponentially, its rate of change is continually increasing.

Now/Next Method

Example 2 showed how to compute y-values. In an exponential growth equation, $y = b \cdot g^x$, you can compute the y-values by substituting for x. However, that method might not always be the fastest.

Numbers in an exponential growth pattern can be displayed on the homescreen of a graphing calculator using the Now/Next method.



Step 1 Type the starting value 48 (the Now) and press ENTER.

Step 2 Multiply by the growth factor 1.5. The calculator represents the Now by Ans (for "answer"). The calculator at the right is now programmed for Now/Next. By pressing ENTER, it automatically performs Now • 1.5 to get the Next term.

Try this on a calculator. Notice how quickly the table values can be displayed by repeatedly pressing ENTER.

STOP **QY**

▶ QY

A Now/Next table is shown below.

x	y	
0	4	
1	?	$\cdot 2.5$
2	?	$\cdot 2.5$
3	?	$\cdot 2.5$

a. What are the three numbers that fill in the missing cells if the growth factor is 2.5?

b. Write an equation of the form $y = b \cdot g^x$ for the ordered pairs (x, y).

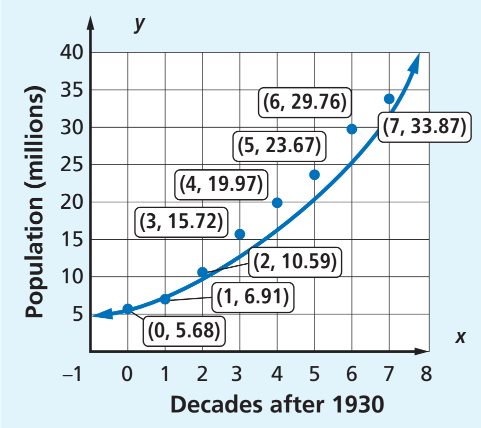
Exponential Population Growth

Other than money calculated using compound interest, few things in the real world grow exactly exponentially. However, exponential growth curves can be used to approximate changes in population. For example, consider the population of California from 1930 to 2000 shown in the table on the next page.

California's population is graphed below. The curve is the graph of the exponential growth model $y = 5.68(1.29)^x$, where y is the population (in millions) x decades after 1930. The growth factor 1.29 was chosen because it is an "average" growth factor for California for the six decades shown. The points lie quite close to the curve, indicating that California's population since 1930 has grown about 29% per decade.

Year	Population
1930	5,677,251
1940	6,907,387
1950	10,586,223
1960	15,717,204
1970	19,971,069
1980	23,667,764
1990	29,760,021
2000	33,871,648

Source: U.S. Census Bureau



Questions

COVERING THE IDEAS

1. A round goby is a bottom-dwelling fish native to Eastern Europe. In 1995, it was found in the Great Lakes, where it is expected to be harmful to already existing habitats. The round goby is known to spawn several times during the summer, and biologists are tracking the growth of its population. Suppose 11 round gobies were in the Great Lakes in 1995 and that their population triples in size each year.
   a. Write an exponential growth equation to describe this situation.
   b. How many round gobies were in the Great Lakes after 2 years?
   c. How many round gobies were in the Great Lakes in 2000?
   d. How many round gobies will there be in the Great Lakes in 2025?

Round gobies have a well-developed sensory system that allows them to feed in complete darkness.

Source: University of Wisconsin Sea Grant

2. Copy the table at the right and complete it to make a Now/Next table for $y = 50 \cdot 1.2^x$.

x	y
0	?
1	?
2	?
3	?

3. Suppose that $3,000 is invested in an account with a 4.5% annual yield.
   a. What is the growth factor?
   b. What will be the value of the account after two years?

4. **True or False** An amount is multiplied by 10 in each of 12 time periods. After the 12 time periods, the original amount will be multiplied by 120.

In 5 and 6, evaluate the expression when $x = 17$, $y = 1.05$, and $z = 1.04$.

5. **a.** x^0 **b.** $x \cdot x^0$ **c.** $(x + x)^0$

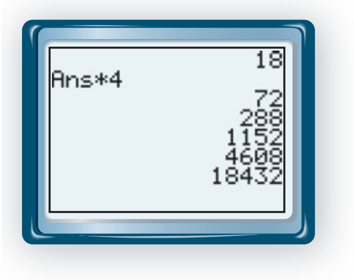
6. **a.** y^0 **b.** $y^0 - z^0$ **c.** $(z - y)^0$

7. Explain why $(-5)^0$ and -5^0 are not equal.

8. Explain how $g^0 = 1$ applies to exponential growth.

9. Let $y = 0.5 \cdot 2^x$.
   a. Make a table for $x = 0, 1, 2, 3, 4$.
   b. Graph the values from Part a.

10. Write an equation of the form $y = b \cdot g^x$ to fit the numbers in the calculator display at the right.



11. Consider the exponential growth model $y = 5.68(1.29)^x$ for California's population on page 407.
   a. How much does the model's value for 1990 deviate from the actual population?
   b. What does the model predict for the population of California in 2020?

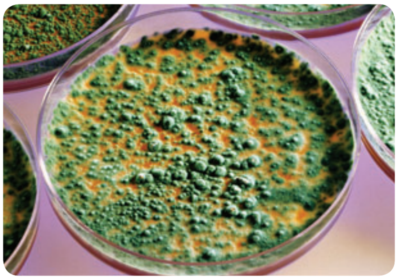
APPLYING THE MATHEMATICS

12. Alexander Fleming discovered penicillin by observing mold growing on Petri dishes. Suppose you are biochemist studying a type of mold that has grown from 3,000 spores to 192,000 spores in one hour. You record the following information.

Time Intervals from Now	Time (min)	Number of Mold Spores
0	0	3,000
1	20	12,000
2	40	48,000
3	60	192,000

• ?
• ?
• ?

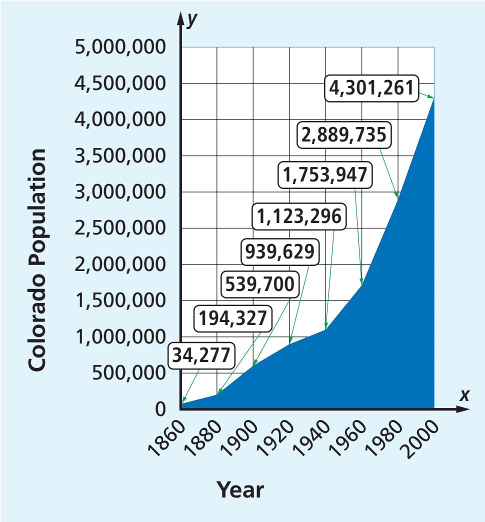
   a. What is the growth factor as the time interval increases by 1?
   b. How many mold spores would there be after 2 hours? Explain how you found your answer.
   c. If this growth rate continues, how many mold spores will there be after x hours have passed?



Before tossing away some old Petri dishes in 1928, Alexander Fleming accidentally discovered a blue mold growing on the culture of a harmful bacteria.

Source: San Jose State University

13. The equation $y = 34{,}277 \cdot 1.04^x$ can be used to model the population y of Colorado x years after 1860. This graph shows Colorado's actual population.



Source: Bureau of the Census

a. What do 1.04 and 34,277 in the equation represent?

b. What does the equation predict for the population of Colorado in 1960? By how much did the actual population deviate from the prediction?

c. Use the model to predict the population of Colorado in 2030.

14. Gossip can be spread quickly in a school. Suppose one person begins spreading the gossip by telling 2 friends. Each friend then tells 2 of his or her different friends. Each person who hears the gossip continues to tell 2 more different friends.

three girls sharing a secret

a. Complete the table below showing the number of new friends and total number of people who have heard the gossip if the pattern continues.

Stage of Gossip	0	1	2	3	4	5	6	7	8	9	10
New Friends Informed	1	2	4	?	?	?	?	?	?	?	?
Total Number of Friends Informed	1	3	7	?	?	?	?	?	?	?	?

b. Make ordered pairs (x, y) from the first two rows of the table and plot the points on a graph.

c. Make ordered pairs (x, y) from the first and third rows of the table and plot the points on a graph.

d. How many stages of gossip will be needed before 800 people in all hear the gossip?

15. a. Graph $y = 2.5^x$ for $x = 0, 1, 2, 3, 4$, and 5.

b. Calculate the rate of change on the graph from $x = 0$ to $x = 1$.

c. Calculate the rate of change on the graph from $x = 4$ to $x = 5$.

d. What do the answers to Parts b and c tell you about this graph?

REVIEW

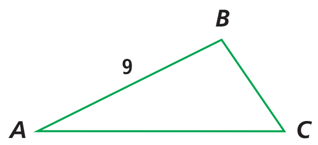
16. Ashley deposits $3,400 in a savings account with an annual yield of 5%. What will be the total amount of money in the account after 8 years? **(Lesson 7-1)**

17. a. Suppose a person's birthday is in July. What is the probability that it is on July 4th?

b. Suppose a person's birthday is in March. What is the probability that it is before the 10th? **(Lesson 5-6)**

18. At one time, the exchange rate for Swiss francs per U.S. dollar was 1.305, meaning that 1 dollar would buy 1.305 francs. With this exchange rate, how many U.S. dollars would 1 Swiss franc buy? **(Lesson 5-4)**

19. In the triangle at the right, side $\overline{AC}$ is 20% longer than side $\overline{AB}$ and side $\overline{BC}$ is 45% shorter than side $\overline{AC}$. If $AB = 9$, find the perimeter of the triangle. **(Lesson 4-1)**



20. **Multiple Choice** Which expression does *not* equal $-(3x - 3y)$? **(Lessons 2-4, 2-1)**

A $3(y - x)$ B $3y + 3x$ C $-3x + 3y$ D $-3(x - y)$

EXPLORATION

21. This exploration will help to explain why 0^0 is undefined. You will examine values of x^0 and 0^x when x is close to 0.

a. Use your calculator to give values of x^0 for $x = 1, 0.1, 0.01, 0.001$, and so on. What does this suggest for the value of 0^0?

b. Use your calculator to give values of 0^x for $x = 1, 0.1, 0.01, 0.001$, and so on. What does this suggest for the value of 0^0?

c. What does your calculator display when you try to evaluate $0^{0?}$ Why do you think it gives that display?

QY ANSWERS

a. 10; 25; 62.5

b. $y = 4 \cdot 2.5^x$