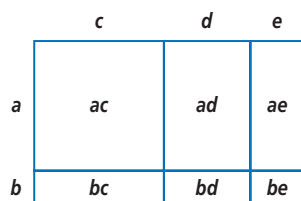# Proofs of the Pythagorean Theorem

▶ **BIG IDEA** There are many ways to deduce the Pythagorean Theorem using algebra.

In this book you have seen how areas of rectangles can picture various forms of the Distributive Property. The idea is to calculate the area of a figure in two different ways. Here is a picture of $(a + b)(c + d + e) = ac + ad + ae + bc + bd + be$.

Mental Math

Tell whether the three numbers can be lengths of sides in a triangle.

a. 5, 13, 5

b. 2, 14, 15

c. 1, 2, 3

	c	d	e
a	ac	ad	ae
b	bc	bd	be

You could also say that this reasoning uses area to prove that $(a + b)(c + d + e) = ac + ad + ae + bc + bd + be$.

We close this book by showing how areas of figures provide proofs of the most famous theorem in geometry, the Pythagorean Theorem. If a and b are the lengths of the legs of a right triangle, and c is the length of its hypotenuse, then $a^2 + b^2 = c^2$.

For these proofs, you need to think of a^2, b^2, and c^2 as the areas of squares whose sides are a, b, and c. This is the form in which the theorem was discovered over 2,500 years ago in many different parts of the world.

These proofs assume that you are familiar with the definitions and area formulas for some common figures. They are:

square: $A = s^2$ 　　　　 rectangle: $A = \ell w$

right triangle: $A = \frac{1}{2}ab$ 　　　 triangle: $A = \frac{1}{2}bh$

trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$

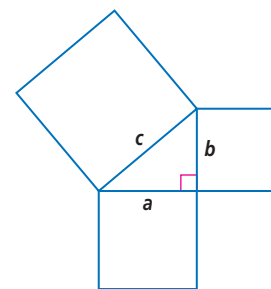The proofs also use the properties of real numbers that you have seen in this course.

Bhaskara's Proof

Bhaskara's proof is a generalization of the idea that you saw in Lesson 8-6. Begin with right triangle DHK with side lengths a, b, and c. Make three copies of the triangle and place them so that quadrilateral $DEFG$ is a square, as shown at the right. In $\triangle DHK$, $\angle DHK$ and $\angle DKH$ are complementary. Since corresponding parts of congruent triangles are congruent, $m\angle GKJ = m\angle DHK$. So $m\angle DKH + m\angle GKJ = 90°$. Thus, $m\angle JKH = 180° - 90° = 90°$. Likewise the other three angles of $HIJK$ are right angles. So, the inside quadrilateral $HIJK$ formed by the four hypotenuses has four right angles and four sides of length c, so it is also a square.

Let A be the area of quadrilateral $DEFG$. Each side of quadrilateral $DEFG$ has length $a + b$. So $A = (a + b)^2$. But the area of $DEFG$ can also be found by adding up the areas of the four right triangles $\left(4 \cdot \frac{1}{2}ab\right)$ and the square in the middle (c^2). So $A = 4 \cdot \frac{1}{2}ab + c^2$. The two values of A must be equal.

$$(a + b)^2 = 4 \cdot \frac{1}{2}ab + c^2$$

Now use the formula for the square of a binomial on the left side and simplify the right side.
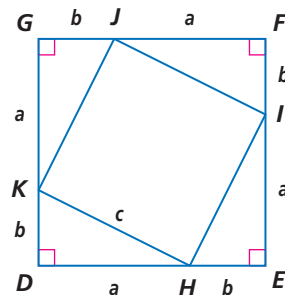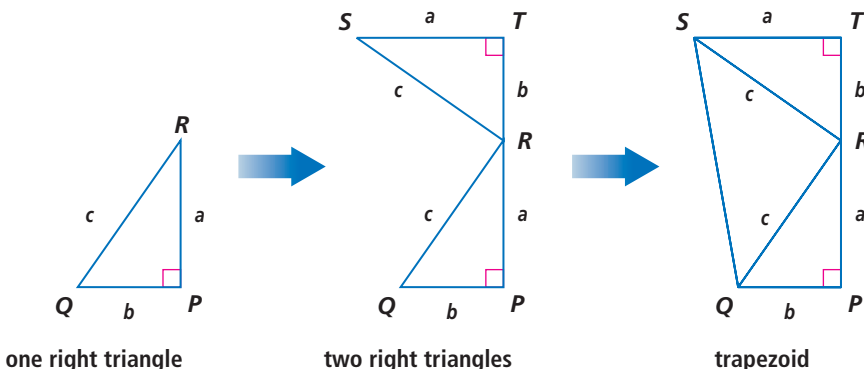
$$a^2 + 2ab + b^2 = 2ab + c^2$$

Add $-2ab$ to each side of the equation.

$$a^2 + b^2 = c^2$$

This is the Pythagorean Theorem.

President Garfield's Proof

This proof of the Pythagorean Theorem was discovered by James Garfield in 1876 while he was a member of the U.S. House of Representatives. Five years later he became the 20th President of the United States.

one right triangle   two right triangles   trapezoid

Major General James Garfield won the 1880 Presidential election by only 10,000 popular votes, defeating Gen. Winfield Scott Hancock.

Source: The White House

President Garfield's proof uses half the figure of the preceding proof. Begin with right triangle PQR as shown on the previous page. With one copy of $\triangle PQR$, create a trapezoid $PQST$ with bases a and b and height $a + b$. The area of any trapezoid is $\frac{1}{2}h(b_1 + b_2)$. Here the height $h = a + b$.

$$\text{Area of } PQST = \frac{1}{2}(a + b)(a + b)$$

But the area of $PQST$ is also the sum of the areas of three right triangles: PQR, RST, and QRS. Look at $\triangle QRS$. Because the sum of the measures of the angles of a triangle is $180°$, $m\angle QRP + m\angle RQP = 90°$. Consequently, $m\angle QRP + m\angle SRT = 90°$. This means that $\angle QRS$ is a right angle and so $\triangle QRS$ is a right triangle. Now add the areas of the three right triangles.

$$\text{Area of } PQST = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$$

The area of the entire trapezoid must be the same regardless of how it is calculated.

$$\frac{1}{2}(a + b)(a + b) = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2.$$

Now multiply both sides of the equation by 2.

$$(a + b)(a + b) = ab + ab + c^2$$

Multiply the binomials on the left side and collect terms on the right side.

$$a^2 + 2ab + b^2 = 2ab + c^2$$

Subtract $2ab$ from each side of the equation and the result is the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

Other Proofs

It takes only one valid proof of a theorem to make it true. Yet in mathematics you will often see more than one proof of a statement, just as you often see more than one way to solve a problem. Alternate methods can help you to understand better how the various parts of mathematics are related. In this lesson, you have seen how areas of triangles, trapezoids, and squares are put together with binomials to prove a statement about the lengths of the three sides of any right triangle. In your next course, likely to be more concerned with geometry than this one, you will see how this theorem is related to similar triangles. Later you will learn how important this theorem is in the study of trigonometry. The algebra you have learned this year is fundamental in these and every other area of mathematics.
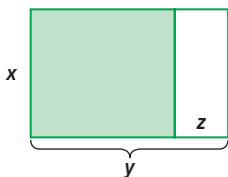
Questions

COVERING THE IDEAS

1. Picture the property that for all positive numbers a, b, and c, $a(b + c) = ab + ac$.

2. Picture the property that for all positive numbers a and b, $a(a + b) = a^2 + ab$.

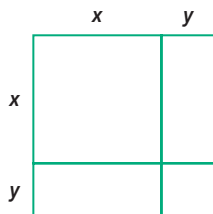**In 3 and 4, refer to Bhaskara's proof of the Pythagorean Theorem.**

3. **a.** Draw the figure of Bhaskara's proof when $a = 6$ and $b = 2$.
   b. What is the area of $DEFG$?
   c. Explain how to get the area of $HIJK$.
   d. What is the value of c?

4. $DH = a$ and $DK = b$ in the figure of Bhaskara's proof.
   a. What is the length of EF?
   b. What is the area of $EFGD$?
   c. What is the area of triangle IJF?
   d. What is the area of $HIJK$?
   e. What is the length of HK in terms of a and b?

5. **a.** Draw a trapezoid whose bases have lengths 1 in. and 2 in., and whose height is 1 in.
   b. What is the area of this trapezoid?

6. Draw a trapezoid with bases b_1 and b_2 and height h. Explain why the area of this trapezoid is $\frac{1}{2}hb_1 + \frac{1}{2}hb_2$.

7. Refer to President Garfield's proof of the Pythagorean Theorem. Let $a = 28$ and $b = 45$.
   a. Find the area of trapezoid $PQST$.
   b. Explain how to get the area of $\triangle RQS$.
   c. What is the value of c?
   d. Does the value of c agree with what you would get using the Pythagorean Theorem?
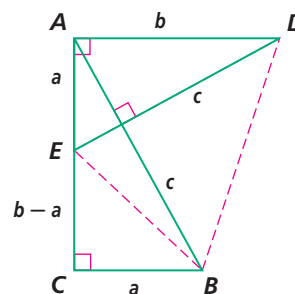
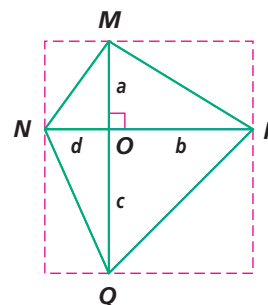## APPLYING THE MATHEMATICS

8. **a.** Find two expressions for the shaded region in the figure below.
   b. What property is illustrated by the answer to Part a?

9. The square below has been split into two smaller squares and two rectangles. What property is pictured?

10. Quadrilateral *MNQP* at the right has perpendicular diagonals. Add the areas of the four triangles to show that the area of *MNQP* is one-half the product of the lengths of its diagonals.

11. A proof of the Pythagorean Theorem published by W.J. Dobbs in 1916 uses the figure at the right. $\triangle ACB$ and $\triangle DAE$ are right triangles, and $AC = b$, $BC = a$, and $AB = c$. Complete each step to show the proof.
   a. What is the length of $\overline{EB}$ in terms of a and b?
   b. Find the area of $\triangle EBC$.
   c. Find the area of *AEBD,* a quadrilateral with perpendicular diagonals, using the formula from Question 10.
   d. Add the areas in Parts b and c to find an expression for the area of *ACBD.*
   e. Use the formula for the area of a trapezoid to express the area of *ACBD.*
   f. Set the formulas from Parts d and e equal to each other to show that $c^2 = a^2 + b^2$.

REVIEW

12. Prove that if the last three digits of a 4-digit number form a number divisible by 8, then the entire number is divisible by 8. **(Lesson 13-6)**

13. **Multiple Choice** Consider the following statement. If the cost of 5 pounds of ice is $2.15, then at the same rate, the cost of 32 ounces of ice is 86 cents. **(Lessons 13-2, 13-1, 5-9)**
   A The statement and its converse are both true.
   B The statement and its converse are both false.
   C The statement is true but its converse is false.
   D The statement is false but its converse is true.

14. a. Find a value of c to complete the square for $4x^2 - 12x + c$.
   b. Use your answer to Part a to solve the equation $4x^2 - 2x = -9 + 10x$. **(Lessons 12-3, 12-2)**

15. Leonardo and Miranda are at an amusement park and are trying to decide in which order they want to ride the 9 roller coasters in the park. (**Lesson 11-7**)

   a. How many different orders can they ride all 9 roller coasters if they ride each coaster one time?

   b. If they only have time to ride six of the roller coasters, how many ways can they do this?

16. Consider the following number puzzle. (**Lessons 8-6, 2-3**)

   Step 1 Choose any whole number.

   Step 2 Square that number.

   Step 3 Add 4 times your original number.

   Step 4 Add 4 to the result of Step 3.

   Step 5 Take the square root of the result of Step 4.

   Step 6 Subtract your original number.

   a. Follow the number puzzle with any whole number. What is your result?

   b. Let x represent the number chosen. Write a simplified expression to represent each step of the puzzle and to show why your result will always be what you found in Part a.

17. A piece of landscaping machinery is valued at $15,000. If the machinery depreciates at a constant rate of 8% per year, what will be its value in 6 years? (**Lesson 7-3**)

Out of the 710 roller coasters in North America, 628 are in the United States.

Source: Roller Coaster Database

EXPLORATION

18. A different kind of proof of the Pythagorean Theorem is called a *dissection proof*. Dissection means cutting the squares on the legs of the right triangle shown on page 823 into pieces and then rearranging these pieces together to fill up the square on the hypotenuse. Find such a proof in a book or on the Internet and explain why it works.