Lesson

# 13-7 Rational Numbers and Irrational Numbers

**Vocabulary**

irrational number

**BIG IDEA** The Distributive Property enables you to prove that repeating decimals represent rational numbers, and divisibility properties enable you to prove square roots of certain integers are irrational.

A number that can be represented by a decimal is a real number. All the real numbers are either rational or irrational. In this lesson, you will see how we know that some numbers are not rational numbers.

**Mental Math**

**A school enrolled 120 freshmen, 110 sophomores, 125 juniors, and 100 seniors. What is the probability that**

**a.** a student at the school is a sophomore?

**b.** a student at the school is a junior or senior?

**c.** a student is not a junior?

## What Are Rational Numbers?

Recall that a *simple fraction* is a fraction with integers in its numerator and denominator. For example, $\frac{2}{3}$, $\frac{5{,}488}{212}$, $\frac{10}{5}$, $\frac{-7}{-2}$, and $\frac{-43}{1}$ are simple fractions.

Some numbers are not simple fractions, but are *equal* to simple fractions. Any mixed number equals a simple fraction. For example, $3\frac{2}{7} = \frac{23}{7}$. Also, any integer equals a simple fraction. For example, $-10 = \frac{-10}{1}$. And any finite decimal equals a simple fraction. For example, $3.078 = 3\frac{78}{1{,}000} = \frac{3{,}078}{1{,}000}$. All these numbers are *rational numbers.* A *rational number* is a number that can be expressed as a simple fraction.

All repeating decimals are also rational numbers. The Example below shows how to find a simple fraction that equals a given repeating decimal.

### Example

**Show that $18.4\overline{23}$ is a rational number.**

**Solution** Let $x = 18.4\overline{23}$. Multiply both sides by $10^n$, where $n$ is the number of digits in the repetend $\overline{23}$. Here there are two digits in the repetend, so we multiply by $10^2$, or 100.

$$x = 18.4\overline{23}$$
$$100x = 1{,}842.3\overline{23}$$

Subtract the top equation from the bottom equation. The key idea here is that the result is no longer an infinite repeating decimal; in this case, after the first decimal place the repeating parts subtract to zero.

$$\begin{array}{rl} 100x = & 1{,}842.3\overline{23} \\ -\ x = & 18.4\overline{23} \\ \hline 99x = & 1{,}823.900 \end{array}$$

Divide both sides by 99.

$$x = \frac{1{,}823.9}{99} = \frac{18{,}239}{990}$$

Since $x = \frac{18{,}239}{990}$, $x$ is a rational number.

STOP QY1

Rational numbers have interesting properties. They can be added, subtracted, multiplied, and divided; and they give answers that are also rational numbers.

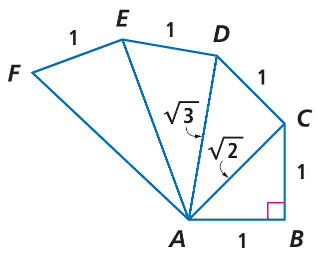
**QY1**

**a.** Divide 18,239 by 990 to check the result of the Example.

**b.** Write $4.1\overline{23}$ as a simple fraction.

## What Are Irrational Numbers?

The ancient Greeks seem to have been the first to discover that there are numbers that are not rational numbers. They called them *irrational*. An **irrational number** is a real number that is not a rational number. Some of the most commonly found irrational numbers in mathematics are the square roots of integers that are not perfect squares. That is, numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, and so on, are irrational. But notice that $\sqrt{4}$ is rational not irrational, because $\sqrt{4} = 2 = \frac{2}{1}$. All these numbers can arise from situations involving right triangles. Examine the array of right triangles shown below.




$\triangle ABC$ is a right triangle with legs of 1 and 1. Use the Pythagorean Theorem to find the side lengths $AC$ and $AD$.

$$AC^2 = AB^2 + BC^2$$
$$AC^2 = 1 + 1$$
$$AC^2 = 2$$
$$AC = \sqrt{2}$$

$\triangle ACD$ is drawn with leg $\overline{AC}$, and another leg $CD = 1$. Use the Pythagorean Theorem.

$$AD^2 = AC^2 + CD^2$$
$$AD^2 = \left(\sqrt{2}\right)^2 + 1^2$$
$$AD^2 = 2 + 1$$
$$AD^2 = 3$$
$$AD = \sqrt{3}$$

STOP QY2

> **QY2**
>
> Find *AE* and *AF* in the figure on the previous page.

## How Do We Know That Certain Numbers Are Irrational?

If you evaluate $\sqrt{2}$ on a calculator, you will see a decimal approximation. One calculator shows 1.414213562. Another shows 1.41421356237. No matter how many decimal places the calculator shows, it is not enough to show the entire decimal because the decimal for $\sqrt{2}$ is infinite and does not repeat.

Is it possible the decimal could repeat after 1,000 decimal places, or after 1 million or 1 billion decimal places? How do we know that the decimal does not repeat? The answer is that we can *prove* the decimal does not repeat, because we can prove that $\sqrt{2}$ is not a rational number. The proof uses some of the ideas of divisibility you have seen in Lessons 13-5 and 13-6. In particular, we use the fact that if a number is even, then its square is divisible by 4. The idea of the proof is to show that there is no simple fraction in lowest terms equal to $\sqrt{2}$.

Here is the proof: Suppose $\sqrt{2}$ is rational. Then there would be two whole numbers $a$ and $b$ with $\sqrt{2} = \frac{a}{b}$ (with the fraction in lowest terms). Then, multiply each side of this equality by itself.

| | |
|---|---|
| $\sqrt{2} \cdot \sqrt{2} = \frac{a}{b} \cdot \frac{a}{b}$ | Multiplication Property of Equality |
| $2 = \frac{a^2}{b^2}$ | Definition of square root; Multiplication of Fractions |
| $2b^2 = a^2$ | Multiply both sides by $b^2$. |

So if you could find two numbers $a$ and $b$ with twice the square of $b$ equal to the square of $a$, then $\sqrt{2}$ would be a rational number. (You can come close. $7^2$ or 49 is one less than twice $5^2$ or 25.)

Notice that since $a^2$ would be twice an integer, $a^2$ would be even. This means that $a$ would be even (because the square of an odd number is odd). Because $a$ would be even, there would be an integer $m$ with $a = 2m$. This means that $a^2 = (2m)^2 = 4m^2$. Substitute in the bottom equation.

$$2b^2 = 4m^2 \quad \text{Substitute } 4m^2 \text{ for } a^2.$$
$$b^2 = 2m^2 \quad \text{Divide both sides by 2.}$$

Now we repeat the argument used above. Because $b^2$ would be twice an integer, $b^2$ would be even. This means that $b$ would have to be even. And because $a$ and $b$ would both be even, the fraction $\frac{a}{b}$ could not be in lowest terms. This shows that what we supposed at the beginning of this proof is not true.

For this reason, it is impossible to find two whole numbers $a$ and $b$ with $\sqrt{2} = \frac{a}{b}$ and with the fraction in lowest terms. Since any simple fraction can be put in lowest terms, it is impossible to find any two whole numbers $a$ and $b$ with $\sqrt{2} = \frac{a}{b}$.

Arguments like this one can be used to prove the following theorem.

**Irrationality of $\sqrt{n}$ Theorem**

If $n$ is an integer that is not a perfect square, then $\sqrt{n}$ is irrational.

Johann Lambert

Today, we now know that there are many irrational numbers. For example, every number that has a decimal expansion that does not end or repeat is irrational. Among the irrational numbers is the famous number π. But the argument to show that π is irrational is far more difficult than the argument used above for some square roots of integers. It requires advanced mathematics, and was first done by the German mathematician Johann Lambert in 1767, more than 2,000 years after the Greeks had first discovered that some numbers were irrational.

There is a practical reason for knowing whether a number is rational or irrational. When a number is rational, arithmetic can be done with it rather easily because it can be represented as a simple fraction. Just work as you do with fractions. But if a number is irrational, then it is generally more difficult to do arithmetic with it. Rather than use its infinite decimal, we often leave it alone and just write π or $\sqrt{3}$, for example.

## Questions

### COVERING THE IDEAS

**In 1–3, find an example of each.**

1. a simple fraction
2. a fraction that is not a simple fraction
3. a rational number

**In 4–6, write the number as a simple fraction.**

**4.** $98.6$ **5.** $0.\overline{84}$ **6.** $14.0\overline{327}$

**7.** **Multiple Choice** Which *cannot* stand for a rational number?

A a terminating decimal

B a simple fraction

C a repeating decimal

D an infinite nonrepeating decimal

**8.** Refer to the proof that $\sqrt{2}$ is irrational.

**a.** If $\sqrt{2}$ were rational, what would $\sqrt{2}$ have to equal?

**b.** **True or False** If the square of an integer is even, then the integer is even.

**c.** **True or False** If an integer is divisible by 2, then its square is divisible by 4.

**d.** In the proof, what characteristic of both $a$ and $b$ shows that the fraction $\frac{a}{b}$ is not in lowest terms?

**In 9–11, tell whether the number is a rational or an irrational number.**

**9.** $\pi$ **10.** $-220$ **11.** $\sqrt{121}$

**12.** Draw a segment whose length is $\sqrt{5}$ units.

**13.** Draw a square whose diagonal has length $\sqrt{338}$ cm.

## APPLYING THE MATHEMATICS

**14.** Is 0 a rational number? Why or why not?

**15.** Is it possible for two irrational numbers to have a sum that is a rational number? Explain why or why not.

**16.** Using the proof in this lesson as a guide, prove that $\sqrt{3}$ is irrational.

**17.** **a.** Draw a segment whose length is $1 + \sqrt{3}$ units.

**b.** Is $1 + \sqrt{3}$ rational or irrational?

**18.** If a circular table has a diameter of 4 cm, is its circumference rational or irrational?

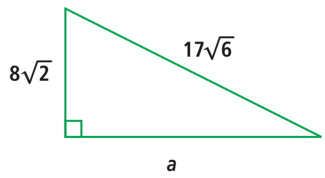
**19.** A diagonal of a square has a length of 42 cm. Find the perimeter of the square. Is the perimeter rational or irrational?

**20.** Determine whether the solutions to the equation $x^2 - 8x - 1 = 0$ are rational or irrational.

**21.** Refer to the right triangle at the right.

**a.** Find the exact value of $a$.

**b.** Is $a$ rational or irrational?



## REVIEW

**In 22–25, consider the spreadsheet below, which was used to compute the value of $f(x) = 3x^3 + 5x^2 - 2x$ for integer values of $x$ from −5 to 5.** (Lessons 12-7, 12-6, 11-4)

**22.** Complete the spreadsheet.

**23.** Graph the function $f$ for $-5 \le x \le 5$.

**24.** Identify all $x$-intercepts.

**25.** Rewrite the equation in factored form.

| ◇ | A | B |
|---|---|---|
| 1 | x | f(x) |
| 2 | -5 | |
| 3 | -4 | -104 |
| 4 | -3 | |
| 5 | -2 | |
| 6 | -1 | 4 |
| 7 | 0 | 0 |
| 8 | 1 | |
| 9 | 2 | 40 |
| 10 | 3 | 120 |
| 11 | 4 | |
| 12 | 5 | 490 |

**26.** Solve $x^3 - 10x^2 + 16x = 0$. (Lessons 12-6, 12-5, 11-4)

**27.** Suppose $20x^2 + 9xy - 20y^2 = (ax + b)(cx + d)$. (Lesson 12-5)

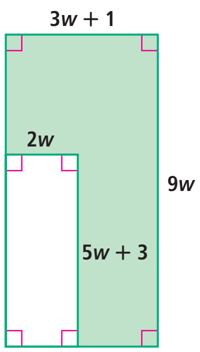
**a.** Find the value of $ad + bc$.

**b.** Find $b$, $c$, and $d$ if $a = 5$.

**28.** Find two numbers whose sum is 30 and whose product is 176. (Lessons 12-4, 11-6, 10-2)

**29.** Expand the expression $\left(\sqrt{25} - \sqrt{x^2}\right)\left(\sqrt{25} + \sqrt{x^2}\right)$. (Lessons 11-6, 8-6)

**30.** Calculate the area of the shaded region. (Lesson 11-3)




**31.** With a stopwatch and a stone, you can estimate the depth of a well. If the stone takes 2.1 seconds to reach the bottom, how deep is the well? Use Galileo's equation, $d = 16t^2$. (Lesson 9-1)

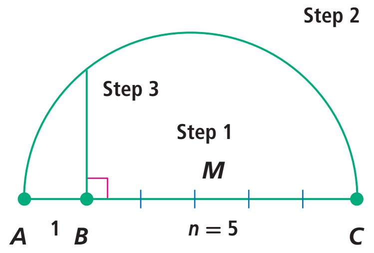
**Dug wells typically used for drinking water are 10 to 30 feet deep.**

Source: U.S. Environmental Protection Agency

## EXPLORATION

32. Because $2 \cdot 5^2$ is one away from $7^2$, 2 is close to $\frac{7^2}{5^2}$. That means that $\sqrt{2}$ is close to $\frac{7}{5}$, or 1.4. Find two other numbers $c$ and $d$ such that $2 \cdot c^2$ is one away from $d^2$. (*Hint:* There is a pair of such numbers with both of them greater than 2 less than 20.) What rational number estimate does that pair give for $\sqrt{2}$?

33. Shown here is a different way to draw a segment with length $\sqrt{n}$ from the one given in the lesson.




**Step 1** Draw a segment $\overline{AB}$ with length 1, and then next to it, a segment $BC$ with length $n$. (In the drawing here, $n = 5$.)

**Step 2** Find the midpoint $M$ of segment $\overline{AC}$. Draw the circle with center $M$ that contains $A$ and $C$. ($\overline{AC}$ will be a diameter of this circle.)

**Step 3** Draw a segment perpendicular to $\overline{AC}$ from $B$ to the circle. This segment has length $\sqrt{n}$. (In our drawing it should have length $\sqrt{5}$.)

a. Try this algorithm to draw a segment with length $\sqrt{7}$.

b. Measure the segment you find.

c. How close is its length to $\sqrt{7}$?

**QY ANSWERS**

**1a.** $18{,}239 \div 990 = 18.\overline{423}$

**b.** $\frac{1{,}373}{333}$

**2.** $AE = 2$, $AF = \sqrt{5}$